Existence of belief-free equilibria in games with incomplete information and known-own payoffs

Johannes Hörner[*], Stefano Lovo[†], Tristan Tomala[†○]

March 24, 2009

Abstract

In this work, we first characterize belief-free equilibrium payoffs in infinitely repeated games with incomplete information. We define a set of payoffs that contains all the belief-free equilibrium payoffs; conversely, any point in the interior of this set is a belief-free equilibrium payoff vector when players are sufficiently patient. This generalizes Hörner and Lovo (2009) who consider the two-player case.

Second, we consider repeated games with known-own-payoffs and study the existence of belief-free equilibria. We prove that if two players have finer information than any other player, and if these two players' information structures are comparable, then belief-free equilibria exist. This extends the 2-player result of Shalev (1994) and provides new conditions for existence of equilibria of undiscounted n-player repeated games with incomplete information.

The talk will emphasize the second issue.

Keywords: repeated game with incomplete information; Harsanyi doctrine; belief-free equilibria.

1 Introduction

Very little is known about existence of equilibria for undiscounted repeated games with incomplete information. Sorin (1983) proved the existence for 2-player games with one fully informed player (one-sided information) and two states of nature. Simon et al. (1995) proved it for general 2-player one-sided games. The 2-player and known-own-payoff case is much easier to handle and existence is proved in Shalev (1994). Renault (2003) proves the existence for 3-player games with two fully informed players and two states of nature. Apart from Shalev, all these papers consider belief-based equilibria. Our results complements this literature in giving a class of n-player games and information structures for which existence holds, with the additional property that the equilibrium is belief-free. That is, the same strategies form an equilibrium of the repeated game, irrespective of the prior beliefs of the players.

2 The model

There is a finite set of players $N = \{1, \ldots, N\}$ and finite state space is $K = \{1, \ldots, K\}$. Each player i has a finite action set A_i and a payoff function $u_i : K \times A \to \mathbb{R}$. Each player is also endowed with a partition I_i of K.

[*]Cowles Foundation, Yale University. johannes.horner@yale.edu.

[†]HEC School of Management, Paris and GREGHEC, 78351 Jouy-en-Josas, France. lovo@hec.fr, [○]tomala@hec.fr.

The repeated game unfolds as follows. In state k, at the beginning of the game, each player receives once and for all the *type* $\theta_i = I_i(k)$. The game is then infinitely repeated, with periods $t = 1, 2, \ldots$ Players use behavior strategies and realized actions profiles are publicly observed (but not necessarily realized payoffs). Conditional on a state, players maximize the expected average discounted sum of payoffs, where the expectation is taken with respect to mixed action profiles. Players use a common discount factor $\delta < 1$. That is, given some outcome $\{a_t\}_{t\in\mathbb{N}}$, player i's payoff in state k is

$$\sum\nolimits_{t\geq 0}(1-\delta)\delta^t u_i(k, a_t).$$

Definition 2.1 *A belief-free equilibrium of the discounted game (resp. of the undiscounted game) is a strategy profile* $\sigma := (\sigma_1, \ldots, \sigma_N)$*, where* $\sigma_i := \{\sigma_{i,\theta_i} : \theta_i \in \Theta_i\}$*, with the property that, for every state* k*,* $(\sigma_{i,I_i(k)})_{i\in N}$ *is a subgame-perfect Nash equilibrium of the discounted (resp. undiscounted) game with stage payoffs* $u(k, \cdot)$*.*

In words, a belief-free equilibrium is a strategy profile that induce subgame perfect equilibria in each repeated game with complete information with known state k.

3 The characterization

We let B_δ (resp. B_∞) be the set of belief-free equilibrium payoffs of the δ-discounted (resp. undiscounted) game. Such a vector $v = (v_i^k)_{i\in N, k\in K}$ is an element of R^{NK}, where v_i^k is the average discounted payoff of player i in state k. The purpose of this paper is to characterize $\lim_{\delta\to 1} B_\delta$.

We write $\Theta := \prod_i \Theta_i$. Given $\theta \in \Theta$, $\kappa(\theta) := \bigcap_{i\in N} \theta_i$ denotes the set of states that are consistent with type profile θ. We also write $\kappa(\theta_{-i}) := \bigcap_{j\neq i} \theta_j$ for the set of states that are consistent with a type profile of all players but one.

Feasibility The payoff vector $v \in \mathbb{R}^{NK}$ is *feasible* if there exists $(\mu_k)_{k\in K} \in (\triangle A)^K$ such that

1. $\forall i \in N, k, k' \in K : I_i(k) = I_i(k') \Rightarrow \mu_k = \mu_{k'}$;
2. $\forall k \in K : v^k = u(k, \mu_k) := \sum_a \mu_k(a) u(k, a)$.

For a type profile θ consistent with k, we write $\mu_\theta = \mu_k$. The interpretation is obvious: given σ, μ_θ is the occupation measure over action profiles in the infinitely repeated game, generated by σ given that types are θ.

Incentive Compatibility If two types θ_i and θ_i' are both consistent with a type profile θ_{-i} of the other players, player i must have an incentive to reveal his true type. Define UD_i as the set of triples $(\theta_i, \theta_i', \theta_{-i}) \in \Theta_i \times \Theta_i \times \Theta_{-i}$ such that $\kappa(\theta_i, \theta_{-i}) \neq \varnothing$ and $\kappa(\theta_i', \theta_{-i}) \neq \varnothing$. The incentive compatibility conditions are

$$\forall i, (\theta_i, \theta_i', \theta_{-i}) \in UD_i, k \in \kappa(\theta_i, \theta_{-i}) : u_i(k, \mu_{\theta_i,\theta_{-i}}) \geq u_i(k, \mu_{\theta_i',\theta_{-i}}). \quad (IC(i, \theta_i, \theta_i', \theta_{-i}))$$

Individual Rationality When player i is publicly recognized as deviating, the other players coordinate their actions and pieces of information to punish him. If their types reveal the state, they simply minimax player i. If player i still holds valuable information, we resort to the minimax strategy for the uninformed player in 2-player repeated games with incomplete information (Aumann and Maschler, 1995), which uses Blackwell's approachability (Blackwell, 1956). For a probability distribution $q \in \triangle\kappa(\theta_{-i})$, we let:

$$\varphi_{i,\theta}(q) = \min_{\alpha \in \prod_j \triangle A_j} \max_{b_i \in A_i} \sum\nolimits_{k \in \kappa(\theta_{-i})} q(k) u_i(k, \alpha_{-i}, b_i).$$

A payoff vector is individually rational if for each player i and each $\theta_{-i} \in \prod\limits_{j \neq i} \Theta_j$,

$$\forall q \in \triangle\kappa(\theta_{-i}) : \sum\nolimits_{k \in \kappa(\theta_{-i})} q(k) v_i^k \geq \varphi_{i,\theta}(q). \qquad (IR(i, \theta_{-i}))$$

Joint Rationality With at least three players, an inconsistent type report might not identify a single deviating player. Let D be the set of type profiles that are compatible with some state of nature and a unilateral deviation. That is, θ is in D if $\kappa(\theta) = \varnothing$ and $\Omega_\theta := \{(i, \theta_i') \mid i \in N, \kappa(\theta_i', \theta_{-i}) \in K\} \neq \varnothing$. If players report their types, and the reported profile was in D, all players know that some player must has lied. Further, the deviating player and the true state of nature are such that $\kappa(\theta_i', \theta_{-i}) \in K$. For each $\theta \in D$, consider the condition

$$\exists \alpha \in \triangle A, \forall (i, \theta_i') \in \Omega_\theta, \forall k \in \kappa\left(\theta_i', \theta_{-i}\right), v_i^k \geq u_i(k, \alpha). \qquad (JR\left(\theta\right))$$

Let $V^* \subset \mathbb{R}^{KN}$ denote the set of payoffs that satisfy IC, IR, and JR. The set V^* may be empty. However, we show that this set characterizes the set of belief-free equilibria (up to its boundary points).

Theorem 3.1 *-Any belief-free equilibrium payoff (of the discounted game or of the undiscounted game) is in V^*.*

-Any interior point of V^* is a belief-free equilibrium payoff of the discounted game for a sufficiently large discount factor.

-Any point in V^* is a belief-free equilibrium payoff of the undiscounted game.

4 Existence for games with known-own-payoffs

We assume that each player knows his payoff function, i.e. $u_i(a, k) = u_i(a, I_i(k))$. We give condition on the information structures that guarantee the existence of belief-free equilibria. We start by the remark that, if a piece of information is held by at least three players, then it will be common knowledge after the announcements, even under a unilateral deviation. This idea is captured by the following notions.

For each pair of states a, b, let $\nu(a, b)$ be the number of players who distinguish a from b. Define the binary relation aRb iff $\nu(a, b) \leq 2$. Let $a \sim b$ iff there is a chain of states $a = a_1, a_2, ..., a_n = b$ such that $a_m R a_{m+1}$. A *majority component* of K is an equivalence class of the relation $\sim$. Note that if A, B are two majority components of K, then if $A \neq B$, for each $a \in A$ and each b in B, $\nu(a, b) \geq 3$. The importance of this notion stems for the following observation: *Let A be a majority component. Then for every true state $k \in A$, if all players, except at most one of them, truthfully announce their types, then it is common knowledge after the announcements that the true state lies in A.*

Now, we give a condition which generalizes the 2-player one-sided information case.

Definition 4.1 *The information is Locally Weakly Embedded if for each state k, there exist two players i, j such that $I_i(k) \subseteq I_j(k) \subseteq I_l(k)$ for each other player l.*

Our main existence result is the following.

Theorem 4.2 *If for each majority component A, the information structure restricted to A is locally weakly embedded, then V^* is non-empty for every game with known-own payoffs.*

The assumption that the most informed players have comparable information is crucial. Assuming that one player is fully informed is not enough to ensure existence, contrary to the two-player case. A counter-example is given in the full paper.

References

[1] R. J. Aumann and M. Maschler. *Repeated games with incomplete information*. With the collaboration of R. Stearns. M.I.T. Press, 1995.

[2] D. Blackwell. An analog of the minmax theorem for vector payoffs. *Pacific Journal of Mathematics*, 65:1–8, 1956.

[3] J. Hörner and S. Lovo. Belief-Free Equilibria in Games With Incomplete Information. *Econometrica*, 77:453-487, 2009.

[4] J. Renault. 3-player repeated games with lack of information on one side, *International Journal of Game Theory*, 30:221–246, 2001 a.

[5] J. Shalev. Nonzero-Sum Two-Person Repeated Games with Incomplete Information and Known-Own Payoffs, *Games and Economic Behavior*, 7:246–259, 1994.

[6] R.S. Simon, S. Spież and H. Toruńczyk. The existence of equilibria in certain games, separation for families of convex functions and a theorem of Borsuk-Ulam type, *Israel Journal of Mathematics*, 92:1–21, 1995.

[7] S. Sorin. Some results on the existence of Nash equilibria for non- zero sum games with incomplete information *International Journal of Game Theory*, 12:193–205, 1983.