Envy-free and Incentive Compatible division of a commodity

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Abstract

This article proposes a new mechanism for allocating a divisible commodity to a number of buyers. Buyers are assumed to behave as price-anticipators rather than as price-takers. The proposed mechanism is as parsimonious as possible, in the sense that it requests participants to report a single-dimensional message instead of an entire utility function, as requested by VCG mechanisms. This article shows that this mechanism yields efficient allocations in Nash equilibria, and moreover, that these equilibria are envy-free. Additionally, this paper presents distinct results that this mechanism is the only simple VCG-like mechanism that both implements efficient Nash equilibria and satisfies the No Envy axiom of fairness. Furthermore, the mechanism's Nash equilibria are proven to satisfy the fairness properties of both Ranking and Voluntary Participation.

1 Introduction

This paper investigates the problem of allocating an infinitely divisible object to a finite number of buyers. Examples of divisible commodity allocation can be found in auctions of Treasury notes (Back and Zender (1993), Keloharju, Nyborg and Rydqvist (2005)), the sale of communication network capacity, the design of electricity markets (Green and Newbery (1992), Ausubel (2006)) or auctions for spectrum licenses (Levin (1966)).

In this paper, we assume that each buyer has a quasi-linear preference and participates in a game promoted by a mechanism. Each participant submits a one-dimensional bid (also known as message or signal) to a mechanism. Once all bids have been collected, the mechanism determines both the allocation of resources and the payment scheme for each participant. Nash equilibrium points are considered to be predictors of the behavior of agents.

This model has received intensive attention from computer scientists interested in designing network capacity allocation mechanisms (Maheswaran and Basar (2005, 2006), Johari and Tsitsiklis (2004), Hajek and Yang (2004), Kelly (1997) and Kelly et al. (1998)). Researchers focused on achieving incentive compatibility of divisible commodity allocation mechanisms have recently constructed some remarkable mechanisms.

SSVCG mechanisms by Johari and Tsitsiklis (2007); g-mechanisms by Yang and Hajek (2005, 2006a); and VCG-Kelly mechanisms (Yang and Hajek (2006b)) have proven to implement efficient Nash equilibria.

The mechanisms discovered by these researchers adopt single dimensional message spaces and differentiated unit prices, similar to the mechanism proposed in this article. They are similar in both spirit and form to the Vickrey-Clarke-Groves mechanisms (VCG mechanisms, Green and Laffont (1979)) except that each individual reports a one-dimensional message, rather than his or her entire utility function. For this reason, this group of mechanisms will be called *pseudo*-VCG mechanisms.[1]

Although research into pseudo-VCG mechanisms has proven that they achieve efficiency and incentive compatibility, very few articles concerning the fairness of pseudo-VCG mechanisms have been written. Since a mechanism designer is concerned with the welfare or happiness of the participants, the question of whether a mechanism's implemented allocation is fair enough to meet every individual's need for justice is an important issue. This normative problem has led to the formation of the *No Envy* axiom, a central standard of fairness in mechanism design theory (Thomson (2007)).

Maskin (1999) and Fleurbaey and Maniquet (1997) show that, for a preference satisfying *monotonic closedness*[2], the No Envy axiom is satisfied if an allocation rule has Nash implementability in addition to *equal treatment of equals*. Unfortunately, quasi-linear preferences are not monotonically closed, rendering Maskin's, and Fluerbaey and Maniquet's promising results inapplicable. Likewise, Zhang (2005) and Feldman et al (2005) study a modified version of the No Envy axiom, *c-approximate envy-freeness*[3], but their results are applicable only to cases of multiple resource allocation. You (2008a) explains that many otherwise-remarkable pseudo-VCG mechanisms fail No Envy tests for every utility profile, or for commonly assumed utility functions. Examining pseudo-VCG mechanisms for the No Envy (or *Envy-freeness*) property in an environment of infinitely divisible goods is a difficult task.

Despite the barrier to success, this paper reaches a positive result. It proposes a group of pseudo-VCG mechanisms that not only implement efficient Nash equilibria, but also satisfy the Envy-free axiom. Furthermore, this group of mechanisms assigns simply-formed payment rules to agents and satisfies the *Ranking* and *Voluntary Participation* fairness properties. We call this group of mechanisms the *Simple Envy-free VCG*

[1] These pseudo-VCG mechanisms have attractive features: informational burden is low in pseudo-VCG mechanisms compared to the size of information in VCG mechanisms, since the latter require reports of infinite dimensional vectors in divisible commodity allocation. Differentiated prices allow these mechanisms to achieve efficiency while mechanisms with uniform price (Johari and Tsitsiklis (2004), Yang and Hajek (2004)) fail to implement efficient allocations.

[2] Let X denote an agent's consumption set with typical elements $a, b, ...,$ and $\mathcal{R}$ denote the domain of admissible preferences over X. We define *Monotonic Closedness* as the follows: $\forall R, R' \in \mathcal{R}$, $\forall a, b \in X$ such that aPb, $\exists R_i'' \in \mathcal{R}$, $\forall c \in X$, (i) $aR'c \Rightarrow aR''c$, $(ii) bRc \Rightarrow bR''c$, and $(iii) \sim (aI''b)$.

[3] c-approximately envy-free is defined as follows: let $\rho(x) = \min_{i \neq j} \frac{u_i(x_i)}{u_i(x_j)}$. When $\rho(x) \geq 1$, the allocation x is known as an envy-free allocation. We call a mechanism c-approximately envy-free if for any x, $\rho(x) \geq c$.

mechanisms (SEF-VCG mechanisms). The SEF-VCG mechanisms are characterized as a group of pseudo-VCG mechanisms in which resource allocation is proportional to signals and payment to each agent is linear in his or her signal. In addition, we confirm that the SEF-VCG mechanisms can be shown as a group of SSVCG mechanisms.

The remainder of the paper is organized in the following manner. In Section 2, we describe the model. In Section 3, we construct the SEF-VCG mechanisms and show their incentive compatibility and fairness properties. Characterizations of the SEF-VCG mechanisms are illustrated in Section 4. In Section 5, we delve into the three most notable of the pseudo-VCG mechanisms mentioned earlier, and discuss the relationship between the three and the SEF-VCG mechanisms, demonstrating how the three mechanisms fail No Envy tests, while the SEF-VCG mechanisms realize envy-free allocations. In the final section, we qualify the need for future research dealing with both the identification of more general classes of Envy-free pseudo-VCG mechanisms, and the investigation of budget imbalances in the pseudo-VCG mechanisms.

2 Model

The objective of a mechanism is to allocate a fixed amount of a divisible resource to a finite number of agents efficiently. Let $n \geq 2$ be the number of agents and the set of agents be denoted as $N = \{1, \cdots, n\}$. Suppose that the total amount of the resource is $C > 0$.

When agent i receives his or her resource share from the mechanism, the monetary value of the share is represented by a utility function, u_i, that is continuous, strictly increasing, concave, and continuously differentiable on $[0, +\infty)$. Let $u_i(0) = 0$ for each $i \in N$.

The mechanism attempts to maximize the sum of agents' utilities through the allocation of a resource. If it achieves the goal by choosing allocation x, that is, $x \in \text{argmax}_{x \in \mathcal{X}} \sum_{i \in N} u_i(x_i)$ where $\mathcal{X} = \{x : \sum_{l \in N} x_l \leq C,\ x_i \geq 0 \quad \text{for all } i \in N\}$, the resource allocation x is said to be *efficient*.

For this model, however, an individual's utility function is unknown to the mechanism, so a method is needed to create efficient allocations. For this, the mechanism requests each agent i to submit a single dimensional bid θ_i such that $\theta_i \in [0, +\infty)$, and then collects these bids $\theta = (\theta_1, \cdots, \theta_n)$. With all collected bids, the mechanism decides the resource allocation to every agent and the payment scheme for each participant. Therefore, a mechanism consists of a triple (Θ, x, t) where Θ is the set of allowable strategies of the form $\theta = (\theta_1, \cdots, \theta_n)$, x is the allocation rule, and t is the payment scheme.

3 The SEF-VCG mechanism

The SEF-VCG mechanisms which this paper is concerned with is constructed in the following way. Resource allocation is determined to be proportional to bids and so the allocation to individual i is $x_i = \frac{\theta_i}{\theta_N}C$ where $\theta_N = \sum_{l \in N} \theta_l$. The payment scheme t_i assigned to each i is $t_i = \theta_i \theta_{N \setminus i} - S_{-i}$ where $\theta_{N \setminus i} = \theta_N - \theta_i, S = \sum_{i \in N} \theta_i^2$, and $S_{-i} = S - \theta_i^2$. Additionally, let $\theta_N^2 = (\theta_N)^2$. Payment of some agents can be negative, which means that they are subsidized through the mechanism by others or directly from the mechanism. In this mechanism, agent i's net utility from submitting θ_i is

$$p_i(\theta_i, \theta_{-i}) = u_i(\frac{\theta_i}{\theta_N}C) - \theta_i \theta_{N \setminus i} + S_{-i}.$$

where $\theta_{-i} = (\theta_1, \cdots, \theta_{i-1}, \theta_{i+1}, \cdots, \theta_n)$ are submitted by others. Let us call this mechanism *the Simple Envy-free VCG mechanism*(SEF-VCG mechanism).

Each agent i tries to maximize his or her net utility by selecting θ_i based on a unilateral decision. We define a *Nash equilibrium* as follows: θ is Nash equilibrium if and only if for every $i \in N$,

$$p_i(\theta_i, \theta_{-i}) \geq p_i(\theta_i', \theta_{-i})$$

for every $\theta_i' \in \mathcal{R}_+$. Nash equilibrium θ is efficient if the resource allocation $x(\theta)$ is efficient.

To see if a Nash equilibrium exists in this mechanism, we consider a case where every agent submits 0 to the mechanism. If agent 1 changes his strategy from 0 to ϵ such that $\epsilon > 0$, then his or her net utility becomes $u_1(C)$, which is positive, while his or her net utility is 0 when staying with $\theta_1 = 0$. Thus, $\theta = 0$ is not a Nash equilibrium.

Now, suppose that agent i submits $\epsilon > 0$ while all others submit 0. Agent i has no incentive to change his or her strategy ϵ to ϵ' such that $\epsilon' \neq \epsilon, > 0$, since his or her net utility remains the same, $u_i(C)$. If agent $j \neq i$ changes his or her strategy from 0 to δ, then his or her net utility becomes $p_j(\delta, \theta_{-j}) = u_j(\frac{\delta}{\delta+\epsilon}C) - \delta\epsilon + \epsilon^2$, while his or her previous utility is $p_j(0, \theta_{-j}) = \epsilon^2$. Thus, if $u_j(\frac{\delta}{\delta+\epsilon}C) - \delta\epsilon \leq 0$, j will keep his or her strategy at 0. Let $f_j(\delta) = u_j(\frac{\delta}{\delta+\epsilon}C) - \delta\epsilon$. Then, $f_j(0) = 0$ and

$$\begin{aligned} f_j'(\delta) &= u_j'(\frac{\delta}{\delta+\epsilon}C)\frac{\epsilon}{(\delta+\epsilon)^2} - \epsilon \\ &= \frac{\epsilon}{(\delta+\epsilon)^2}[u_j'(\frac{\delta}{\delta+\epsilon}C) - (\delta+\epsilon)^2]. \end{aligned}$$

$u_j'(\frac{\delta}{\delta+\epsilon}C) - (\delta+\epsilon)^2$ is nonincreasing in δ and $\frac{\epsilon}{(\epsilon+\delta)^2} > 0$. Thus, $u_j(\frac{\delta}{\delta+\epsilon}C) - \delta\epsilon \leq 0$, if and only if $u_j'(0) - \epsilon^2 \leq 0$. Consequently, if $u_j'(0) \leq \epsilon^2$ for every $j \neq i$, then $\theta_i = \epsilon > 0$, $\theta_j = 0$ for all $j \neq i$ is Nash equilibrium. There are infinitely many Nash equilibria having this form.

However, equilibria which give the entirety of a resource to one agent are typically not efficient. Consider the following: there are two agents and a resource with a total amount equal to 1. The preference of these participants are described as $u_1(x) = 5x$ and $u_2(x) = 2x$. Suppose that agent 1 submits $\theta = 0$ and agent 2 submits $\theta_2 > 0$. If $\theta_2^2 \geq 5$, this is an inefficient Nash equilbrium since agent 2 takes all the resource. Certainly, for the case in which there is only one agent with a positive strategy, Nash equilibria may exist, but they are generally not efficient. As long as all other agents $j's$ have finite $u_j'(0)$, $j \neq i$, agent i has an opportunity to take all the entirety of the resource resulting in an inefficient equilibrium. Thus, to prevent an inefficient equilibrium, for each agent, there should be at least one other agent with $u'(0) = +\infty$. From this example, we can create the following assumption that ensures that there are at least two agents whose strategies are positive.

Assumption 1. $u_i'(0) = \infty$ for at least two agents.

With assumption 1, we show that the SEF-VCG mechanism has Nash equilibria and that all of its equilibria are efficient.

Proposition 1. *The SEF-VCG mechanism has Nash equilibria and they are efficient.*

Proof. The net utility of agent i with strategy θ_i when others submit θ_{-i} is

$$p_i(\theta_i, \theta_{-i}) = u_i(\frac{\theta_i}{\theta_N}C) - \theta_i\theta_{N\backslash i} + S_{-i}.$$

Note that for each agent i, $\theta_{N\backslash i} \neq 0$. Given θ_{-i}, agent i tries to maximize $p_i(\theta_i, \theta_{-i})$ given θ_{-i} and p_i is continuous and concave in θ_i. Therefore, the first order condition (FOC) is sufficient and necessary condition to find Nash equilibria. The condition is

$$\begin{aligned} u_i'(\frac{\theta_i}{\theta_N}C)\frac{\theta_{N\backslash i}}{\theta_N^2}C - \theta_{N\backslash i} &= 0 \quad \text{if } \theta_i > 0 \\ u_i'(\frac{\theta_i}{\theta_N}C)\frac{\theta_{N\backslash i}}{\theta_N^2}C - \theta_{N\backslash i} &\leq 0 \quad \text{if } \theta_i = 0. \end{aligned}$$

Since $\theta_{N\backslash i} > 0$, this conditions equals

$$\begin{aligned} u_i'(\frac{\theta_i}{\theta_N}C) &= \frac{\theta_N^2}{C} \quad \text{if } \theta_i > 0 \\ u_i'(\frac{\theta_i}{\theta_N}C) &\leq \frac{\theta_N^2}{C} \quad \text{if } \theta_i = 0. \end{aligned}$$

Let $\mu = \frac{\theta_N^2}{C}$ and $x_i = \frac{\theta_i}{\theta_N}C$ for $\forall i \in N$. Then the FOC is rewritten as

$$\begin{aligned} u_i'(x_i) &= \mu \quad \text{if } x_i > 0 \\ u_i'(x_i) &\leq \mu \quad \text{if } x_i = 0. \end{aligned}$$

Thus, θ is a Nash equilibrium if and only if for all $i \in N$, we have

$$\begin{aligned} u_i'(x_i) = \mu \quad &\text{if } x_i > 0 \\ u_i'(x_i) \leq \mu \quad &\text{if } x_i = 0 \end{aligned}$$

where $\mu = \frac{\theta_N^2}{C}$, $x_i = \frac{\theta_i}{\theta_N}C$, and $\sum_{i\in N} x_i \leq C$. we know allocation x^* is efficient if and only if it satisfies $x^* \in \text{argmax}_{x\in\mathcal{X}} \sum_{i\in N} u_i(x_i)$. Since $\sum_{i\in N} u_i(x_i)$ is continuous in x and $\mathcal{X}$ is compact, efficient allocations exist. Also, $\sum_{i\in N} u_i(x_i)$ is concave, so the sufficient and necessary, FOC is

$$\begin{aligned} u_i'(x_i^*) &= \lambda \quad \text{if } x_i^* > 0 \\ u_i'(x_i^*) &\leq \lambda \quad \text{if } x_i^* = 0 \end{aligned}$$

where $\lambda > 0$. We can show that $\mu = \lambda$.

For the sake of contradiction, suppose that $\mu > \lambda$. Denote an allocation at equilibrium by x and an efficient allocation by x^*. Choose i such that $x_i > 0$. Then, $u_i'(x_i) = \mu > \lambda \geq u_i'(x_i^*)$. This implies $x_i < x_i^*$, so $C - x_i = \sum_{j\neq i} x_j > C - x_i^* = \sum_{j\neq i} x_j^*$. If this is the case, there should be $j \neq i$ such that $x_j > x_j^*$ and we have $u_j'(x_j) \leq u_j'(x_j^*)$. Since $x_j^* \geq 0$, we have $x_j > 0$ and $\mu = u_j'(x_j) \leq u_j'(x_j^*) \leq \lambda$. Hence, $\mu \leq \lambda$ and this contradicts to the previous assumption. Therefore, $\mu = \lambda$. We conclude two FOC's are indeed the same, so that θ is Nash if and only if $x = x(\theta)$ is an efficient allocation. The existence of efficient allocations also guarantees that of Nash equilibria. Therefore, Nash equilibria exist and they are efficient, as desired. ■

The SEF-VCG mechanism may have multiple Nash equilibria depending on utility functions. An example of multiple equilibria is given as follows:

Example 1. Multiple Nash equilibria

Let the number of agents be two. If each agent's utility function has a constant slope over a part of the domain as we see in the following graph, we can have multiple equilibria. Let u_1 and u_2 have the same constant slope over $[x_1, \frac{C}{2}]$ and $[\frac{C}{2}, x_2]$, respectively, where $x_1 + x_2 = C$. Then, we have $u_1'(x_1) = u_2'(x_2)$, and $x = (x_1, x_2)$ is a Nash equilibrium allocation. Therefore, there is a pair of equilibrium strategies θ_1, θ_2 which satisfies $x_1 = \frac{\theta_1}{\theta_1+\theta_2}C$ and $x_2 = \frac{\theta_2}{\theta_1+\theta_2}C$. Likewise, if $Q_1 \in [x_1, \frac{C}{2}]$ and $Q_2 \in [\frac{C}{2}, x_2]$ with $Q_1 + Q_2 = C$, we again have $u_1'(Q_1) = u_2'(Q_2)$, and there is a pair of equilibrium strategies θ_1' and θ_2' which satisfies $Q_1 = \frac{\theta_1'}{\theta_1'+\theta_2'}C$ and $Q_2 = \frac{\theta_2'}{\theta_1'+\theta_2'}C$. We can find infinitely many equilibria in this example.

The SEF-VCG mechanism is not only efficient, but also satisfies fairness properties. If an agent receives a bigger share of the resource than the other agents, he or she has to pay a greater amount than the others. This notion is represented as *Ranking* (RK): if $x_i \leq x_j$, then $t_i \leq t_j$ for any $i \neq j \in N$. Individuals are not forced to participate in the mechanism if they would be made worse off through participation. If equilibrium allocation satisfies this property, the mechanism is said to satisfy *voluntary participation*. We assume that $u_i(0) = 0$ for any $i \in N$ and $x_i = t_i = 0$ if agent i doesn't submit any bid. Then, *Voluntary Participation* is expressed as: if each agent $i \in N$ has net utility $p_i(\theta)$ which is nonnegative at equilibrium θ, the mechanism satisfies Voluntary Participation (VP).

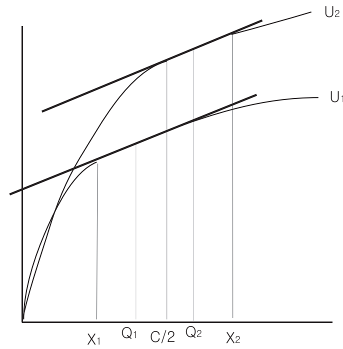

Figure 1: An Example of Multiple Equilibria

Additionally, the mechanism is *envy-free*, or satisfies *No Envy*, if no agent envies any others at equilibrium. Agent i doesn't envy agent j if his own equilibrium allocation of resource and payment gives net utility at least as high as his net utility from the case in which he receives agent j's share and payment allocation instead. The envy free state of agent i against agent j is written as $u_i(x_i) - t_i \geq u_i(x_j) - t_j$ at equilibrium allocation. If this relationship holds for every pair in N, the mechanism is envy free.

Proposition 2. *(i) The SEF-VCG mechanism satisfies Ranking, (ii) achieves Voluntary Participation, and (iii) guarantees No Envy.*

Proof. (i) Suppose that $x_i \leq x_j$, which is equivalent to $\theta_i \leq \theta_j$. Remember that $t_i = \theta_i \theta_{N \setminus i} - S_{-i}$. Then,

$$\begin{aligned} t_i - t_j &= \theta_i \theta_{N\setminus i} - S_{-i} - \theta_j \theta_{N \setminus j} + S_{-j} \\ &= \theta_N(\theta_i - \theta_j). \end{aligned}$$

Thus $t_i \leq t_j$ if and only if $\theta_i \leq \theta_j (x_i \leq x_j)$.

(ii) Since $p_i(\theta_i, \theta_{-i})$ is concave in θ_i, it is sufficient to check if $p_i(0, \theta_{-i}) \geq 0$. We see $p_i(0, \theta_{-i}) = S_{-i} > 0$ and so VP holds.

(iii) Agent i doesn't envy j if and only if $u_i(x_i) - u_i(x_j) \geq t_i - t_j$ at the equilibrium allocation (x, t). This inequality equals $u_i(x_i) - u_i(x_j) \geq \theta_N(\theta_i - \theta_j)$ which can be rewritten as

$$u_i(x_i) - u_i(x_j) \geq \frac{\theta_N^2}{C}(x_i - x_j)$$

since $\theta_i = x_i \frac{\theta_N}{C}$. If $x_i = x_j$, then $t_i = t_j$ and there is no envy. If $x_i < x_j$, there is no envy if and only if $\frac{u_i(x_i) - u_i(x_j)}{x_i - x_j} \leq \frac{\theta_N^2}{C}$. Due to the concavity of u_i, we have the equation $\frac{u_i(x_i) - u_i(x_j)}{x_i - x_j} \leq u_i'(x_i)$ and the equilibrium condition gives $u_i'(x_i) \leq \frac{\theta_N^2}{C}$. Thus, no envy condition holds. If $x_i > x_j$, then $x_i > 0$ and, at equilibrium, $u_i'(x_i) = \frac{\theta_N^2}{C}$. No envy holds if and only if $\frac{u_i(x_i) - u_i(x_j)}{x_i - x_j} \geq \frac{\theta_N^2}{C}$. Concave u_i implies $\frac{u_i(x_i) - u_i(x_j)}{x_i - x_j} \geq u_i'(x_i)$ and therefore, envy free condition holds. ■

However, the SEF-VCG mechanism yields a Budget Deficit (BD) or budget balance.

Proposition 3. *The SEF-VCG mechanism yields a budget deficit which can range from 0 to $C\lambda(n-1)$ where λ is the market clearing price for price taking buyers. When every agent has the same utility function so that each agent submits the same strategy, $\theta = (\alpha, \cdots, \alpha)$, the mechanism's budget is balanced.*

Proof. The mechanism collects $\sum_{i \in N} t_i$ and we have

$$\begin{aligned}
\sum_{i \in N} t_i &= \sum_{i \in N} [\theta_i \theta_{N \setminus i} - S_{-i}] \\
&= \sum_{i \in N} [\theta_i \theta_N - \theta_i^2 - S_{-i}] \\
&= \sum_{i \in N} [\theta_i \theta_N - S] \\
&= \sum_{i \in N} \theta_i \theta_N - nS \\
&= \theta_N^2 - nS \\
&= (\sum_{i \in N} \theta_i)^2 - n \sum_{i \in N} \theta_i^2 \\
&\leq 0.
\end{aligned}$$

The last inequality holds due to the Cauchy-Schwartz inequality. Thus, the SEF-VCG mechanism always yields a budget deficit.

Budget deficit (BD) $= n \sum_{i \in N} \theta_i^2 - (\sum_{i \in N} \theta_i)^2$. Substituting $\theta_i = x_i \frac{\theta_N}{C}$,

$$\begin{aligned}
BD &= n(\sum_{i \in N} x_i^2) \frac{\theta_N^2}{C} - \frac{\theta_N^2}{C} (\sum_{i \in N} x_i)^2 \\
&= n(\sum_{i \in N} x_i^2) \frac{\theta_N^2}{C^2} - \theta_N^2.
\end{aligned}$$

There is a $i \in N$ with $x_i > 0$ and so $u_i'(x_i) = \frac{\theta_N^2}{C}$.

$$\begin{aligned}
BD &= \frac{n}{C} u_i'(x_i) \sum_{i \in N} x_i^2 - C u_i'(x_i) \\
&= \frac{n}{C} u_i'(x_i) [\sum_{i \in N} x_i^2 - \frac{C^2}{n}] \\
&= \frac{n}{C} \lambda [\sum_{i \in N} x_i^2 - \frac{C^2}{n}].
\end{aligned}$$

Observations:

(a) If $u_i = u_j$ for all $i \neq j$ and $i, j \in N$, then $\theta_i = \theta_j$ for all $i \neq j$ and $x_i = \frac{C}{n}$ for all $i \in N$. Then, it is easy to check that the mechanism has a balanced budget.

(b) Note that the supremum of $\sum_{i \in N} x_i^2$ for $x \in \mathcal{X}$ is achieved at extreme points of $x \in \mathcal{X}$. Then, we have

$$\begin{aligned} BD &= \frac{n}{C}\lambda[\sum_{i \in N} x_i^2 - \frac{C^2}{n}] \\ &\leq \frac{n}{C}\lambda(C^2 - \frac{C^2}{n}) \\ &= \lambda C(n-1). \end{aligned}$$

■

4 Generalized SEF-VCG mechanism

We can generalize the SEF-VCG mechanism by modifying its payment scheme. Depending on specific parameters of a generalized Simple mechanism, we can reduce the possibility of a large budget deficit to a negligible amount.

Definition 1. The *SEF-VCG*(k, γ) is a mechanism which allocates a resource proportionally to strategies such that $x_i = \frac{\theta_i}{\theta_N} C$ for each $i \in N$ and requests agent i to pay $t_i = k\theta_i\theta_{N\setminus i} - kS_{-i} + \gamma$ where $k, \gamma > 0$.

According to this definition, the previously discussed Simple mechanism is renamed SEF-VCG(1,0). We can show that any SEF-VCG(k, γ) mechanism shares the same properties to the SEF-VCG(1,0).

Theorem 1. *A SEF-VCG*(k, γ) *mechanism Nash implements efficient allocations and satisfies Ranking and No Envy. In addition, if* $\gamma = 0$*, the mechanism satisfies Voluntary Participation and its budget deficit will equal the SEF-VCG(1,0) mechanism's.*

If $\gamma \neq 0$, Voluntary Participation may fail, but the mechanism may have a budget surplus. Since the proof of Theorem 1 is basically the same as that of the SEF-VCG(1,0) mechanism, we will omit it here.

We can show that the generalized SEF-VCG mechanism is characterized by the combination of efficient Nash implementation proportional to agents' strategies and No Envy fairness. Consider the following assumptions on a mechanism (Θ, x, t).

Assumption S1. There are at least two agents. The set of strategies Θ equals $\mathcal{R}_+^N$, and the allocation is proportional to submitted strategies: $x_i = \frac{\theta_i}{\theta_N} C$ if $\theta \neq 0$ and $x_i = 0$ if $\theta = 0$.

Assumption S2. The symmetric payment by agent i is the sum of a variable price in θ_i and a fixed price independent of θ_i: $t_i = g_i(\theta) + h(\theta_{-i})$ where $g_i(\theta)$ is affine in θ_i, i.e., $g_i(\theta) = \theta_i \alpha(\theta_{-i}) + \beta(\theta_{-i})$.

Assumption S3. For any utility profiles $u_1, \cdots, u_n$ such that each u_i is strictly increasing, concave, continuous and continuously differentiable for all $i \in N$, a Nash equilibrium exists and it is efficient.

Assumption S4. The mechanism's Nash equilibria are envy-free.

Proposition 4. *Suppose (Θ, x, t) is a mechanism satisfying Assumptions $S1 - S4$. Then it is a SEF-VCG(k,γ) mechanim for some constants k and γ where $k, \gamma > 0$.*

Proof. With these assumptions, agent i's net utility is $p_i(\theta_i, \theta_{-i}) = u_i(\frac{\theta_i}{\theta_N}C) - g_i(\theta) - h(\theta_{-i})$. p_i is concave in θ_i since g_i is linear in θ_i. For Nash equilibrium, θ is Nash if and only if we have

$$
\begin{aligned}
u_i'(\frac{\theta_i}{\theta_N}C)\frac{\theta_{N\setminus i}}{\theta_N^2}C &= g_i'(\theta) \quad \text{if } \theta_i > 0 \\
u_i'(\frac{\theta_i}{\theta_N}C)\frac{\theta_{N\setminus i}}{\theta_N^2}C &\leq g_i'(\theta) \quad \text{if } \theta_i = 0
\end{aligned}
$$

where $g_i'(\theta) = \frac{\partial g_i(\theta)}{\partial \theta_i}$. With $x_i = \frac{\theta_i}{\theta_N}C$, this FOC condition equals

$$
\begin{aligned}
u_i'(x_i) &= g_i'(\theta)\frac{\theta_N^2}{\theta_{N\setminus i}C} \quad \text{if } \theta_i > 0 \\
u_i'(x_i) &\leq g_i'(\theta)\frac{\theta_N^2}{\theta_{N\setminus i}C} \quad \text{if } \theta_i = 0.
\end{aligned}
$$

Since every Nash equilibrium is efficient, we should have

$$
\lambda = g_i'(\theta)\frac{\theta_N^2}{\theta_{N\setminus i}C}
$$

where λ is the market clearing price for price taking buyers and $\lambda > 0$. Since $g_i(\theta) = \theta_i \alpha(\theta_{-i}) + \beta(\theta_{-i})$ for $\alpha(\theta_{-i}) > 0$, $g_i'(\theta) = \alpha(\theta_{-i})$. Again, $\lambda = \alpha(\theta_{-i})\frac{\theta_N^2}{\theta_{N\setminus i}C}$ for all $i \in N$ implies $\alpha(\theta_{-i}) = k\theta_{N\setminus i}$ where k is a positive constant. Thus, $t_i = k\theta_{N\setminus i}\theta_i + \beta(\theta_{-i}) + h(\theta_{-i})$. Now any equilibria should be envy free, i.e., we have

$$
u_i(x_i) - u_i(x_j) \geq t_i - t_j = k(\theta_{N\setminus i}\theta_i - \theta_{N\setminus j}\theta_j) + h(\theta_{-i}) - h(\theta_{-j}) + \beta(\theta_{-i}) - \beta(\theta_{-j})
$$

for any $i \neq j$. From $x_i = \frac{\theta_i}{\theta_N}C$, $x_i - x_j = (\theta_i - \theta_j)\frac{C}{\theta_N}$. If $\theta_i = \theta_j$, then $x_i = x_j$, $t_i = t_j$ and there is no envy. If $\theta_i < \theta_j$, then $x_i < x_j$ and in this case, No Envy holds if and only if

$$
\frac{u_i(x_i) - u_i(x_j)}{x_i - x_j} \leq k\frac{(\theta_{N\setminus i}\theta_i - \theta_{N\setminus j}\theta_j)}{\theta_i - \theta_j}\frac{\theta_N}{C} + \frac{[h(\theta_{-i}) - h(\theta_{-j}) + \beta(\theta_{-i}) - \beta(\theta_{-j})]}{\theta_i - \theta_j}\frac{\theta_N}{C}.
$$

By concavity and the equilibrium condition, $\frac{u_i(x_i)-u_i(x_j)}{x_i-x_j} \leq u_i'(x_i) \leq \lambda = \frac{k\theta_N^2}{C}$. Thus, the envy-free condition holds if and only if

$$\frac{k\theta_N^2}{C} \leq k\frac{(\theta_{N\setminus i}\theta_i - \theta_{N\setminus j}\theta_j)}{\theta_i - \theta_j}\frac{\theta_N}{C} + \frac{[h(\theta_{-i}) - h(\theta_{-j}) + \beta(\theta_{-i}) - \beta(\theta_{-j})]}{\theta_i - \theta_j}\frac{\theta_N}{C}.$$

In the case such that $\theta_i > \theta_j \geq 0$, we have $x_i > 0$ and No Envy holds if and only if

$$\frac{u_i(x_i) - u_i(x_j)}{x_i - x_j} \geq k\frac{(\theta_{N\setminus i}\theta_i - \theta_{N\setminus j}\theta_j)}{\theta_i - \theta_j}\frac{\theta_N}{C} + \frac{[h(\theta_{-i}) - h(\theta_{-j}) + \beta(\theta_{-i}) - \beta(\theta_{-j})]}{\theta_i - \theta_j}\frac{\theta_N}{C}.$$

Concave u_i and an equilibrium condition implies $\frac{u_i(x_i)-u_i(x_j)}{x_i-x_j} \geq u_i'(x_i) = \lambda$, so envy-free condition equals

$$\frac{k\theta_N^2}{C} \geq k\frac{(\theta_{N\setminus i}\theta_i - \theta_{N\setminus j}\theta_j)}{\theta_i - \theta_j}\frac{\theta_N}{C} + \frac{[h(\theta_{-i}) - h(\theta_{-j}) + \beta(\theta_{-i}) - \beta(\theta_{-j})]}{\theta_i - \theta_j}\frac{\theta_N}{C}.$$

Therefore, No Envy holds if and only if

$$\begin{aligned}
\frac{k\theta_N^2}{C} &= k\frac{(\theta_{N\setminus i}\theta_i - \theta_{N\setminus j}\theta_j)}{\theta_i - \theta_j}\frac{\theta_N}{C} + \frac{[h(\theta_{-i}) - h(\theta_{-j}) + \beta(\theta_{-i}) - \beta(\theta_{-j})]}{\theta_i - \theta_j}\frac{\theta_N}{C} \\
&= k\frac{\theta_N}{C}\frac{\theta_N(\theta_i - \theta_j) - (\theta_i^2 - \theta_j^2)}{\theta_i - \theta_j} + \frac{[h(\theta_{-i}) - h(\theta_{-j}) + \beta(\theta_{-i}) - \beta(\theta_{-j})]}{\theta_i - \theta_j}\frac{\theta_N}{C} \\
&= k\frac{\theta_N}{C}[\sum_{l\neq i,j}\theta_l] + \frac{[h(\theta_{-i}) - h(\theta_{-j}) + \beta(\theta_{-i}) - \beta(\theta_{-j})]}{\theta_i - \theta_j}\frac{\theta_N}{C}
\end{aligned}$$

for any $i \neq j$ such that $i, j \in N$. This is equivalent to

$$k\theta_N = k\sum_{l\neq i,j}\theta_l + \frac{[h(\theta_{-i}) - h(\theta_{-j}) + \beta(\theta_{-i}) - \beta(\theta_{-j})]}{\theta_i - \theta_j}$$

and again, in the same way, to

$$[h(\theta_{-i}) + \beta(\theta_{-i})] - [h(\theta_{-j}) + \beta(\theta_{-j})] = k(\theta_i^2 - \theta_j^2).$$

Then, $h(\theta_{-i}) + \beta(\theta_{-i}) = -kS_{-i} + \gamma$ where γ is an arbitrary constant. Therefore, we have $t_i = k\theta_i\theta_{N\setminus i} - kS_{-i} + \gamma$. ■

5 Discussion

As we briefly mentioned earlier, most of the currently developed allocation mechanisms for a divisible commodity fail to meet fairness properties such as the No Envy axiom. They are mainly concerned with incentive compatibility and efficiency. In this section, we introduce three special pseudo-VCG mechanisms with a uniform pricing mechanism and discuss about their efficiency and fairness properties.[4] These groups are the Kelly,

[4]For detailed discussion, refer to You (2008a)

VCG-Kelly, g-mechanisms, and SSVCG mechanisms.

For submitted bids or strategies θ, the Kelly mechanism determines resource allocation x such that for each $i \in N$, $x_i = \frac{\theta_i}{\theta_N} C$ if $\theta \neq 0$ or $x_i = 0$ if $\theta = 0$. The mechanism requests agent i to pay $t_i = \theta_i$. Thus, each agent i's net utility is

$$\begin{aligned} p(\theta_i, \theta_{-i}) &= u_i(\frac{\theta_i}{\theta_N} C) - \theta_i \quad \text{if } \theta \neq 0 \\ & \quad u_i(0) \quad \text{if } \theta = 0. \end{aligned}$$

With this allocation and payment rules, the Kelly mechanism performs efficient allocation when participants are price takers. However, the Kelly mechanism and its general version of $\mathcal{D}$ class mechanisms fail to implement efficient allocations when agents behave strategically.[5] Moreover, these mechanisms fail to provide fair allocations except that they apply a uniform price for every participant. The Kelly mechanism always generates envy among agents for very common utility profiles.

Due to the fact that we allow price discrimination among different users in the same way that VCG mechanisms do, we can distribute a divisible resource efficiently using a VCG-like mechanism when we face strategic players. These mechanisms are called *pseudo VCG mechanisms* and three groups of pseudo VCG mechanisms are introduced here: VCG-Kelly, g-mechanisms, and SSVCG mechanisms.

A g-mechanism allocates a resource proportionally to bids, but uses a continuous version of a pivotal payment scheme such that $t_i = C\theta_{N\backslash i} \int_0^{\theta_i} \frac{g(t;\theta_{-i})}{(t+\theta_{N\backslash i})^2} dt$ if $\theta_{N\backslash i} \neq 0$ or $t_i = 0$ if $\theta_{N\backslash i} = 0$ for each $i \in N$. Here it is assumed that $g(\theta) : \mathcal{R}_+^n \to \mathcal{R}_+$ is a continuous, nondecreasing function such that $g(c\theta)$ is a strictly increasing function from $\mathcal{R}_+$ onto $\mathcal{R}_+$ whenever $\theta \neq 0$. Thus, the net utility of agent i is

$$\begin{aligned} p_i(\theta_i, \theta_{-i}) &= u_i(\frac{\theta_i}{\theta_N} C) - C\theta_{N\backslash i} \int_0^{\theta_i} \frac{g(t;\theta_{-i})}{(t+\theta_{N\backslash i})^2} dt \quad \text{if } \theta_{N\backslash i} \neq 0 \\ & \quad u_i(0) \quad \text{otherwise.} \end{aligned}$$

Any g-mechanisms assign efficient allocations at unique Nash equilibrium for strictly concave utility profiles. Yang and Hajek (2006a) discuss the g-mechanism's efficiency and convergence to equilibrium. Unfortunately, when a g-mechanism sets its generator function as $g(\theta) = \frac{\theta_N^2}{C}$, the mechanisms fails the No Envy test for every utility profile. Furthermore, if $g(\theta) = \frac{\theta_N}{C}$ or $g(\theta) = \frac{\sqrt{\theta_N}}{2\sqrt{C}}$, the mechanism produces envy for very common utility profiles.

A SSVCG mechanism uses a surrogate function $\bar{u}(x, \theta)$ with some regular assumptions and an allocation rule determined by the surrogate function. The resource allocation x is an argument to maximize $\sum_{i \in N} \bar{u}(x_i, \theta_i)$ over the feasible set $\mathcal{X}$. Its payment scheme for agent i is $t_i = -\sum_{j \neq i} \bar{u}(x_j(\theta), \theta_j) + h_i(\theta_{-i})$ where $h_i(\theta_{-i})$ is an arbitrary function

[5] For detailed discussion, refer to Jahari and Tsitsiklis (2004, 2007).

independent of θ_i, which we will refer to as its residual payment scheme. Thus, agent i has a net utility written as

$$p_i(\theta_i, \theta_{-i}) = u_i(x_i(\theta)) + \sum_{i \in N} \bar{u}(x_j(\theta), \theta_j) - h_i(\theta_{-i}).$$

As Johari and Tsitsiklis (2007) discuss, the SSVCG mechanisms achieve efficient Nash implementation and they encompass many mechanisms because their allocation and payment scheme change according to the choice of surrogate functions.

Indeed, VCG-Kelly mechanisms (Yang and Hajek (2006b)) are specific cases of SSVCG mechanisms. The VCG-Kelly mechanisms set a surrogate function $\bar{u}(x_i, \theta_i) = \theta_i f_i(x_i)$ for each $i \in N$ under slightly different assumptions for a function f, such that f_i's are strictly increasing, strictly concave, and twice differentiable over $\mathcal{R}_+$. Instead of using the general form of residual payment scheme $h(\theta_{-i})$ as SSVCG mechanisms do, VCG-Kelly mechanisms specify payment to be a pivotal scheme in terms of their surrogate functions as follows: each agent $i \in N$ pays

$$t_i = \max_{x \in \mathcal{X}, x_i = 0} \sum_{j \in N, j \neq i} \theta_j f_j(x_j, \theta_j) - \sum_{j \in N, j \neq i} \theta_j f_j(x(\theta), \theta_j)$$

where $x(\theta) = \text{argmax}_{x \in \mathcal{X}} \sum_{i \in N} \bar{u}(x_i, \theta_i)$. Therefore, they share the main properties of SSVCG mechanisms.

Unfortunately, many SSVCG mechanisms fail to have the envy-free property. Whether the No Envy axiom holds or fails really depends on the surrogate function $\bar{u}$ of SSVCG mechanisms. If a specific form of surrogate function such as $\bar{u}(x, \theta) = \theta \log x$ or $\bar{u}(x, \theta) = \sqrt{\theta x}$ is chosen by a SSVCG mechanism, equilibrium allocations generate envy for very common utility profiles no matter what the residual payment scheme $h(\theta_{-i})$ is set as. Even if it is the case that a surrogate function is carefully chosen, the next task is to find a residual payment function $h(\theta_{-i})$ that meets the envy-free property.

Of course, there are some SSVCG mechanisms which satisfy the envy-free axiom. If we set a surrogate function of SSVCG mechanisms to be $\bar{u}(x, \theta) = -k\frac{\theta^2}{x}C$ and residual payment scheme to be $h(\theta_{-i}) = -k\theta^2_{N\backslash i} - kS_{-i} + \gamma$ where $\theta^2_{N\backslash i} = (\theta_{N\backslash i})^2$, we can check easily that the SSVCG mechanisms become the SEF-VCG(k, γ) mechanisms.

The SEF-VCG(k, γ) mechanisms are remarkable since they not only Nash implement efficient division but also satisfy strong fairness properties such as Ranking, No Envy, and Voluntary Participation. Also, with a SEF-VCG mechanism, a designer can choose the extent of the budget deficit or surplus as he or she desires. Additionally, the payment scheme of the SEF-VCG(k, γ) mechanisms are very simple compared to g-mechanisms or VCG-Kelly mechanisms.

6 Conclusion

Identifying a broad group of efficient and envy free pseudo-VCG mechanisms is a task that lies ahead. It would be especially promising to find them in the SSVCG mechanisms. Another path for future research concerns problems with budget imbalances in the SEF-VCG mechanisms similar to other pseudo-VCG mechanisms. To counteract this, comparing the SEF-VCG mechanisms to others in terms of relative efficiency loss[6] may be a reasonable solution. A final direction for future study is finding incentive-compatible and fair mechanisms for division of an economic bad such as pollution.

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