Games with Type Indeterminate Players

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Abstract

The Type Indeterminacy model[4] is a theoretical framework that uses some elements of quantum formalism to model the constructive preference perspective suggested by Kahneman and Tversky. In this paper we extend the TI-model from simple to strategic decision-making. In Part I we introduce TI-games by means of an example. We investigate a 2X2 game with the pre-play of a cheap-talk promise game. We show in a numerical example that in the TI-model the promise game can have impact on next following behavior when the standard classical model predicts no impact whatsoever. The TI approach differs from other behavioral approaches in identifying the source of the effect of cheap-talk promises in the intrinsic indeterminacy of the players' type. In Part II, we formulate some basic concepts for the analysis of games with type indeterminate players. We develop the theory in close connection with the standard approach to game of incomplete information à la Harsanyi. We show an equivalence between static games of incomplete information and static TI-games. We extend this equivalence result to dynamic commuting TI-games. Finally, we develop a new solution concept for non-commuting dynamic TI-games. It differs from the Perfect Bayesian equilibrium by the rule used for updating beliefs. The updating rule captures the novelty brought about by Type Indeterminacy namely that in addition to affecting information and payoffs, the action of a player impacts on the profile of types and thus on future actions. We provide an example showing that strategies that form a Perfect Bayesian equilibrium are not part of any Perfect TI-equilibrium.

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[4] The TI-model is due to A. Lambert-Mogiliansky, S. Zamir and H Zwirn (2003, 2009).

1 Introduction

This paper belongs to a very recent and rapidly growing literature where formal tools of Quantum Mechanics are proposed to explain a variety of behavioral anomalies in social sciences and in psychology (see e.g., [14], [5], [6], [8], [11], [16], [12], [27]). To many people it may appear unmotivated or artificial to turn to Quantum mechanics when investigating human behavioral phenomena. However, the founders of QM, including Bohr [1] and Heisenberg [21] early recognized the similarities between the two fields. The similarity stems from the fact that in both fields *the object of investigation cannot (always) be separated from the process of investigation*.[5] Quantum Mechanics and in particular its mathematical formalism was developped to respond to that epistemological challenge (see the introduction in [2] for a enlighting presentation).

The use of quantum formalism in game theory was initiated by Eisert et al. [15] who study how the extension of classical moves to quantum ones can affect the analysis of a game.[6] Another example is La Mura [28] who investigates correlated equilibria with quantum signals in classical games. Whether and when the use of quantum strategies (or strategies using quantum signals) can bring something truly novel to game theory has been discussed in Levine [30] and in Brandenburger [9]. Our approach is different from the so-called quantum game approach. It is based on the idea that players' preferences(types) (rather than the strategies they can choose) can feature non-classical (quantum) properties. This idea is formalized in the Type Indeterminacy (TI) model of decision-making introduced by Lambert-Mogiliansky, Zamir and Zwirn [27]. The TI-model has been proposed as a theoretical framework for modelling the KT(Kahneman–Tversky)–man, i.e., for the "constructive preference perspective".[7] Most of the critics developed by Levine does not apply. Yet, does Type Indeterminacy bring novel elements to Game Theory? Interestingly, this issue is closely related to the so-called hidden variable argument in Physics.[8] At some very general level Game Theory allows for contextual types, but in most applications (e.g., in economics)

[5] in the words of Bohr "the impossibility of a sharp separation between the behavior of atomic object and the interaction with the measuring instruments which serves to define the condition under which the phenomena appears". In psychology investigating a person's emotional state affects the state of the person. In social sciences "revealing" one's preferences in a choice can affect those preferences: "*There is a growing body of evidence that supports an alternative conception according to which preferences are often constructed – not merely revealed – in the elicitation process. These constructions are contingent on the framing of the problem, the method of elicitation, and the context of the choice". [23] p.525.*

[6] From a game-theoretical point of view the approach consists in changing the strategy spaces, and thus the interest of the results lies in the appeal of these changes.

[7] See the quote in footnote 5.

[8] As well-known when no restriction are put on the hidden variables all quantum phenomena can be reproduced. But this is no longer true as soon as one requires some desirable properties e.g., locality or independence between the observer and the observed system see [9] for a discussion in the context of quantum games.

a clear separation between the definition of players (the payoff functions) and the (path of) interaction is maintained. In section 5, we discuss the question in view of generic impossibility theorems and we revisit our lead example of Part 2. We conclude that the TI-game approach does have something novel to contribute to economics. We also want to argue that TI-games contributes to Game Theory by proposing a fruitful and tractable way of formalizing contextual types. .

A main interest with TI-game is that the Type Indeterminacy hypothesis may modify the way we think about games. Indeed, a major implication of the TI-hypothesis is to extend the field of strategic interactions. This is because actions (or more precisely Game Situations (GS)[9]) impact not only on the payoffs of other players but also on the profile of types of the players i.e., who the players *are*. In a TI-model, players do not have a deterministic, exogenously given, type (preferences). The types change along the game together with the Game Situations (which are modelled as measurements of the type). We illustrate this in the example of Part 1 showing that an initially non-cooperative player can be (on average) turned into a rather cooperative one by confronting him with a "tough" player in a pre-play cheap-talk promise game.

The paper is divided into two parts. In part 1 we introduce some central features of TI-games. For this purpose we investigate, in two different settings, a 2x2 game with options, to cooperate and to defect and we refer to it as a Prisoner Dilemma, PD[10]. In the first setting, the players move simultaneously and the game is played once. In the second setting, the simultaneous move PD game is preceded by a promise exchange game. One objective is to illustrate how the TI approach can provide an explanation to why cheap-talk promises matter.[11] There exists a substantial literature on cheap talk communication games (see for instance [24] for a survey). The approach in our paper does *not* belong to the literature on communication games. The cheap-talk promise exchange stage is used to illustrate the possible impact of pre-play interaction. Various behavioral theories have also been proposed to explain the impact of cheap-talk promises. They most often rely on very specific assumptions amounting to adding ad-hoc elements to the utility function (e.g., a moral cost for breaking promises) or emotional communication [18]. Our approach provides an explanation relying on a fundamental structure of the model i.e., the quantum indeterminacy of players' type. Another advantage of our approach is that, as earlier mentioned, the type indeterminacy hypothesis also explains a variety of so called behavioral anomalies such as framing effects, cognitive dissonance [27], the disjunction effect [7] or the inverse fallacy [17].

In part II we develop basic concepts and solutions of maximal information TI-games. From a formal

[9] Game Situations are situations where the players must choose an action in a strategic context. In TI-games, they are modelled as operators.

[10] This is for convenience, as we shall see that the game is not perceived as a true PD by all possible types of a player.

[11] Cheap talk promises are promises that can be broken at no cost.

point of view the one single novelty is that we substitute the Harsanyi type space with a Hilbert space of types. We find that much of conventional game theory can be maintained. In particular the standard equilibrium concepts are applicable with non substantial modifications in static games and games where the player only move once (simple signaling games). The first novel results appear in multi-stage non-commuting games and they are linked to updating. We formulate an updating rule consistent with the algebraic structure of the type space of TI-games. We show that this rule gives new content (beyond the informational one) to pooling respectively separating behavior. In particular, this allows for a player's action to impact upon the future type of his opponent which enlarges the scope of strategic interaction. The intuition is that when the best-reply to an action implies some pooling, some indeterminacy is preserved and the probabilities for next-following choices will be marked by interference effects which are the signature of indeterminacy. In contrast when a best-reply to an action separates between the types of the opponent, interference effect are destroyed. We define a Perfect TI-game equilibrium and demonstrate in an example that the set of Perfect Bayesian equilibrium and Perfect TI-game equilibria do not coincide.

Part I

Type Indeterminacy in Strategic Decision-making: An Example[12]

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2 A TI-model of strategic decision-making

In the TI-model a simple "decision situation" is represented by an *observable*[13] called a *DS*. A decision-maker is represented by his state or *type*. A type is a vector $|t_i\rangle$ in a Hilbert space. The measurement of the observable corresponds to the act of choosing. Its outcome, the chosen item, actualizes an *eigentype*[14] of the observable (or a *superposition*[15] of eigentypes if the measurement is coarse). It is information about the preferences (type) of the agent. For instance consider a TI-model where the agent has preferences over sets of three items, i.e. he can rank any 3 items from the most preferred to the least preferred. Any choice experiment involving three items is associated with six possible eigentypes corresponding to the six rankings of the items. Supppose the agent is initially in a fully indeterminate state.[16] Suppose next that when confronted with the *DS* corresponding to "choosing one element out of the $\{a, b, c\}$", he chooses a. According to the TI-model his type is modified by the act of choosing, i.e., it is projected onto some superposition of the rankings $[a > b > c]$ and $[a > c > b]$. Because the TI-model allows for *DS* that do not commute, the change in type has implications that go beyond classical information updating. For a detailed exposition of the TI-model see [27]. How does this simple scheme change when we are dealing with strategic decision-making?

We denote by GS (for Game Situation) an observable that measures the type of an agent in a strategic situation, i.e., in a situation where the outcome of the choice, in terms of the agent's utility, depends

[12] Part 1 is heavily inspired by a joint work with Jerry Busemeyer from Indiana University in Quantum Interaction 2009.

[13] An observable is a linear operator that operates on the state of a system. .

[14] The eigentypes are the types associate with the eigenvalues of the observable i.e., the possible outcomes of the measurement of the *DS*.

[15] A superposition is a linear combination of the form $\sum \lambda_i |t_i\rangle$; $\sum \lambda_i^2 = 1$ where the t_i are possible states/types of the players.

[16] A state where all of the six potential rankings have positive coefficient of superposition.

on the choice of other agents as well. The interpretation of the outcome of the measurement is that the chosen action is a *best reply* against the opponents' expected action. This interpretation parallels the one in the simple decision context. There, we interpret the chosen item as the *preferred one* in accordance with an underlying assumption of rationality i.e., the agent maximizes his utility (he chooses what he prefers). The notion of revealed preferences (we shall use the term "actualized" rather than revealed[17]) and a fortiori of "actualized best-reply" is problematic however. A main issue here is that a best-reply is a response to an *expected* play. When the expected play involves subjective beliefs there may be a problem as to the observability of the preferences. This is in particular so if subjective beliefs are quantum properties.[18] But in the context of maximal information games (see below for precise definition) probabilities are objective which secures that the actualized best-reply is well-defined.

TI-games are games with type indeterminate players, i.e., games characterized by uncertainty. In particular, players do not know the payoff of other players. The standard (classical) approach to incomplete information in games is due to Harsanyi. It amounts to transforming the game into a game of imperfect information where Nature moves at the beginning of the game and selects, for each player, one among a multiplicity of possible types (payoff functions). A player's own type is his private information. But in a TI-game the players may not even know their *own payoff*. This is true even in TI-game of *maximal information* where the initial types are pure types.[19]

Types and eigentypes We use the term *type* to refer to the *quantum pure state* of a player. A pure type is maximal information about the player i.e., about his payoff function. But because of (intrinsic) indeterminacy, the type is *not* complete information about the payoff function in all games simultaneously not even to the player himself (see [11] for a systematic investigation of (non-classical) indeterminacy with application to Social Sciences).

In a TI-game we also speak about the *eigentypes* of any specific game M, these are *complete information* about the payoff functions *in a specific static game* M. Any eigentype of a player knows his own M-game payoff function but he may not know that of the other players. The eigentypes of a TI-game M

[17] The expression revealed preferences implicitely assumes that the prefrences pre-existed the measurement and that they are uncovered by the measurement. A central feature of the TI-model is precisely to depart from that assumption. Preferences do not pre-exist the measurement. Preferences are in a state of potentials that can be actualized by the measurement.

[18] If subjective beliefs and preferences are quantum properties that do not commute then they cannot be measured simultaneously.

[19] Pure types provide maximal information about a player. But in a context of indeterminacy, there is an irreducible uncertainty. It is impossible to know all the type characteristics of a player with certainty. For a discussion about pure and mixed types see Section 3.2 in [11].

are identified with their payoff function in that game.

So we see that while the Harsanyi approach only uses a single concept, i.e., that of type and it is identified both with the payoff function and with the player. In any specific TI-game M, we must distinguish between the type which is identified with the player and the eigentypes (of M) which are identified with the payoff functions in game M. A helpful analogy is with multiple-selves models (see e.g., [31] and [19]). In multiple-selves models, we are most often dealing with two "levels of identity". These two levels are identified with short-run impulsive selves on the one side and a long-run "rational self" on the other side. In our context we have two levels as well: the level of the player (the type) and the level of the selves (the eigentypes) which are to be viewed as potential incarnations of the player *in a specific game*. In a TI-model a player is described as a superposition of (simultaneous) selves.

A central assumption that we make is that the reasoning leading to the determination of the best-reply is performed at the level of the eigentypes of the game. This key assumption deserves some clarification. What we do is to propose that players are involved in some form of parallel reasoning: all the active (with non-zero coefficient of superposition) eigentypes perform their own strategic thinking. Another way to put it is that we assume that the player is able to reason from different perspectives. Note that this is not as demanding as it may at first appear. Indeed we are used in standard game theory to the assumption that players are able to put themselves "in the skin" of other players to think out how those will play in order to be able to best-respond to that.

As in the basic TI-model, the outcome of the act of choosing, here a *move,* is information about the (actualized) type of the player and the act of choosing *modifies* the type of the player e.g., from some initial superposition it "collapses" onto a specific eigentype of the game under consideration (see next section for concrete examples).

Finally, we assume that each player is an independent system i.e., there is no entanglement between players.[20]

We next investigate an example of a maximal information two-person game. The objective is to introduce some basic features of TI-games in a simple context and to illustrate an equivalence and some distinctions between the Bayes-Harsanyi approach and the TI-approach.

2.1 A single interaction

Consider a 2X2 symmetric game, M, and for concreteness we call the two possible actions cooperate (C) and defect (D) (as in a Prisoner's Dilemma game but as we shall see below for certain types, it is a

[20] In future research we intend to investigate the possibility of entanglement between players.

coordination game) and we define the preference types of game M also called the M-eigentypes as follows:

θ_1 : prefers to cooperate whatever he expects the opponent to do;

θ_2 : prefers to cooperate if he expects the opponent to cooperate with probability $p > q$ (for some $q \leq 1$) otherwise he prefers to defect;

θ_3 : prefers to defect whatever he expects the opponent to do.

An example of these types is in the payoff matrices below where we depict the row player's payoff:

$$\theta_1 : \begin{pmatrix} & C & D \\ C & 10 & 5 \\ D & 0 & 0 \end{pmatrix}, \quad \theta_2 : \begin{pmatrix} & C & D \\ C & 10 & 0 \\ D & 6 & 8 \end{pmatrix}, \quad \theta_2 : \begin{pmatrix} & C & D \\ C & 0 & 0 \\ D & 10 & 5 \end{pmatrix}$$

We shall now proceed to investigate this simultaneous move TI-game. We note immediately that θ_1 and θ_3 are non-strategic while θ_2 is, i.e., his best-reply will depend on what he expects the opponent to do. The initial types are generally not eigentypes of the game under consideration. Let player 1 be described by the superposition

$$|t_1\rangle = \lambda_1 |\theta_1\rangle + \lambda_2 |\theta_2\rangle + \lambda_3 |\theta_3\rangle, \sum \lambda_i^2 = 1. \tag{1}$$

We shall first be interested in the optimal play of player 1 when he interacts with a player 2 of different eigentypes. Suppose he interacts with a player 2 of eigentype θ_1. Using the definitions of the eigentypes θ_i above and (1), we know by Born's rule[21] that with probability $\lambda_1^2 + \lambda_2^2$[22] player 1 plays C (because θ_2's best-reply to θ_1 is C) and he collapses on the (superposed) type $|t_1'\rangle = \frac{\lambda_1}{\sqrt{\lambda_1^2+\lambda_2^2}} |\theta_1\rangle + \frac{\lambda_2}{\sqrt{\lambda_1^2+\lambda_2^2}} |\theta_2\rangle$. With probability λ_3^2 player 1 plays D and collapses on the eigentype θ_3. If instead player 1 interacts with a player 2 of type θ_3 then with probability λ_1^2 he plays C and collapses on the eigentype θ_1 and since $\theta_2' s$ best-reply to θ_3 is *D,* with probability $\lambda_2^2 + \lambda_3^2$ he plays D and collapses on type $|t"_1\rangle = \frac{\lambda_2}{\sqrt{\lambda_3^2+\lambda_2^2}} |\theta_2\rangle + \frac{\lambda_3}{\sqrt{\lambda_3^2+\lambda_2^2}} |\theta_3\rangle$.

We note that the probabilities for player 1's moves depends on the opponent's type and corresponding expected play - as usual. More interesting is that, as a consequence, the *resulting type* of player 1 also depends on the type of the opponent. This is because in a TI-model the act of choice is a measurement that operates on the type and changes it. We interpret the resulting type as the initial type modified by the measurement. In a one-shot context, this is just an interpretation since formally it cannot be

[21] The calculus of probability in Quantum Mechanics is defined by Born's rule according to which the probability for the different eigentypes is given by the square of the coefficients of superposition.

[22] As in the original TI-model, the coefficients of superposition are real numbers and not complex numbers as in Quantum Mechanics. We motivation for that can be found in **??**.

distinguished from a classical informational interpretation where the resulting type captures our revised beliefs about player 1 (when our initial beliefs are given by (1)).

We now consider a case when player 2's type is indeterminate as well:

$$|t_2\rangle = \gamma_1 |\theta_1\rangle + \gamma_2 |\theta_2\rangle + \gamma_3 |\theta_3\rangle, \sum \gamma_i^2 = 1. \tag{2}$$

From the point of view of the eigentypes of a player (the θ_i), the situation can be analyzed as a standard situation of incomplete information. We consider two examples:

Example 1 *Let $\lambda_1^2 \geq q$, implying that the eigentype θ_2 of player 2 cooperates and let $\gamma_1^2 + \gamma_2^2 \geq q$ so the eigentype θ_2 of player 1 cooperates as well.*

Example 2 *Let $\lambda_1^2 \geq q$ so the eigentype θ_2 of player 2 cooperates but now let $\gamma_1^2 + \gamma_2^2 < q$ so here the eigentype θ_2 of player 1 prefers to defect.*

In Example 1 the types θ_1 and θ_2 of both players pool to cooperate. So in particular player 1's resulting type is a superposition of $|\theta_1\rangle$ and $|\theta_2\rangle$ with probability $(\lambda_1^2 + \lambda_2^2)$ and it is the eigentype $|\theta_3\rangle$ with probability λ_3^2. In Example 2, player 1's eigentypes θ_2 and θ_3 pool to defect so player 1's resulting type is a superposition of $|\theta_2\rangle$ and $|\theta_3\rangle$ with probability $\lambda_2^2 + \lambda_3^2$ and $|\theta_1\rangle$ with probability λ_1^2. So we see again how the resulting type of player 1 varies with the initial (here superposed) type of his opponent.

Definition

A pure static TI-equilibrium of a game M with eigentypes $\theta_i \in E^1 = E^2 = E$, with action set $A = \{a_1, a_2\}$ and strategy sets $S_1 = S_2 = S : E \to S$ and initial types $\left(\left|t_1^{t=0}\right\rangle, \left|t_2^{t=0}\right\rangle\right)$ is

i. A profile of pure strategies $(s_1^*, s_2^*) \in S \times S$ such that each one of the $M-$eigentypes of each player maximizes his expected utility given the (superposed) type of his opponent and the strategies played by the opponent's eigentypes:

$$s_1^* \left(\theta_i^1\right) = \arg \max_{s_1' \cdot \in S_1} \sum_{\theta_i^2; \gamma_i > 0} \gamma_i^2 u_i \left(s_1', s_2^* \left(\theta_i^2\right), \left(\theta_i^1, \theta_i^2\right)\right) \tag{3}$$

and similarly for player 2.

ii. A corresponding profile of resulting types, one for each player and each action:A corresponding profile of resulting types (t_1', t_2'), one for each player

$$|\, t_1' |\, a_i\rangle = \sum_{\theta_i; s_1^* \left(\theta_i^1\right) = a_i^1} \lambda_i' \left|\theta_i^1\right\rangle$$

where $\lambda'_i = \frac{\lambda_i}{\sqrt{\sum_{j \neq i} (\lambda^1_j)^2 (s^*_1(\theta^1_j) = a^1_i)}}$ *and* a_i *is the action played by player 1. Similarly for* $\left| t'_2 \right| a^2_i \rangle$.

For concreteness we shall now solve for the TI-equilibrium of this game for the following initial types

$$|t_1\rangle = \sqrt{.7}\,|\theta_1\rangle + \sqrt{.2}\,|\theta_2\rangle + \sqrt{.1}\,|\theta_3\rangle\,, \tag{4}$$

$$|t_2\rangle = \sqrt{.2}\,|\theta_1\rangle + \sqrt{.6}\,|\theta_2\rangle + \sqrt{.2}\,|\theta_3\rangle\,. \tag{5}$$

Given the payoff matrices above, the threshold probability q that rationalizes the play of C for the eigentype θ_2 is $q = .666$. For the ease of presentation, we let $q = .7$. We know that the θ_2 of player 2 cooperates since $\lambda^2_1 = .7 \geq q$ and so does the θ_2 of player 1 since $\gamma^2_1 + \gamma^2_2 = .8 > q$.

In the TI-equilibrium of this game player 1 plays C with probability .9 and collapses on $|t'_1\rangle = \frac{\sqrt{.7}}{\sqrt{.7+.2}}\,|\theta_1\rangle + \frac{\sqrt{.2}}{\sqrt{.7+..2}}\,|\theta_2\rangle$ and with probability .1 player 1 plays D and collapses on $|\theta_3\rangle$. Player 2 plays C with probability .8 and collapses on $|t'_2\rangle = \frac{\sqrt{.4}}{\sqrt{.4+.4}}\,|\theta_1\rangle + \frac{\sqrt{.4}}{\sqrt{.4+.4}}\,|\theta_2\rangle$ and with probability .2, he plays D and collapses on $|\theta_3\rangle$.

We note that the *mixture actually played* by player 1 (.9C, .1D) is *not* the best reply of any of his eigentypes. The same holds for player 2. The eigentypes are the "real players" and they play pure strategies.

We end this section with a comparison of the TI-game approach with the standard incomplete information treatment of this game where the square of the coefficients of superposition in (1) and (2) are interpreted as players' beliefs about each other. The sole substantial distinction is that in the Bayes-Harsanyi setting the players privately learn their own type *before* playing while in the TI-model they learn it in the process of playing. A player is thus in the same informational situation as his opponent with respect to his own play. However under our assumption that all the reasoning is done by the eigentypes, the classical approach and the TI-approach are indistinguishable. They yield the same equilibrium outcome. The distinction is merely interpretational.

Statement 1

The equilibrium predictions TI-model of a simultaneous one-move game are the same as those of the corresponding Bayes-Harsanyi model.

A formal proof of Statement 1 can be found in Part II.

This central equivalence result should be seen as an achievement which provides support for the hypotheses that we make to extend the basic TI-model to strategic decision-making. Indeed, we do want the non-classical model to deliver the same outcome in a simultaneous one-move context.[23] We next move

[23] We know that quantum indeterminacy cannot be distinguished from incomplete information in the case of a single measurement. A simultaneous one-move game corresponds to two single measurements performed on two non-entangled systems.

to a setting where one of the players is involved in a sequence of moves. This is the simplest setting in which to introduce the novelty brought about by the type indeterminacy hypothesis.

2.2 A multi-stage TI-game

In this section we introduce a new interaction involving player 1 and a third player, a promise exchange game.[24] We assume that the GS representing the promise game do not commute with the GS representing the game M (described in the previous section).[25] Player 1 and 3 play a promise game where they choose between either making a non-binding promise to cooperate with each other in game M or withholding from making such a promise. Our objective is to show that playing a promise exchange game - with a third player - can increase the probability for cooperation (decrease the probability for defection) between the player 1 and 2 in a next following game *M.* Such an impact of cheap-talk promises is related to experimental evidence reported in Frank (1988)

We shall compare two situations called respectively protocol I and II. In protocol 1 player 1 and 2 play game *M.* In protocol II we add a third player, 3, and we have the following sequence of events:

step 1 Player player 1 and 3 play a promises exchange game N, described below.

step 2 Player 1 and 2 play M.

step 3 Player 1 and 3 play M.[26]

The promise exchange game

At *step 1*, player 1 and 3 have to simultaneously select one of the two announcements: "I promise to play cooperate", denoted, P, and "I do not promise to play cooperate" denoted $no - P$. The promises are cheap-talk i.e., breaking them in the next following games has no implications for the payoffs i.e., at step 2 or step 3.

There exists three eigentypes in the promise exchange game:

τ_1 : prefers to never make cheap-talk promises - let him be called the "honest type";

τ_2 : prefers to make a promise to cooperate if he believes the opponent cooperates with probability $p \geq q$ (in which case he cooperates whenever he is of type θ_2 or θ_1 or any superposition of the 2). Otherwise he makes no promises - let him be called the "sincere type";

[24] The reason for introducing a third player is that we want to avoid any form of signaling. The exercise could be done with only two players but the comparison between the classical and the TI-model would be less transparent.

[25] To each game we associate a collection of GS each of which measures the best reply a possible type of the opponent.

[26] The reason why we have the interaction at *step 3* is essentially to motivate the promise exchange game. Our main interest will focus on the interaction at *step 2*.

τ_3 : prefers to promise that he will cooperate whatever he intends to do - he can be viewed as the "opportunistic type".

Information assumptions

We make the following assumptions about players' information in the multi-stage game:

i. All players know the statistical correlations (conditional probabilities) between the eigentypes of the two (non-commuting) games.[27]

ii. At *step 2,* player 2 knows that player 1 has interacted with player 3 but he does not know the outcome of the interaction.

We note that ii. implies that we are not dealing with an issue of strategic communication between player 1 and 2. No message is being received by player 2.

The classical model

We first establish that in the classical setting we have the same outcome in protocol I and at *step 2* of protocol II. We already know from Statement 1 that the predictions of a TI model of game M are the same as the prediction of the classical Bayes-Harsanyi model of the corresponding incomplete information game.

We investigate in turn how the interaction between player 1 and 3 at *step 1* affects the incentives and/or the information of player 1 and 2 at *step 2.* Let us first consider the case of player 1. In a classical setting, player 1 knows his own type, so he learns nothing from the promise exchange stage. Moreover the announcement he makes is not payoff relevant to his interaction with player 2. So the promise game has no direct implication for his play with player 2. As to player 2, the question is whether he has reason to update his beliefs about player 1. Initially he knows $|t_1\rangle$ from which he derives his beliefs about player 1's equilibrium play in game M. By our informational assumption (i) he also knows the statistical correlations between the eigentypes of the two games from which he can derive the expected play conditional on the choice at the promise stage. He can write the probability of e.g., the play of D using the conditional probability formula:

$$p(D) = p(P)\, p(D|\, P) + p(no - P)\, p(D|\, no - P)\,. \tag{6}$$

He knows that player 1 interacted with 3 but he does *not* know the outcome of the interaction. Therefore he has no new element from which to update his information about player 1. We conclude that the introduction of the interaction with player 3 at *step 1* leaves the payoffs and the information in the game M unchanged. Hence, expected behavior at *step 2* of protocol II is the same as in protocol I.

[27] So in particular they can compute the correlation between the *plays* in the different games.

The TI-model

Recall that the GS representing the promise game do not commute with the GS representing the game M. We now write eq.(1) and (2) in terms of the eigentypes of game N, i.e., of the promise stage eigentypes:

$$|t_1\rangle = \lambda_1' |\tau_1\rangle + \lambda_2' |\tau_2\rangle + \lambda_3' |\tau_3\rangle \text{ and } |t_3\rangle = \gamma_1' |\tau_1\rangle + \gamma_2' |\tau_2\rangle + \gamma_3' |\tau_3\rangle .$$

Each one of the $N-$eigentype can in turn be expressed in terms of the eigentypes of game M :

$$\begin{aligned} |\tau_1\rangle &= \delta_{11} |\theta_1\rangle + \delta_{12} |\theta_2\rangle + \delta_{13} |\theta_3\rangle \\ |\tau_2\rangle &= \delta_{21} |\theta_1\rangle + \delta_{22} |\theta_2\rangle + \delta_{23} |\theta_3\rangle \\ |\tau_3\rangle &= \delta_{31} |\theta_1\rangle + \delta_{32} |\theta_2\rangle + \delta_{33} |\theta_3\rangle \end{aligned} \tag{7}$$

where the δ_{ij} are the elements of the basis transformation matrix.[28] Assume that player 3 is (initially) of type θ_3 with probability close to 1, we say he is a "tough" type. We shall investigate the choice of between P and *no-P* of player 1 i.e., the best response of the eigentypes τ_i of player 1.

By definition of the τ_i type, we have that τ_1 always plays *no-P* and τ_3 always play *P*. Now by assumption, player 3 is of type θ_3 who never cooperates. Therefore, by the definition of τ_2, player 1 of type τ_2 chooses not to promise to cooperate, he plays *no−* P.

This means that at *step 1* with probability $\lambda_1'^2 + \lambda_2'^2$ player 1 plays *no−* P and collapses on $\left|\widehat{t}_1\right\rangle = \frac{\lambda_1'}{\sqrt{(\lambda_1^{2\prime}+\lambda_2^{2\prime})}} |\tau_1\rangle + \frac{\lambda_2'}{\sqrt{(\lambda_1^{2\prime}+\lambda_2^{2\prime})}} |\tau_2\rangle$. With probability $\lambda_3'^2$ he collapses on $|\tau_3\rangle$.

We shall next compare player 1's propensity to defect in protocol I with that propensity in protocol II. For simplicity we shall assume the following correlations: $\delta_{13} = \delta_{31} = 0$, meaning that the honest type τ_1, never systematically defects and that the opportunistic guy τ_3 never systematically cooperates.

Player 1's propensity to defect in protocol I

We shall consider the same numerical example as before i.e., given by (4) and (5) so in particular we know that θ_2 of player 1 cooperates so $p(D\,||t_1\rangle) = \lambda_3^2$. But our objective in this section is to account for the indeterminacy due to the fact that in protocol I the promise game is *not* played. We have

$$|t_1\rangle = \lambda_1' |\tau_1\rangle + \lambda_2' |\tau_2\rangle + \lambda_3' |\tau_3\rangle ,$$

[28] A basis transformation matrix links the eigentypes of the two GO M and N :

$$\begin{pmatrix} \langle\tau_1|\,\theta_1\rangle = \delta_{11} & \langle\tau_1|\,\theta_2\rangle = \delta_{12} & \langle\tau_1|\,\theta_3\rangle = \delta_{13} \\ \langle\tau_2|\,\theta_1\rangle = \delta_{21} & \langle\tau_2|\,\theta_2\rangle = \delta_{22} & \langle\tau_2|\,\theta_3\rangle = \delta_{23} \\ \langle\tau_3|\,\theta_1\rangle = \delta_{31} & \langle\tau_3|\,\theta_2\rangle = \delta_{32} & \langle\tau_3|\,\theta_3\rangle = \delta_{33} \end{pmatrix}.$$

using the formulas in (7) we substitute for the $|\tau_i\rangle$

$$\begin{aligned} |t_1\rangle &= \lambda_1' \left(\delta_{11} |\theta_1\rangle + \delta_{12} |\theta_2\rangle + \delta_{13} |\theta_3\rangle\right) + \lambda_2' \left(\delta_{21} |\theta_1\rangle + \delta_{22} |\theta_2\rangle + \delta_{23} |\theta_3\rangle\right) \\ & +\lambda_3' \left(\delta_{31} |\theta_1\rangle + \delta_{32} |\theta_2\rangle + \delta_{33} |\theta_3\rangle\right). \end{aligned}$$

Collecting the terms we obtain

$$\begin{aligned} |t_1\rangle &= \left(\lambda_1'\delta_{11} + \lambda_2'\delta_{21} + \lambda_3'\delta_{31}\right) |\theta_1\rangle + \left(\lambda_1'\delta_{12} + \lambda_2'\delta_{22} + \lambda_3'\delta_{32}\right) |\theta_2\rangle + \\ & \left(\lambda_{13}'\delta + \lambda_2'\delta_{23} + \lambda_3'\delta_{33}\right) |\theta_3\rangle. \end{aligned}$$

We know from the preceding section that both $|\theta_1\rangle$ and $|\theta_2\rangle$ choose to cooperate so

$$p\left(D\,||t_1\rangle\right) = p\left(|\theta_3\rangle\,||t_1\rangle\right).$$

Using $\delta_{13} = 0$, we obtain the probability for player 1's defection in protocol I*:*

$$p\left(D\,||t_1\rangle\right)_M = \left(\lambda_2'\delta_{23} + \lambda_3'\delta_{33}\right)^2 = \lambda_2^{2\prime}\delta_{23}^2 + \lambda_3^{2\prime}\delta_{33}^2 + 2\lambda_2'\delta_{23}\lambda_3'\delta_{33}. \quad (8)$$

Player 1's propensity to defect in protocol II

When the promise game is being played, i.e., the measurement N is performed, we can (as in the classical setting) use the conditional probability formula to compute the probability for the play of D

$$p\left(D\,||t_1\rangle\right)_{MN} = p\left(P\right)p\left(D|\,P\right) + p\left(no - P\right)p\left(D|\,no - P\right). \quad (9)$$

Let us consider the first term: $p\left(P\right)p\left(D|\,P\right)$. We know that $p\left(P\right) = p\left(|\tau_3\rangle\right) = \lambda_3^{2\prime}$. We are now interested in $p\left(D|\,P\right)$ or $p\left(D\,|\tau_3\rangle\right)$. $|\tau_3\rangle$ writes as a superposition of the θ_i with θ_1 who never defects, θ_3 who always defect while θ_2's propensity to defect depends on what he expects player 2 to do. We cannot take for granted that player 2 will play in protocol II as he plays in protocol I. Instead we assume for now that eigentype θ_2 of player 2 chooses to cooperate (as in protocol I) because he expects player 1's propensity to cooperate to be no less than in protocol I. We below characterize the case when this expectation is correct. Now if θ_2 of player 2 chooses to cooperate so does θ_2 of player 1 and $p\left(D\,|\tau_3\rangle\right) = \delta_{33}^2$ so

$$p\left(P\right)p\left(D|\,P\right) = \lambda_3^{2\prime}\delta_{33}^2$$

We next consider the second term of (9). The probability $p\left(no - P\right)$ is $\left(\lambda_1^{2\prime} + \lambda_2^{2\prime}\right)$ and the type of player 1 changes, he collapses on $\left|\widehat{t}_1\right\rangle = \frac{\lambda_1'}{\sqrt{\left(\lambda_1^{2\prime}+\lambda_2^{2\prime}\right)}}|\tau_1\rangle + \frac{\lambda_2'}{\sqrt{\left(\lambda_1^{2\prime}+\lambda_2^{2\prime}\right)}}|\tau_2\rangle$. Since we consider a case when θ_2 of player 1 cooperates, the probability for defection of type $|\widehat{t_1}\rangle$ is $\left(\frac{\lambda_1'}{\sqrt{(\lambda_1^{2\prime}+\lambda_2^{2\prime})}}\right)^2 \delta_{13}^2 + \left(\frac{\lambda_2'}{\sqrt{(\lambda_1^{2\prime}+\lambda_2^{2\prime})}}\right)^2 \delta_{23}^2$. Recalling that $\delta_{13} = 0$, we obtain that $p\,(no - P)\, p\,(D|\, no - P)$ is equal to

$$\left(\lambda_1^{2\prime} + \lambda_2^{2\prime}\right) \left(\frac{\lambda_2'}{\sqrt{\left(\lambda_1^{2\prime} + \lambda_2^{2\prime}\right)}}\right)^2 \delta_{23}^2 = \lambda_2^{2\prime}\delta_{23}^2$$

which gives

$$p\,(D\,||t_1\rangle\,)_{MN} = \lambda_2^{2\prime}\delta_{23}^2 + \lambda_3^{2\prime}\delta_{33}^2. \tag{10}$$

Comparing formulas in (8) and (10) :

$$p\,(D\,||t_1\rangle\,)_{MN} - p\,(D\,||t_1\rangle\,)_M = -2\lambda_2'\delta_{23}\lambda_3'\delta_{33} \tag{11}$$

which can be negative or positive because the interference terms only involves amplitudes of probability i.e., the square roots of probabilities. The probability to play defect decreases (and thus the probability for cooperation increases) when player 1 plays a promise stage whenever $2\lambda_2'\delta_{23}\lambda_3'\delta_{33} > 0$. In that case the expectations of player 2 are correct and we have that the θ_2 type of both players cooperate which we assumed in our calculation above.[29]

Result 1: *When player 1 meets a tough player 3 at step 1, the probability for playing defect in the next following M game is not the same as in the M game alone,* $p\,(D\,||t_1\rangle\,)_M - p\,(D\,||t_1\rangle\,)_{MN} \neq 0$.

It is interesting to note that $p\,(D\,||t_1\rangle\,)_{MN}$ is the same as in the classical case. It can be obtained from the same conditional probability formula.

In order to better understand our *Result 1,* we now consider a case when player 1 meets with a "soft" player 3, i.e., a θ_1 type, at *step 1.*

The soft player 3 case

In this section we show that if the promise stage is an interaction with a soft player 3 there is no effect of the promise stage on player 1's propensity to defect and thus no effect on the interaction at *step 2.*

Assume that player 3 is (initially) of type θ_1 with probability close to 1. What is the best reply of the N-eigentypes of player 1, i.e., how do they choose between P and *no-P*? By definition we have that τ_1 always plays *no-P* and τ_3 always play *P.* Now by the assumption we just made player 3 is of type θ_1 who always cooperates so player 1 of type τ_2 chooses to promise to cooperate, he plays P.

[29] For the case the best reply of the θ_2 types changes with the performence of the promise game, the comparison between the two protocols is less straightforward.

This means that at *t=1* with probability $\lambda_1'^2$ he collapses on $|\tau_1\rangle$ and with probability $\lambda_2'^2 + \lambda_3'^2$ player 1 plays P and collapses on $\left|\widehat{t}_1\right\rangle = \frac{\lambda_2'}{\sqrt{\lambda_2^{2\prime}+\lambda_3^{2\prime}}}\left|\tau_2\right\rangle + \frac{\lambda_3'}{\sqrt{\lambda_2^{2\prime}+\lambda_3^{2\prime}}}\left|\tau_3\right\rangle$. We shall compute the probability to defect of that type.[30] We write the type vector $\left|\widehat{t}_1\right\rangle$ in terms of the M-eigentypes,

$$\begin{aligned}\left|\widehat{t}_1\right\rangle &= \left(\frac{\lambda_2'}{\sqrt{\lambda_2'^2+\lambda_3'^2}}\right)\left(\delta_{21}\left|\theta_1\right\rangle + \delta_{22}\left|\theta_2\right\rangle + \delta_{23}\left|\theta_3\right\rangle\right) \\ &\quad + \left(\frac{\lambda_3'}{\sqrt{\lambda_2'^2+\lambda\prime_3^2}}\right)\left(\delta_{31}\left|\theta_1\right\rangle + \delta_{32}\left|\theta_2\right\rangle + \delta_{33}\left|\theta_3\right\rangle\right)\end{aligned}$$

As we investigate player 1's M-eigentypes' best reply, we again have to make an assumption about player 2's expectation. And the assumption we make is that he believes that player 1's propensity to defect is unchanged, so as in protocol I the θ_2 of both players cooperate and only θ_3 defects. We have

$$p\left(D\left|\left|\widehat{t}_1\right\rangle\right.\right)_{MN} = \left[\frac{\lambda_2'}{\sqrt{\lambda_2'^2+\lambda_3'^2}}\delta_{23} + \frac{\lambda_3'}{\sqrt{\lambda_2'^2+\lambda_3'^2}}\delta_{33}\right]^2$$

$$p\left(D\left|\left|\widehat{t}_1\right\rangle\right.\right)_{MN} = \frac{1}{\lambda_2^{2\prime}+\lambda_3^{2\prime}}\left[\lambda\prime_2^2\delta_{23}^2 + \lambda\prime_3^2\delta_{33}^2 + 2\lambda_2'\lambda_3'\delta_{23}\delta_{33}\right]$$

The probability for defection is thus

$$p\left(D\left|\left|t_1\right\rangle\right.\right)_{MN} = P\left(\tau_1\right)p\left(D\left|\left|\tau_1\right\rangle\right.\right) + P\left(\widehat{t}_1\right)p\left(D\left|\left|\widehat{t}_1\right\rangle\right.\right) =$$

$$0 + \left(\lambda_2^{2\prime}+\lambda_3^{2\prime}\right)\frac{1}{\lambda_2^{2\prime}+\lambda_3^{2\prime}}\left[\lambda_2^{2\prime}\delta_{23}^2 + \lambda_3^{2\prime}\delta_{33}^2 + 2\lambda_2'\lambda_3'\delta_{23}\delta_{33}\right] = \lambda_2^{2\prime}\delta_{23}^2 + \lambda_3^{2\prime}\delta_{33}^2 + 2\lambda_2'\lambda_3'\delta_{23}\delta_{33}.$$

Comparing with eq. (8) of protocol I we see that here

$$p\left(D\left|\left|t_1\right\rangle\right.\right)_M = p\left(D\left|\left|t_1\right\rangle\right.\right)_{MN}$$

There is NO effect of the promise stage. This is because the interference terms are still present. We note also that player 2 was correct in his expectation about player 1's propensity to defect.

Result 2

If player 1's move at step 1 does not separate between the N-eigentypes that would otherwise interfere in the determination of his play of D at step 2 then $p\left(D\left|\left|t_1\right\rangle\right.\right)_M = p\left(D\left|\left|t_1\right\rangle\right.\right)_{MN}$.

Let us try to provide an intuition for our two results. In the absence of a promise stage (protocol I) both the sincere and opportunistic type coexist in the mind of player 1. Both these two types have a positive propensity to defect. When they coexist they interfere positively(negatively) to reinforce(weaken) player 1's propensity to defect. When playing the promise exchange game the two types may either separate

[30] Recall that τ_1 never defects.

or not. They separate in the case of a tough player 3. Player 1 collapses either on a superposition of the honest and sincere type (and chooses *no-P*) or on the opportunistic type (and chooses P). Since the sincere and the opportunistic types are separated (by the first measurement, game N) there is no more interference. In the case of a soft player 3 case, the play of the promise game does not separate the sincere from the opportunistic guy, they both prefer P. As a consequence the two Neigentypes interfere in the determination of outcome of the next following M game as they do in protocol I.

In this example we demonstrated that in a TI-model of strategic interaction, a promise stage does make a difference for players' behavior in the next following performance of game M. The promise stage makes a difference because it may destroy interference effects that are present in protocol I.

Quite remarkably the distinction between the predictions of the classical and the TI-game only appears in the *absence* of the play of a promise stage (with a tough player). Indeed the probability formula that applies in the TI-model for the case the agent undergoes the promise stage (10) is the same as the conditional probability formula that applies in the standard classical setting.

The cheap-talk promise paradox

When promises that have no commitment or informational value affect behavior, we may speak about a cheap talk paradox (with respect to established theory). In particular we may have the case that despite the fact that all types pool to make cheap-talk promises (we only have non-revealing equilibria), they nevertheless affect subsequent play. Our paper does not exactly address this case. This is because on the one hand playing the promise game always separates between the τ_1 and τ_3. On the other hand the promises are not communicated to player 2. Yet, because the analysis focuses on the separation between τ_2 and τ_3 (and by its information assumption avoids Bayesian updating with respect to τ_1), it suggests two possible explanations for why cheap-talk promises may matter:

1. Unobserved separation

Here the idea is that the promise game actually does trigger separation between types (like in protocol 2 with a taught type). Reaching the promise response is more difficult for the reciprocating type τ_2 than for the opportunistic τ_3. It takes longer time to do the reasoning. The act of playing breaks the indeterminacy of player 1 but that is not observed by player 2.[31] In that case the TI-model's predictions in the next following PD game are not the same with a promise exchange pre-play compared as or without pre-play. We have an impact of cheap-talk promises.

2. Observed pooling

[31] The idea is that player 2's observation is too coarse: he does not know whether reaching the promise decision took long time or not.

The second line of explanation of the paradox follows a different logic. It relies on the observation that if the observer has the classical model in mind, his predictions are incorrect. When he confronts his predictions in protocol 2 (which are the same as his predictions in protocol 1) with the actual outcome of protocol 2, he notes a difference. This is because simply he did not account for the interference effects. So here the explanation is not that pooling in cheap-talk promises changes behavior but that there is an error in the modeling of the pooling outcome.

Part II
Basic Concepts and Solutions of TI-games

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3 Static TI-games of maximal information

TI-games are games with type indeterminate players, i.e., games characterized by uncertainty, players have incomplete information about important aspects of the game. It is therefore natural to look for the classical counter-part in games with incomplete information.

In this section we deal exclusively with static games. Using the terminology introduced in section 2 of Part 1, we denote by GS (for Game Situation) an operator (called observable in Physics) that measures the type of an agent in a strategic situation i.e., in a situation where the outcome of the choice, in terms of the agent's utility, depends on the choice of other agents as well. The notion of GS applies exclusively to game situations that can be associated with a first-kind measurement.[34] The outcome of a GS i.e., of a *move,* is information about the type of the player.[35] This is because we interpret the outcome of the measurement so that the chosen action is a *best reply* against the opponent's expected action. Possible problems with the notion of actualized best-reply due to the role of players' beliefs are discussed in Part 1 p. 5. But with maximal information games, expectation is computed with objective probabilities. A GS does not generally give maximal information about the type of the player.[36] In particular, this is the case when some of the possible types pool in their best-reply. Such a GS is a coarse measurement which preserves some indeterminacy with implications beyond the pure informational aspects as we shall see below.

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[34]First-kindness is a property of reproducibility. If one performs a measurement on a system and obtains a result, then one will get the same result if one performs again that measurement on the same system immediately after. First-kindness does not entail that the first outcome is obtained when repeating a measurement if other measurements are performed on the system in between.

[35]We assume throughout that each player is an independent system i.e., there is no quantum entanglement between players.

[36]In a TI context there is a distinction between maximal and complete information. See the subsection "type an eigentype".

Uncertainty When some players do not know the payoff function of other players the game is said of incomplete information. The standard approach, due to Harsanyi, is to transform the game into a game of imperfect information where Nature moves at the beginning of the game and selects, for each player, one among a multiplicity of possible types. A player's own type is his private information. Generally, the Harsanyi type includes a player's information and beliefs. But in this paper we let the term Harsanyi type exclusively refer to the payoff function while information is dealt with separately. This is because we shall be dealing with TI-games of *maximal information* where all players are represented by pure types (eigentype or superposition of eigentypes, see below for a definition).[37] As we shall see this implies that we are dealing with objective uncertainty. However in such a TI-game some players may not even know their *own payoff function*. Can we nevertheless extend the Harsanyi approach to TI-games? We shall argue that the TI-paradigm gives new content to Harsanyi's approach. What is a fictitious Nature's move in Harsanyi's setting becomes a real move (a measurement) with substantial implications. And the theoretical multiplicity of types of a player becomes a substantial multiplicity of "selves".[38]

Types and eigentypes

We use the term *type* as the term *quantum pure state*. A (pure) type $|t\rangle \in \mathbb{T}$[39] where $\mathbb{T}$ is a Hilbert space, is maximal information about the player i.e., about his payoff function[40]. But because of (intrinsic) indeterminacy (see below), a type does not provide complete information about the payoff function in all possible GS not even to the player himself.

In a TI-game we also speak about the *eigentypes* of a game M, $e_i(M) \in E(M)$, $E(M) \subset \mathbb{T}$. The term eigentype parallels the term eigenstate of a system. It is a state associated with one of the possible eigenvalues of an observable. An eigentype is thus the type associated with one of the possible outcomes of a GS (or more correctly of a complete set of commuting GS associated with a game).[41] The eigentypes are truly *private and complete information* about the payoff functions *in a specific static game M*. Any eigentype of a player knows his own M-game payoff function but he may not know that of the other players. The eigentypes of a TI-game M are identified with their payoff function in that game.

Let us be more precise as to the distinction between the Harsanyi approach to uncertainty and the Type Indeterminacy approach. Harsanyi assumes that the player's type are independently drawn from

[37] For a discussion about pure and mixed types (states) see Section 3.2 in Danilov and Lambert-Mogiliansky 2008.

[38] There exists various models of multiple selves in the literature. The original idea is due to Strotz (1956) and is used to model time inconsistent behavior in dynamic utility maximization

[39] We use Dirac ket notation $|\rangle$ to denote a vector in a Hilbert space.

[40] We do not include any private information or beliefs in our definition of a type or of an eigentype.

[41] A GS corresponds to a specific strategic decision situation involving one or more other players. A complete set of commuting GS provides information about a type's behavior for any possible play of any opponent. It is complete description of his preferences in this game and we identify it with a payoff function.

a joint distribution $p(\theta_1, ..., \theta_n)$ where $\theta_i \in \Theta$ is some Boolean space with finite number of elements. A Harsanyi type summarizes all possible type characteristics of a player.[42] The joint distribution is common knowledge among all the players. Upon receiving information from Nature about one's own type, a player updates his beliefs about others' type. The conditional probabilities $p(\boldsymbol{\theta}_{-i}; \theta_i)$ constitute the initial beliefs of player i and these beliefs are updated according Bayes' rule along the game in view of new information.

In TI-games, the types $|t_i\rangle$ are (vectors) elements of a (finite dimensional) Hilbert space $\mathbb{T}$. In a maximal information game the initial type of the players are pure types (i.e., not mixed) and they are common knowledge among players. For any static game M, a type t_i can be expressed in terms of the eigentypes $|e_i(M)\rangle \in E(M) \subset \mathbb{T}$, with $|e_i(M)\rangle \perp |e_j(M)\rangle$, $i \neq j$. An eigentype summarizes the type characteristics *relevant to a particular game.* The uncertainty relevant to game M is given by the expression of the initial type t_i in terms of (a superposition) the eigentypes $e_i(M)$. In contrast to a Harsanyi type, an eigentype does not summarize *all* the possible type characteristics. Consider a multi-move game: M followed by N, the TI-model allows for the case when the type characteristics relevant to M respective N are "incompatible" in the sense that they cannot be revealed (actualized) simultaneously.[43] This is the source of intrinsic indeterminacy i.e., of an uncertainty not due to incomplete information but to the impossibility of actualizing type characteristics belonging to two non-commuting *GS*. The classical Bayes-Harsanyi model corresponds to the special case of the TI-model when all *GS* commute. As a consequence of non-commutativity, in a TI-game the players' type change as the game proceeds.[44] As the game proceeds players update their information but they do not generally follow Bayes' rule.[45] The updating rule reflects the (non- Boolean) structure of the type space, see B1-B2 below.

Assumption 1

In a TI-game all strategic reasoning is done by the eigentypes of the players.

This key assumption has been commented in Part 1, p.6. As earlier mentioned it is consistent with standard game theory which assumes that players are able to adopt the perspective of other players. This is necessary to figure out how other players will play in order to best-respond to their play. In the context of dynamic TI-games Assumption 1 has non-trivial implications for utility maximization as we shall see next.

[42] A Harsanyi type usually includes the player's private information but we do not have any private information here.
[43] The tensor product set $E(M) \otimes E(N)$ does not exist when M and N are incompatible.
[44] See [11] for an explanation of the links between the incompatibility of observables and the change on the type.
[45] See [13] for a discussion on updating in a non-classical context

Utility In the context of type indeterminacy we have to be careful with the notion of utility. A question arises because the type of the agent changes along the path of actions with the decisions that are made: which type's utility is it that matters to the agent's decisions? We shall adopt the following principle: at each step the utility of an action profile is evaluated by the potential *current* eigentypes. In a multi-stage context, these types are of course well aware that there is a continuation game and they do care about the utility of the player's future selves. We shall assume that the utility of the future selves is accounted *for as evaluated by these future selves.* More precisely: in each period each potential eigentype of a player maximizes his current utility plus the expected utility of the resulting type conditional on "surviving" the current stage.[46] This approach secures that the current eigentypes internalize the impact of their own decision on the future selves of the player. We adopt the convention of giving the same weight to the utility of all relevant types along the path. The utility of the whole path is the sum of the utilities of each period's action profile evaluated by the resulting type in that period.[47]

3.1 Static TI-game equilibrium

The standard equilibrium concept of static incomplete information game is the *Bayesian Equilibrium.* It relies on the Harsanyi type approach to uncertainty described above. Let the set of players be N, $N = \{1, 2\}$, the players are indexed by $j = 1, 2$. The sets of pure strategies are denoted S_j (we shall limit ourselves to *pure* strategies) with $s_j : \Theta \to A_j$ where A_j is the action set. The payoff functions are $u_j\left(s_1, s_2, \theta^1, \theta^2\right)$. If player j knew the strategies of the other player as a function of his type, player j could use his beliefs $p\left(\theta^{-j}; \theta^j\right)$ to compute his expected payoff. This is the idea behind the concept of Bayesian equilibrium.

A Bayesian equilibrium in a game of incomplete information with a finite number of types θ_i for each player j, prior distribution $p(.)$, and pure strategy spaces S_j, is a Nash equilibrium where each player j maximizes his utility conditional on θ_i^j[48]

$$s_j\left(\theta_i^j\right) = \arg\max_{s'_j \cdot \in S_j} \sum_{\theta_i^{-j}} p\left(\theta_i^{-j} \middle| \theta_i^j\right) u_j\left(s'_j, s_{-j}\left(\theta_i^{-j}\right), \left(\theta_i^j, \theta_i^{-j}\right)\right). \tag{12}$$

[46] This means that the resulting type must include that eigentype among the potential eigentypes resulting from the first move. If that was not the case, that eigentype's utility would not enter the player's utility.

[47] We note that this avoids doubble accounting. The eigentype does takes into account the impact of an action for the future when making his choice between actions. But in each period only the utility of the (expected) current action is accounted for in the computation of the utility of the path.

[48] The formulation in terms of utility conditional on the type is equivalent to the unconditional formulation because all types have positive probability.

A first equivalence result

As we wrote earlier, in a TI-game a player is not identified with his payoff function so when it comes to equilibrium analysis we cannot proceed as in the classical case. Yet, under Assumption 1 all the strategic reasoning is done by the eigentypes (the selves) who know their payoff functions just as the players know their own payoff function in the Bayes-Harsanyi model. The eigentypes are the "real players" and we shall see that under this assumption a static TI-game looks very much like an incomplete information game and a static TI-equilibrium can be defined as a Nash equilibrium of the original two-player game expanded to the eigentypes of each player.

Let A_j be a finite set of actions available to player $j = 1, 2$. Each player is represented by his type $|t_j\rangle \in \mathbb{T}$. For any game situation M we have $E_j(M) \subset \mathbb{T}$, where $E_j(M) = \left\{ \left| e_1^j(M) \right\rangle, ..., \left| e_k^j(M) \right\rangle \right\}$ is the set of eigentypes of player j in GS M. In the static context of this section we can delete the qualifier in parenthesis and write $\left| e_1^j \right\rangle$ and E_j. A pure strategy for player 1 is a function $s_1 \in S_1,\ s_1 : E^1 \to A_1$.

The initial type vector of player $j = 1$ can be expressed in terms of the eigenvectors of M:

$$|t_1\rangle = \sum_1^k \lambda_i \left| e_i^1 \right\rangle, \ \sum_1^k \lambda_i^2 = 1. \tag{13}$$

The initial common knowledge beliefs about the eigentypes are given by the types $|t_j\rangle$ according to Born's rule

$$prob\left(e_i^1 \middle| |t_1\rangle \right) = \lambda_i^2.$$

We call e_i^1 a *potential* eigentype of player 1 iff $\lambda_i > 0$.[49]

In the Bayes-Harsanyi model, Nature moves first and selects the (eigen)type of each player who is privately informed about it. In a TI-game uncertainty is (partially) resolved by the measurement i.e., the actual act of playing. So before playing, the player does not know his own payoff function (i.e., his eigentype) only his initial (superposed) type (eq. 13). However, each one of his selves (we use the terms self and eigentype interchangeably) knows his own payoff function. We assume that the potential selves of a player all have the same information about the opponent's type. Now if the selves know the strategy of the eigentypes of their opponent, they can compute their expected payoff using the information encapsulated in the (superposed) initial type of his opponent.

Definition

[49] We note that this is equivalent to an incomplete information representation with player $2's$ initial beliefs about 1 given by

$$\sum_1^k p\left(e_i^1 \middle| e_j^2 \right) = p\left(e_i^1 \middle| e_j^2 \right) = \left(\lambda_i^1 \right)^2 \text{for all } e_j.$$

A pure strategy TI-equilibrium of a two-player static game M with initial types $|t_1\rangle = \sum \lambda_i |e_i^1\rangle$ *and* $|t_2\rangle = \sum \gamma_i |e_i^2\rangle$ *is*

i. A profile of pure strategies (s_1^*, s_2^*) *with*

$$s_1^*(e_i^1) = \arg\max_{s_1' \cdot \in S_1} \sum_{e_i^2; \gamma_i > 0} \gamma_i^2 u_i(s_1', s_2^*(e_i^2), (e_i^1, e_i^2)) \tag{14}$$

for all $e_i^1; \lambda_i > 0$ $i = 1, ..k$ *and similarly for player 2.*

ii. A corresponding profile of resulting types (t_1', t_2'),

$$|t_1'| a_i\rangle = \sum_{e_i; s_1^*(e_i^1) = a_i^1} \lambda_i' |e_i^1\rangle$$

where $\lambda_i' = \frac{\lambda_i}{\sqrt{\sum_{j \neq i} \lambda_j^2 (s_1^*(e_j^1) = a_i^1)}}$ *and* a_i *is the action played by player 1. Similarly for* $|t_2'| a_i^2\rangle$.

The first part (i) says that each of the potential eigentypes of each player maximizes his expected utility given the (superposed) type of his opponent and the strategies played by the opponent's potential eigentypes. It is very similar to the definition of a Bayesian equilibrium strategy profile except that the probabilities for the opponent's eigentypes are given by the initial superposed type instead of a joint probability distribution.[50] In the classical case, a player is identified with an eigentype. Hence, maximizing the eigentype's utility is the same as maximizing the player's utility. In a TI-game, a player is a superposition of eigentypes. The question may arises as to whether when each of the potential eigentype maximizes his own utility, the utility of the player also is maximized. In a static context, this issue is rather unproblematic. The (expected) utility of the, possibly superposed, resulting type $|t_1'| a_i\rangle$, is simply a convex combination of the (potential) eigentypes' utilities which is increasing in all its components.

The second part of the definition (ii) captures the fact that in a TI-game the players' type is modified by their play. The rule governing the change in the type is given by the von Neuman-Luder's projection postulate (see below for a detailed exposition). In a static setting it is equivalent to Bayesian updating *within* the set E_j. We return to this issue in details below in connection with updating in multi-stage games.

Proposition 1 *A pure strategy TI-equilibrium profile of a static maximal information game M with eigentypes* $(e_1, .., e_k)$ *and initial types* $|t_i^j\rangle = \sum_1^{k_j} \lambda_i^j |e_i^j\rangle$ *is equivalent to a Bayesian pure strategy equilib-*

[50] Note also that the best-reply condition (14) applies to potential eigentypes i.e., with a strictly positive coefficient of superposition in the initial type only (in the standard classical approach all types have positive probability).

rium profile of a game with type space $E_j = \left\{ e_1^j, .., e_k^j \right\}$ *and common beliefs given by the distributions* $p\left(e_i^1 \middle| t_1 \right) = \lambda_i^2,\ p\left(e_i^2 \middle| t_2 \right) = \gamma_i^2$.

The proof follows immediately from the definitions.

In a TI-equilibrium, each eigentype of each player, if called to play, plays a best-reply to the (superposed) type of his opponent. Nature moves at the time of the actual play, selects an eigentype who already knows his best-reply to the type of the other player. From the point of view of an eigentype the situation is equivalent to that of a player in an incomplete information game à la Harsanyi. The eigentype knows his payoff function and can compute his best-reply to the expected eigentype of his opponent. The fact that in the classical setting Nature moves before the beginning of the game while Nature moves at the time of playing in the TI-setting has no implication for equilibrium analysis under the assumption that all the strategic reasoning is done by the eigentypes.

4 Multi-stage TI-game

In the classical context, the static Bayesian equilibrium notion extends to multi-move games when no observation is made between the moves. In such a case the different moves can be merged into a compound move. For instance consider a situation where the players must first choose between Right and Left and thereafter without having observed the opponent's play, choose between Up and Down. This game can just as well be expressed as a static game with action set $A = \{RU, RD, LU, LD\}$. The actions are merged. This is generally not true in a TI-context (see Part 1 for an example where a move matters even when it is not observed).

When it comes to games composed of more than one step for at least one player, the crucial issue for TI-games is whether the corresponding GS commute with each other or not.

Commutativity of *GS*

We say that two GS M and N commute if they share a common set of eigentypes $E = E(M) \otimes E(N)$.[51] Another way to express this is to say that the type are *separable* $|t\rangle = |t\rangle_M \otimes |t\rangle_N$. This is the standard definition of commuting observables. For the case all GS commute, the Type Indeterminacy representation of uncertainty about other players' type is equivalent the classical Harsanyi representation.[52]

Definition

[51] When the two GS do not commute the tensor product set $E(M) \otimes E(N)$ does not exist.

[52] For a general result establishing the equivalence of the classical and the non-classical measurement theory in a multiple commuting measurement context see (see ALM, HZ, SZ 2006 (JMP 2009) or Danilov Lambert-Mogiliansky 2008).

A commuting multi-stage TI-game is such that for each player all the GS he may face (in and out of the equilibrium path) commute with each other. Otherwise we say that the multi-stage game is non-commuting.

If there is no observation between two commuting moves we can merge the two GS into one compound GS with outcome set $(a_i(M), a_i(N))$. We are in a fully classical situation and the static equilibrium concept applies. More generally commuting multi-stage TI-games are not distinguishable from classical Bayesian games whether the actions are observed or not. This is because the von Neuman-Luder's postulate for deriving the resulting types at each stage is then equivalent with the Bayesian principle for updating beliefs. We postpone the proof of this statement.

4.1 Simple signaling games

In standard game theory, the notion of multi-stage games includes games where each player only plays once but in a sequence, e.g., simple signaling games. Simple signaling game qualify as multi-stage games because updating takes place. The corresponding equilibrium concept is that of Perfect Bayesian Equilibrium.

Updating

Generally, updating in non-classical measurement theory is a complex issue (see [13]). However, in the case of Hilbert space models, the earlier mentioned von Neumann-Luder postulate tells us exactly where a measurement takes the state and accordingly how information should be updated. Let the initial type vector be an arbitrary

$$|t_1\rangle = \sum_{i=1}^{k} \lambda_i \left|e_i^1\right\rangle$$

Suppose that the measurement of M yields action a_1^1 which is a best-reply against t_2 for e_1^1 and e_3^1 (the GS is a coarse measurement). According the von Neuman-Luder postulate, the initial type vector $|t_1\rangle$ is projected onto the eigenspace spanned by $\left|e_1^1\right\rangle$ and $\left|e_3^1\right\rangle$). After the play of a_1^1 player 1's type is given by

$$|t_1'\rangle = \frac{\lambda_1}{\sqrt{(\lambda_1)^2 + (\lambda_3^1)^2}} \left|e_1^1\right\rangle + \frac{\lambda_3}{\sqrt{(\lambda_1)^2 + + (\lambda_3)^2}} \left|e_3^1\right\rangle.$$

The resulting type $|t_1'\rangle$ is a superposition of the two eigentypes who pooled in choosing a_1^1.[53] Given this new type, the updated probability for say e_1^1 is given by Born's rule to be the square of the coefficient of superposition:

[53] In [27] we discuss the behavioral hypothesis behind our interpretation of pooling in choice as a the outcome of a coarse measurement.

$$prob\left(e_1^1 \middle| \left|t_1'\right\rangle\right) = \left(\frac{\lambda_1}{\sqrt{\lambda_1^2+\lambda_3^2}}\right)^2 = \frac{\lambda_1^2}{\lambda_1^2+\lambda_3^2}.$$

As earlier noticed within E_1 the von Neuman-Luder postulate induces Bayesian updating for appropriately defined initial beliefs.

Consider a signaling two-player TI-game with action sets A_1 and A_2. A pure strategy for player 1 who moves first is $s_1 : E_1 \to A_1$, a strategy for player 2 $s_2 : E_2 \times A_1 \to A_2$.

Definition

A pure strategy signaling TI-equilibrium with initial types are $|t_1\rangle = \sum \lambda_i \left|e_i^1\right\rangle$, $|t_2\rangle = \sum \gamma_i \left|e_i^2\right\rangle$ *and where player 1 moves first to select* a_1 *is characterized by :*

P1 $s_1^*\left(e_i^1\right) = \arg\max_{s_i' \in S_1} \sum_{e_j^2} \gamma_j^2 u_i^1\left(s_1', s_2^*\left(e_j^2\right), e_i^1, e_j^2\right)$, *for all* $e_i^1 \in E_1$

P2 $s_2^*\left(e_i^2\right) = \arg\max_{s_i' \in S_2} \sum_{e_i^1} \mu\left(e_i^1 \middle| a_1\right) u_i^2\left(s_2', a_1, e_i^2, e_i^1\right)$ *for all* $e_i^2 \in E_2$

B $\mu\left(e_i^1 \middle| a_1\right) = \frac{\lambda_i^2}{\sum_j \lambda_j^2\left(s_1^*\left(e_j^1\right)=a_1\right)}$ *or any probability if* $\sqrt{\sum_j \lambda_j^2\left(s_1^*\left(e_j^1\right)=a_1\right)} = 0.$

Proposition 2 *A signaling TI-game where each player makes a single move in a sequence, is equivalent to a classical signaling game for appropriately defined prior beliefs. The eigentypes are identified with the Harsanyi type. The pure equilibrium strategy profile of a signaling TI-game is a Perfect Bayesian equilibrium.*

The proof follows from the definition of the signaling TI-equilibrium.

Again we obtain an equivalence result. This should not surprise us because signaling games only involves a single measurement of each player's type (as in a static TI-game). And we know that within a set of eigentypes E_i, the von Neuman-Luder's postulate implies Bayesian updating.

4.2 Multi-move games with observed actions

In the previous section we noted that commuting multi-stage TI-games without observation are not distinguishable from static TI-game both of which are equivalent to static incomplete information games. This result extends further to commuting dynamic games with observed action.

Result

Commuting dynamic TI-games are not distinguishable from classical games in terms of equilibrium predictions.

This result follows directly from a general result proving the equivalence between the quantum and the classical models with respect to the predictions in a context where all measurements are commuting (see e.g., [11] and [27] for a derivation of this result in a Social Sciences context). The intuition is that the type space representing commuting type characteristics in the TI-context has a standard Boolean algebraic structure.

Non-commuting multi-stage TI-games: strategic manipulation of players' type

We are interested in multi-stage game with observed actions where in each period t the players simultaneously choose their action which are revealed at the end of the period.

We let as before the set of player be $N = \{1, 2\}$. And we consider a two-stage game $t = 1, 2$ where the first stage corresponds to the Game Situation $G(h^0)$ where h^0 is the history at date 0. The second stage's GS is $G(h^1)$ which does not commute with $G(h^0)$. Let $A_j(h^t)$ be a finite set of actions available to player $j = 1, 2$ after history $h^t = (\mathbf{a}^0, ..., \mathbf{a}^{t-1})$ where $\mathbf{a}_t = (a_t^1, a_t^2)$ is the vector of date-t actions. A pure behavioral strategy s maps the set of possible histories $\mathcal{H}^t$ and relevant eigentypes into the action spaces: $s_i^t : \mathcal{H}^t \times E_1(G(h^t)) \times E_2(G(h^t)) \to A_1^t$. Player $i's$ payoff is $u_i(h^T, \mathbf{t}_1^T)$ where $\mathbf{t}_1^T$ is the vector of (initial and) resulting types $\mathbf{t}_1^{T=2} = (t_1^0, t_1^1, t_1^2)$ for simplicity we assume that only the own type vector affects utility. As we defined utility, it is the sum of the utility of the profile of actions up to date T evaluated by the resulting type of the corresponding period.

Each player is represented by his type $|t_j^t\rangle \in \mathbb{T}$. As before we have $E_j(G(h^0)) \subset \mathbb{T}$, where we write $E_j(G(h^0)) = E_j(h^0) = \left\{e_1^j(h^0), ..., e_k^j(h^0)\right\}$ is the set of eigentypes of player j in $G(h^0)$. And $E_j(h^1) \subset \mathbb{T}$, where $E_j(h^1) = \left\{e_1^j(h^1), ..., e_k^j(h^1)\right\}$ is a set of eigentypes of the GS associated with history h^1.

The standard classical equilibrium concept is that of Perfect Bayesian equilibrium. It is an equilibrium where all the strategies yield a Bayes Nash equilibrium not only for the whole game but for all continuation game starting in each period after every possible history h^t. To make the continuation games into a true game one must specify the beliefs at the start of each continuation game. We shall proceed similarly and characterize the properties of the equilibrium updating rule consistent with the Type Indeterminacy hypothesis.

The new feature of non-commuting TI-games is that two consecutive GS are associated with two distinct sets of eigentypes and that the set $E(h^0) \times E(h^1)$ does not exist. This implies that updating cannot be purely Bayesian. Instead it will involve two steps including quantum probability calculus. As we shall see this later feature opens up for strategic "manipulation" of the opponent's type.

Step 1

Let $\left(\left|t_1^0\right\rangle, \left|t_2^0\right\rangle\right)$ be the initial common knowledge pure type vector, $\left|t_1^0\right\rangle = \sum_i \lambda_i \left|e_i\left(h^0\right)\right\rangle$ and $\left|t_2^0\right\rangle = \sum_i \gamma_i \left|e_i\left(h^0\right)\right\rangle$. The eigentypes of both players share the initial beliefs

$$B0: \ \mu\left(e_i^1\left(h^0\right)\right) = \lambda_i^2, \ \mu\left(e_i^2\left(h^0\right)\right) = \gamma_i^2. \tag{15}$$

The first step is as described in the static TI-game in Sect.5. For initial types $\left(\left|t_1^0\right\rangle, \left|t_2^0\right\rangle\right)$, period 1 actions $\left(a_1^1, a_2^1\right)$ and strategy profile (s_1, s_2), the von Neuman-Luder projection postulate is used to compute the resulting type vector $\left(\left|t_1^1\left(h^1\right)\right\rangle, \left|t_2^1\left(h^1\right)\right\rangle\right)$ in terms of $G\left(h^0\right)$ eigenvectors:

$$\left|t_1\left(h^1\right)\right\rangle = \sum_i \lambda_i' \left|e_i\left(h^0\right)\right\rangle \tag{16}$$

where the sum is taken over i such that $s_1\left(e_i^1\left(h^0\right)\right) = a_1^1$ and $\lambda_i' = \frac{\lambda_i}{\sqrt{\sum_k \lambda_k^2\left(s_1\left(e_k^1(h^0)\right)=a_1^1\right)}}$. The beliefs in term of the eigentypes of $G\left(h^0\right)$ are

$$B1: \ \mu\left(e_i^1\left(h^0\right)\middle| a_1\right) = \lambda_i'^2. \tag{17}$$

As we earlier noted this is equivalent to Bayesian updating (for appropriately defined initial beliefs) which is thus applied whenever it is possible.[54]

Step 2

The second step involves a "translation" of the types resulting from the first actions in eq. (16) so that we express them in terms of the eigentypes of $G\left(h^1\right)$. This operation is necessary to be able to compute the equilibrium strategy in the subgames following h^1. Indeed the potential $G\left(h^1\right)-$eigentypes of each player must reason using their expectation about the opponent's play. But that expected play is computed from the best-replies of the opponent's $G\left(h^1\right)-$eigentypes and from their relative probability weights in the type vector resulting from period 1. The translation is performed using a basis transformation matrix $B_{G(h^0)G(h^1)}$ that links the two non-commuting *GS*.[55] We can express each eigentype of $G\left(h^0\right)$ in terms of the eigentypes of $G\left(h^1\right)$: $\left|e_i\left(h^0\right)\right\rangle = \sum_j \delta_{ij}\left|e_j\left(h^1\right)\right\rangle$ where δ_{ij} are the elements of the basis transformation matrix $\delta_{ij} = \left\langle e_j\left(h^1\right)\middle| e_i\left(h^0\right)\right\rangle$.[56] Collecting the terms we can write

$$\left|t_1\left(h^1\right)\right\rangle = \sum_j \left(\sum_i \lambda_i'\delta_{ij}\right)\left|e_j^1\left(h^1\right)\right\rangle$$

[54] $\mu\left(e_{iG(h_0)}\middle| a_1\right)$ is any probability if $\sqrt{\sum_{jG(h_0)\neq iG(h_0)} \lambda_j^2\left(s_1^*\left(e_{jG(h_0)}^1\right) = a_1\right)} = 0$.

[55] The transformation links the two non-commuting *GS*. It is indepedent of the player's identity.

[56] We note that the basis transformation matrix does not depends on the individual players, it is the same for both.

The probability for eigentype $e_1^1\left(h^1\right)$ of player 1 at date $t = 1$ is

$$B2: \mu\left(e_1^1\left(h^1\right)\middle| a_1\right) = \left(\sum_i \lambda_i' \delta_{1i}\right)^2 \tag{18}$$

This is the **crucial formula** that captures all the key distinction between the classical and the quantum approach. B2 is *not* a conditional probability formula where the δ_{ij}^2 are statistical correlations between the eigentypes at the two stages. The player is a non-separable system with respect to $G\left(h^0\right)$ and $G\left(h^1\right)$.[57] As a consequence, the updated beliefs are given by the square of a sum (implying cross terms) and not the sum of squares.

To see this, consider a simple symmetric example and set $k = 2$, so $E\left(h^0\right) = \left\{e_1\left(h^0\right), e_2\left(h^0\right)\right\}$ and $E\left(h^1\right) = \left\{e_1\left(h^1\right), e_2\left(h^1\right)\right\}$. Similarly, let $A\left(h^0\right) = \{a_1, a_2\}$ and $A\left(h^1\right) = \{b_1, b_2\}$. Consider a strategy profile such that the eigentypes of player 1 have the same best-reply to the expected play of player 2 ,i.e., they pool, they both choose a_1. In that case there is no updating $\left|t_1\left(h^0\right)\right\rangle = \left|t_1\left(h^1\right)\right\rangle = \lambda_1\left|e_1^1\left(h^0\right)\right\rangle + \lambda_2\left|e_2^1\left(h^0\right)\right\rangle$. Assume next that at stage 2 $e_1^1\left(h^1\right)$ and $e_2^1\left(h^0\right)$ choose respectively b_1 and b_2 in response to player 2's expected play. The ex-ante (at date h^0) probability for the play of b_1 by player 1 is

$$\begin{aligned} \mu\left(e_1^1\left(h^1\right)\right) &= \left(\lambda_1\delta_{11} + \lambda_2\delta_{21}\right)^2 \\ &= \lambda_1^2\delta_{1j}^2 + \lambda_2^2\delta_{2j}^2 + 2\lambda_1\lambda_2\delta_{1j}\delta_{2j}. \end{aligned} \tag{19}$$

Consider now another strategy profile where player 1's eigentypes do not have the same best-reply to the expected play of player 2, i.e., they separate at stage 1. Then, $\left|t_1\left(h^0\right)\right\rangle \neq \left|t_1\left(h^1\right)\right\rangle$. We have $\left|t_1\left(h^1\right)\right\rangle = \left|e_1\left(h^0\right)\right\rangle$ with probability λ_1^2 and $\left|t_1\left(h^1\right)\right\rangle = \left|e_2\left(h^0\right)\right\rangle$ with probability λ_2^2. We assume for simplicity that, at stage 2, the h^1-eigentypes separate as before in response to player 2's expected play. The ex-ante probability for the play of b_1 is the probability for each of the possible resulting type times the correlation with the eigentype of $G\left(h^1\right)$ that chooses b_1 :

$$\mu\left(e_1^1\left(h^1\right)\right) = \lambda_1^2\delta_{11}^2 + \lambda_2^2\delta_{21}^2. \tag{20}$$

Hence, although *player 1's strategy at stage 2 does not depend on* h^1 : it is simply that eigentype $e_1^1\left(h^1\right)$ plays b_1, his actual expected play at stage 2 does! The probability for b_1 is not the same in (19) and (20) because the type of player 1 has been modified by his best-reply to player 2. Consequently when player 2 always (for all h^1) prefers player 1 to play b_1 he should clearly try to induce player 1's eigentypes to pool at stage 1 if the term $2\lambda_1^1\lambda_2^1\delta_{1j}\delta_{2j}$ is positive (and to separate if the term is negative).

[57] This means that $\left|t_1^1\right\rangle$ cannot be written as a tensor product type composed of $\left|t^1\right\rangle_{GS(h_0)}$ and $\left|t^1\right\rangle_{GS(h_1)}$ or equivalently that the tensor product set $E\left(h_0\right) \otimes E\left(h_1\right)$ does not exist.

Thus, when considering a move a player must account not only for the best-reply of his opponent as usual but also for the induced resulting type of the opponent. More precisely, type indeterminacy gives a new strategic content to pooling respectively separating moves, a content that goes beyond the informational one. When some eigentypes of a player pool, that player remains indeterminate (superposed) with respect to those eigentypes. This conserved indeterminacy implies that in the next following (non-commuting) *GS,* these superposed eigentypes *may interact with each other* producing interference effects that affect the probabilities for the different play. Property B2 of the $\mu()$ function secures that the players takes into account the impact of a play on the resulting profile of types.

For each player i, history h^t, eigentype $e_j^i(h^t)$ and alternative strategy s'

$$P : u_i\left(s|\, h^t, e_j^i\left(h^t\right), \mu\left(., h^t\right)\right) \geq u_i\left(s'|\, h^t, e_j^i\left(h^t\right), \mu\left(., h^t\right)\right)$$

Definition 1 *A Perfect TI-equilibrium is a pair* (s, μ) *that satisfies conditions P and B0-B2 above.*

Next follows an example showing that a Perfect TI-game equilibrium is not necessarily a Perfect Bayesian equilibrium.

Example

We have 2 players and the following sequence of moves:

stage 1 both players choose simultaneously between Left or Right and

stage 2 player 1 chooses between Up and Down which ends the game. The payoffs are obtained at the end.

Let $E\left(h^0\right) = \{\theta_1, \theta_2, \theta_3\}$ with θ_1 : always prefers to go Left (stubborn or S-type); θ_2 : prefers to do the same as the opponent (coordination or C-type) so if e.g., 2 is expected to go Left with 50% or more chance θ_2 chooses to go Left and Right if the expectation is less than 50% ; $\theta_{3:}$ prefers to do the contrary of the opponent (anti-coordination or AC-type) following the same logic in his best response to expected play as θ_2.

The initial types of the players are

$$|t_1\rangle = \sum_{i=1}^{3} \lambda_i \,|\theta_i\rangle \;\text{ and }\; |t_2\rangle = \sum_{i=1}^{3} \gamma_i \,|\theta_i\rangle\,.$$

for concreteness we set $\lambda_1 = \lambda_2 = \lambda_3 = \sqrt{.33}$ and $\gamma_1 \simeq 0$, $\gamma_2 \simeq \sqrt{0.4}$ and $\gamma_3 = \sqrt{.6}$.

We first consider a case when stage 1 is played alone. There is a pure strategy equilibrium where the eigentypes of each player play as follows:

Player 1 :θ_1^1 plays L, θ_2^1, R, and θ_3^1 plays L, which we write $\left\{\left(\theta_1^1, L\right), \left(\theta_2^1, R\right), \left(\theta_3^1, L\right)\right\}$ and for player 2 we have $\left\{\left(\theta_1^2, L\right), \left(\theta_2^2, L\right), \left(\theta_3^2, R\right)\right\}$.

We check that this is an equilibrium. Player 1 plays L with prob $\lambda_1^2 + \lambda_3^2 > .5$ so player 2's AC-type best-responds with R with probability $\gamma_3^2 = .6 > .5$ so player 1 C-type best-responds with R with probability λ_2^2 while player $2'$s C-type plays L with probability $\gamma_2^2 = .4$. So all eigentypes do play a best-reply to the opponent expected play.

Now consider the whole game where both stages are played. We shall assume that player 1's preferences between Up and Down are the same whatever happened before (at stage 1). So from the point of view of player 1, $GS\left(h^1\right)$ is the same for all h^1 and it is a simple decision situation *DS*. Player 1's preferences and his play does not depend on player 2 but only on his type in $GS\left(h^1\right)$. To simplify we consider only two types: τ_1 who prefers to play Up and τ_2 who prefers to play Down We assume that the *DS* $(GS\left(h^1\right))$ is an operator that does not commute with the $GS\left(h^0\right)$ which means that the eigentype θ_i can be expressed in terms of the eigentypes of the *DS*:

$$\left|\theta_1\right\rangle = \alpha_1\left|\tau_1\right\rangle + \alpha_2\left|\tau_2\right\rangle; \ \left|\theta_2\right\rangle = \beta_1\left|\tau_1\right\rangle + \beta_2\left|\tau_2\right\rangle; \ \left|\theta_3\right\rangle = \delta_1\left|\tau_1\right\rangle + \delta_2\left|\tau_2\right\rangle \tag{21}$$

with δ_1 close to 0 to simplify the presentation.

Assume further that Player 2 (end)payoff depends critically on player 1's stage 2 decision: his payoff of Up *for any path of stage 1* is between 100 and 150 and his payoff of D is 0 for *all* paths of stage 1.

We next show that a Bayesian equilibrium of the classical version of the game is no longer an equilibrium in the TI-game version. In a classical version of the game player 1's type characteristics (preferences) are fixed, they are given by the initial move by Nature. As usual we take the square of the coefficients of superposition to describe the corresponding classical incomplete information model. And we obtain that the ex-ante probability for U is $\text{Prob}(U) = \lambda_1^2\alpha_1^2 + \lambda_2^2\beta_1^2$. It *does not* depend on the play at stage 1 so the equilibrium depicted above is part of a Bayesian equilibrium for the whole game.

What about the TI-model? First, we show that with the equilibrium of stage 1 depicted above we obtain the same outcome as in the classical case. We note that the equilibrium depicted above Player 1's resulting type is $\left|t_1'\right\rangle = \frac{\lambda_1}{\sqrt{\lambda_1^2+\lambda_3^2}}\left|\theta_1\right\rangle + \frac{\lambda_3}{\sqrt{\lambda_1^2+\lambda_3^2}}\left|\theta_3\right\rangle$ with probability $\left(\lambda_1^2 + \lambda_3^2\right)$, and $\left|t_1'\right\rangle = \left|\theta_2\right\rangle$ with probability λ_2^2. Using the expressions in (21) the ex-ante probability for Up equal to

$$\left(\lambda_1^2 + \lambda_3^2\right)\left(\frac{\lambda_1^2}{\lambda_1^2 + \lambda_3^2}\alpha_1^2 + \frac{\lambda_2^2}{\lambda_1^2 + \lambda_3^2}\beta_2^2\right) = \lambda_1^2\alpha_1^2 + \lambda_2^2\beta_1^2.$$

But we claim that under some condition, namely if $2\lambda_1\alpha_1\lambda_2\beta_1 > 0$, *this is no longer an equilibrium of the TI-game.* Player 2 can do better. He can increase the probability for the play of U when he realizes that his play at stage 1 influences the type of his opponent (and thus his subsequent expected play). More specifically player 2's θ_3^2 eigentype realizes that if he plays L then the θ_2^1 of player 1 will choose L and will pool with θ_1^1 (θ_3^1 choose R). The resulting type of player 1 is then $|t_1'\rangle = \frac{\lambda_1}{\sqrt{\lambda_1^2+\lambda_2^2}}|\theta_1\rangle + \frac{\lambda_2}{\sqrt{\lambda_1^2+\lambda_2^2}}|\theta_2\rangle$ with probability $\lambda_1^2+\lambda_2^2$ and θ_3 with probability λ_3^2. This implies an ex-ante probability for the play of U equal to (recall θ_3 never plays U)

$$prob(U;|t_1'\rangle) = \lambda_1^2\alpha_1^2 + \lambda_2^2\beta_1^2 + 2\lambda_1\alpha_1\lambda_2\beta_1 > \lambda_1^2\alpha_1^2 + \lambda_2^2\beta_1^2.$$

In the classical Bayes-Harsanyi version of this game, player 2 has no means of influencing player 1's play at stage 2. This play depends exclusively on the type of player 1, which is known to player 1 from the beginning of the game.

In the TI-version of this game, the expected move of player 2 at stage 1 induces a measurement of player 1's indeterminate type. Different expected moves of player 2 induce different measurement because the eigentypes of player 1 best-reply to player 2's expected play. The impact of the measurement on the type of player 1 is to induce some patterns of separation (or pooling) between eigentypes. In our example when indeterminacy is preserved (by pooling) between θ_1^1, θ_2^1, this gives rise to interference effects in the determination of the probability for *Up.* By construction player 2 strictly (whatever his eigentype) prefers player 1 to play *U.* So player 2 optimal strategy boils down to maximizing the probability for the play of *U.* In our example θ_1^1 and θ_2^1 can interfere positively to increase the probability for the play of U while θ_1^1, θ_3^1, or θ_2^1 and θ_3^1 cannot (because $\delta_1 = 0$). Hence, the optimal strategy of player 2 is to induce θ_1^1 and θ_2^1 to pool at stage 1. This is achieved by the proposed first stage strategy profile $\{(\theta_1^1, L), (\theta_2^1, L), (\theta_3^1, R)\}$ and $\{(\theta_1^2, L), (\theta_2^2, L), (\theta_3^2, L)\}$ with the second stage play being computed mechanically and this is the only Perfect TI–equilibrium of this game.

We note that the proposed strategy at stage 1 $\{(\theta_1^1, L), (\theta_2^1, L), (\theta_3^1, R)\}$ and $\{(\theta_1^2, L), (\theta_2^2, L), (\theta_3^2, L)\}$ are not part of a Bayesian equilibrium of the whole game.

Theorem 1 *In any dynamic TI-game involving more than one action for each player, a Perfect TI-game equilibrium strategy profile is a Perfect Bayesian equilibrium of the corresponding incomplete information dynamic game if and only if the TI-game is a commuting TI-game. Otherwise, the two equilibrium concepts do not coincide.*

We have identified the distinction between the Bayes Harsanyi and the TI-game approaches in the rule for updating beliefs. In the classical approache, it captures learning. While in TI-games, it also implies an opportunity to influence upon the (resulting) type profile. This influence takes the particular form of inducing a pattern of separation and pooling of the currently relevant eigentypes.

5 "Do TI-games have any news for economists?" - The Hidden Variable argument

The title of this section echoes the one of a paper by Levine "Quantum Qames have no News for Econmists"[58]. Can the TI-game approach bring something truly novel to economics in the sense that it cannot be replicated using standard instruments of game theory?

As mentioned in the Introduction, the question of this Section is closely related to the issue of hidden variables in Quantum Mechanics. A main characteristics of quantum theory is its non deterministic character witnessing of the fact that a quantum system cannot be fully determined in the sense that the result of any experiment can be given with certainty. This is the key feature that distinguishes a classical system from a quantum system. This feature of the theory has been heavily criticized "God does not play dices" famously said Einstein. The critics meant that the uncertainty characterizing quantum theory was an expression of its incompleteness, i.e., there were other variables, hidden ones, not accounted for by the theory. As well-known the hidden variable argument was definitely rejected in Physics by Aspect's experiments which exhibited unambiguous quantum entanglement between two particles.(give references).

In our context, the issue of entanglement does not arise however. In fact, as a first step we chose to assume that there is no entanglement (between players). Yet, the TI-model exhibits an essential feature of quantum systems namely Bohr's complementarity of properties(give references). In a TI-model, the type characteristics of a players are complementary in the sense that they cannot be simultaneously actualized (revealed). This is captured by the algebraic structure of the type space including the non-commutativity of the (measurement) operators, the *GS.* There are well-known proofs including that of von Neuman [32], Jauch and Piron [22] and Kochen and Specker [25] showing that it is impossible to embed the algebraic structure of the properties of a quantum system (represented by self-adjoint Hilbert space operators) into the commutative algebra of real-valued functions on a phase space of hidden variables.[59]For an presentation of these proofs see [4]. It is worthwhile noting that the arguments are purely mathematical

[58] We remind that quantum games and TI-games are not the same see introduction.

[59] While the earlier proofs were addressing a relatively limited class of hidden variables, this is not the case of the later ones.

i.e., they do not appeal to any particular physical phenomena. Nevertheless hidden variable theories have been developped see e.g. Bohm and Bub (see [4]). But those theories depart from the classical interpretation of a physical system. In fact, they do not recognize quantum properties as properties pertaining to a system but as properties of a system *in a measurement context.* The hidden variables are multi-valued, they are contextual. So just as Quantum Mechanics, those theories rest on a radical revision of basic principles of physical explanation. Bohm et al. do not contradict the earlier mentioned impossibility results: It is not possible to build a hidden variable theory that recovers quantum theory in the sense that if we knew the (one true) value of the hidden variables we would be able to predict the outcome of any experiment with certainty.

At some very general level Game Theory puts no restriction as to the definition of types. In particular a player's type can depend upon the whole history of past interactions. This truly means that Game Theory is allowing for contextual types, i.e., for the case when type characteristics do not pertain to players but to players in a context. At that level of generality Game Theory is consistent with the approach developped by Bohm. However not much can be done at that level of generality. In nearly all applications assumptions are made allowing for a clear separation between the type of the players and the context in which players interact. Take for instance bargaining games, often players are characterized by a discount factor "how impatient they are" or an "acceptance threshold". The characteristics that define the types are not contextual e.g., the discount factor is exogenously given. This means that the classical paradigm is adopted when modelling economic behavior: players' type (payoff function) is defined separately from the interaction. Thus in economics, the earlier mentioned impossibility results apply: the predictions of TI-games cannot generally be replicated in an extended model with (non-contextual) hidden variables.

In their paper [3] Brandenburger and Yanofsky list a number of properties that hidden variables should satisfied in order to qualify as acceptable candidates for building a theory in the classical sense. We below revisit our example in view of the properties relevant to our context.

5.1 Example revisisted

Let us revisit our example from the previous section "looking for hidden variables" or equivalently trying to provide a standard game theoretical model that yields the same equilibrium predictions as our TI-game.

Let q denote the probability for the play of Up, our Perfect Baysian equilibrium strategy profile is $\left[\left\{\left(\theta_1^1, L\right), \left(\theta_2^1, R\right), \left(\theta_3^1, L\right), q\right\}, \left\{\left(\theta_1^2, L\right), \left(\theta_2^2, L\right), \left(\theta_3^2, R\right)\right\}\right]$ with $q = \lambda_1^2\alpha_1^2 + \lambda_2^2\beta_1^2$. .

Our Perfect TI-equilibrium is $\left[\left\{\left(\theta_1^1, L\right), \left(\theta_2^1, L\right), \left(\theta_3^1, R\right), \right\}, \left\{\left(\theta_1^2, L\right), \left(\theta_2^2, L\right), \left(\theta_3^2, L\right), \left(\tau_1, U\right), \left(\tau_2, D\right)\right\}\right]$ where the probability that player 1 turns out to be a τ_1 type which prefers Up is $prob(U|\, t_1') =$

$\lambda_1^2\alpha_1^2 + \lambda_2^2\beta_1^2 \ + 2\lambda_1\alpha_1\lambda_2\beta_1$.

Comparing the two equilibria, we see that

- the strategies of both players at stage 1 are not the same in the PBE and the PTIE

- the stage 2 expected play of player 1 is not the same in the PBE as in the PTIE.

Below we attempt to modify the classical variant of game by introducing a hidden variable so as to recover the predictions of the TI-game i.e., the PTIE behavior.

Case 1

Consider a case where player 1's payoff depends on the state of the world. We introduce some uncertainty for player 1 relevant to the decision as to whether it is better to play U or D at stage 2. In a standard context this could be modelled as follows. Whatever the type of player 1 $\left(i.e., \theta_1^1 \ - \theta_3^1\right)$,[60] he faces an uncertainty at stage 2 which in the absence of additional information makes him play mixed so the probability for Up is $\lambda_1^2\alpha_1^2 + \lambda_2^2\beta_1^2$ (and for Down $\left(1 - \lambda_1^2\alpha_1^2 + \lambda_2^2\beta_1^2\right)$. The figures are obviously chosen so as allow for direct comparison with the equilibria (*PBE* and *PTIE*) depicted in the example of the previous section.

Assume next that player 2 has information about the state of the world relevant to player 1. Player 2 can try to transmit some information through his own play at stage 1. For some reason, player 1 believes that if player 2 plays R, it signals that he should rather play D while if he plays L it may signal that the play of U is more profitable.

The first point we want to make is that the profile $\left[\left\{\left(\theta_1^1, L\right), \left(\theta_2^1, R\right), \left(\theta_3^1, L\right)\right\}, \left\{\left(\theta_1^2, L\right), \left(\theta_2^2, L\right), \left(\theta_3^2, R\right)\right\}\right]$ is no longer part of an equilibrium. Indeed in that equilibrium player 2 plays R with probability .6 which prompts a decrease in the probability for the play of Up at stage 2. But as we know player 2 always prefers player 1 to play Up. That is whatever the information about the state of the world that player 2 has he wants to let player 1 believe that he should play Up.

In this game player 2 can affect the play of player 1 with his play at stage 1 as in the TI-game. Since it is optimal for him to maximize the probability for the play of Up, he will never play R and the first stage PTIE equilibrium strategy profile $\left[\left\{\left(\theta_1^1, L\right), \left(\theta_2^1, L\right), \left(\theta_3^1, R\right)\right\}, \left\{\left(\theta_1^2, L\right), \left(\theta_2^2, L\right), \left(\theta_3^2, L\right)\right\}\right]$ becomes part of the PBE. Now the question is whether player 1 should modify his beliefs when he observes L. Our point is that he knows that whatever the state of the world player 2 wants him to believe he should increase his play of Up which he might do when observing L. But this means that the play of L is not informative since player 2 choose L whatever the state of the world. Consequently player 1 does not update his beliefs after observing L and he plays Up with probability $q < q'$. Hence we do not recover the *PTIE*.

[60] In this classical variant of the model, there are only 3 types and some uncertainty about the state of the world.

Case 2.

Consider another hidden variable model. In this model, player 1 learns something about the state of the world of relevance for stage 2 as he plays stage 1. More precisely what counts is what the cooperative type θ_2^1 plays. If the cooperative type plays R, player 1 learns (on avarage) nothing. But if the cooperative type plays L, player 1 may learn that Up is more likely to be profitable. Again in this model player 2 can influence player 1's play by inducing a best reply of L from the cooperative type. This happens in the PTIE first stage strategy profile. In contrast with case 1, here player 2 can increase the expected play of Up, because if when θ_2^1 plays L player 1 learns some reliable information that prompts an increase in his propensity to play Up. By properly callibrating the impact of the signal, we could fit the PTIE equilibrium predictions.

This hidden variable model with "learning" is thus able to reproduce the TI-game predictions. There is a problem however namely it violates the so-called λ-independence property [3]. "This property says that the nature of the particle (here the agent defined by his preferences) - as determined by the value of the hidden variable - does not depend on the experiment conducted. There is, in a sense, no dependence between the observer and the observed system."

Recall now in TI-games different play of player 2 corresponds to different experiments, GS, that measure the type of player 1 (because he best-replies). Now in our model the value of the hidden variable is not unique. In one experiment i.e., when player 1 best-replies to $\left\{\left(\theta_1^2, L\right), \left(\theta_2^2, L\right), \left(\theta_3^2, R\right)\right\}$ it yields a value which convey no information about the state of the world i.e., no change in preferences. In the other experiment where agent 1 best-replies to $\left\{\left(\theta_1^2, L\right), \left(\theta_2^2, L\right), \left(\theta_3^2, L\right)\right\}$ it takes another value which prompts a change in preferences with respect to Up and Down. We are dealing with a multiple valued hidden variable à la Bohm which is not consistent with a standard classical model.

Clearly our investigation is not exhaustive in any sense but it suggests that reasonable hidden variable models may be difficult to find when requesting that the hidden variables satisfy some desirable properties.

6 Possible fields of application for TI-games

We have learned that TI-games may bring forth new results in the context of multi-stage games or when a game is preceded by some form of "pre-play". We conjecture that the Type Indeterminacy approach may bring new light on the following issues:

- Players' choice of selection principle in multiple equilibria situation;

In Camerer [10], the author reports about experiments where a pre-play auction impacts on the principle of selection among multiple equilibria in a coordination game. A pre-play auction for the right to play a coordination game tends to push toward the payoff-dominant equilibrium compared with the no pre-play case. In a TI-game, preferences with respect to the equilibrium selection criteria can be modified by pre-play.

- The selection of a reference point;

According to experiments (see) playing a contest before an ultimatum game can affect the equilibrium offer and acceptance threshold. In a TI-game the pre-play of a contest may change the preferences of the players with respect to what they feel entitled to in an ultimatum game played next.

- The sunk cost fallacy;

According to numerous experiments and casual evidence people seem to be the victims of the sunk cost fallacy. In an experiment, people who were offered a year subscription to the theater showed (on average) a greater propensity to go to the theater than people who were not offered subscription. In a TI-game the decision to purchase a subscription may modify people's valuation of theater plays.

- Path-dependency;

A single (little probable) move can radically modify the type of a player with significative implications for the path of future play.

We believe that that TI-games have a rich potential to explain a variety of puzzles in (sequential) interactive situations and to give new impulses to game theory.

7 Concluding remarks

This paper constitutes a first step in the development of a theory of games with type indeterminate players. In a first part we introduce TI-games by means of an example where we show how in a TI-game cheap-talk promises can have impact on subsequent play when standard theory predicts no impact. Compared with conventional game theory the TI approach amounts to substituting the standard Harsanyi type space for a Hilbert space. We show that this has no implication for the analysis of static games. In contrast in a multi-move context, we must define an updating rule consistent with the algebraic structure of our type space. We show that for non-commuting TI-games, it implies that players can manipulate each others' type thereby extending the field of strategic interaction. Using the new updating rule we define an equilibrium concept similar to the Perfect Bayesian equilibrium. We call it Perfect TI-equilibrium. We provide an example showing how the two concepts differ.

We discuss whether TI-game have any news for economist in light of the discussion on hidden variables in Physics. We conclude that at its most general level game theory is consistent with any form of contextual type including those corresponding to quantum indeterminacy. However most applications in particular in economics adopt the classical paradigm which posits that players can be defined independently from the interaction to which they participate. In that context, TI-games do bring in novelties that cannot always be reproduced when extending the game with classical signals.

A restriction so far is that we confine ourselves to objective beliefs. Nevertheless we learned that a coherent theory can be built with Type Indeterminate players and that it generates new equilibrium predictions that cannot be replicated extending the model with standard hidden variables. Our next step will be to introduce subjective beliefs which in the context of Type Indeterminacy opens up for new fascinating issues (as suggest in the brief discussion on p.6).

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