Appointment Games in Fixed-Route Traveling Salesman Problems and The Shapley Value*

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March 30, 2009

Abstract

Starting from her home, a service provider visits several customers, following a predetermined route, and returns home after all customers are visited. The problem is to find a fair allocation of the total cost of this tour among the customers served. A transferable-utility cooperative game can be associated with this cost allocation problem. We introduce a new class of games, which we refer as the *fixed-route traveling salesman games with appointments.* We study the Shapley Value in this class and show that it is in the core. Our first characterization of the Shapley value involves a property which requires that sponsors do not benefit from mergers, or splitting into a set of sponsors. Our second theorem involves a property which requires that the cost shares of two sponsors who get connected are equally effected. We also show that except for our second theorem, none of our results for appointment games extend to the class of routing games (Potters et al, 1992).

Keywords : fixed-route traveling salesman games, routing games, appointment games, the Shapley value, the core, transferable-utility games, merging and splitting proofness, equal impact, networks, cost allocation.

1 Introduction

Finding the least-costly route that visits a given set of locations and returns to the starting location, the so called "traveling salesman problem (TSP)" is one of the most well-known combinatorial optimization problems in operations research. A vide variety of problems can be modelled as a TSP or one of its extensions.[1] In several of these problems, the cost of the tour has to be allocated among the customers visited (sponsors). This kind of a cost allocation problem in a TSP was first investigated by Fishburn and Pollack (1983). Some examples where a cost allocation problem arises include a salesman (repairman, cable guy, parcel delivery guy etc.) visiting his customers, a professor invited by several universities for

*The first draft of this paper was written when I was a Ph.D student in the University of Rochester. I am grateful to William Thomson for his guidance and advice.

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[1]For instance, location routing, closed-loop material flow system design in production settings, sequencing jobs in a flexible manufacturing environmet, post box collection, stochastic vehicle routing, grocery shopping, scheduling of home deliveries of online shopping, robotic travel problems like soldering or drilling operations on printed circuit boards, sequencing local genome maps to produce a global map, planning the order in which a satellite interferometer studies a sequence of stars, seriation problems in archeology, etc.

seminars, passengers using shuttle buses or car-pooling, and distribution planning situations such as delivery of supplies to grocery stores by a manufacturer.[2]

In some of the above examples, the traveler may need to follow a route that is not necessarily the least costly one. We study the so called "*fixed-route traveling salesman problems*" where the route is fixed according to the restrictions in the agenda of the traveler. Here, starting from her home (main office, factory, or depot), a service provider visits several customers, following a predetermined route, and returns home after all customers are visited. Each customer is to be visited exactly once but home can be visited more than once, which may be necessary, for instance, when the service provider needs to replenish her supplies after visiting a group of customers and before visiting the rest. Another reason may be that the traveler has appointments to meet with the customers and there is a considerable waiting time between two consecutive appointments. Then, in between those appointments, she would go home and wait there.

Various factors other than the cost may affect the route. Some of the sponsors may need to be visited before the others due to the urgency of their needs, their higher priority status, or the availability of their free times for a visit. For example, a professor may have to visit several universities in the order specified by their available seminar dates or a service provider has to visit her customers according to their appointments. In some cases, it is not possible to visit a location before visiting certain others. For instance, an employer may need to pick up some files from some offices and submit them to other offices to get them signed and there is an authority structure according to which signatures must be collected. Other examples include a communication network where the flow of information has to follow the specified network structure[3] or a product which has to be processed in several departments in a firm according to the stage of its development (e.g. it can not be sent to the marketing department before quality control).

Our goal is to find a fair distribution of the total cost generated in a fixed-route TSP among the sponsors. One way to solve this distribution problem is to associate a *cooperative game with transferable utilities* (TU-game) with the cost allocation problem.[4] A TU-game is a pair (N, v) where N is a finite set of agents and $v : 2^N \to \mathbb{R}$ is a characteristic function which assigns to each coalition $S \subseteq N$, a value $v(S)$ such that $v(\emptyset) = 0$. In the current context, $v(S)$ represents the cost of the tour in which only the members of S are served by the service provider. Potters et al (1992) formulate a TU-game associated with a fixed-route TSP as follows: for each coalition $S \subset N$, $v(S)$ is defined as the cost of the original route restricted to S, where the salesman visits the members of S in the same order as they were visited in the original route over N, skipping all agents in $N \backslash S$.[5] They refer to these games as *routing games.* Note that if the salesman and all sponsors live along the coast of an island and the

[2]For a case study of the cost allocation problem concerning the transportation of gas and gas oil to the customers of Norsk Hydro, see Engevall et al (1998).

[3]For example, consider a network in which a central office sends information (or papers/products to be processed) to several satellite offices which need to send back the processed information. There is a fixed order of satellite offices that must be respected: office i's information has to be obtained by the central office before office $i+1$. Each satellite office i can only communicate with the central office and office $i+1$. Hence, she can send the information to the central office via the office next to her.

[4]Examples of other cost allocation problems where cooperative game theory is used include airplane landing fees [9], water resource planning [11], telephone billing rates [1], and investment in electric power [5].

[5]Potters et al (1992) also studied TSPs where the route is not fixed. They introduced the *traveling salesman games,* where the value of a coalition is the cost of a least costly tour over the members of that coalition. The salesman is allowed to visit any agent more than once and he is free to visit the agents in a coalition in any order he wishes as long as the cost of the trip is minimized. See [3], [8], [12], [14].

travel costs are proportional to the Euclidean distances, then the least costly tour for a subset of sponsors is the one specified in a routing game [2].

We introduce a new class of games which we refer as the class of *fixed-route traveling salesman games with appointments* (here after, *appointment games*). Consider the case in which each sponsor in N makes an appointment to meet the traveler at a specified time. After all the appointments are made, suppose the members of $S \subseteq N$ decide to hire the traveler without cooperating with the sponsors in $N \backslash S$. That is, the members of S together will pay $v(S)$ to the traveler. This can be thought as if all the sponsors in $N \backslash S$ cancel their appointments. The permissible route over S is the one where the traveler still visits the sponsors in S according to their original appointments. So, the traveler follows the original route, skipping the sponsors who are not in S, and *when she skips a sponsor, she goes home from where she goes to the next unvisited sponsor in S.*[6] The value of a coalition S, is the cost of this permissible route over S.

Our formulation of permissible routes over coalitions makes sense in several TSPs where the service provider makes appointments with the customers which can not be changed in a short notice of time. Hence, if some appointments are cancelled, the remaining ones can not be rescheduled. Also, suppose that when a traveler visits a sponsor, she has to spend a considerable period of time to complete her service for that sponsor. In that case, if an appointment is cancelled, then the traveler has to wait a lengthy period of time till the next appointment. Hence, it is not feasible for the traveler to go to the next sponsor immediately. Hence, when the traveler skips a sponsor, she goes home where she waits till it is time for the next appointment. For instance, consider a professor who wants to visit universities in different cities at specific dates as a visiting professor. When she visits a university, suppose she has to stay there for a few weeks. If a university cancels its appointment, instead of visiting the next university in the route right away, the professor goes back to her home and waits there until the appointed date for the next university arrives.

Several papers discuss the "core" in traveling salesman games and routing games (see [2], [3], [8], [12], [14]). Here, we study another well-known solution, the "Shapley value" (Shapley, 1971). In general, the Shapley value is computationally complex. However, in appointment games, we show that this is not the case. We also show that under a mild condition on the costs, the class of appointment games is convex, hence, in this class, the Shapley value is in the core. Moreover, the Shapley value may be an appealing alternative to core since it is always non-empty, single-valued, and is the unique solution satisfying certain desirable properties. Characterizations of the Shapley value in general networks are provided by Myerson (1997) and Jackson and Wolinsky (1996). Kar (2002) characterizes the Shapley value in minimum cost spanning tree games. For the TSP games with appointments, we present two characterizations of the Shapley value. The first one involves a strategic property called *merging and splitting proofness* which requires that sponsors who follow each other on a route should not gain by merging or a sponsor should not gain by splitting into several sponsors located next to each other. Our second characterization involves a property which requires that when two sponsors get connected, they are effected equally. This characterization is in the same spirit of Kar's characterization of the Shapley value in minimum cost spanning tree games.

We also analyze the Shapley value in the class of routing games. In this class, the Shapley value does not have a simple form as it does in the appointment games. In general, the class of routing games is not convex, hence, it is not certain whether the Shapley value is in the

[6] Note that an appointment game would coincide with a routing game if for each pair of sponsors $\{i, j\}$, the cost of traveling between i and j is equal to cost of traveling from i to home and from home to j.

core or not. We show that it is not. We also show that our first theorem doesn't extend to this class. However, we extend our second theorem to the class of routing games.

In Section 2, the model is described. The results for the appointment games are presented in Section 3. Section 5 involves the results for the routing games. All proofs are in the Appendix.

2 The Model

2.1 The Economy

Let $N = \{1, 2, ..., n\}$ with $|N| = n$ be an ordered list of sponsors and 0 be home. Without loss of generality, we assume that the sponsors are visited in the same order as they appear in N. Let $N^0 \equiv N \cup \{0\}$ and for each $S \subseteq N$, let $S^0 \equiv S \cup \{0\}$. A route $r = (i_1, i_2, ..., i_M)$ is an ordered list of the agents (sponsors and home) to be visited by a "*traveler*" such that

(i) the route starts from home and ends at home (i.e. $i_1 = i_M = 0$),

(ii) each sponsor is visited exactly once,

(iii) home can be visited more than once,

(iv) after sponsor $i \in N$ is visited, either home or sponsor $i + 1$ is to be visited (i.e. the relative order of sponsors in r respect their order in N).

For each pair $\{i, j\} \subseteq N^0$, i is *connected* to j on a route r (denoted as $i \succ_r j$), if after i, the next agent visited is $j : r = (0, ..., i, j, ..., 0)$.

For each $\{i, j\} \subseteq N^0$, let $c_{i,j} \geq 0$ be the cost of traveling between agents i and j. Let $c_i \equiv c_{0,i}$ be the cost of traveling between home and sponsor i. The cost of a route r is $c(r) = \sum_{\{i,j\} \subseteq N^0 :\, i \succ_r j} c_{i,j}$.

Let $\mathbf{c} = \{c_{i,j} : \{i, j\} \subseteq N^0\}$. An *economy* is given by $\mathbf{e} = \langle N, \mathbf{c}, r\rangle$. Let the domain of all economies be $\mathcal{E}$.

A sponsor set $S = \{l, l + 1, ..., m - 1, m\} \subseteq N$ is a *connected set on* r if and only if $0 \succ_r l \succ_r l+1 \succ_r ... \succ_r m-1 \succ_r m \succ_r 0$. Let $\mathcal{S}_{\mathbf{e}}$ be the set of all connected sets in economy $\mathbf{e}$.

In order to visualize the problem, we can associate a graph with each $\mathbf{e} = \langle N, \mathbf{c}, r\rangle \in \mathcal{E}$. The elements of N^0 are called *nodes,* 0 being the *source*. A *link* between nodes i and j (denoted as l_{ij}) is a direct path between them. Let $l_i \equiv l_{0i}$ be the link between home and i. Let $L = \{l_{ij} : \{i, j\} \subseteq N^0\}$ be the set of all links between all agents. A graph g over N^0 is a subset of L. The graph associated with $\mathbf{e} = \langle N, \mathbf{c}, r\rangle \in \mathcal{E}$ is $g(\mathbf{e}) = \{l_{ij} : \{i, j\} \subseteq N^0$ and $i \succ_r j\}$ where each link l_{ij} in $g(\mathbf{e})$ is associated with weight $c_{i,j}$.

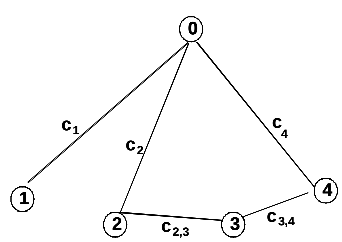

Figure 1

Example 1. Let $\mathbf{e} = \langle N, \mathbf{c}, r\rangle$ with $r = (i_1, i_2, i_3, ..., i_7) = (0, 1, 0, 2, 3, 4, 0)$. The route r describes a trip where starting from 0 (home), the traveler visits sponsor 1, then goes back

home. From home, she visits sponsors 2, 3, and 4, in that order, and returns home and completes the tour.
Here, the connected sets are $S = \{1\}$ and $S' = \{2, 3, 4\}$. Hence, $\mathcal{S}_{\mathbf{e}} = \{S, S'\}$. Sponsor 1 is an end-point sponsor in S, sponsors 2 and 4 are end-point sponsors in S', and 3 is an interior sponsor in S'. The cost of the route is $c(r) = 2c_1 + c_2 + c_{2,3} + c_{3,4} + c_4$. The associated graph $g(\mathbf{e})$ is as in Figure 1.

2.2 Appointment Games

Let $\mathbf{e} = \langle N, \mathbf{c}, r\rangle \in \mathcal{E}$ and $S \subset N$. Let the *permissible route over* S (denoted as r_S) be as follows:

Starting from home, the traveler first visits the smallest numbered sponsor in S, let us call this sponsor j_1. Suppose, in the original route r, after visiting sponsor j_1, the traveler visits agent (home or a sponsor) $i \in N^0$ (i.e. $j_1 \succ_r i$). If $i \in S^0$, then in route r_s, the traveler goes to i right after visiting j_1 (i.e. $j_1 \succ_{r_S} j_2 \equiv i$). If $i \notin S$, then it is as if i has cancelled her appointment. In this case, in r_S, after visiting j_1, the traveler goes home and she waits there till it is time to attend the next outstanding appointment with the agents in $S \backslash \{j_1\}$. That is, if $j_1 \succ_r i$ and $i \notin S$, then $j_1 \succ_{r_S} 0 \succ_{r_S} l$ where $l = \min\{k : k \in S \text{ and } k > j_1\}$). A similar procedure is followed until all the sponsors in S are visited, then the traveler returns home. Note that each time after the traveler visits home, the next agent she visits is the smallest numbered agent in S that has not been visited so far.

Formally, for some $T \geq |S|$, let $r_S = (0, j_1, j_2, ..., j_T, 0)$ be such that:

(i) for each $t \in \{1, .., T\}$, $j_t \in S^0$, and for each $i \in S$, there is a unique $t \in \{1, .., T\}$ such that $i \equiv j_t$ on r_S,
(ii) $j_1 = \min\limits_{i \in S} i$ and $j_T = \max\limits_{i \in S} i$,
(iii) for each $j_t \in S$ with $t \in \{1, 2, .., T\}$ and each $i \in N$ such that $j_t \succ_r i$, if $i \in S^0$, then $j_t \succ_{r_S} j_{t+1} \equiv i$, otherwise $j_t \succ_{r_S} j_{t+1} \equiv 0$, and
(iv) for each $j_t \equiv 0$ with $t \in \{2, .., T-1\}$, we have $j_t \succ_{r_S} \min\{k : k \in \text{ and } k > j_{t-1}\}$.

Let $\mathbf{e} = \langle N, \mathbf{c}, r\rangle \in \mathcal{E}$. For each $S \subseteq N$, let $\mathbf{c}_S = \{c_{i,j} \geq 0 : \{i, j\} \subseteq S^0\}$. The economy restricted to S with respect to r_S is $\mathbf{e}_S = \langle S, \mathbf{c}_S, r_S\rangle \in \mathcal{E}$.

For each $\mathbf{e} = \langle N, \mathbf{c}, r\rangle \in \mathcal{E}$, the *fixed-route traveling salesman game with appointments* (in short, *appointment game*) *associated with* $\mathbf{e}$ is $V_{\mathbf{e}} = (N, v_{\mathbf{e}})$ where $v_{\mathbf{e}} : 2^N \to \mathbb{R}_+$ is such that for each $S \subseteq N$, $v_{\mathbf{e}}(S) = c(r_S)$. Let $\{V_{\mathbf{e}} : \mathbf{e} \in \mathcal{E}\}$ be the class of appointment games.

Example 2. Let $\mathbf{e} = \langle N, \mathbf{c}, r\rangle$ where $r = (0, 1, 0, 2, 3, 4, 5, 0, 6, 0, 7, 8, 9, 0)$. Let $S = \{1, 4, 5, 6, 7, 9\}$. Then, $r_S = (0, 1, 0, 4, 5, 0, 6, 0, 7, 0, 9, 0)$. Here, $7 \succ_r 8$ but $8 \notin S$ (i.e. 8 cancelled her appointment). Thus, after visiting 7, the traveler goes home from where she goes to sponsor 9.
The graphs $g(\mathbf{e})$ and $g(\mathbf{e}_S) = \{l_{ij} : \{i, j\} \subseteq S^0 \text{ and } i \succ_{r_S} j\}$ are as in Figures 2a and 2b.

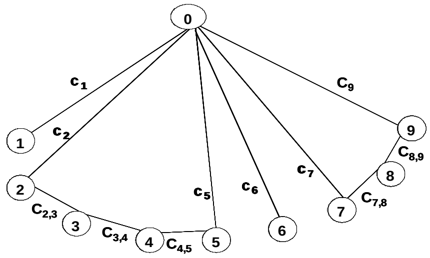

Figure $2a$: $g(\mathbf{e})$

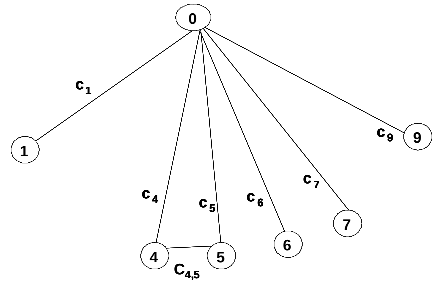

Figure $2b$: $g(\mathbf{e}_S)$

2.3 The Shapley Value

For each $\mathbf{e} = \langle N, \mathbf{c}, r\rangle \in \mathcal{E}$, to determine a cost allocation vector $\mathbf{x} = (x_1, x_2, ..., x_n) \in \mathbb{R}^n_+$, we have two options. First option is to define a rule that selects an allocation for each economy directly. Second one is to associate a TU-game with each economy, and define a rule that selects an allocation for the TU-game associated with the economy. In this paper, we follow the later approach.

Let $V = (N, v)$ be a TU-game where $v : 2^N \to \mathbb{R}$ is a characteristic function such that $v(\emptyset) = 0$. A *solution* F is a mapping that associates with each $V = (N, v)$, an allocation vector $\mathbf{x} = (x_1, x_2, ..., x_n) \in \mathbb{R}^n_+$ where $\sum x_i = v(N)$. An example of a solution is the *Shapley value, SV:* $V = (N, v)$ and each $i \in N$,

$$SV_i(V) = \sum_{S \subseteq N \setminus \{i\}} \frac{|S|!(n - |S| - 1)!}{n!}[v(S \cup \{i\}) - v(S)].$$

In general, the Shapley value is computationally complex since we need to calculate the marginal contribution of each agent to each possible coalition. But, on the class of appointment games, it turns out that the Shapley value has a simple form (see the Appendix for the derivation of the Shapley Value). Let $\mathbf{e} = \langle N, \mathbf{c}, r\rangle \in \mathcal{E}$, $i \in N$, and $S_i \subseteq N$ be the connected set such that $i \in S_i$.

• If $S_i = \{i\}$, then

$$SV_i(V_{\mathbf{e}}) = 2c_i.$$

• If $S_i \cap \{i-1, i+1\} = j$, then

$$SV_i(V_{\mathbf{e}}) = \frac{3c_i + c_{i,j} - c_j}{2}.$$

• If $\{i-1, i+1\} \subseteq S_i$, then

$$SV_i(V_{\mathbf{e}}) = \frac{1}{2}(2c_i + c_{i-1,i} + c_{i,i+1} - c_{i-1} - c_{i+1}).$$

Note that in the appointment games, the Shapley value of a sponsor is independent of the costs of the links between those agents who are not connected to her. Also, a change in the cost of a sponsor to connect home affects only herself and the sponsors who are connected to her.

3 Characterizations of the Shapley Value in Appointment Games

In a cost allocation problem, the *core* of a TU-game $V = (N, v)$ is the set of vectors $\mathbf{x} \in \mathbb{R}^n_+$ such that for each $S \subseteq N$, $\sum_{i \in S} x_i \leq v(S)$ and $\sum_{i \in N} x_i = v(N)$. If an allocation $\mathbf{x} \in \mathbb{R}^n_+$ is in the core of a game V, then no coalition of sponsors has an incentive to leave the grand coalition N. In general, the core can be empty. Potters et al (1992) state that in the class of routing games, if the route r chosen for the grand coalition is a least-costly tour and triangle inequalities hold for the agents (i.e. for each pair $\{i, j\} \subseteq N^0$, $c_i + c_j \geq c_{i,j}$), then the core is non-empty.

In appointment games, a weaker condition is sufficient for the core to be non-empty: the triangle inequalities hold for those sponsors who follow each other on route r. Formally, for each r and each pair $\{i, j\} \subseteq N$ such that $i \succ_r j$, $c_i + c_j \geq c_{i,j}$. Let $\mathcal{E}_T$ be the set of economies in which this condition holds. Actually, on $\mathcal{E}_T$, we achieve more than the non-emptiness of the core. Here, we also have the convexity of the appointment games[7] and hence, by Theorem 7 of Shapley (1971), the Shapley value is an element of the core.

Proposition 1. *On the domain $\mathcal{E}_T$, appointment games are convex and the Shapley value is in the core.*

In the rest of the paper, unless stated otherwise, the results hold on both of the domains $\mathcal{E}$ and $\mathcal{E}_T$.

Although, the core compares the sum of cost shares in each coalition with the value of that coalition, the following two axioms are concerned with only the grand coalition N and singleton coalitions, respectively.

Efficiency: For each $V{=}(N, v)$, $\sum_{i \in N} F_i(V) = v(N)$.

Individual Rationality: For each $V{=}(N, v)$, $F_i(V) \leq v(\{i\})$.

The following axiom states that connected sets should not cross-subsidize each other.

Respect of Connected Sets: For each $\mathbf{e} = \langle N, \mathbf{c}, r\rangle \in \mathcal{E}$ and each connected set $S \in \mathcal{S}_\mathbf{e}$,

$$\sum_{i \in S} F_i(V_\mathbf{e}) = v_\mathbf{e}(S).$$

Note that since the cost of a route is the sum of the permissible routes over the connected sets (i.e. $c(r) = \sum_{S \in \mathcal{S}_\mathbf{e}} c(r_S)$), we have for each $\mathbf{e} \in \mathcal{E}$, $v_\mathbf{e}(N) = \sum_{S \in \mathcal{S}_\mathbf{e}} v_\mathbf{e}(S)$. Hence, *Core* implies *Respect of Connected Sets* which in turn implies *Efficiency.*

Suppose a group of consecutive sponsors K form a coalition and act as a single sponsor $k \in K$ (i.e. K merge into k). One may require that the cost allocation should be immune to such strategic activity. Requiring immunity of the cost allocation to a merger by a group of sponsors that follow each other on a route, rather than any group of sponsors, is more intuitive especially if non-consecutive sponsors can not effectively cooperate and merge due to the geographical distance between them.

[7]Note that in a cost allocation problem, a TU-game $V = (N, v)$ is convex if for each $i \in N$ and $S \subseteq T \subset N \backslash \{i\}$, $v(S \cup \{i\}) - v(S) \geq v(T \cup \{i\}) - v(T)$.

If K merges and acts like a single sponsor $k \in K$, then we assume that as a group, K is willing to pay the service provider up to $v(K)$. Also, after the merger, no element of K can behave on their own and form coalitions with sponsors outside K, however all the sponsors in K, as a group, can join with other sponsors. Hence, in effect, by requiring K to act as a single entity, we are imposing restrictions on which coalitions can form. The resulting restricted TU-game can be defined as follows.

Let $\mathbf{e} = \langle N, \mathbf{c}, r\rangle \in \mathcal{E}$ and $K \subseteq N$ be such that $K = \{k, k+1, k+2, ..., l\}$ for some $1 \leq k < l \leq n$. Let $\widehat{v} : 2^{(N \setminus K) \cup \{k\}} \to \mathbb{R}_+^{n-|K|+1}$ be such that
- $\widehat{v}(\{k\}) = v_{\mathbf{e}}(K)$;
- for each $S \subseteq N \setminus K$; $\widehat{v}(S) = v_{\mathbf{e}}(S)$, and
- for each $S \subseteq N \setminus K$; $\widehat{v}(S \cup \{k\}) = v_{\mathbf{e}}(S \cup K)$.

We refer $\widehat{V} = ((N \setminus K) \cup \{k\}, \widehat{v})$ as the TU-game obtained from $V_{\mathbf{e}}$ when K merges into a single sponsor k.[8]

The following axiom states that whether we work with

Merging and Splitting Proofness: For each $\mathbf{e} = \langle N, \mathbf{c}, r\rangle \in \mathcal{E}$, each $K \subseteq S$ for some $S \in \mathcal{S}_{\mathbf{e}}$, and each $\widehat{V}$ as described above,

$$F_k(\widehat{V}) = \sum_{i \in K} F_i(V_{\mathbf{e}}).$$

Consider the difference between the value of the coalition consisting of only one sponsor and her cost share in the grand coalition. This difference measures how much a sponsor benefits from cooperating with the other sponsors rather than being alone. The following fairness axiom requires that in a two-sponsor TU-game, the sponsors should equally benefit from cooperation. In a sense, in two-sponsor games, we require the sponsors to have equal bargaining powers when it comes to sharing the benefits from cooperation.

Equal Benefit: For each $V = (N, v)$ with $N = \{i, j\}$,

$$v(\{i\}) - F_i(V) = v(\{j\}) - F_j(V).$$

It is easy to see that if a solution satisfies *Efficiency* and *Equal Benefit*, then in 2-sponsor economies, it coincides with the Shapley value.

Lemma 1. *A solution F satisfies Efficiency and Equal Benefit if and only if for each $V = (N, v)$ with $n = 2$, $F(V) \equiv SV(V)$.*

Let S be a connected set in $\mathbf{e} = \langle N, \mathbf{c}, r\rangle$. Now, suppose all the sponsors which do not belong to S leave after paying their cost shares. The rest of the sponsors have to cover the remaining part of the total cost. The reduced economy which only includes the sponsors in S is equivalent to $\mathbf{e}_S = \langle S, \mathbf{c}_S, r_S\rangle \in \mathcal{E}$. Note that since S is a connected set, the cost of serving the sponsors in S is same both in the original and the reduced economies. Hence, whether the sponsors in S cooperate with the grand coalition or not should not effect their cost shares. In other words, the sponsors in S should not be affected when the other sponsors leave the economy.

Consistency over Connected Sets: For each $\mathbf{e} = \langle N, \mathbf{c}, r\rangle \in \mathcal{E}$, each $S \in \mathcal{S}_{\mathbf{e}}$, and each $i \in S$,

$$F_i(V_{\mathbf{e}}) = F_i(V_{\mathbf{e}_S}).$$

[8] Note that the choice of k as the representative of K is arbitrary.

The following Lemma states some logical relations between the axioms introduced so far.

Lemma 2. *a) Efficiency, Merging and Splitting Proofness, and Equal Benefit together imply Respect of Connected Sets.*
b) Efficiency, Individual Rationality, and Merging and Splitting Proofness together imply Respect of Connected Sets.
c) Efficiency and Consistency over Connected Sets together imply Respect of Connected Sets.

Our main characterization states that there is only one solution if one requires no cross-subsidization among connected sets, immunity to manipulability by mergers, and equal bargaining power in two-person games: the Shapley value.

Theorem 1. *The Shapley value is the only solution which satisfies Respect of Connected Sets, Merging and Splitting Proofness, and Equal Benefit.*

By Lemma 2 and Theorem 1, the Shapley value can also be characterized as follows:

Corollary 1. *The Shapley value is the only solution which satisfies Efficiency, Merging and Splitting Proofness, and Equal Benefit.*

Consider the following two economies where the only difference between them is that there are two sponsors i and $i+1$ such that the service provider visits them consecutively in one economy, and via home in the other. In such a situation, one may require that other things being equal, when two sponsors become connected, their cost shares should be affected equally. This requirement is similar to the "equal-gains principle" of Myerson (1977) and "equal bargaining power" of Jackson and Wolinsky (1996).

We weaken this condition further by allowing the possibility that there may be other changes in the respective routes in these two economies. However, we do not allow any changes on the links between $i-1$ and i; and $i+1$ and $i+2$. Formally, let $\{\mathbf{e}, \mathbf{e}'\} \subset \mathcal{E}$ where $\mathbf{e} = \langle N, \mathbf{c}, r\rangle$ and $\mathbf{e}' = \langle N, \mathbf{c}, r'\rangle$ be such that there exists $i \in N\backslash\{n\}$ where
(i) $i \succ_r i+1$ and $i \succ_{r'} 0 \succ_{r'} i+1$,
(ii) $i-1 \succ_{r'} i$ if and only if $i-1 \succ_r i$, and
(ii) $i+1 \succ_{r'} i+2$ if and only if $i+1 \succ_r i+2$,

Weak Equal Impact : For each pair $\{\mathbf{e}, \mathbf{e}'\} \subset \mathcal{E}$ as described above,

$$F_i(V_{\mathbf{e}}) - F_i(V_{\mathbf{e}'}) = F_{i+1}(V_{\mathbf{e}}) - F_{i+1}(V_{\mathbf{e}'}).$$

The next result states that requiring no-cross subsidization between connected sets and equal treatment of sponsors when they get connected is also enough to characterize the Shapley value.

Theorem 2. *The Shapley value is the only solution which satisfies Respect of Connected Sets and Weak Equal Impact.*

Theorem 2 is in similar spirit to Myerson's characterization of the Shapley value in general networks. However, our framework and the domain of allocations are different that of Myerson (1977).

Comparison of Theorems 1 and 2 is interesting: keeping *Respect of Connected Sets*, we can replace *Merging and Splitting Proofness,* and *Equal Benefit* with *Weak Equal Impact*, and we still obtain the Shapley value, even though *Merging and Splitting Proofness* and *Equal Benefit* do not together imply *Weak Equal Impact.* The following corollary to Theorem 2 is obtained by using the logical relations between several axioms laid out in Lemma 2.

Corollary 2. *a) The Shapley value is the only solution which satisfies Efficiency, Merging and Splitting Proofness, Equal Benefit, and Weak Equal Impact*
b) On $\mathcal{E}_T$, the Shapley value is the only solution which satisfies Efficiency, Individual Rationality, Merging and Splitting Proofness, and Weak Equal Impact.[9]
c) The Shapley value is the only solution which satisfies Efficiency, Consistency over Connected Sets, and Weak Equal Impact.

The results in this Section shows that several different combinations of axioms characterize the Shapley value. Hence, Shapley value generally seems to be the answer regardless of what different requirements one expects from a solution. The fact that the Shapley value is also in the core of appointment games adds to its desirability. The following Table summarizes the results of this Section.

	Eff	*IR*	*RCS*	*Cons*	*MSP*	*WEI*	*EB*
Thm1			•		•		•
Cor1	•				•		•
Thm2			•			•	
Cor2 a	•				•	•	•
Cor2 b (on $\mathcal{E}_T$)	•	•			•	•	
Cor2 c	•			•		•	

Eff: Efficiency, IR: Individual Rationality, RCS: Respect of Connected Sets, Cons: Consistency over Connected Sets, MSP: Merging and Splitting Proofness, WEI: Weak Equal Impact, EB: Equal Benefit.

4 Characterization of the Shapley Value in Routing Games

In this section, we analyze the Shapley value in the class of routing games (TU-games proposed by Potters et al (1992) to study the fixed-route TSP, see the Introduction).

Let $\mathbf{e} = \langle N, \mathbf{c}, r\rangle \in \mathcal{E}$ and $S \subset N$. The permissible route over S in a routing game is the one where the traveler follows the order in which agents are listed on r, skipping all the sponsors who are absent in S. Let r^*_S be the resulting route over S. The *routing game* associated with $\mathbf{e}$ is $V^*_{\mathbf{e}} = (N, v^*_{\mathbf{e}})$ where $v^*_{\mathbf{e}} : 2^N \to \mathbb{R}_+$ is such that for each $S \subseteq N$, $v^*_{\mathbf{e}}(S) = c(r^*_S)$.

Example 3. Let $\mathbf{e} = \langle N, \mathbf{c}, r\rangle$ where $r = (0, 1, 0, 2, 3, 4, 5, 0, 6, 0, 7, 8, 9, 0)$. Let $S = \{1, 4, 5, 6, 7, 9\}$. Then, $r^*_S = (0, 1, 0, 4, 5, 0, 6, 0, 7, 9, 0)$ and $v^*_{\mathbf{e}}(S) = 2c_1 + c_4 + c_{4,5} + c_5 + 2c_6 + c_7 + c_{7,9} + c_9$.

The axioms in Section 3 can be stated for routing games just by replacing all $V_{\mathbf{e}}$ with $V^*_{\mathbf{e}}$, $v_{\mathbf{e}}(.)$ with $v^*_{\mathbf{e}}(.)$, and r_S with r^*_S.

The difference in the calculation of the value of a coalition between an appointment game and a routing game changes some of the results we derived in Section 3. First of all, in the class of routing games, the Shapley value doesn't reduce into a simple formula as it does in the class of appointment games. Also, in this class, the Shapley value is not in the core even on the domain $\mathcal{E}_T$. Moreover, Theorem 1 no longer holds in the class of routing games since the Shapley value violates *Merging and Splitting Proofness.*

[9]It is easy to see that on $\mathcal{E} \backslash \mathcal{E}_T$, the Shapley value violates *Individual Rationality*. Hence, this result holds only on $\mathcal{E}_T$.

Proposition 2. *i) In the class of routing games, the Shapley value is not in the core.*
ii) In the class of routing games, the Shapley value violates Merging and Splitting Proofness.

The good news is that Theorem 2 extends to the class of routing games. However, we need to strengthen the *Weak Equal Impact* axiom as follows: let $\{\mathbf{e}, \mathbf{e}'\} \subset \mathcal{E}$ where $\mathbf{e} = \langle N, \mathbf{c}, r\rangle$ and $\mathbf{e}' = \langle N, \mathbf{c}, r'\rangle$ be such that there exists $i \in N\backslash\{n\}$ where

(i) $i \succ_r i+1$ and $i \succ_{r'} 0 \succ_{r'} i+1$, and

(ii) for $S_i \in \mathcal{S}_\mathbf{e}$ with $i \in S_i$, for each $k \in S_i\backslash\{i\}$ and $l \in S_i\backslash\{i+1\}$, $k \succ_r l$ if and only if $k \succ_{r'} l$.

Hence, on the route r, the traveler goes from i to $i+1$ directly, but on r' she travels from i to $i+1$ via home. That is, on the graph $g(\mathbf{e})$, i and $i+1$ has a direct link between them but not on the graph $g(\mathbf{e}')$. There may be other changes between the routes r and r', as long as those changes only concern the sponsors that do not belong to S_i where S_i is the connected set that includes i and $i+1$ in economy $\mathbf{e}$. That is, the links between the sponsors in other connected sets are allowed to change.

Strong Equal Impact : For each pair $\{\mathbf{e}, \mathbf{e}'\} \subset \mathcal{E}$ as described above,

$$F_i(V^*_\mathbf{e}) - F_i(V^*_{\mathbf{e}'}) = F_{i+1}(V^*_\mathbf{e}) - F_{i+1}(V^*_{\mathbf{e}'}).$$

Theorem 3. *In the class of routing games, the Shapley value is the only solution which satisfies Respect of Connected Sets and Strong Equal Impact.*

Note that in the class of routing games, the Shapley value still satisfies *Efficiency* and *Consistency over Connected Sets.* Hence, Corollary 2c extends to the routing games:

Corollary 3. *The Shapley value is the only solution which satisfies Efficiency, Consistency over Connected Sets, and Strong Equal Impact.*

5 Appendix

5.1 Derivation of the Shapley Value in Appointment Games

Let $\mathbf{e} = \langle N, \mathbf{c}, r\rangle \in \mathcal{E}$, $i \in N$, and $S_i \subseteq N$ be the connected set such that $i \in S_i$. For each $S \subseteq N$, let $|S| = s$ and $f(s) = \frac{s!(n-s-1)!}{n!}$.

• If $S_i = \{i\}$, then since for each $S \subseteq N\backslash\{i\}$, $v_\mathbf{e}(S \cup \{i\}) - v_\mathbf{e}(S) = v_\mathbf{e}(\{i\}) = 2c_i$, we have

$$SV_i(V_\mathbf{e}) = 2c_i.$$

• If $S_i \cap \{i-1, i+1\} = j$, then since
for each $S \subseteq N\backslash\{i\}$ such that $j \in S$, $v_\mathbf{e}(S \cup \{i\}) - v_\mathbf{e}(S) = c_i + c_{i,j} - c_j$, and
for each $S \subseteq N\backslash\{i\}$ such that $j \notin S$, $v_\mathbf{e}(S \cup \{i\}) - v_\mathbf{e}(S) = 2c_i$, we have

$$\begin{aligned}
SV_i(V_{\mathbf{e}}) &= \sum_{S\subseteq N\backslash\{i\}} f(s)\left(v_{\mathbf{e}}(S\cup\{i\}) - v_{\mathbf{e}}(S)\right) \\
&= \sum_{S\subseteq N\backslash\{i\}:j\in S} f(s)\left(c_i + c_{i,j} - c_j\right) + \sum_{S\subseteq N\backslash\{i,j\}} f(s)\left(2c_i\right) \\
&= \left(c_i + c_{i,j} - c_j\right)\sum_{s=1}^{n-1}\binom{n-2}{s-1} f(s) + 2c_i\sum_{s=0}^{n-2}\binom{n-2}{s} f(s) \\
&= \left(c_i + c_{i,j} - c_j\right)\frac{1}{2} + (2c)_i\frac{1}{2} \\
&= \frac{3c_i + c_{i,j} - c_j}{2}.
\end{aligned}$$

Here, $\binom{n-2}{s-1}$ is the number of $(s-1)$-combinations from the set $N\backslash\{i,j\}$. It gives us the number of subsets of $N\backslash\{i\}$ that contains j and has s number of sponsors: to find such subsets, we need to pick $s-1$ sponsors from the set $N\backslash\{i,j\}$. Similar interpretation applies to $\binom{n-2}{s}$ and all other binomial coefficients from now on.

• If $\{i-1,i+1\} \subseteq S_i$, then since
for each $S \subseteq N\backslash\{i\}$ such that $\{i-1,i+1\} \subseteq S$, $v_{\mathbf{e}}(S\cup\{i\}) - v_{\mathbf{e}}(S) = c_{i-1,i} + c_{i,i+1} - c_{i-1} - c_{i+1}$,
for each $S \subseteq N\backslash\{i\}$ such that $S \cap \{i-1,i+1\} = j$, $v_{\mathbf{e}}(S\cup\{i\}) - v_{\mathbf{e}}(S) = c_i + c_{i,j} - c_j$, and
for each $S \subseteq N\backslash\{i\}$ such that $S \cap \{i-1,i+1\} = \emptyset$, $v_{\mathbf{e}}(S\cup\{i\}) - v_{\mathbf{e}}(S) = 2c_i$, we have

$$\begin{aligned}
SV_i(V_{\mathbf{e}}) &= \sum_{S\subseteq N\backslash\{i\}:\{i-1,i+1\}\subseteq S} f(s)\left(c_{i-1,i} + c_{i,i+1} - c_{i-1} - c_{i+1}\right) \\
&\quad + \sum_{S\subseteq N\backslash\{i\}:\{i-1,i+1\}\cap S=i-1} f(s)\left(c_i + c_{i-1,i} - c_{i-1}\right) \\
&\quad + \sum_{S\subseteq N\backslash\{i\}:\{i-1,i+1\}\cap S=i+1} f(s)\left(c_i + c_{i,i+1} - c_{i+1}\right) + \sum_{S\subseteq N\backslash\{i\}:\{i-1,i+1\}\cap S=\emptyset} f(s)\left(2c_i\right) \\
&= \left(c_{i-1,i} + c_{i,i+1} - c_{i-1} - c_{i+1}\right)\sum_{s=2}^{n-1}\binom{n-3}{s-2} f(s) + \left(c_i + c_{i-1,i} - c_{i-1}\right)\sum_{s=1}^{n-2}\binom{n-3}{s-1} f(s) \\
&\quad + \left(c_i + c_{i,i+1} - c_{i+1}\right)\sum_{s=1}^{n-2}\binom{n-3}{s-1} f(s) + 2c_i\sum_{s=0}^{n-3}\binom{n-3}{s} f(s) \\
&= \left(c_{i-1,i} + c_{i,i+1} - c_{i-1} - c_{i+1}\right)\frac{1}{3} + \left(c_i + c_{i-1,i} - c_{i-1}\right)\frac{1}{6} + \left(c_i + c_{i,i+1} - c_{i+1}\right)\frac{1}{6} + (2c_i)\frac{1}{3} \\
&= \frac{1}{2}(2c_i + c_{i-1,i} + c_{i,i+1} - c_{i-1} - c_{i+1}).
\end{aligned}$$

5.2 Proofs of the Results in Section 3

Proof of Proposition 1: Let $\mathbf{e} = \langle N, \mathbf{c}, r\rangle \in \mathcal{E}_T$, $i \in N$, and $S \subseteq T \subset N\backslash\{i\}$. Let $K = \{j \in N\backslash\{i\} : \text{either } i \succ_r j \text{ or } j \succ_r i\}$.[10] Note that on $\mathcal{E}_T$, for each $j \in K$,

$$c_i + c_j \geq c_{i,j}. \tag{1}$$

[10]If $0 \succ_r i \succ_r 0$, then $K = \emptyset$. If $i-1 \succ_r i \succ_r 0$, then $K = \{i-1\}$. If $0 \succ_r i \succ_r i+1$, then $K = \{i+1\}$. If $i-1 \succ_r i \succ_r i+1$, then $K = \{i-1, i+1\}$.

We need to show that

$$v_{\mathbf{e}}(S \cup \{i\}) - v_{\mathbf{e}}(S) \geq v_{\mathbf{e}}(T \cup \{i\}) - v_{\mathbf{e}}(T).^{11} \tag{2}$$

There are 6 possible cases. We will show that in each case, (2) holds.

1. $K \cap S = \emptyset$. Then, $v_{\mathbf{e}}(S \cup \{i\}) - v_{\mathbf{e}}(S) = 2c_i$.
a) $K \cap T = \emptyset$. Then, $v_{\mathbf{e}}(T \cup \{i\}) - v_{\mathbf{e}}(T) = 2c_i$. Hence, (2) holds.
b) $K \cap T = \{j\}$. Then, $v_{\mathbf{e}}(T \cup \{i\}) - v_{\mathbf{e}}(T) = c_{j,i} + c_i - c_j$. Hence, by (1), (2) holds.
c) $K \cap T = \{i-1, i+1\}$. Then, $v_{\mathbf{e}}(T \cup \{i\}) - v_{\mathbf{e}}(T) = c_{i-1,i} + c_{i,i+1} - c_{i-1} - c_{i+1}$. Hence, by (1), (2) holds.

2. $K \cap S = \{j\}$. Then, $v_{\mathbf{e}}(S \cup \{i\}) - v_{\mathbf{e}}(S) = c_{j,i} + c_i - c_j$.
a) $K \cap T = \{j\}$. Then, $v_{\mathbf{e}}(T \cup \{i\}) - v_{\mathbf{e}}(T) = c_{j,i} + c_i - c_j$. Hence, (2) holds.
b) $K \cap T = \{i-1, i+1\}$. Then, $v_{\mathbf{e}}(T \cup \{i\}) - v_{\mathbf{e}}(T) = c_{i-1,i} + c_{i,i+1} - c_{i-1} - c_{i+1}$. Hence, by (1), (2) holds.

3. $K \cap S = K \cap T = \{i-1, i+1\}$. Then, $v_{\mathbf{e}}(S \cup \{i\}) - v_{\mathbf{e}}(S) = v_{\mathbf{e}}(T \cup \{i\}) - v_{\mathbf{e}}(T) = c_{i-1,i} + c_{i,i+1} - c_{i-1} - c_{i+1}$. Hence, (2) holds.

□

Proof of Lemma 1: Let F satisfy *Efficiency* and *Equal Benefit*. Let $V = (N, v)$ be such that $N = \{i, j\}$. By *Equal Benefit*, (i) $F_i(V) - F_j(V) = v(\{i\}) - v(\{j\})$. By *Efficiency*, (ii) $v(N) = F_i(V) + F_j(V)$. By (i) and (ii),

$$F_i(V) = \frac{1}{2}\left[v(N) + v(\{i\}) - v(\{j\})\right] = SV_i(V). \tag{3}$$

□

Proof of Lemma 2:

a) Let F satisfy the first 3 axioms listed in Lemma 2a. Let $\mathbf{e} = \langle N, \mathbf{c}, r\rangle \in \mathcal{E}$ and $\mathcal{S}_{\mathbf{e}} = \{S_1, S_2, ..., S_T\}$ for some $T \leq n$. The proof is by induction.

∗*Base Step:* Let S_1 merge into a single sponsor denoted by 1 and $N \backslash S_1$ merge into a single sponsor denoted by n. Let $V^1 = (\{1, n\}, v^1)$ be the TU-game obtained from $V_{\mathbf{e}}$ by this merger. Thus, $v^1(\{1\}) = v_{\mathbf{e}}(S_1)$, $v^1(\{n\}) = v_{\mathbf{e}}(N \backslash S_1)$, and $v^1(\{1, n\}) = v_{\mathbf{e}}(N)$. Note that since S_1 is a connected set, $v_{\mathbf{e}}(N) = v_{\mathbf{e}}(S_1) + v_{\mathbf{e}}(N \backslash S_1)$. These equalities and Lemma 1 together imply

$$\begin{aligned} F_1(V^1) &= \tfrac{1}{2}\left[v^1(\{1, n\}) + v^1(\{1\}) - v^1(\{n\})\right], \\ &= v_{\mathbf{e}}(S_1). \end{aligned} \tag{4}$$

By *Merging and Splitting Proofness*, $F_1(V^1) = \sum_{i \in S_1} F_i(V_{\mathbf{e}})$. This equality and (4) together imply $\sum_{i \in S_1} F_i(V_{\mathbf{e}}) = v_{\mathbf{e}}(S_1)$.

∗*Induction Step:* Let $k \leq T$. Assume that for each $t \in \{1, ..., k-1\}$, $\sum_{i \in S_t} F_i(V_{\mathbf{e}}) = v_{\mathbf{e}}(S_t)$.We will prove that $\sum_{i \in S_k} F_i(V_{\mathbf{e}}) = v_{\mathbf{e}}(S_k)$.

Let $\{S_1, S_2, ..., S_k\}$ merge into a single sponsor denoted by k, and $\{S_{k+1}, .., S_T\}$ merge into a single sponsor denoted by n. Let $V^k = (\{k, n\}, v^k)$ be the TU-game obtained from $V_{\mathbf{e}}$ by this merger. Thus, $v^k(\{k\}) = v_{\mathbf{e}}(\overset{k}{\underset{t=1}{\cup}} S_t) = \sum\limits_{t=1}^{k} v_{\mathbf{e}}(S_t)$, $v^k(\{n\}) = v_{\mathbf{e}}(\overset{T}{\underset{t=k+1}{\cup}} S_t) = \sum\limits_{t=k+1}^{T} v_{\mathbf{e}}(S_t)$, and $v^k(\{1,n\}) = v_{\mathbf{e}}(N) = \sum\limits_{t=1}^{k} v_{\mathbf{e}}(S_t) + \sum\limits_{t=k+1}^{T} v_{\mathbf{e}}(S_t)$.

These equalities and Lemma 1 together imply

$$F_k(V^k) \quad = \quad v_{\mathbf{e}}(S_k) + \sum_{t=1}^{k-1} v_{\mathbf{e}}(S_t). \tag{5}$$

By *Merging and Splitting Proofness*, $F_k(V^k) = \sum\limits_{i \in S_k} F_i(V_{\mathbf{e}}) + \sum\limits_{t=1}^{k-1} \sum\limits_{i \in S_t} F_i(V_{\mathbf{e}})$. This equality, (5), and the induction hypothesis together imply $\sum\limits_{i \in S_k} F_i(V_{\mathbf{e}}) = v_{\mathbf{e}}(S_k)$.

$*$*Conclusion Step:* By the Base and Induction steps, for each $t \leq T$, $\sum\limits_{i \in S_t} F_i(V_{\mathbf{e}}) = v_{\mathbf{e}}(S_t)$. This completes the proof.

b) Let F satisfy the first 3 axioms listed in Lemma 2b. Suppose, by contradiction, that F does not satisfy *Respect of Connected Sets*. Then, there are $\mathbf{e} = \langle N, \mathbf{c}, r \rangle \in \mathcal{E}$ and $\{S', S''\} \subseteq \mathcal{S}_{\mathbf{e}}$ such that $\sum\limits_{i \in S''} F_i(V_{\mathbf{e}}) < v_{\mathbf{e}}(S'')$ and (i) $\sum\limits_{i \in S'} F_i(V_{\mathbf{e}}) > v_{\mathbf{e}}(S')$. Such S' and S'' exist since by *Efficiency*, $\sum\limits_{S \in \mathcal{S}_{\mathbf{e}}} \left(\sum\limits_{i \in S} F_i(V_{\mathbf{e}}) \right) = v_{\mathbf{e}}(N) = \sum\limits_{S \in \mathcal{S}_{\mathbf{e}}} v_{\mathbf{e}}(S)$.

Now, let S' merge into a single sponsor denoted by $s' \in S'$. Let $\widehat{V} = ((N \backslash S') \cup \{s'\}, \widehat{v})$ be the TU-game obtained from $V_{\mathbf{e}}$ by this merger. Thus, (ii) $\widehat{v}(\{s'\}) = v_{\mathbf{e}}(S')$. By *Merging and Splitting Proofness*, (iii) $F_{s'}(\widehat{V}) = \sum\limits_{i \in S'} F_i(V_{\mathbf{e}})$. By *Individual Rationality*, (iv) $F_{s'}(\widehat{V}) \leq \widehat{v}(\{s'\})$. By (ii), (iii), and (iv), $\sum\limits_{i \in S'} F_i(V_{\mathbf{e}}) \leq v_{\mathbf{e}}(S')$ which contradicts (i).

c) Let $\mathbf{e} = \langle N, \mathbf{c}, r \rangle \in \mathcal{E}$ and $S \in \mathcal{S}_{\mathbf{e}}$. Consider $\mathbf{e}_S = \langle S, \mathbf{c}_S, r_S \rangle \in \mathcal{E}$. By *Efficiency*, (i) $\sum\limits_{i \in S} F_i(V_{\mathbf{e}_S}) = v_{\mathbf{e}_S}(S)$. By *Consistency over Connected Sets*, for each $i \in S$, (ii) $F_i(V_{\mathbf{e}}) = F_i(V_{\mathbf{e}_S})$. Since S is a connected set, $v_{\mathbf{e}_S}(S) = v_{\mathbf{e}}(S)$. Hence, by (i) and (ii), $\sum\limits_{i \in S} F_i(V_{\mathbf{e}}) = v_{\mathbf{e}}(S)$. That is, F satisfies *Respect of Connected Sets*. ∎

Proof of Theorem 1:

- First, we prove that the Shapley value is the only solution that satisfies the axioms listed in Theorem 1.

Let F satisfy those axioms and $\mathbf{e} = \langle N, \mathbf{c}, r \rangle \in \mathcal{E}$. We will show that for each $S \in \mathcal{S}_{\mathbf{e}}$ and each $i \in S$, $F_i(V_{\mathbf{e}}) = SV_i(V_{\mathbf{e}})$.

If $n = 2$, by Lemma 1, $F = SV$. Let $n > 2$. By *Respect of Connected Sets*, for each $S \in \mathcal{S}_{\mathbf{e}}$ with $|S| = 1$ and each $i \in S$, we have $F_i(V_{\mathbf{e}}) = v_{\mathbf{e}}(\{1\}) = SV_i(V_{\mathbf{e}})$.

Now, let $S \in \mathcal{S}_{\mathbf{e}}$ be such that $|S| \geq 2$ and $S = \{l, l+1, ..., m\}$ for some $\{l, m\} \subseteq N$. Let $K_1 = \{i \in N : i < l\}$ and $K_2 = \{i \in N : i > m\}$.[12] The proof is in three parts.

[12]If $l = 1$, then $K_1 = \emptyset$ and if $m = n$, then $K_2 = \emptyset$.

Part 1: This part of the proof is by induction.

$*$*Base Step:* Let K_1 and $\{l\}$ merge into a single sponsor denoted by l. Let K_2 and $\{l+1, l+2, ...m\}$ merge into a single sponsor denoted by n. Let $V^l = (\{l,n\}, v^l)$ be the TU-game obtained from $V_{\mathbf{e}}$ by this merger. Thus,
$v^l(\{l\}) = v_{\mathbf{e}}(\{1,...,l\}) = c(r_{K_1}) + 2c_l$,
$v^l(\{n\}) = v_{\mathbf{e}}(\{l{+}1,...,n\}) = c_{l+1} + \sum_{t=l+1}^{m-1} c_{t,t+1} + c_m + c(r_{K_2}) = (c(r_S) - c_l - c_{l,l+1} + c_{l+1}) + c(r_{K_2})$,
and
$v^l(\{l,n\}) = v_{\mathbf{e}}(N) = c(r_{K_1}) + c(r_S) + c(r_{K_2})$.[13]

These equalities and Lemma 1 together imply

$$\begin{aligned} F_l(V^l) &= \tfrac{1}{2}\left[v^l(\{l,n\}) + v^l(\{l\}) - v^l(\{n\})\right], \\ &= c(r_{K_1}) + \tfrac{1}{2}(3c_l + c_{l,l+1} - c_{l+1}). \end{aligned} \tag{6}$$

By *Respect of Connected Sets,*

$$\sum_{i \in K_1} F_i(V_{\mathbf{e}}) = c(r_{K_1}) \text{ and } \sum_{i \in K_2} F_i(V_{\mathbf{e}}) = c(r_{K_2}). \tag{7}$$

By *Merging and Splitting Proofness,*

$$F_l(V^l) = \sum_{i \in K_1} F_i(V_{\mathbf{e}}) + F_l(V_{\mathbf{e}}). \tag{8}$$

By equalities (6), (7), and (8),

$$F_l(V_{\mathbf{e}}) = \frac{1}{2}(3c_l + c_{l,l+1} - c_{l+1}) = SV_l(V_{\mathbf{e}}). \tag{9}$$

$*$*Induction Step:* Let $l < l+k < m$. Assume that, for each $i \le l+k-1$, $F_i(V_{\mathbf{e}}) = SV_i(V_{\mathbf{e}})$. We will prove that $F_{l+k}(V_{\mathbf{e}}) = SV_{l+k}(V_{\mathbf{e}})$.

Let K_1 and $S_1 = \{l, .., l+k\}$ merge into a single sponsor denoted by $l+k$. Let K_2 and $S \backslash S_1$ merge into a single sponsor denoted by n. Let $V^{l+k} = (\{l+k, n\}, v^{l+k})$ be the TU-game obtained from $V_{\mathbf{e}}$ by this merger. Thus,
$v^{l+k}(\{l+k\}) = v_{\mathbf{e}}(\{1,...,l+k\}) = c(r_{K_1}) + c_l + \sum_{t=l}^{l+k-1} c_{t,t+1} + c_{l+k}$,
$v^{l+k}(\{n\}) = v_{\mathbf{e}}(\{l+k+1,...,n\}) = (c(r_S) - c_l - \sum_{t=l}^{l+k} c_{t,t+1} + c_{l+k+1}) + c(r_{K_2})$, and
$v^{l+k}(\{l+k,n\}) = v_{\mathbf{e}}(N)$.

These equalities and Lemma 1 together imply

$$\begin{aligned} F_{l+k}(V^{l+k}) &= \tfrac{1}{2}\left[v^{l+k}(\{l+k,n\}) + v^{l+k}(\{l+k\}) - v^{l+k}(\{n\})\right], \\ &= c(r_{K_1}) + \tfrac{1}{2}(2c_l + 2\sum_{t=l}^{l+k-1} c_{t,t+1} + c_{l+k} + c_{l+k,l+k+1} - c_{l+k+1}). \end{aligned} \tag{10}$$

By *Merging and Splitting Proofness,*

[13] Note that K_1 is a union of connected sets and so is K_2. That is, $K_1 = \cup\{S' : S' \in \mathcal{S}_{\mathbf{e}} \backslash (S \cup K_2)\}$ and similarly for K_2. Hence, $v_{\mathbf{e}}(N) = v_{\mathbf{e}}(K_1) + v_{\mathbf{e}}(S) + v_{\mathbf{e}}(K_2)$.

$$F_{l+k}(V^{l+k}) = \sum_{i \in K_1} F_i(V_\mathbf{e}) + \sum_{i=l}^{l+k-1} F_i(V_\mathbf{e}) + F_{l+k}(V_\mathbf{e}). \tag{11}$$

By the induction hypothesis and equalities (10) and (11),

$$F_{l+k}(V_\mathbf{e}) = (2c_{l+k} + c_{l+k-1,l+k} + c_{l+k,l+k+1} - c_{l+k-1} - c_{l+k+1})/2 = SV_{l+k}(V_\mathbf{e}).$$

This concludes the induction step.
$*$*Conclusion Step:* By the Base and Induction steps, for each $l \leq l+k < m$, we have $F_{l+k}(V_\mathbf{e}) = SV_{l+k}(V_\mathbf{e})$.

Part 2: Let K_1 and $\{l, .., m\}$ merge into a single sponsor denoted by m. Let K_2 merge into a single sponsor denoted by n. Let $V^m = (\{m, n\}, v^m)$ be the TU-game obtained from $V_\mathbf{e}$ by this merger. Thus,
$v^m(\{m\}) = v_\mathbf{e}(\{1, ..., m\}) = c(r_{K_1}) + c(r_S)$,
$v^m(\{n\}) = v_\mathbf{e}(\{m+1, ..., n\}) = c(r_{K_2})$, and
$v^m(\{m, n\}) = v_\mathbf{e}(N)$.

These equalities and Lemma 1 together imply

$$\begin{aligned} F_m(V^m) &= \tfrac{1}{2}\left[v^m(\{m,n\}) + v^m(\{m\}) - v^m(\{n\})\right], \\ &= c(r_{K_1}) + c(r_S). \end{aligned} \tag{12}$$

By *Merging and Splitting Proofness*,

$$F_m(V^m) = \sum_{i \in K_1} F_i(V_\mathbf{e}) + \sum_{i=l}^{m-1} F_i(V_\mathbf{e}) + F_m(V_\mathbf{e}). \tag{13}$$

By Part 2 and equalities (12) and (13),

$$F_m(V_\mathbf{e}) = (3c_m + c_{m-1,m} - c_{m-1})/2 = SV_m(V_\mathbf{e}).$$

Part 3: By repeating the proofs in Part 1 and 2 for each $S \in \mathcal{S}_\mathbf{e}$ we obtain that for each $i \in S$, $F_i(V_\mathbf{e}) = SV_i(V_\mathbf{e})$. This completes the proof.

- Now, we show that the Shapley value satisfies the axioms listed in Theorem 1.

Respect of Connected Sets:
Let $\mathbf{e} = \langle N, \mathbf{c}, r\rangle \in \mathcal{E}$ and $S \in S_\mathbf{e}$ be such that $S = \{l, l+1, l+2, ..., m\}$ for some $\{l, m\} \subseteq N$. Then, $SV_l(V_\mathbf{e}) = \frac{3c_l + c_{l,l+1} - c_{l+1}}{2}$, $SV_m(V_\mathbf{e}) = \frac{3c_m + c_{m,m-1} - c_{m-1}}{2}$, and for each $i \in S \backslash \{l, m\}$, $SV_i(V_\mathbf{e}) = \frac{1}{2}(2c_i + c_{i-1,i} + c_{i,i+1} - c_{i-1} - c_{i+1})$. Then, $\sum_{i \in S} SV_i(V_\mathbf{e}) = c_l + \sum_{i=l}^{m-1} c_{i,i+1} + c_m = v_\mathbf{e}(S)$. Hence, the Shapley value satisfies *Respect of Connected Sets.* □

Merging and Splitting Proofness:
Let $\mathbf{e} = \langle N, \mathbf{c}, r\rangle \in \mathcal{E}$ and $K \subseteq N$ be such that $K = \{k, k+1, k+2, ..., l\}$ for some $1 \leq k < l \leq n$. Let $\widehat{V} = ((N \backslash K) \cup \{k\}, \widehat{v})$ be the TU-game obtained from $V_\mathbf{e}$ when K merges into k. Hence, $\widehat{v}(\{k\}) = v_\mathbf{e}(K)$; and for each $S \subseteq N \backslash K$, $\widehat{v}(S) = v_\mathbf{e}(S)$ and $\widehat{v}(S \cup \{k\}) = v_\mathbf{e}(S \cup K)$.

Note that K may contain some connected sets. We partition K as follows. For some $1 \leq M \leq |K|$, let $P_K = \{K_1, K_2, ..., K_M\}$ be the partitioning of K such that

* for each $m \in \{1, M-1\}$, each $i \in K_m$, and each $j \in K_{m+1}$, we have $i < j$,
* for each $m \in \{1, M\}$, $K_m \subseteq S$ for some $S \in \mathcal{S}_{\mathbf{e}}$, and
* for each $m \notin \{1, M\}$, $K_m \in \mathcal{S}_{\mathbf{e}}$.

For example, if $r = (0, 1, 2, 3, 4, 0, 5, 6, 0, 7, 0, 8, 9, 0)$ and $K = \{3, 4, .., 8\}$, then $P_K = \{\{3, 4\}, \{5, 6\}, \{7\}, \{8\}\}$.

Note that there are $n - |K| + 1$ agents in the game $\widehat{V}$. For each $S \subseteq N$, let $|S| = s$ and $g(s) = \frac{s!(n-|K|-s)!}{(n-|K|+1)!}$. The following four cases are possible.

a) $P_K \subseteq \mathcal{S}_{\mathbf{e}}$: That is, for each $1 \leq m \leq M$, $K_m \in \mathcal{S}_{\mathbf{e}}$. Then, for each $S \subseteq N \backslash K$,

$$\widehat{v}(S \cup \{k\}) - \widehat{v}(S) = \widehat{v}(\{k\}) = v_{\mathbf{e}}(K) = \sum_{m=1}^{M} v_{\mathbf{e}}(K_m). \tag{14}$$

Hence, $SV_k(\widehat{V}) = \widehat{v}(\{k\})$. By *Respect of Connected Sets*, for each $K_m \in P_K$, $\sum_{i \in K_m} SV_i(V_{\mathbf{e}}) = v_{\mathbf{e}}(K_m)$. These equalities and (14) together imply that $SV_k(\widehat{V}) = \sum_{i \in K} SV_i(V_{\mathbf{e}})$.

b) $P_K \backslash \mathcal{S}_{\mathbf{e}} = \{K_1\}$: That is, except for K_1, each $K_m \in P_K$ is a connected set. Note that this case covers the possibility that $P_K = \{K\} = \{K_1\}$. Let $K_1 = \{k_1, k_1 + 1, ..., l_1\}$. Then, for each $S \subseteq N \backslash K$ such that $k_1 - 1 \notin S$, (14) holds. For each $S \subseteq N \backslash K$ such that $k_1 - 1 \in S$,

$$\widehat{v}(S \cup \{k\}) - \widehat{v}(S) = (c_{k_1-1,k_1} + \sum_{t=k_1}^{l_1-1} c_{t,t+1} + c_{l_1} - c_{k_1-1}) + \sum_{m=2}^{M} v_{\mathbf{e}}(K_m). \tag{15}$$

Then,

$SV_k(\widehat{V}) = \sum_{s=1}^{n-|K|} \binom{n-|K|-1}{s-1} g(s)[c_{k_1-1,k_1} + \sum_{t=k_1}^{l_1-1} c_{t,t+1} + c_{l_1} - c_{k_1-1} + \sum_{m=2}^{M} v_{\mathbf{e}}(K_m)]+$
$\sum_{s=0}^{n-|K|-1} \binom{n-|K|-1}{s} g(s)[c_{k_1} + \sum_{t=k_1}^{l_1-1} c_{t,t+1} + c_{l_1} + \sum_{m=2}^{M} v_{\mathbf{e}}(K_m)]$

Note that $\sum_{s=1}^{n-|K|} \binom{n-|K|-1}{s-1} g(s) = \sum_{s=0}^{n-|K|-1} \binom{n-|K|-1}{s} g(s) = \frac{1}{2}$. Hence,

$SV_k(\widehat{V}) = \frac{1}{2}(2c_{l_1} + 2\sum_{t=k_1}^{l_1-1} c_{t,t+1} + c_{k_1-1,k_1} + c_{k_1} - c_{k_1-1}) + \sum_{m=2}^{M} v_{\mathbf{e}}(K_m)$
$= \sum_{i \in K_1} SV_i(V_{\mathbf{e}}) + \sum_{m=2}^{M} \sum_{i \in K_m} SV_i(V_{\mathbf{e}})$
$= \sum_{i \in K} SV_i(V_{\mathbf{e}})$.

c) $P_K \backslash \mathcal{S}_{\mathbf{e}} = \{K_M\}$: That is, except for K_M, each $K_m \in P_K$ is a connected set. Let $K_M = \{k_M, k_M + 1, ..., l_M\}$. Then, for each $S \subseteq N \backslash K$ such that $l_M + 1 \notin S$, (14) holds. For each $S \subseteq N \backslash K$ such that $l_M + 1 \in S$,

$$\widehat{v}(S \cup \{k\}) - \widehat{v}(S) = (c_{k_M} + \sum_{t=k_M}^{l_M-1} c_{t,t+1} + c_{l_M,l_M+1} - c_{l_M+1}) + \sum_{m=1}^{M-1} v_{\mathbf{e}}(K_m) \tag{16}$$

Then,

$$SV_k(\widehat{V}) = \sum_{s=1}^{n-|K|} \binom{n-|K|-1}{s-1} g(s)[c_{k_M} + \sum_{t=k_M}^{l_M-1} c_{t,t+1} + c_{l_M,l_M+1} - c_{l_M+1} + \sum_{m=1}^{M-1} v_{\mathbf{e}}(K_m)] +$$

$$\sum_{s=0}^{n-|K|-1} \binom{n-|K|-1}{s} g(s)[c_{k_M} + \sum_{t=k_M}^{l_M-1} c_{t,t+1} + c_{l_M} + \sum_{m=1}^{M-1} v_{\mathbf{e}}(K_m)]$$

$$= \frac{1}{2}(2c_{k_M} + 2\sum_{t=k_M}^{l_M-1} c_{t,t+1} + c_{l_M,l_M+1} + c_{l_M} - c_{l_M+1}) + \sum_{m=1}^{M-1} v_{\mathbf{e}}(K_m)$$

$$= \sum_{i\in K_M} SV_i(V_{\mathbf{e}}) + \sum_{m=1}^{M-1} \sum_{i\in K_m} SV_i(V_{\mathbf{e}})$$

$$= \sum_{i\in K} SV_i(V_{\mathbf{e}}).$$

d) $P_K\backslash\mathcal{S}_{\mathbf{e}} = \{K_1, K_M\}$: That is, except for K_1 and K_M, each $K_m \in P_K$ is a connected set. Then, for each $S \subseteq N\backslash K$ such that $\{k_1-1,\, l_M+1\}\cap S = \emptyset$, (14) holds. For each $S \subseteq N\backslash K$ such that $k_1 - 1 \in S$ and $l_M + 1 \notin S$, (15) holds. For each $S \subseteq N\backslash K$ such that $l_M + 1 \in S$ and $k_1 - 1 \notin S$, (16) holds. For each $S \subseteq N\backslash K$ such that $\{k_1-1,\, l_M+1\} \subseteq S$,

$$\widehat{v}(S\cup\{k\})-\widehat{v}(S) = (c_{k_1-1,k_1}+\sum_{t=k_1}^{l_1-1} c_{t,t+1}+c_{l_1}-c_{k_1-1})+(c_{k_M}+\sum_{t=k_M}^{l_M-1} c_{t,t+1}+c_{l_M,l_M+1}-c_{l_M+1})+\sum_{m=2}^{M-1} v_{\mathbf{e}}(K_m).$$

Then,

$$SV_k(\widehat{V}) = \sum_{s=0}^{n-|K|-2} \binom{n-|K|-2}{s} g(s) \sum_{m=1}^{M} v_{\mathbf{e}}(K_m) + \sum_{s=1}^{n-|K|-1} \binom{n-|K|-2}{s-1} g(s)[c_{k_1-1,k_1} + \sum_{t=k_1}^{l_1-1} c_{t,t+1} + c_{l_1} -$$

$$c_{k_1-1} + \sum_{m=2}^{M} v_{\mathbf{e}}(K_m)] +$$

$$\sum_{s=1}^{n-|K|-1} \binom{n-|K|-2}{s-1} g(s)[c_{k_M} + \sum_{t=k_M}^{l_M-1} c_{t,t+1} + c_{l_M,l_M+1} - c_{l_M+1} + \sum_{m=1}^{M-1} v_{\mathbf{e}}(K_m)] +$$

$$\sum_{s=2}^{n-|K|} \binom{n-|K|-2}{s-2} g(s)[c_{k_1-1,k_1} + \sum_{t=k_1}^{l_1-1} c_{t,t+1} + c_{l_1} - c_{k_1-1}$$

$$+c_{k_M} + \sum_{t=k_M}^{l_M-1} c_{t,t+1} + c_{l_M,l_M+1} - c_{l_M+1} + \sum_{m=2}^{M-1} v_{\mathbf{e}}(K_m)]$$

Note that $\sum_{s=0}^{n-|K|-2} \binom{n-|K|-2}{s} g(s) = \sum_{s=2}^{n-|K|} \binom{n-|K|-2}{s-2} g(s) = \frac{1}{3}$ and $\sum_{s=1}^{n-|K|-1} \binom{n-|K|-2}{s-1} g(s) = \frac{1}{6}$. Hence,

$$= \frac{1}{2}(2c_{l_1} + 2\sum_{t=l_1}^{k_1-1} c_{t,t+1} + c_{k_1-1,k_1} + c_{k_1} - c_{k_1-1}) + \sum_{m=2}^{M-1} v_{\mathbf{e}}(K_m) +$$

$$\frac{1}{2}(2c_{k_M} + 2\sum_{t=k_M}^{l_M-1} c_{t,t+1} + c_{l_M,l_M+1} + c_{l_M} - c_{l_M+1})$$

$$= \sum_{i\in K_1} SV_i(V_{\mathbf{e}}) + \sum_{m=2}^{M-1} \sum_{i\in K_m} SV_i(V_{\mathbf{e}}) + \sum_{i\in K_M} SV_i(V_{\mathbf{e}})$$

$= \sum_{i\in K} SV_i(V_{\mathbf{e}})$.

In all the possible cases, we showed that $SV_k(\widehat{V}) = \sum_{i\in K} SV_i(V_{\mathbf{e}})$. Therefore, the Shapley value satisfies *Merging and Splitting Proofness.* □

Equal Benefit:
Let V=(N, v) with $N = \{1, 2\}$. For each $i \in N$, $SV_i(V_{\mathbf{e}}) = \frac{1}{2}\left[v(N){+}v(\{i\}){-}v(\{j\})\right]$ where $j = N\backslash\{i\}$. Hence, $SV_1(V_{\mathbf{e}}) - v(\{1\}) = \frac{1}{2}\left[v(N){-}v(\{1\}){-}v(\{2\})\right] = SV_2(V_{\mathbf{e}}) - v(\{2\})$. Therefore, the Shapley value satisfies *Equal Benefit.* □

■

Proof of Theorem 2:

- First, we show that the Shapley value is the only solution which satisfies the axioms in Theorem 2.

Let F satisfy those axioms. Let $\mathbf{e} = \langle N, \mathbf{c}, r\rangle \in \mathcal{E}$. We will show that for each $S \in \mathcal{S}_{\mathbf{e}}$ and each $i \in S$, $F_i(V_{\mathbf{e}}) = SV_i(V_{\mathbf{e}})$.

Step 1: Let $\mathbf{e}^1 = \left\langle N, \mathbf{c}, r^1\right\rangle$ where for each $S \in \mathcal{S}_{\mathbf{e}^1}$, $|S| = 1$. By *Respect of Connected Sets,* for each $i \in N$, $F_i(V_{\mathbf{e}^1}) = SV_i(V_{\mathbf{e}^1}) = 2c_i$.
Step 2: Let

$$E^2 \;=\; \{\mathbf{e}^2{\in}\mathcal{E} : \mathbf{e}^2 = \left\langle N, \mathbf{c}, r^2\right\rangle \text{ is such that for each } S \in \mathcal{S}_{\mathbf{e}^2},\ |S| \leq 2\}$$

We will show that for each $\mathbf{e}^2{\in}E^2$ and each $i \in N$, $F_i(V_{\mathbf{e}^2}) = SV_i(V_{\mathbf{e}^2})$.

Let $i \in N$. First, suppose that $\{i\} \in \mathcal{S}_{\mathbf{e}^2}$. Then, by *Respect of Connected Sets,* $F_i(V_{\mathbf{e}^2}) = SV_i(V_{\mathbf{e}^2}) = 2c_i$. Next, suppose that $\{i, j\} \in \mathcal{S}_{\mathbf{e}^2}$ where $j \in \{i-1, i+1\}$. Without loss of generality, let $j = i + 1$.

Now, consider the economy $\mathbf{e}^1 = \left\langle N, \mathbf{c}, r^1\right\rangle$ where for each $S \in \mathcal{S}_{\mathbf{e}^1}$, $|S| = 1$. Hence, each sponsor in $\mathbf{e}^1$ constitutes a singleton connected set.

Note that $0 \succ_{r^1} i \succ_{r^1} 0 \succ_{r^1} i+1 \succ_{r^1} 0$ and $0 \succ_{r^2} i \succ_{r^2} i+1 \succ_{r^2} 0$. Since F satisfies *Weak Equal Impact,*

$$F_i(V_{\mathbf{e}^2}) - F_i(V_{\mathbf{e}^1}) = F_{i+1}(V_{\mathbf{e}^2}) - F_{i+1}(V_{\mathbf{e}^1}). \tag{17}$$

By *Respect of Connected Sets,* for each $i \in N$, $F_i(V_{\mathbf{e}^1}) = 2c_i = SV_i(V_{\mathbf{e}^1})$. This equality and (17) together imply

$$F_i(V_{\mathbf{e}^2}) - F_{i+1}(V_{\mathbf{e}^2}) = SV_i(V_{\mathbf{e}^1}) - SV_{i+1}(V_{\mathbf{e}^1}). \tag{18}$$

Since SV satisfies *Weak Equal Impact,*

$$SV_i(V_{\mathbf{e}^2}) - SV_i(V_{\mathbf{e}^1}) = SV_{i+1}(V_{\mathbf{e}^2}) - SV_{i+1}(V_{\mathbf{e}^1}). \tag{19}$$

Equalities (18) and (19) together imply

$$F_i(V_{\mathbf{e}^2}) - SV_i(V_{\mathbf{e}^2}) = F_{i+1}(V_{\mathbf{e}^2}) - SV_{i+1}(V_{\mathbf{e}^2}) = \gamma(\mathbf{e}^2). \tag{20}$$

By *Respect of Connected Sets*, $F_i(V_{\mathbf{e}^2}) + F_{i+1}(V_{\mathbf{e}^2}) = SV_i(V_{\mathbf{e}^2}) + SV_{i+1}(V_{\mathbf{e}^2}) = v_{\mathbf{e}}(\{1,2\})$. Hence, $[F_i(V_{\mathbf{e}^2}) - SV_i(V_{\mathbf{e}^2})] + [F_{i+1}(V_{\mathbf{e}^2}) - SV_{i+1}(V_{\mathbf{e}^2})] = 2\gamma(\mathbf{e}^2) = 0$. Therefore, $F_i(V_{\mathbf{e}^2}) = SV_i(V_{\mathbf{e}^2})$.

Step 3: Let $\widehat{S} \in \mathcal{S}_{\mathbf{e}}$. Without loss of generality, suppose $\widehat{S} = \{1, 2, ..., \widehat{s}\}$. We will show that for each $i \leq \widehat{s} - 1$,

$$F_i(V_{\mathbf{e}^2}) - SV_i(V_{\mathbf{e}^2}) = F_{i+1}(V_{\mathbf{e}^2}) - SV_{i+1}(V_{\mathbf{e}^2}).$$

For each $i \leq \widehat{s} - 1$, let $\mathbf{e}_i = \langle N, \mathbf{c}, r_i \rangle$ be such that

a) if $i = 1$, then $1 \succ_{r_1} 0 \succ_{r_1} 2 \succ_{r_1} 3 \succ_{r_1} 0$,

b) if $i > 1$, then $0 \succ_{r_i} i-1 \succ_{r_i} i \succ_{r_i} 0 \succ_{r_i} i+1 \succ_{r_i} i+2 \succ_{r_i} 0$,

c) for each $j \in N \backslash \{i-1, i, i+1, i+2\}$, we have $\{j\} \in \mathcal{S}_{\mathbf{e}}$.

By *Weak Equal Impact*, for each $i \leq \widehat{s} - 1$,

$$F_i(V_{\mathbf{e}}) - F_i(V_{\mathbf{e}_i}) = F_{i+1}(V_{\mathbf{e}}) - F_{i+1}(V_{\mathbf{e}_i}). \tag{21}$$

Note that for each $i \leq \widehat{s} - 1$, we have $\mathbf{e}_i \in E^2$. Hence, by Step 2, for each $i \in N$, $F_i(V_{\mathbf{e}_i}) = SV_i(V_{\mathbf{e}_i})$. This equality and (21) together imply

$$F_i(V_{\mathbf{e}}) - F_{i+1}(V_{\mathbf{e}}) = SV_i(V_{\mathbf{e}_i}) - SV_{i+1}(V_{\mathbf{e}_i}). \tag{22}$$

Since SV satisfies *Weak Equal Impact*,

$$SV_i(V_{\mathbf{e}}) - SV_i(V_{\mathbf{e}_i}) = SV_{i+1}(V_{\mathbf{e}}) - SV_{i+1}(V_{\mathbf{e}_i}). \tag{23}$$

Equalities (22) and (23) together imply

$$F_i(V_{\mathbf{e}}) - SV_i(V_{\mathbf{e}}) = F_{i+1}(V_{\mathbf{e}}) - SV_{i+1}(V_{\mathbf{e}_i}) = \gamma(\mathbf{e}). \tag{24}$$

By *Respect of Connected Sets*, $\sum_{i \in \widehat{S}} F_i(V_{\mathbf{e}}) = \sum_{i \in \widehat{S}} SV_i(V_{\mathbf{e}}) = v_{\mathbf{e}}(\widehat{S})$. This equality and (24) together imply

$$\sum_{i \in \widehat{S}} [F_i(V_{\mathbf{e}}) - SV_i(V_{\mathbf{e}})] = \widehat{s}\, \gamma(\mathbf{e}) = 0.$$

Thus, $\gamma(\mathbf{e}) = 0$ and for each $i \in \widehat{S}$, $F_i(V_{\mathbf{e}}) = SV_i(V_{\mathbf{e}})$. By repeating Step 3, for each $S \in \mathcal{S}_{\mathbf{e}}$, we prove that for each $S \in \mathcal{S}_{\mathbf{e}}$ and each $i \in S$, $F_i(V_{\mathbf{e}}) = SV_i(V_{\mathbf{e}})$. This completes the proof.

- Now, we show that the Shapley value satisfies the axioms listed in Theorem 2.

By Theorem 1, the Shapley value satisfies *Respect of Connected Sets*. Let us show that it satisfies *Weak Equal Impact*. Let $\{\mathbf{e}, \mathbf{e}'\} \subset \mathcal{E}$ where $\mathbf{e} = \langle N, \mathbf{c}, r \rangle$ and $\mathbf{e}' = \langle N, \mathbf{c}, r' \rangle$ be such that there exists $i \in N \backslash \{n\}$ where

(i) $i \succ_r 0 \succ_r i+1$ and $i \succ_{r'} i+1$, and

(ii) $i-1 \succ_{r'} i$ if and only if $i-1 \succ_r i$, and

(ii) $i+1 \succ_{r'} i+2$ if and only if $i+1 \succ_r i+2$.

There are four cases to consider:

Case 1: $i-1 \succ_r i \succ_r 0 \succ_r i+1 \succ_r i+2$. Then, $SV_i(V_{\mathbf{e}}) = \frac{1}{2}(3c_i + c_{i-1,i} - c_{i-1})$, $SV_i(V_{\mathbf{e}'}) = \frac{1}{2}(2c_i + c_{i-1,i} + c_{i,i+1} - c_{i-1} - c_{i+1})$, $SV_{i+1}(V_{\mathbf{e}}) = \frac{1}{2}(3c_{i+1} + c_{i+1,i+2} - c_{i+2})$, and $SV_{i+1}(V_{\mathbf{e}'}) = \frac{1}{2}(2c_{i+1} + c_{i,i+1} + c_{i+1,i+2} - c_i - c_{i+2})$. Hence,

$$SV_i(V_{\mathbf{e}}) - SV_i(V_{\mathbf{e}'}) = \frac{1}{2}(c_i + c_{i+1} - c_{i,i+1}) = SV_{i+1}(V_{\mathbf{e}}) - SV_{i+1}(V_{\mathbf{e}'}). \tag{25}$$

Case 2: $0 \succ_r i \succ_r 0 \succ_r i+1 \succ_r i+2$. Then, $SV_i(V_{\mathbf{e}}) = 2c_i$, $SV_i(V_{\mathbf{e}'}) = \frac{1}{2}(3c_i + c_{i,i+1} - c_{i+1})$; and $SV_{i+1}(V_{\mathbf{e}})$ and $SV_{i+1}(V_{\mathbf{e}'})$ are as in Case 1. It is easy to check that equality 25 still holds in Case 2.

Case 3: $0 \succ_r i \succ_r 0 \succ_r i+1 \succ_r 0$. Then, $SV_i(V_{\mathbf{e}})$ and $SV_i(V_{\mathbf{e}'})$ are as in Case 2; and $SV_{i+1}(V_{\mathbf{e}}) = 2c_{i+1}$, $SV_{i+1}(V_{\mathbf{e}'}) = \frac{1}{2}(3c_{i+1} + c_{i,i+1} - c_i)$. Then, again equality 25 holds.

Case 4: $i-1 \succ_r i \succ_r 0 \succ_r i+1 \succ_r 0$. Then, $SV_i(V_{\mathbf{e}})$ and $SV_i(V_{\mathbf{e}'})$ are as in Case 1; and $SV_{i+1}(V_{\mathbf{e}})$ and $SV_{i+1}(V_{\mathbf{e}'})$ are as in Case 3. Then, equality 25 still holds. ■

5.3 Independence of Axioms

Independence of the axioms in Theorem 1

• The following Dictatorial solution satisfies *Respect of Connected Sets* and *Merging and Splitting Proofness*, but not *Equal Benefit*.

For each $\mathbf{e} = \langle N, \mathbf{c}, r\rangle \in \mathcal{E}$ and each $i \in N$,

if $\mathcal{S}_{\mathbf{e}} = \{N\}$, then

$$D_i(V_{\mathbf{e}}) = \begin{cases} v_{\mathbf{e}}(N) & \text{if } i = \arg\min\{j \in N\}, \\ 0 & \text{otherwise.} \end{cases}$$

if $|\mathcal{S}_{\mathbf{e}}| > 1$, then $F_i(V_{\mathbf{e}}) = SV_i(V_{\mathbf{e}})$.

• The following solution satisfies *Merging and Splitting Proofness* and *Equal Benefit*, but not *Respect of Connected Sets*.

For each $\mathbf{e} = \langle N, \mathbf{c}, r\rangle \in \mathcal{E}$ and each $i \in N$, $F_i(V_{\mathbf{e}}) = \frac{1}{v_{\mathbf{e}}(N)} SV_i(V_{\mathbf{e}})$.

• The following solution satisfies *Respect of Connected Sets* and *Equal Benefit*, but not *Merging and Splitting Proofness*.

For each $\mathbf{e} = \langle N, \mathbf{c}, r\rangle \in \mathcal{E}$, each $i \in N$, and each $S_i \in \mathcal{S}_{\mathbf{e}}$ with $i \in S_i$,

$$F_i(V_{\mathbf{e}}) = \begin{cases} SV_i(V_{\mathbf{e}}) & \text{if } n \leq 2, \\ \frac{c_i}{\sum_{j \in S_i} c_j} v_{\mathbf{e}}(S) & n > 2. \end{cases}$$

Independence of the axioms in Theorem 2

• The following solution satisfies *Respect of Connected Sets* but not *Weak Equal Impact*.

For each $\mathbf{e} = \langle N, \mathbf{c}, r\rangle \in \mathcal{E}$, each $i \in N$, and each $S_i \in \mathcal{S}_{\mathbf{e}}$ with $i \in S_i$,

$$P_i(V_{\mathbf{e}}) = \frac{c_i}{\sum_{j \in S_i} c_j} v_{\mathbf{e}}(S).$$

• The following solution satisfies *Weak Equal Impact* but not *Respect of Connected Sets*.

Let $\lambda \geq 0$. For each $\mathbf{e} = \langle N, \mathbf{c}, r \rangle \in \mathcal{E}$ and each $i \in N$,

$$F_i^\lambda(V_\mathbf{e}) = SV_i(V_\mathbf{e}) - \lambda.$$

5.4 Proof of the Result in Section 4

Proof of Proposition 2: First, we demonstrate the calculation of the Shapley value for a 3-sponsor economy. Suppose $N = \{1, 2, 3\}$ and $r = (0, 1, 2, 3, 0)$. For each $S \subseteq N$, let $|S| = s$ and $f(s) = \frac{s!(n-s-1)!}{n!}$. Then,

$f(s)$	$S : 1 \notin S$	$v^*_\mathbf{e}(S \cup \{1\}) - v^*_\mathbf{e}(S)$	$S : 2 \notin S$	$v^*_\mathbf{e}(S \cup \{2\}) - v^*_\mathbf{e}(S)$	$S : 3 \notin S$	$v^*_\mathbf{e}(S \cup \{3\}) - v^*_\mathbf{e}(S)$
2/6	$\emptyset$	$2c_1$	$\emptyset$	$2c_2$	$\emptyset$	$2c_3$
1/6	$\{2\}$	$c_1 + c_{1,2} - c_2$	$\{1\}$	$c_2 + c_{1,2} - c_1$	$\{1\}$	$c_3 + c_{1,3} - c_1$
1/6	$\{3\}$	$c_1 + c_{1,3} - c_3$	$\{3\}$	$c_2 + c_{2,3} - c_3$	$\{2\}$	$c_3 + c_{2,3} - c_2$
2/6	$\{2, 3\}$	$c_1 + c_{1,2} - c_2$	$\{1, 3\}$	$c_{1,2} + c_{2,3} - c_{1,3}$	$\{1, 2\}$	$c_3 + c_{2,3} - c_2$

Since $SV_i(V^*_\mathbf{e}) = \sum_{S \subseteq N \setminus \{i\}} f(s)[v^*_\mathbf{e}(S \cup \{i\}) - v^*_\mathbf{e}(S)]$, we have

$$\begin{aligned} SV_1(V^*_\mathbf{e}) &= \frac{4}{3}c_1 - \frac{1}{2}c_2 - \frac{1}{6}c_3 + \frac{1}{2}c_{1,2} + \frac{1}{6}c_{1,3} \\ SV_2(V^*_\mathbf{e}) &= c_2 - \frac{1}{6}c_1 - \frac{1}{6}c_3 + \frac{1}{2}c_{1,2} - \frac{1}{3}c_{1,3} + \frac{1}{2}c_{2,3} \\ SV_3(V^*_\mathbf{e}) &= \frac{4}{3}c_3 - \frac{1}{2}c_2 - \frac{1}{6}c_1 + \frac{1}{6}c_{1,3} + \frac{1}{2}c_{2,3}. \end{aligned}$$

Let $\mathbf{e} = \langle N, \mathbf{c}, r \rangle \in \mathcal{E}$ where $c_1 = 30$, $c_2 = 6$, $c_3 = 15$, $c_{1,2} = 25$, $c_{1,3} = 16$, $c_{2,3} = 20$, and $r = (0, 1, 2, 3, 0)$. Note that $\mathbf{e} \in \mathcal{E}_T$. We have $SV_1(V^*_\mathbf{e}) = \frac{149}{3}$, $SV_2(V^*_\mathbf{e}) = \frac{47}{3}$, and $SV_3(V^*_\mathbf{e}) = \frac{74}{3}$.

(i) Now, $SV_1(V^*_\mathbf{e}) + SV_2(V^*_\mathbf{e}) = \frac{196}{3}$ and $v^*_\mathbf{e}(\{1, 2\}) = c_1 + c_{1,2} + c_2 = 61$. Since $SV_1(V^*_\mathbf{e}) + SV_2(V^*_\mathbf{e}) > v^*_\mathbf{e}(\{1, 2\})$, $SV(V^*_\mathbf{e})$ is not in the core of $\mathbf{e}$.

(ii) Let sponsors 1 and 2 merge into a single sponsor denoted by k. Let $\widehat{V} = ((N \setminus \{1, 2\}) \cup \{k\}, \widehat{v})$ be the TU-game obtained from $V^*_\mathbf{e}$ by this merger. Thus, $\widehat{v}(\{k\}) = c_1 + c_{1,2} + c_2$, $\widehat{v}(\{k, 3\}) - \widehat{v}(\{3\}) = c_1 + c_{1,2} + c_{2,3} - c_3$. Then, $SV_k(\widehat{V}) = \frac{1}{2}(2c_1 + c_2 - c_3 + 2c_{1,2} + c_{2,3}) = \frac{121}{2}$. Since, $SV_k(\widehat{V}) \neq SV_1(V^*_\mathbf{e}) + SV_2(V^*_\mathbf{e})$, SV is not *Merging and Splitting Proof.*
Note that to show that SV is not *Merging and Splitting Proof,* similar examples can be constructed for any number of sponsors more than 3, any structure of connected sets, or any sets of sponsors who merge. ■

Proof of Theorem 3:

- First, we show that the Shapley value is the only solution which satisfies the axioms in Theorem 2.

Let F satisfy those axioms. Let $\mathbf{e} = \langle N, \mathbf{c}, r \rangle \in \mathcal{E}$. For each $1 \leq t \leq n$, let

$$E^t = \{\mathbf{e}^t \in \mathcal{E} : \mathbf{e}^t = \langle N, \mathbf{c}, r^t \rangle \text{ is such that for each } S \in \mathcal{S}_{\mathbf{e}^t},\ |S| \leq t\}.$$

Suppose that $\mathbf{e} \in E^T$ for some $1 \leq T \leq n$. For each $i \in N$, let $S_i(\mathbf{e}^t)$ be the connected set that i belongs to in economy $\mathbf{e}^t$.

For each $i < n$, let

$$\begin{aligned} E_i^t \;\; = \;\; & \{\mathbf{e}_i^t \in \mathcal{E} : \mathbf{e}_i^t = \langle N, \mathbf{c}, r_i^t \rangle \text{ is such that for each } j \notin S_i(\mathbf{e}^t),\ \{j\} \in \mathcal{S}_{\mathbf{e}_i^t}, \\ & i \succ_{r_i^t} 0 \succ_{r_i^t} i+1, \text{ and for each } k \in S_i(\mathbf{e}^t) \backslash \{i\} \text{ and } l \in S_i(\mathbf{e}^t) \backslash \{i+1\},\ k \succ_{r_i^t} l \text{ if and only if } k \succ_{r^t} l. \end{aligned}$$

That is, on route r_i^t, each sponsor that do not belong to $S_i(\mathbf{e}^t)$ is a singleton connected set, the traveler goes from i to $i+1$ via home, and the other sponsors in $S_i(\mathbf{e}^t)$ are visited on route r_i^t exactly the same way they were visited on route r^t. In other words, to obtain $g(\mathbf{e}_i^t)$, we modify $g(\mathbf{e}^t)$ by removing the link between i and $i+1$ and making each sponsor $j \notin S_i(\mathbf{e}^t)$ a singleton connected set.

By induction on t, we will show for each $t \leq T$, each $\mathbf{e}^t \in E^t$, and each $i \in N$, $F_i(V^*_{\mathbf{e}^t}) = SV_i(V^*_{\mathbf{e}^t})$.

Step 1 and 2: These steps are same as Steps 1 and 2 in the proof of Theorem 2 (just replace *Weak Equal Impact* with *Strong Equal Impact*).
Step 3: Consider $\mathbf{e}^3 \in E^3$.

For each $i \in N$ with $\{i\} \in \mathcal{S}_{\mathbf{e}^3}$, by *Respect of Connected Sets*,.$F_i(V^*_{\mathbf{e}^3}) = SV_i(V^*_{\mathbf{e}^3}) = 2c_i$. Now, we will show that for each $\widehat{S} \in \mathcal{S}_{\mathbf{e}^3}$ with $|\widehat{S}| > 1$ and each $\{i, i+1\} \subseteq \widehat{S}$,

$$F_i(V^*_{\mathbf{e}^3}) - SV_i(V^*_{\mathbf{e}^3}) = F_{i+1}(V^*_{\mathbf{e}^3}) - SV_{i+1}(V^*_{\mathbf{e}^3}).$$

Note that $i \succ_{r^3} i+1$ and $i \succ_{r_i^3} 0 \succ_{r_i^3} i+1$; and for each $k \in S_i(\mathbf{e}^3) \backslash \{i\}$ and $l \in S_i(\mathbf{e}^3) \backslash \{i+1\}$, $k \succ_{r_i^3} l$ if and only if $k \succ_{r^3} l$. Hence, by *Strong Equal Impact*,

$$F_i(V^*_{\mathbf{e}^3}) - F_i(V^*_{\mathbf{e}_i^3}) = F_{i+1}(V^*_{\mathbf{e}^3}) - F_{i+1}(V^*_{\mathbf{e}_i^3}). \tag{26}$$

Note that $\mathbf{e}_i^3 \in E^1 \cup E^2$. Hence, by Steps 1 and 2, for each $i \in N$, $F_i(V^*_{\mathbf{e}_i^3}) = SV_i(V^*_{\mathbf{e}_i^3})$. This equality and (26) together imply

$$F_i(V^*_{\mathbf{e}^3}) - F_{i+1}(V^*_{\mathbf{e}^3}) = SV_i(V^*_{\mathbf{e}_i^3}) - SV_{i+1}(V^*_{\mathbf{e}_i^3}). \tag{27}$$

Since SV satisfies *Strong Equal Impact*,

$$SV_i(V^*_{\mathbf{e}^3}) - SV_i(V^*_{\mathbf{e}_i^3}) = SV_{i+1}(V^*_{\mathbf{e}^3}) - SV_{i+1}(V^*_{\mathbf{e}_i^3}). \tag{28}$$

Equalities (27) and (28) together imply

$$F_i(V^*_{\mathbf{e}^3}) - SV_i(V^*_{\mathbf{e}^3}) = F_{i+1}(V^*_{\mathbf{e}^3}) - SV_{i+1}(V^*_{\mathbf{e}_i^3}) = \gamma(\mathbf{e}_i^3). \tag{29}$$

By *Respect of Connected Sets*, $\sum\limits_{i \in \widehat{S}} F_i(V^*_{\mathbf{e}^3}) = \sum\limits_{i \in \widehat{S}} SV_i(V^*_{\mathbf{e}^3}) = v^*_{\mathbf{e}}(\widehat{S})$. That is,

$$\sum_{i \in \widehat{S}} [F_i(V^*_{\mathbf{e}^3}) - SV_i(V^*_{\mathbf{e}^3})] = |\widehat{S}|\ \gamma(\mathbf{e}) = 0.$$

Thus, for each $\widehat{S} \in \mathcal{S}_{\mathbf{e}^3}$ with $|\widehat{S}| > 1$ and $i \in \widehat{S}$, $F_i(V^*_{\mathbf{e}^3}) = SV_i(V^*_{\mathbf{e}^3})$.

Step T: Assume that for each $t < T$ and each $i \in N$, $F_i(V^*_{\mathbf{e}^t}) = SV_i(V^*_{\mathbf{e}^t})$. We will show that for each $i \in N$, $F_i(V^*_{\mathbf{e}^T}) = SV_i(V^*_{\mathbf{e}^T})$.

For each $i \in N$ with $\{i\} \in \mathcal{S}_{\mathbf{e}^T}$, by *Respect of Connected Sets*,.$F_i(V^*_{\mathbf{e}^T}) = SV_i(V^*_{\mathbf{e}^T}) = 2c_i$. Now, we will show that for each $\widehat{S} \in \mathcal{S}_{\mathbf{e}^T}$ with $|\widehat{S}| > 1$ and each $\{i, i+1\} \subseteq \widehat{S}$,

$$F_i(V^*_{\mathbf{e}^T}) - SV_i(V^*_{\mathbf{e}^T}) = F_{i+1}(V^*_{\mathbf{e}^T}) - SV_{i+1}(V^*_{\mathbf{e}^T}).$$

Note that $i \succ_{r^T} i+1$ and $i \succ_{r_i^T} 0 \succ_{r_i^T} i+1$; and for each $k \in S_i(\mathbf{e}^T)\backslash\{i\}$ and $l \in S_i(\mathbf{e}^T)\backslash\{i+1\}$, $k \succ_{r_i^T} l$ if and only if $k \succ_{r^T} l$. Hence, by *Strong Equal Impact*,

$$F_i(V^*_{\mathbf{e}^T}) - F_i(V^*_{\mathbf{e}_i^T}) = F_{i+1}(V^*_{\mathbf{e}^T}) - F_{i+1}(V^*_{\mathbf{e}_i^T}). \tag{30}$$

Note that $\mathbf{e}_i^T \in \bigcup\limits_{t \leq T-1} E^t$. Hence, by the induction hypothesis, for each $i \in N$, $F_i(V^*_{\mathbf{e}_i^T}) = SV_i(V^*_{\mathbf{e}_i^T})$. This equality and (30) together imply

$$F_i(V^*_{\mathbf{e}^T}) - F_{i+1}(V^*_{\mathbf{e}^T}) = SV_i(V^*_{\mathbf{e}_i^T}) - SV_{i+1}(V^*_{\mathbf{e}_i^T}). \tag{31}$$

Since SV satisfies *Strong Equal Impact*,

$$SV_i(V^*_{\mathbf{e}^T}) - SV_i(V^*_{\mathbf{e}_i^T}) = SV_{i+1}(V^*_{\mathbf{e}^T}) - SV_{i+1}(V^*_{\mathbf{e}_i^T}). \tag{32}$$

Equalities (31) and (32) together imply

$$F_i(V^*_{\mathbf{e}^T}) - SV_i(V^*_{\mathbf{e}^T}) = F_{i+1}(V^*_{\mathbf{e}^T}) - SV_{i+1}(V^*_{\mathbf{e}_i^T}) = \gamma(\mathbf{e}^T). \tag{33}$$

By *Respect of Connected Sets*, $\sum\limits_{i \in \widehat{S}} F_i(V^*_{\mathbf{e}^T}) = \sum\limits_{i \in \widehat{S}} SV_i(V^*_{\mathbf{e}^T}) = v^*_{\mathbf{e}}(\widehat{S})$. This equality and (33) together imply

$$\sum_{i \in \widehat{S}} [F_i(V^*_{\mathbf{e}^T}) - SV_i(V^*_{\mathbf{e}^T})] = |\widehat{S}| \ \gamma(\mathbf{e}^T) = 0.$$

Thus, $\gamma(\mathbf{e}^T) = 0$ and for each $\widehat{S} \in \mathcal{S}_{\mathbf{e}^T}$ with $|\widehat{S}| > 1$ and $i \in \widehat{S}$, $F_i(V^*_{\mathbf{e}^T}) = SV_i(V^*_{\mathbf{e}^T})$. This concludes the proof.

- Now, we show that the Shapley value satisfies the axioms listed in Theorem 3.

Respect of Connected Sets:
Let $\mathbf{e} = \langle N, \mathbf{c}, r \rangle \in \mathcal{E}$ and $\widehat{S} \in \mathcal{S}_{\mathbf{e}}$. Let $\widehat{\mathbf{e}} = \langle N, \widehat{\mathbf{c}}, r \rangle$ be such that for each $j \in N\backslash\widehat{S}$, j is a null sponsor in the sense that $\widehat{c}_j = 0$ and $\widehat{c}_{j,l} = 0$ for each $l \in N$. For each null sponsor $j \in N\backslash\widehat{S}$, since her marginal contribution to any coalition is zero, $SV_j(V^*_{\widehat{\mathbf{e}}}) = 0$. Note that due to the definition of the permissible routes in a routing game, for each $i \in \widehat{S}$ and each $S \subseteq N\backslash\{i\}$,

$$v^*_{\mathbf{e}}(S \cup \{i\}) - v^*_{\mathbf{e}}(S) = v^*_{\mathbf{e}}((S \cap \widehat{S}\backslash\{i\}) \cup \{i\}) - v^*_{\mathbf{e}}((S \cap \widehat{S}\backslash\{i\})) =$$
$$v^*_{\widehat{\mathbf{e}}}((S \cap \widehat{S}\backslash\{i\}) \cup \{i\}) - v^*_{\widehat{\mathbf{e}}}((S \cap \widehat{S}\backslash\{i\})) = v^*_{\widehat{\mathbf{e}}}(S \cup \{i\}) - v^*_{\widehat{\mathbf{e}}}(S).$$

Hence, for each $i \in \widehat{S}$, $SV_i(V^*_{\mathbf{e}}) = SV_i(V^*_{\widehat{\mathbf{e}}})$. By *Efficiency* the Shapley value, $\sum\limits_{i \in \widehat{S}} SV_i(V^*_{\widehat{\mathbf{e}}}) = \widehat{c}(r) = v^*_{\widehat{\mathbf{e}}}(\widehat{S})$. All together, $\sum\limits_{i \in \widehat{S}} SV_i(V^*_{\mathbf{e}}) = v^*_{\widehat{\mathbf{e}}}(\widehat{S})$. Hence, the Shapley value satisfies *Respect of Connected Sets*.

Strong Equal Impact:

Let $\{\mathbf{e}, \mathbf{e}'\} \subset \mathcal{E}$ where $\mathbf{e} = \langle N, \mathbf{c}, r\rangle$ and $\mathbf{e}' = \langle N, \mathbf{c}, r'\rangle$ be such that there exists $i \in N\backslash\{n\}$ where

(i) $i \succ_r i+1$ and $i \succ_{r'} 0 \succ_{r'} i+1$, and

(ii) for $S_i \in \mathcal{S}_\mathbf{e}$ with $i \in S_i$, for each $k \in S_i\backslash\{i\}$ and $l \in S_i\backslash\{i+1\}$, $k \succ_r l$ if and only if $k \succ_{r'} l$.

Let $\widehat{S} = S_i\backslash\{i, i+1\}$. Note the following: for each $S \subseteq N\backslash\{i, i+1\}$,

(a) $v^*_\mathbf{e}(S \cup \{i\}) - v^*_\mathbf{e}(S) = v^*_{\mathbf{e}'}(S \cup \{i\}) - v^*_{\mathbf{e}'}(S)$,

(b) $v^*_\mathbf{e}(S \cup \{i+1\} \cup \{i\}) - v^*_\mathbf{e}(S \cup \{i+1\}) = v^*_\mathbf{e}(\widehat{S} \cup \{i+1\} \cup \{i\}) - v^*_\mathbf{e}(\widehat{S} \cup \{i+1\})$,

(c) $v^*_{\mathbf{e}'}(S \cup \{i+1\} \cup \{i\}) - v^*_{\mathbf{e}'}(S \cup \{i+1\}) = v^*_{\mathbf{e}'}(\widehat{S} \cup \{i+1\} \cup \{i\}) - v^*_{\mathbf{e}'}(\widehat{S} \cup \{i+1\})$,

(d) $v^*_\mathbf{e}(\widehat{S} \cup \{i+1\}) = v^*_{\mathbf{e}'}(\widehat{S} \cup \{i+1\})$, and

(e) $v^*_\mathbf{e}(S \cup \{i+1\}) - v^*_\mathbf{e}(S) = v^*_{\mathbf{e}'}(S \cup \{i+1\}) - v^*_{\mathbf{e}'}(S)$,

(f) $v^*_\mathbf{e}(S \cup \{i+1\} \cup \{i\}) - v^*_\mathbf{e}(S \cup \{i\}) = v^*_\mathbf{e}(\widehat{S} \cup \{i+1\} \cup \{i\}) - v^*_\mathbf{e}(\widehat{S} \cup \{i\})$,

(g) $v^*_{\mathbf{e}'}(S \cup \{i+1\} \cup \{i\}) - v^*_{\mathbf{e}'}(S \cup \{i\}) = v^*_{\mathbf{e}'}(\widehat{S} \cup \{i+1\} \cup \{i\}) - v^*_{\mathbf{e}'}(\widehat{S} \cup \{i\})$,

(h) $v^*_\mathbf{e}(\widehat{S} \cup \{i\}) = v^*_{\mathbf{e}'}(\widehat{S} \cup \{i\})$.

By (a), (b), (c), and (d)

$$SV_i(V^*_\mathbf{e}) - SV_i(V^*_{\mathbf{e}'}) =$$

$$\sum_{S \subseteq N\backslash\{i,i+1\}} f(s)[(v^*_\mathbf{e}(S\cup\{i+1\}\cup\{i\}) - v^*_\mathbf{e}(S\cup\{i+1\})) - (v^*_{\mathbf{e}'}(S\cup\{i+1\}\cup\{i\}) - v^*_{\mathbf{e}'}(S\cup\{i+1\}))]$$

$$= \sum_{S \subseteq N\backslash\{i,i+1\}} f(s)[(v^*_\mathbf{e}(\widehat{S}\cup\{i+1\}\cup\{i\}) - v^*_\mathbf{e}(\widehat{S}\cup\{i+1\})) - (v^*_{\mathbf{e}'}(\widehat{S}\cup\{i+1\}\cup\{i\}) - v^*_{\mathbf{e}'}(\widehat{S}\cup\{i+1\}))]$$

$$= \sum_{S \subseteq N\backslash\{i,i+1\}} f(s)[(v^*_\mathbf{e}(\widehat{S} \cup \{i+1\} \cup \{i\}) - (v^*_{\mathbf{e}'}(\widehat{S} \cup \{i+1\} \cup \{i\})].$$

Similarly, by (e), (f), (g), and (h),

$$SV_{i+1}(V^*_\mathbf{e}) - SV_{i+1}(V^*_{\mathbf{e}'}) = \sum_{S \subseteq N\backslash\{i,i+1\}} f(s)[(v^*_\mathbf{e}(\widehat{S} \cup \{i+1\} \cup \{i\}) - (v^*_{\mathbf{e}'}(\widehat{S} \cup \{i+1\} \cup \{i\})].$$

Therefore, $SV_i(V^*_\mathbf{e}) - SV_i(V^*_{\mathbf{e}'}) = SV_{i+1}(V^*_\mathbf{e}) - SV_{i+1}(V^*_{\mathbf{e}'})$ and the Shapley value satisfies *Strong Equal Impact.*

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7 Appendix for referees

Here, we depict the calculation of Shapley values we used in the proof of Theorem 1 where we showed the Shapley value is merging and splitting proof.

In part (b) and (c), we used the following expressions whose simplifications are as follows:

- $\sum_{s=1}^{n-|K|} \binom{n-|K|-1}{s-1} f(s) = \sum_{s=1}^{n-|K|} \frac{(n-|K|-1)!}{(s-1)!(n-|K|-s)!} \frac{s!(n-|K|+1-s-1)!}{(n-|K|+1)!} = \sum_{s=1}^{n-|K|} \frac{s}{(n-|K|+1)(n-|K|)}$

  $= \frac{1}{(n-|K|+1)(n-|K|)} \frac{(n-|K|)(n-|K|+1)}{2} = 1/2$

- $\sum_{s=0}^{n-|K|-1} \binom{n-|K|-1}{s} f(s) = \sum_{s=0}^{n-|K|-1} \frac{(n-|K|-1)!}{(s)!(n-|K|-s-1)!} \frac{s!(n-|K|-s)!}{(n-|K|+1)!}$

$= \sum_{s=0}^{n-|K|-1} \frac{(n-|K|-s)}{(n-|K|+1)(n-|K|)} = \sum_{s=0}^{n-|K|-1} \frac{1}{(n-|K|+1)} - \frac{1}{(n-|K|+1)(n-|K|)} \sum_{s=0}^{n-|K|-1} s$

$= \frac{(n-|K|)}{(n-|K|+1)} - \frac{1}{(n-|K|+1)(n-|K|)} \frac{(n-|K|-1)(n-|K|)}{2} = \frac{2(n-|K|)-(n-|K|-1)}{2(n-|K|+1)} = 1/2$

In part (d), we used the following expressions:

- $\sum_{s=0}^{n-|K|-2} \binom{n-|K|-2}{s} f(s) = \sum_{s=0}^{n-|K|-2} \frac{(n-|K|-2)!}{(s)!(n-|K|-s-2)!} \frac{s!(n-|K|-s)!}{(n-|K|+1)!}$

  $= \frac{(n-|K|-2)!}{(n-|K|+1)!} \sum_{s=0}^{n-|K|-2} (n-|K|-s)(n-|K|-s-1)$

  Let $x = n - |K| + 1$

  Then,

  $\frac{(n-|K|-2)!}{(n-|K|+1)!} \sum_{s=0}^{n-|K|-2} (n-|K|-s)(n-|K|-s-1) = \frac{(x-3)!}{x!} \sum_{s=0}^{x-3} \left[(x^2 - 3x + 2) + s(3-2x) + s^2\right]$

  $= \frac{(x-3)!}{x!} \left[\sum_{s=0}^{x-3} (x^2 - 3x + 2) + (3-2x) \sum_{s=0}^{x-3} s(3-2x) + (3-2x)s^2\right]$

  $= \frac{(x-3)!}{x!} \left[(x^2 - 3x + 2)(x-2) + \frac{(x-3)(x-2)}{2}(3-2x) + \frac{(x-3)(x-2)(2x-5)}{6}\right]$

  $= \frac{(x-3)!}{x!} \frac{(x-2)(x-1)(3x-6-2x+6)}{3} = \frac{1}{3}$

- $\sum_{s=1}^{n-|K|-1} \binom{n-|K|-2}{s-1} f(s) = \sum_{s=1}^{n-|K|-1} \frac{(n-|K|-2)!}{(s-1)!(n-|K|-s-1)!} \frac{s!(n-|K|-s)!}{(n-|K|+1)!}$

  $= \sum_{s=1}^{n-|K|-1} \frac{s(n-|K|-s)}{(n-|K|+1)(n-|K|)(n-|K|-1)}$

  $= \sum_{s=1}^{n-|K|-1} \frac{s}{(n-|K|+1)(n-|K|-1)} - \frac{1}{(n-|K|+1)(n-|K|)(n-|K|-1)} \sum_{s=1}^{n-|K|-1} s^2$

$= \frac{(n-|K|-1)(n-|K|)}{2(n-|K|+1)(n-|K|-1)} - \frac{1}{(n-|K|+1)(n-|K|)(n-|K|-1)} \frac{(n-|K|-1)(n-|K|)(2n-2|K|-1)}{6}$

$= \frac{(n-|K|)}{2(n-|K|+1)} - \frac{1}{(n-|K|+1)} \frac{(2n-2|K|-1)}{6} = \frac{3(n-|K|)}{6(n-|K|+1)} - \frac{1}{(n-|K|+1)} \frac{(2n-2|K|-1)}{6}$

$= \frac{3n-3|K|-2n+2|K|+1}{6(n-|K|+1)} = \frac{n-|K||+1}{6(n-|K|+1)} = 1/6$

- $\sum_{s=2}^{n-|K|} \binom{n-|K|-2}{s-2} f(s) = \sum_{s=2}^{n-|K|-2} \frac{(n-|K|-2)!}{(s-2)!(n-|K|-s)!} \frac{s!(n-|K|+1-s-1)!}{(n-|K|+1)!}$

$$= \sum_{s=2}^{n-|K|} \frac{s(s-1)}{(n-|K|+1)(n-|K|)(n-|K|-1)} = \frac{1}{(n-|K|+1)(n-|K|)(n-|K|-1)} (\sum_{s=2}^{n-|K|} s^2 - \sum_{s=2}^{n-|K|} s)$$

$$= \frac{1}{(n-|K|+1)(n-|K|)(n-|K|-1)} \{ [\frac{(n-|K|)(n-|K|+1)(2n-2|K|+1)}{6} - 1] - [\frac{(n-|K|)(n-|K|+1)}{2} - 1] \}$$

$$= \frac{1}{(n-|K|+1)(n-|K|)(n-|K|-1)} \{ \frac{(n-|K|)(n-|K|+1)(2n-2|K|+1)}{6} - \frac{3(n-|K|)(n-|K|+1)}{6} \}$$

$$= \frac{1}{(n-|K|+1)(n-|K|)(n-|K|-1)} \frac{(n-|K|)(n-|K|+1)(2n-2|K|+1-3)}{6}$$

$$= \frac{1}{(n-|K|+1)(n-|K|)(n-|K|-1)} \frac{(n-|K|)(n-|K|+1)2(n-|K|-1)}{6} = 1/3$$