Robustness of Intermediate Agreements and Bargaining Solutions*

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Abstract

Most real-life bargaining resolves gradually; two parties reach intermediate agreements without knowing the whole range of possibilities. These intermediate agreements serve as disagreement points in subsequent rounds. Cooperative bargaining solutions ignore this dynamics and therefore can yield accurate predictions only if they are robust to its specification. We identify robustness criteria that four of the best-known bargaining solutions, Nash, Kalai-Smorodinsky, Proportional and Discrete Raiffa, satisfy. We show that "robustness of intermediate agreements" plus well-known and plausible additional axioms provide the first characterization of the Discrete Raiffa solution and novel axiomatizations of the other three solutions. Hence, we provide a unified framework for comparing these solutions' bargaining theories.

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1 Introduction

Nash's bargaining problem is a pair (S, d) where $S \subset \mathbb{R}^2$ is a convex and compact utility possibility set and d is the disagreement point, the utility allocation that results if no

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agreement is reached by both parties. Since Nash (1950)'s seminal solution and axioms, various other solution concepts and related reasonable axioms have been proposed.

From our paper's point of view, Kalai (1977) proposed a very notable axiom, the Step-by-Step Negotiation (SSN) axiom. In SSN, parties know that they will face two nested sets in sequence, first the smaller set and then the augmented one. But parties do not necessarily know how either S or T looks. SSN then requires that the bargaining outcome is invariant under decomposition of the negotiation stages. Thus, the solution outcome of the initial bargaining set can perfectly function as an intermediate agreement. One very important reason SSN is a very notable axiom is that most real-life bargaining resolves gradually indeed; two parties reach intermediate agreements without knowing the whole range of possibilities.[1] Cooperative bargaining solutions ignore this dynamics and therefore can yield accurate predictions only if they are robust to its specification.

In the context of intermediate agreements, SSN provides a significant robustness criterion. SSN's robustness criterion, however, is too strong. One would not expect parties to reach intermediate agreements in S - and especially the solution outcome of S as an intermediate agreement - even if they knew that the ultimate T would contain S unless some additional things about the relationship between S and T were also known. In other words, it requires too much robustness of the solution outcome of S to serve as the intermediate agreement when T is encountered afterwards without a specific knowledge of any additional relationship between S and T.

Here, we identify weaker robustness criteria that four of the best-known bargaining solutions, Nash, Kalai-Smorodinsky, Proportional and Discrete Raiffa, satisfy. One minor weakening of SSN's robustness criterion would be to still expect an intermediate agreement in S to serve as an intermediate agreement in both S and T whenever S is contained in the ultimate T but that intermediate agreement not necessarily being the solution outcome of S. That is, the presence of certain points in S to serve as intermediate agreements in S as well as in an ultimate T can be robust to any augmentation of S, which is the first robustness criterion below. The robustness criteria following it become much more specific, as will be elaborated on shortly:

[1] In a single-person decision making as well, uncertainty resolves gradually. A shopper in a typical supermarket faces more than 200 varieties of cookies, soups and cereal (Schwarz, 2004). Marketing and economics literatures provide well-established analysis and evidence that consumers do not consider all brands in a given market at once before making a purchase decision and that the set of brands they consider changes in time as they learn more about that product since they later include brands that they were initially unaware of (Chiang, Chib and Narasimhan, 1999; Goeree, 2008). See Huberman and Regev (2001) regarding financial decisions involving plans among hundreds of available funds, and Dawes and Brown (2004) regarding university choice. Also see Rubinstein and Salant (2006) for a theoretical model in which individuals encounter alternatives sequentially.

1. S is included in T

2. S and T have the same *ideal payoffs* - i.e., each party's highest possible individually rational payoff is the same in both S and T - and S and T need not include one another

3. parties expect to receive the same relative payoff gains in S and T, and S and T need not include one another

4. parties expect to make the same relative concessions in S and T, and S and T need not include one another.

The second robustness criterion states that the presence of certain points in S to serve as intermediate agreements in S as well as in another T (where S and T need not include each other) can be robust to both S and T sharing the same ideal point. Likewise, the third and fourth robustness criteria state that the presence of certain points in S to serve as intermediate agreements in S as well as in another T (where S and T need not include each other) can be robust to parties expecting either to have the same relative gains (in Criterion 3) or relative concessions (in Criterion 4) in both S and T.

The above four robustness criteria are conditions of our four "Robustness of Intermediate Agreements" (RIA) axioms. The relevant RIA axiom - combined with some other well-known and plausible axioms - will lead to the first known axiomatization of the Discrete Raiffa solution[2] as well as novel axiomatizations of the Nash, Kalai/Smorodinsky and Proportional solutions. Our results can be briefly summarized as follows:

(1) The Discrete Raiffa solution is characterized by the Midpoint Domination (MD) axiom, an RIA axiom, and the Independence of Non-Midpoint Dominating Alternatives (INMD) axiom.

(2) The Nash solution is characterized by MD, an RIA axiom, and the Disagreement Point Continuity (DCONT) axiom.

(3) The Kalai/Smorodinsky solution is characterized by MD, an RIA axiom, DCONT and the Strong Disagreement Point Monotonicity (SDM).

(4) The Proportional solutions are characterized by an RIA axiom, DCONT, the Weak Pareto Optimality (WPO), the Pareto Continuity (PCONT) axiom and the Strong Individual Rationality (SIR) axiom.[3]

[2] The desirable feature of Raiffa's contribution, namely its description of a bargaining process, nevertheless led researchers to seek - and provide characterizations of - a "continuous" version of the Raiffa solution (Livne, 1989, and Peters and van Damme, 1991).

[3] The following table further helps summarizing our results:

Hence, we provide a unified framework for comparing these solutions' bargaining theories.

We will now provide an example in which two agents face an uncertain negotiation prospect such that writing a contingent binding agreement regarding any of two potential utility possibility sets is not possible. Consider a country with two major political parties that can win the up-coming elections. Depending on which party wins the elections, the new government will provide to firms in major sectors either (1) tax breaks or (2) trade protection. Suppose that there are two major automobile companies in that country with sufficient production synergies between them. Given the prevailing recession in the country, it is taken for granted that these firms will be allowed to collude in the product market tacitly regardless of the policy that will be pursued after the elections.

Observe that each policy will generate a different utility possibility set between the two automobile companies which are the players in the above example. Suppose the "tax break" policy with tacit collusion will generate a particular utility possibility set S and the "trade protection" policy with tacit collusion the set T. Their initial disagreement point d involves their current profits. If these companies choose to wait by doing nothing until the new government gets elected and announces its policy, they will be obtaining d in the meantime (they cannot write an overt contingent binding agreement since overt collusion is prohibited by law). But they can reach an intermediate agreement $d' > d$ in the meantime instead. The intermediate agreement d' may involve forming a research joint venture (RJV); such an RJV can pave the way for their tacit collusion regardless of the actual policy that the new government will announce in the future.[4]

Note that, since an intermediate agreement d' may possibly affect the bargaining

DR	an RIA axiom	MD		INMD
N	an RIA axiom	MD	DCONT	
KS	an RIA axiom	MD	DCONT	SDM
P	an RIA axiom		DCONT	PCONT+WPO+SIR

[4]Another example considers two high-tech research engineers who work in the same field. In the future, the government will adopt either of the two possibilities: (1) to subsidize some expensive strategic equipment or (2) to provide generous progressive tax deductions to individuals' earnings that will accrue in research teams. Here too, each policy will generate a different utility possibility set between these two engineers who are the players. Suppose the "subsidy" policy regarding the major strategic equipment will generate a particular utility possibility set S and the "tax deduction" policy the set T. Their initial disagreement point d involves their current earnings. If these individuals choose to wait by doing nothing until the actual policy is announced, they will be obtaining d in the meantime. But they can reach an interim outcome $d' > d$ in the meantime instead. The interim outcome d' may involve renting an office space together; they can use the joint office space regardless of whether they will use the expensive strategic equipment together without forming a research team or they will start working as a research team without buying the expensive strategic equipment, which would not payoff without the subsidy.

solution outcome of any potential utility possibility set differently than the initial disagreement point d would, the ideal situation would be that the agents agree on an intermediate agreement $d' > d$ which is also *neutral*: that is, if both agents adhere to the same particular sharing rule (in the form of a bargaining solution), regardless of the utility possibility set realized, it would not matter whether later they move to the bargaining solution outcome of the realized utility possibility set from (i) the intermediate agreement d' or from (ii) their initial disagreement point d. Otherwise - i.e., if the intermediate agreement affects their future bargaining solution outcome differently than the initial disagreement point would - at least one of the agents would not be willing to agree to the intermediate agreement.[5]

In the next section, we provide a brief review of relevant literature. In Section 3, we define some basic solutions and axioms. In Section 4, we propose and motivate our RIA axioms. Section 5 provides characterizations of the Discrete Raiffa, Nash, Kalai/Smorodinsky and Proportional solutions. The final section concludes.

2 Relevant Literature

The significance and fundamental role of the basic bargaining problem[6] was recognized as early as 1881 by Edgeworth, and for a very long period of time it was notoriously deemed to lack a clear solution.[7] In 1950, Nash proposed a seminal framework which allowed a unique feasible outcome to be selected as the solution of a given bargaining problem. Nash (1950) also provided the first axiomatic bargaining solution, characterized by four axioms - namely by WPO, Symmetry (SYM), Scale Invariance (SI), and Independence of Irrelevant Alternatives (IIA). Raiffa (1953) later criticized the Nash solution (and especially the IIA axiom) and proposed another solution which essentially described a discrete bargaining process but has never been characterized axiomatically since. A couple of decades later, Kalai and Smorodinsky (1975) raised similar criticisms and characterized a new solution concept which, as the Discrete Raiffa solution did, placed

[5] When uncertainty will not resolve quickly (or will re-surface frequently), at least one of the parties may be better off using a series of intermediate agreement in time instead of committing to long-term binding agreements. This may also be the case if each feasible intermediate agreement is not neutral.

[6] Binmore (1994, p. 21): "much negotiation in real life ... create[s] a surplus that would otherwise be unavailable ... If you have a fancy house to sell that is worth \$2m to you and \$3m to me, then ... a surplus of \$1m is available for us to split."

[7] See Roth (1979b, p. 5).

significant emphasis on the parties' ideal payoffs.[8]

Initially all bargaining solutions that were characterized axiomatically had a crucial *independence* or *monotonicity* axiom (pioneered by Nash, 1950, and Kalai and Smorodinsky, 1975, respectively).[9] The second generation characterizations of these solution concepts shifted the focus to *changes in the disagreement payoffs* as well as to the consideration of *uncertain disagreement points* (pioneered by Thomson, 1987, and Chun and Thomson, 1990, respectively). These axioms, however, typically did not describe any bargaining process.[10] By adding a description of the bargaining process into the bargaining framework, Nash (1953)'s Demand Game established a new research agenda, which has been commonly referred to as the Nash program (see Binmore, 1998). It utilizes the strategic (non-cooperative) approach to provide non-cooperative foundations for axiomatic (cooperative) bargaining solution concepts by describing an explicit bargaining process.[11]

Later, MD (Sobel, 1981) and the Step-by-Step Negotiation (SSN) axiom (Kalai, 1977)[12] embedded a bargaining process via reaching intermediate agreements.[13] How-

[8] There have been other solution concepts characterized axiomatically since then: the Egalitarian solution (Kalai, 1977; Roth, 1979a), the Equal Sacrifice solution (Aumann and Maschler, 1985; Chun, 1988), the Perles/Maschler solution (Perles and Maschler, 1985), the Equal Area solution (Anbarci, 1993; Anbarci and Bigelow, 1994), the Average Payoff solution (Anbarci, 1995)), and the Dictatorial solutions (Bigelow and Anbarci, 1993).

[9] When the solution outcome is irresponsive to the changes in the bargaining set, that axiom is coined as the independence axiom; when at least one of solution payoffs may be altered following a change in the bargaining set, it is dubbed as the monotonicity axiom. Indeed, the Nash, Kalai/Smorodinsky, the Perles/Maschler solution, the Equal Area solution, the Average Payoff solution have been all initially characterized by SYM, WPO, SI and an independence or monotonicity axiom.

[10] Both generations of characterizations were essential since a bargaining problem consists of a bargaining set and a disagreement point, i.e., (S, d).

[11] This strand of research produced very interesting work accompanying the cooperative bargaining solution concepts, starting with Nash (1953). It was followed by Moulin (1984), Binmore et al (1986), Howard (1992), Anbarci (1993), Anbarci and Boyd III (2009) among others. Outside of the Nash bargaining framework, in the cooperative game theory (with Transferable Utility in Characteristic Form Games), Gul (1989) provided non-cooperative foundations for the Shapley Value as well.

[12] Moulin (1983) used MD to characterize the Nash solution, and Kalai (1977) used SSN to characterize Proportional solutions (the Egalitarian and Dictatorial solutions are special cases of the latter).

[13] Intermediate agreements help eliminate the most lop-sided and/or inefficient portions of a utility possibility set S, which at least one of the parties would strongly dislike (in effect, the meta-bargaining models of van Damme, 1986, and Anbarci and Yi, 1992, too pertain to eliminations of such portions of S deemed undesirable by at least one of the parties). Anbarci, Skaperdas and Syropoulos (2002) consider a setup in which each party can invest in his disagreement payoff to avoid such portions of S and to improve his solution outcome payoff.

ever, MD and SSN have not been generalized subsequently to give rise to a class of axioms which would be instrumental in characterizing some other prominent solution concepts. This paper, with the exception of one instance, does not utilize any 'independence', 'monotonicity' or 'disagreement-point-change' axioms. It, instead, aims to highlight the crucial role of robustness of intermediate agreements in a unified way by proposing such a class of axioms.

3 Basic Definitions

A two-person *bargaining problem* is a pair (S, d), where $S \subset \mathbb{R}^2$ is the *set of utility possibilities* that the players can achieve through cooperation and $d \in S$ is the *disagreement point*, which is the utility allocation that results if no agreement is reached by both parties. It is assumed that (1) S is compact, convex and comprehensive,[14] and (2) $x > d$ for some $x \in S$.[15] Let Σ be the class of all two-person bargaining problems. Unless stated otherwise, our results will consider bargaining problems in Σ.[16]

A bargaining problem (S, d) is *symmetric* if $d_1 = d_2$ and $(x_1, x_2) \in S$ implies $(x_2, x_1) \in S$. Define $IR(S, d) \equiv \{x \in S | x \geq d\}$, $WPO(S) \equiv \{x \in S | \forall x' \in \mathbb{R}^2$ and $x' > x \Rightarrow x' \notin S\}$ and $PO(S) \equiv \{x \in S | \forall x' \in \mathbb{R}^2$ and $x' \geq x \Rightarrow x' \notin S\}$. Denote the ideal point of (S, d) as $b(S, d) = (b_1(S, d), b_2(S, d))$, where $b_i(S, d) = \sup\{x_i | x \in IR(S, d)\}$. The midpoint of (S, d) is denoted by $m(S, d) = \frac{1}{2}(b(S, d) + d)$. A solution is a function $f : \Sigma \to \mathbb{R}^2$ such that for all $(S, d) \in \Sigma$, $f \in S$.

The disagreement point set of (S, d) with respect to f, $D(S, d, f) = \{x \in IR(S, d) | f(S, x) = f(S, d)\}$, is the set of all points x in S (weakly) dominating d such that if we replace the initial disagreement point d with x and keep the utility feasibility set S unchanged, we can still reach the same bargaining solution outcome. $D(S, d, f)$ will be a key element of our analysis in this paper.

Next, we list some basic axioms that have been commonly used in the literature.

Weak Pareto Optimality (WPO) $f(S, d) \in WPO(S)$.

Symmetry (SYM) If (S, d) is symmetric, then $f_1(S, d) = f_2(S, d)$.

Scale Invariance (SI) $T = (T_1, T_2) : \mathbb{R}^2 \to \mathbb{R}^2$ is a positive affine transformation if $T(x_1, x_2) = (a_1 x_1 + b_1, a_2 x_2 + b_2)$ for some positive constant a_i and constant b_i. We require that for such a transformation T, $f(T(S), T(d)) = T(f(S, d))$.

[14] A set $S \subset \mathbb{R}^2$ is said to be comprehensive if $x, z \in S$ implies that $y \in S$ for all $x \leq y \leq z$.

[15] Given $x, y \in \mathbb{R}^2$, $x > y$ if $x_i > y_i$ for each i, and $x \geq y$ if $x_i \geq y_i$ for each i.

[16] A more restrictive class of bargaining problems is as follows: A bargaining problem (S, d) is *smooth* if S admits a unique supporting hyperplane at each utility vector on its boundary. $\Sigma^s \subset \Sigma$ denotes the class of all smooth problems.

Individual Rationality (IR) $f(S,d) \geq d$.

Strong Individual Rationality (SIR) $f_i(S,d) \geq d_i$, with strict inequality whenever $x_i > d_i$ for some $x \in S$ with $i = 1, 2$.

Independence of Non-individually Rational Alternatives (INIR) $f(S,d) = f(IR(S,d),d)$.

Disagreement Point Monotonicity (DM) If d and e are in S with $e_i = d_i$ and $e_j > d_j$, then $f_j(S,e) \geq f_j(S,d)$, for $i,j = 1,2$ with $i \neq j$.

Strong Disagreement Point Monotonicity (SDM) As DM with "$>$" instead of "$\geq$", but only if such a point $f(S,e)$ exists.

Disagreement Point Continuity (DCONT) For every bargaining set S and every sequence $d^1, d^2, \ldots$ in S, if $\lim_{n\to\infty} d^n = d \in S$ (in the Hausdorff topology), then $\lim_{n\to\infty} f(S,d^n) = f(S,d)$.

Pareto Continuity (PCONT) For all sequences $\{(S^n,d)\}$ in Σ, if $WPO(S^n)$ converges to $WPO(S)$ in the Hausdorff topology and $(S,d) \in \Sigma$, then $\lim_{n\to\infty} f(S^n,d) = f(S,d)$.

Midpoint Domination (MD) $f(S,d) \geq m(S,d)$.

MD requires that any reasonable agreement Pareto dominates the outcome of the random dictatorship. Note that the relationship between MD and the next axiom is just like the relationship between IR and INIR axioms. IR and INIR are based on d while MD and INMD are based on m.

Independence of Non-Midpoint-Dominating Alternatives (INMD) Suppose $(S,d),(T,d) \in \Sigma$. If $IR(S,m(S,d)) = IR(T,m(T,d))$, then $f(S,d) = f(T,d)$.

Observe that, if the condition $IR(S,m(S,d)) = IR(T,m(T,d))$ holds, then $m(S,d) = m(T,d)$ and $b(S,d) = b(T,d)$. It states that parties should focus only on the alternatives dominating the midpoint in their negotiations, and those alternatives below the midpoint should not influence the bargaining outcome. It is also easy to verify that INMD is satisfied by the Nash, Kalai/Smorodinsky, Discrete Raiffa and Dictatorial solutions.

We introduce four prominent solution concepts.

Definition 1 *The Nash solution* N : *For each* $(S,d) \in \Sigma$, $N(S,d) = \arg\max\{(x_1 - d_1)(x_2 - d_2) | x \in IR(S,d)\}$.

Definition 2 *The Kalai/Smorodinsky solution* KS : *For each* $(S,d) \in \Sigma$, $KS(S,d) = \max\{u \in S | \text{there exists } \alpha \in [0,1] \text{ such that } u = \alpha b(S,d) + (1-\alpha)d\}$.

Definition 3 *The Discrete Raiffa solution* DR : *For each* $(S,d) \in \Sigma$, *consider a non-decreasing sequence* $\{m_t\} \in S$ *with* $m_0 = m(S,d)$ *and* $m_t = m(S, m_{t-1})$, *then* $DR(S,d) = \lim_{t\to\infty} m_t$.

Definition 4 *The Proportional solutions P : For each $(S,d) \in \Sigma$,there are strictly positive constants p^1 and p^2 such that $f(S,d) = d + \lambda(S,d)p$ where $p = (p^1, p^2)$ and* $\lambda(S,d) = \max\{t | tp \in S - d\}$.

4 The Robustness of Intermediate Agreements (RIA) Axioms

Consider the *Step-by-step Negotiations* (SSN) axiom of Kalai (1977):

Step-by-step Negotiations (SSN) A solution f satisfies (SSN) if whenever (S,d), $(T,d) \in \Sigma$, $T \subset S$, and $(S - f(T,d), 0) \in \Sigma$, then $f(S,d) = f(T,d) + f(S - f(T,d), 0)$.

In SSN, parties know that they will face two nested sets in sequence, first the smaller set and then the augmented one. But parties do not necessarily know how either S or T looks. [17] SSN has a strong robustness content, as argued in the Introduction. Kalai (1977) demonstrated that, combined with WPO and SIR, it is sufficient to uniquely characterize the Proportional solutions (including the Egalitarian and Dictatorial solutions).

Next, we propose four fairly intuitive axioms that are closely related to SSN but have weaker robustness requirements. Recall from the last section that $D(S,d,f)$ represents the set of all common intermediate agreements x in S dominating d such that, if we replace the initial disagreement point d with x, we can still reach the same bargaining outcome. To link our axioms to SSN conceptually, we restate SSN as follows:

Given two bargaining problems $(S,d),(T,d) \in \Sigma$, $D(T,d,f) \supset \{f(S,d)\}$ whenever $S \subset T$, $f(S,d) \geq d$ and $(T, f(S,d)) \in \Sigma$.

For a given bargaining problem (T,d), SSN requires that $D(T,d,f)$ is not only nonempty, but also contains $f(S,d)$ for *all* bargaining problems $(S,d) \in \Sigma$ with $S \subset T$ and $(T, f(S,d)) \in \Sigma$. In other words, $f(S,d)$ can serve as an intermediate agreement in reaching $f(T,d)$. But as we argued in the Introduction, one would not expect parties to reach intermediate agreements in S - and especially the solution outcome of S as an intermediate agreement - even if they knew that the ultimate T would contain S unless some additional things about the relationship between S and T were also known.

[17] Kalai (1977) emphasized the advantage of this interim outcome as follows:

> This principle is observed in actual negotiations (e.g., Kissinger's step-by-step) and it is attractive since it makes the implementation of a solution easier. It is also attractive because we can view every bargaining situation that we encounter in life as a first step in a sequence of predictable or unpredictable bargaining situations that may still arise. Thus the outcome of the current bargaining situation will be the threat point for the future ones.

The first axiom below, RIA-Inclusion, is a weaker version of SSN. As in SSN, in RIA-Inclusion too parties know that they will face two nested sets in sequence, first the smaller set and then the augmented one; likewise parties do not necessarily know how either S or T looks. But RIA-Inclusion only requires that the disagreement point set, $D(T, d, f)$, of the larger utility possibility set, T, to include the disagreement point set, $D(S, d, f)$, of the smaller set, S. As is the case with all RIA axioms, we want the parties to be able to reach an intermediate agreement, $d' > d$, meanwhile. As mentioned before, this intermediate agreement, d', should be neutral in that the agents should be able to move from the intermediate agreement, d', or from d to either $f(S, d)$ or $f(T, d)$.

1. Robustness of Intermediate Agreements with Inclusion (RIA-Inclusion) Suppose $(S, d), (T, d) \in \Sigma$ with $S \subset T$. Then $D(S, d, f) \subset D(T, d, f)$.[18]

In other words, RIA-Inclusion does not require that the solution outcome, $f(S, d)$, of the smaller set S necessarily be an intermediate agreement. It only states that in this case parties will be willing to reach a neutral intermediate agreement $d' > d$ instead of sticking to the status quo d. Thus, the presence of certain points in S to serve as intermediate agreements in S as well as in an ultimate T can be robust to any augmentation of S.

Now, consider two parties facing a bargaining situation where - unlike SSN and RIA-Inclusion above - S and T need not include one another. Apart from the disagreement outcome d, assume that parties also know the maximal utility each of them can receive (i.e., the ideal point) from bargaining, but are uncertain about the resulting Pareto optimal frontier from all possible underlying compromises. In this case too, they will be willing to reach a neutral intermediate agreement instead of sticking to the status quo. Accordingly we must have $\cap_{(S,d)\in\Sigma^{b,d}} D(S, d, f) \backslash \{d\} \neq \emptyset$, where $\Sigma^{b,d}$ is the collection of all bargaining problems in Σ with ideal point b and disagreement point d. The RIA stated below is a weaker version of this requirement:[19]

2. Robustness of Intermediate Agreements in the (d, b)-Box (RIA-Box) For all (S, d) and (T, d) in Σ with $b(S, d) = b(T, d)$, we have $(D(S, d, f) \cup \{f(S, d)\}) \cap (D(T, d, f) \cup \{f(T, d)\}) \backslash \{d\} \neq \emptyset$.

RIA-Box only requires that each pair of bargaining problems with the same disagreement point and ideal point have a non-empty intersection of their disagreement point sets union the final outcomes.[20] Thus, this robustness criterion states that the presence

[18] To see that RIA-Inclusion is weaker than SSN, pick any $\widetilde{d} \in D(S, d, f)$ (if it exists). SSN implies that $f(T, \widetilde{d}) = f(T, f(S, \widetilde{d})) = f(T, f(S, d)) = f(T, d)$.

[19] The terms $\{f(S, d)\}$ and $\{f(T, d)\}$ can be dropped if we work on an enlarged domain by allowing d to be on the boundary of the bargaining set.

[20] This axiom can easily be modified to hold for all $(S, d) \in \Sigma$ instead of for only two bargaining problems $(S, d), (T, d) \in \Sigma$. Such a modification would surely make its intuition stronger but would also

of certain points in S to serve as intermediate agreements in S as well as in another T can be robust to both S and T sharing the same ideal point.

The agents know d and b but they do not know whether they will face S or T. Therefore, they can not reach a final outcome yet. Again, we want the parties to be able to reach an intermediate agreement, $d' > d$, meanwhile. This intermediate agreement, d', should be neutral in that whether S or T is realized tomorrow, the agents should be able to move from the intermediate agreement, d', or d to $f(S, d)$ or $f(T, d)$. So, any such intermediate agreement, d', needs to be a common intermediate agreement in $D(S, d, f) \cap D(T, d, f)$ for a possible S or T.

Our third and fourth robustness criteria state that the presence of certain points in S to serve as intermediate agreements in S as well as in another T (where, like in the case of RIA-Box axiom, S and T need not include each other) can be robust to parties expecting either to have the same relative gains or relative concessions in both S and T. As such they bring up the issue of parties' relative bargaining powers.

It may not be very clear *ex-ante* what kind of economic and non-economic factors will determine a party's bargaining power relative to that of the other. Nevertheless, it should be clear from an *ex-post* point of view that one party's gain relative to the other in a negotiation must be monotone increasing with their bargaining power. This simple idea inspires our first definition of bargaining power in different contexts. It is as follows: For any $x, y \in \mathbb{R}^2$ and $x \neq y$, let $l[x, y]$ be the line segment connecting x and y, and $\theta(x, y)$ be the gradient (slope) of $l[x, y]$. Suppose the bargaining solution outcome is $f(S, d) \geq d$ for a given bargaining problem (S, d), then the gradient $\theta(d, f(S, d))$, which measures the relative gains in bargaining, could be a good index of bargaining power (See Figure 1).

$\theta(d, f(S, d)) = 0$ implies Agent 1 has complete bargaining power, $\theta(d, f(S, d)) = \infty$ implies that Agent 2 has complete bargaining power, and Agent 1's bargaining power is monotone decreasing with θ. If $\theta(d, f(S, d)) = \theta(d, f(T, d))$, then parties receive the same relative gains over two bargaining problems (S, d) and (T, d) (See Figure 2).

3. Robustness of Intermediate Agreements with Identical Relative Gains (RIA-Gains) Suppose $(S, d), (T, d) \in \Sigma$. If (i) $f(S, d) \in IR(S, d)\backslash\{d\}$ and $f(T, d) \in IR(T, d)\backslash\{d\}$ and (ii) $\theta(d, f(S, d)) = \theta(d, f(T, d))$, then $D(S, d, f) \cap D(T, d, f)\backslash\{d\} \neq \emptyset$.

In this axiom too agents try to reach an intermediate agreement before uncertainty is resolved as to whether S or T will take place later. The intuition of this axiom is very simple: (1) the slope of the line $l[d, f]$ connecting d and f can serve as a measure of parties' relative bargaining power, and (2) beginning with the same disagreement point

make it a mathematically stronger axiom.

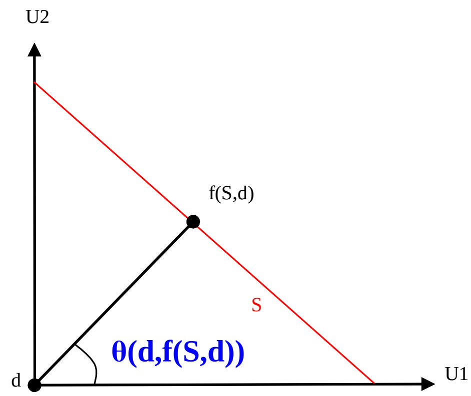

FIGURE 1

Identical Relative Gains

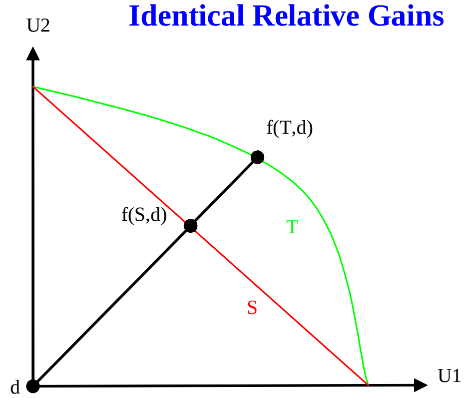

FIGURE 2

d, if two parties perceive (correctly) that they are going to receive the same relative gains in two bargaining problems (S, d) and (T, d), then there exists at least one allocation d' in $S \cap T$ that is agreeable to both parties as an intermediate agreement. As in the RIA-Box axiom, here too this intermediate agreement, $d' > d$, should be neutral in that whether S or T is realized tomorrow, the agents should be able to move from the intermediate agreement, d', or d to $f(S, d)$ or $f(T, d)$. So, any such intermediate agreement, d', needs to be a common intermediate agreement in $D(S, d, f) \cap D(T, d, f)$ for a possible S or T.

Our last RIA axiom, RIA-Concessions, is built on relative concessions by parties. Suppose two parties have reached an agreement after some negotiation. This outcome can be viewed as a compromise balancing the concessions made by the two parties. A particular party's concessions are feasible outcomes - measured in a particular way that both parties agree on - that a party prefers to the negotiated outcome. Thus, concessions can be considered as bargaining chips of parties; possessing more bargaining chips in a particular negotiation would then yield more bargaining power to a party.

Suppose that the parties' environment now changes in such a way that new potential outcomes mostly beneficial to Agent 1 have been added to the feasible set. Then from both parties' point of view, maintaining the original compromise in the face of the changed environment would amount to Agent 1's making relatively more concessions than before. This in turn would lend more bargaining power to Agent 1 in the new environment. If the initial compromise was reached via "balancing" the concessions made by one party against those made by the other, then maintaining the original payoff ratio will result in an "imbalance" of concessions and therefore at least one party - Agent 1 - will think that the original payoff ratio should not remain intact any more.

But if the new environment is such that new potential outcomes are equally beneficial to both parties (via the way parties agree to measure concessions), then maintaining the original payoff ratio can still result in an outcome balancing parties' relative concessions and thus their relative bargaining powers. In that regard, $\theta(b(T, f(S, d)), f(S, d))$ can measure parties' relative concessions and relative bargaining powers in $(T, f(S, d))$ with respect to $f(S, d)$. As in RIA-Box and RIA-Gains, S and T need not include one another.

4. Robustness of Intermediate Agreements with Identical Relative Concessions (RIA-Concessions) Suppose $(S, d), (T, d) \in \Sigma$. If (i) $f(S, d) \in IR(S, d)\backslash\{d\}$ and $f(T, d) \in IR(T, d)\backslash\{d\}$ and (ii) $\theta(b(T, f(S, d)), f(T, d)) = \theta(b(T, f(S, d)), f(S, d))$, then $D(S, d, f) \cap D(T, d, f)\backslash\{d\} \neq \emptyset$; moreover, $f_i(T, d) = b_i(T, d)$ for some i only if $b(T, d) \in T$.

The above axiom requires that parties should *not* expect that their bargaining power will change if added relative concessions are the same as before, i.e., when $\theta(b(T, f(S, d)),$

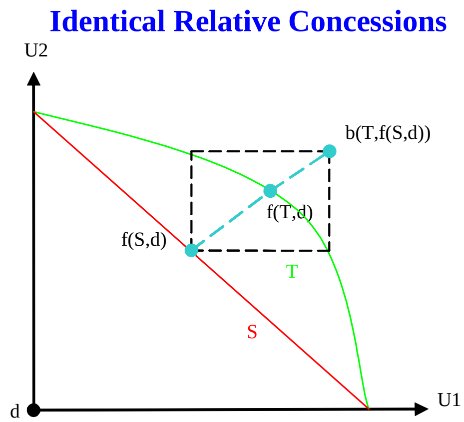

FIGURE 3

$f(T,d)) = \theta(b(T, f(S,d)), f(S,d))$ (See Figure 3).[21]

5 Characterizations of Discrete Raiffa, Nash, Kalai/Smorodinsky, and Proportional Solutions

5.1 The Discrete Raiffa Solution

Note that RIA-Box is satisfied by all Proportional solutions (i.e., including Egalitarian and Dictatorial solutions) as well as by the Discrete Raiffa solution. It is also known that MD is satisfied by the Nash, Kalai/Smorodinsky, Discrete Raiffa, Equal Area, Average Payoff solutions. As mentioned before, INMD is satisfied by the Nash, Kalai/Smorodinsky, Discrete Raiffa and Dictatorial solutions. The following is the first axiomatic characterization of the Discrete Raiffa solution. All proofs in this section are relegated to Appendix, Part A.

Proposition 1 *DR is the unique solution satisfying INMD, MD and RIA-Box.*

Thus, given INMD and MD, if two parties, whenever facing an uncertain bargaining circumstance with two possible underlying bargaining problems with the same disagreement point d and ideal point b, are willing to reach intermediate agreements, then the bargaining outcome must be DR.

[21] The last requirement "$f_i(T,d) = b_i(T,d)$ for some i only if $b(T,d) \in T$" is there to guarantee that $b(T,x) \neq f(T,d)$ will hold for all $x \in IR(T,d) \backslash \{f(T,d)\}$; otherwise $\theta(b(T, f(S,d)), f(T,d))$ may not be well-defined. This condition can be dropped if we restrict the domain of bargaining problems to be non-level or replace DCONT by PCONT in characterizing KS.

5.2 The Nash Solution

Recall our RIA-Gains axiom. It is easy to see that it is satisfied by all Proportional solutions and the Nash solution. MD is satisfied by a significant number of solutions, as mentioned above. DCONT, which even more innocuous, is satisfied by all known solution concepts.

RIA-Gains is closely related to the axiom of *Disagreement Point Convexity* introduced by Peters and Van Damme (1991):

Disagreement Point Convexity (DPC) $f(S, \alpha d + (1-\alpha) f(S,d)) = f(S,d)$ for all $\alpha \in (0,1)$.

DPC requires that $D(S,d,f) \supset l(d, f(S,d))$. If the premises of RIA-Gains hold, then DPC implies that $D(S,d,f) \cap D(T,d,f) \supset l(d, \min\{f(S,d), f(T,d)\})$.[22] Therefore DPC implies RIA-Gains, but not vice versa. Consider the $\epsilon-$egalitarian solution, $\epsilon - E$, such that (1) if $E_1(S,d) - d_1 = E_2(S,d) - d_2 \geq \epsilon$, it assigns $(E_1(S,d) - \epsilon, E_2(S,d) - \epsilon)$, where $\epsilon > 0$. (2) if $E_1(S,d) - d_1 = E_2(S,d) - d_2 < \epsilon$, it assigns d.[23] $\epsilon - E$ satisfies DCONT and RIA-Gains, but violates DPC.

Proposition 2 *N is the unique solution satisfying DCONT, MD and RIA-Gains.*[24]

Thus, given DCONT and MD, if two parties, whenever facing an uncertain bargaining circumstance with two possible underlying bargaining problems with the same disagreement point d, are willing to reach intermediate agreements as long as they expect to receive the same relative gains over these two possible bargaining problems, then the bargaining outcome must be N, the compromise that maximizes the product of their bargaining gains.

Remark 1 *Peters and Van Damme (1991) demonstrate that N is the unique solution satisfying INIR, SIR, DCONT, SYM, SI and DPC. The following proposition improves their result.*

Proposition 3 *N is the unique solution satisfying INIR, SIR, DCONT, SYM, SI and RIA-Gains.*

Proof. *It is straightforward to show that SIR, DCONT, SYM, SI and RIA-Gains imply DPC.* ■

[22] Note that $\min\{f(S,d), f(T,d)\}$ is well-defined when $\theta(d, f(S,d)) = \theta(d, f(T,d))$.

[23] $E(S,d)$ stands for the Egalitarian solution.

[24] DCONT is merely a technical condition and can be dropped if we modify the axiom of RIA-Gains slightly. Please see Appendix, Part B.

5.3 The Kalai/Smorodinsky Solution

SDM is satisfied by the Kalai/Smorodinsky solution as well as the Equal Area and Average Payoff solutions. We already elaborated on the number of solutions satisfying MD and DCONT. Using these axioms together with RIA-Concessions yields a characterization of the Kalai/Smorodinsky solution:

Proposition 4 *KS is the unique solution satisfying SDM, DCONT, MD and RIA-Concessions.*[25][26]

Given SDM, DCONT, and MD, if two parties, whenever facing an uncertain bargaining circumstance with two possible underlying bargaining problems with the same disagreement point d, are willing to reach intermediate agreements as long as they expect to take the same relative concessions over these two possible bargaining problems, then the bargaining outcome must be KS.

Remark 2 *It can readily be seen that the axiom of MD can be replaced by PO.*

Proposition 5 *KS is the unique solution satisfying SDM, DCONT, PO and RIA-Concessions.*

5.4 Proportional Solutions

DCONT, PCONT, SIR and WPO are satisfied by all known solution concepts. As mentioned before, RIA-Inclusion is weaker than SSN. By using DCONT, PCONT, SIR, WPO and RIA-Inclusion together, we obtain the following result:

Proposition 6 *Proportional solutions are the only class of solutions satisfying DCONT, PCONT, SIR, WPO and RIA-Inclusion.*

Given DCONT, PCONT, SIR and WPO, if two parties, whenever facing an uncertain bargaining circumstance with two possible underlying bargaining problems with the same disagreement point d, are willing to reach intermediate agreements as long as the

[25] As mentioned before, DCONT is merely a technical condition. It can be dropped if we modify the axiom of RIA-Concessions slightly. Please see Appendix B.

[26] Note that SDM, instead of its weaker version, DM, is required in the characterization of KS. However, even though N does not satisfy SDM in Σ, it does satisfy it in Σ^s nevertheless. Hence, clearly one cannot distinguish KS, N, and DR from each other - at least in Σ^s - solely on the basis of SDM.

two bargaining problems S and T are nested, then the bargaining outcome must be proportional.[27]

We can now fully summarize our results:

DR	RIA-Box	MD		INMD
N	RIA-Gains	MD	DCONT	
KS	RIA-Concessions	MD	DCONT	SDM
P	RIA-Inclusion		DCONT	PCONT+WPO+SIR

That is, beside MD and RIA-Box, the Discrete Raiffa solution's axiomatic characterization uses only one more axiom, INMD. Beside MD and RIA-Gains, the Nash solution's characterization too uses only one more axiom, DCONT. Beside MD and RIA-Concessions, the Kalai/Smorodinsky solution's characterization uses DCONT as well as SDM. Beside RIA-Inclusion, the Proportional solutions' characterization uses WPO, DCONT, PCONT and SIR.

6 Conclusion

Although there were previous non-unified attempts that tried to bring bargaining process into Nash's bargaining problem (via the SSN axiom of Kalai, 1977, and the MD axiom of Sobel, 1981), previous characterizations of bargaining solutions typically relied on crucial axioms entailing changes in the utility possibility set and in the disagreement point, and did not describe any bargaining process. In this paper, we relax the strong robustness criterion of SSN and thereby highlight the crucial role robustness of intermediate agreements and bargaining process play further in a unified way. By describing circumstances under which such intermediate agreements can be obtained, our Robustness of Intermediate Agreements (RIA) axioms portray a bargaining process.

A major accomplishment of our framework is the axiomatic characterization of the Discrete Raiffa solution, which had eluded researchers before. All of our novel characterizations except for that of Proportional solutions, involve both MD and an RIA axiom, both of which pertain to different aspects of a bargaining process; characterization of Proportional solutions does not employ MD.

[27]Rachmilevitch (2009) recently wrote an interesting note on our paper, which at that time had not incorporated characterization of Proportional solutions yet. Rachmilevitch used IR, SYM, a slightly weaker version of DCONT, 'Translation Invariance' and a new axiom, 'Interim Improvement', to characterize the Egalitarian solution. His Interim Improvement axiom and our RIA-Inclusion axioms do not imply one another.

This unified approach used here aims to bridge the axiomatic and strategic approaches in bargaining. The use of robustness of intermediate agreements is certainly one fruitful way of bringing these two approaches together. Future research may identify further fruitful ways in that direction.

7 Appendix

7.1 Part A: Proofs

Proof of Proposition 1. It is obvious that DR satisfies these three axioms. Suppose f satisfies INMD, MD and RIA-Box and we show that $f = DR$. Pick any $(S,d) \in \Sigma$. $m(S,d) \in S$ by convexity of S. If $m(S,d) \in PO(S)$, then by MD $f(S,d) = m(S,d) = DR(S,d)$. Now suppose $m(S,d) \notin PO(S)$. $m(S,d) \notin WPO(S)$ by convexity of S again. Hence $(S, m(S,d)) \in \Sigma$. To show that $f(S,d) = DR(S,d)$ in this case, it is sufficient to show $f(S,d) = f(S, m(S,d))$. Consider a bargaining problem (T,d) where $T = conv\{d, (d_1, b_2(S,d)), (b_1(S,d), d_2)\}$.[28] MD implies that (i) $f(T,d) = m(S,d)$, and (ii) $D(T,d,f) = l[d, m(S,d))$. By RIA-Box, there exists a common intermediate agreement $a \in l[d, m(S,d)) \cup \{m(S,d)\}$ such that $f(S,d) = f(S,a)$. INMD excludes all points below $m(S,d)$ to be a common intermediate agreement. Hence, $a = m(S,d)$. ■

Proof of Proposition 2. It is obvious that N satisfies these three axioms. We will show that, if f satisfies these three axioms, then it must be $f = N$. The proof is based on the following nice characterization of the Nash solution by De Clippel (2007).

Lemma 1 *(Theorem 1, de Clippel (2007)) N is the unique solution satisfying MD and DPC.*

With this Lemma in hand, it is sufficient to show that DCONT, MD and RIA-Gains imply DPC. Pick any (S,d) in Σ and let $f(S,d)$ be its solution. MD implies $f(S,d) > d$. Consider a bargaining problem (T^ε, d) with $T^\varepsilon = conv\{d, (2f_1(S,d) - d_1 - \frac{\varepsilon}{f_2(S,d)-d_2}, d_2), (d_1, 2f_2(S,d) - d_2 - \frac{\varepsilon}{f_1(S,d)-d_1})\}$. MD implies that (i) $f(T^\varepsilon, d) = (f_1(S,d) - \frac{\varepsilon}{2(f_2(S,d)-d_2)}, f_2(S,d) - \frac{\varepsilon}{2(f_1(S,d)-d_1)})$, which in turn implies that $\theta(d, f(S,d)) = \theta(d, f(T^\varepsilon, d))$, and (ii) $D(T^\varepsilon, d, f) = l[d, (f_1(S,d) - \frac{\varepsilon}{2(f_2(S,d)-d_2)}, f_2(S,d) - \frac{\varepsilon}{2(f_1(S,d)-d_1)}))$. RIA-Gains tells us that at least one point $a^1 \in l(d, (f_1(S,d) - \frac{\varepsilon}{2(f_2(S,d)-d_2)}, f_2(S,d) - \frac{\varepsilon}{2(f_1(S,d)-d_1)}))$ is in the disagreement point set of (S,d) with respect to f. Starting at a^1 as a new disagreement point and repeating the argument above gives us a strictly increasing sequence $\{a^n\}$ such that $a^n \in D(S,d,f)$ $\forall n$. $\lim_{n\to\infty} a^n = (f_1(S,d) -$

[28] "conv" denotes "the convex hull of."

$\frac{\varepsilon}{2(f_2(S,d)-d_2)}, f_2(S,d) - \frac{\varepsilon}{2(f_1(S,d)-d_1)})$ by RIA-Gains and DCONT. Invoking DCONT and RIA-Gains again gives us $(f_1(S,d) - \frac{\varepsilon}{2(f_2(S,d)-d_2)}, f_2(S,d) - \frac{\varepsilon}{2(f_1(S,d)-d_1)}) \in D(S,d,f)$. Ranging ε from 0 to $2(f_1(S,d)-d_1)(f_2(S,d)-d_2)$ gives us DPC. ■

Proof of Proposition 4. It is straightforward to see that KS satisfies these four axioms. Suppose f satisfies SDM, DCONT, MD and RIA-Concessions; then we will show that $f = KS$ must hold. The proof consists of two steps:

(I) If $T = conv\{d, (d_1+a, d_2), (d_1, d_2+c), (d_1+a, d_2+c)\}$ for some $a, c > 0$, then $f(T,d) = b(T,d) = (d_1+a, d_2+c)$. We show it by contradiction. Suppose to the contrary that $f(T,d) \neq (d_1+a, d_2+c)$. MD implies $f(T,d) \geq m(T,d) > d$. Denote $L(f(T,d), (d_1+a, d_2+c))$ to be the straight line going through $f(T,d)$ and (d_1+a, d_2+c), and define $\eta \equiv \inf\{x \geq m(T,d) |\ x \in L(f(T,d), (d_1+b, d_2+c))\}$. η is well-defined as the partial order $\geq$ in $\mathbb{R}^2$ induces a linear order in $L(f(T,d), (d_1+b, d_2+c))$. There are two possible cases:

(i) $\eta = m(T,d)$. Consider a new bargaining problem (W,d) with $W = conv\{d, (d_1 + a, d_2), (d_1, d_2+c)\}$. Notice that $b(T,d) = b(W,d) = (d_1+a, d_2+c)$. MD implies (a) $f(W,d) = (d_1 + \frac{a}{2}, d_2 + \frac{c}{2}) = m(T,d)$, and (b) $D(W,d,f) = l[d, m(T,d))$. Accordingly we have $\theta(b(T, f(W,d)), f(T,d)) = \theta(b(T, f(W,d)), f(W,d))$, and there exists $y^1 \in l(d, m(T,d))$ such that $f(T,d) = f(T,y^1)$ and $f(W,d) = f(W,y^1)$ by RIA-Concessions. Repeatedly applying RIA-Concessions we get a strictly increasing sequence $\{y^i\}$ with $y^i \in l(d, m(T,d))$ such that $f(T,d) = f(T,y^i)$ and $f(W,d) = f(W,y^i)$ for all i. It can be shown that $\lim y^i = m(T,d)$ by DCONT and RIA-Concessions; consequently $f(T,d) = f(T, m(T,d))$ by DCONT. Taking $m(T,d)$ as a new disagreement point and iteratively applying the equation $f(T,d) = f(T, m(T,d))$ shows that $f(T,d) = (d_1+a, d_2+c)$, contradicting our premise that $f(T,d) \neq (d_1+a, d_2+c)$.

(ii) If $\eta \neq m(T,d)$, then either $\eta = (\alpha, m_2(T,d))$ for some $\alpha \in (m_1(T,d), d_1+a]$ or $\eta = (m_1(T,d), \beta)$ for some $\beta \in (m_2(T,d), d_2+c]$. Without loss of generality, assume $\eta = (\alpha, m_2(T,d))$ for some $\alpha \in (m_1(T,d), d_1+a]$. There are two sub-cases:

Case 1. $\alpha \in (m_1(T,d), d_1+a)$. Consider a new bargaining problem (Φ, d) with $\Phi = conv\{d, (2\alpha - d_1, d_2), (d_1, d_2+c)\}$. Following the same steps as in (i) we have $\eta \in D(T,d,f)$, then take η as a new disagreement point and iteratively apply the equation $f(T,\eta) = f(T, m(T,\eta))$ concluding that $f(T,d) = (d_1+a, d_2+c)$.

Case 2. $\alpha = d_1+a$. Consider a new bargaining problem (Ψ, d) with $\Psi = conv\{d, (d_1 + 2a, d_2), (d_1, d_2+c)\}$. $m(\Psi,d) = \eta$ but note that $(T,\eta) \notin \Sigma$. Nevertheless, by the same token as in (i) we are still able to get a strictly increasing sequence $\{x^i\}$ with $\lim_{i\to\infty} x^i = \eta$ such that $x^i \in D(T,d,f)$ for all i. $f_1(T,d) = d_1+a$ and $f_2(T,d) \geq \frac{1}{2} m_2(T, m(T,d))$ by MD. Taking x^i sufficiently close to η as a new disagreement point and invoking the standard limiting argument (recursively) shows that $f_2(T,d) = d_2 + c$. Therefore $f(T,d) = (d_1 + a, d_2 + c)$.

(II) Pick any (S,d) in Σ. If $b(S,d) \in S$, then $IR(S,d) = conv\{d, (d_1, b_2(S,d)), (b_1(S,d), d_2), b(S,d)\}$. Therefore $f(S,d) = b(S,d) = KS(S,d)$ from (I). Assume now $b(S,d) \notin S$. $f(S,d) \geq m(S,d)$ by MD; moreover, we show that $f(S,d) \in PO(S)$. Suppose to the contrary that $f(S,d) \notin PO(S)$. Consider a bargaining problem (W,d) with $W = conv\{d, (2f_1(S,d) - d_1, d_2), (d_1, 2f_2(S,d) - d_2)\}$. Then $f(W,d) = f(S,d)$ by MD. Hence $\theta(b(S, f(W,d)), f(S,d)) = \theta(b(S, f(W,d)), f(W,d))$. Repeatedly invoking RIA-Concessions and DCONT gives us a strictly increasing sequence $\{x^i\}$ with $\lim_{i\to\infty} x^i = f(S,d)$ such that $x^i \in D(S,d,f)$ for all i. Consequently, $f(S,d)$ must be in $PO(S)$ by MD.

Define $\Gamma \equiv \{x \in IR(S,d) |\ \theta(b(S,x), f(S,d)) = \theta(b(S,x), x) \text{ and } x \leq f(S,d)\}$. Since $f_i(S,d) \neq b_i(S,d)$, $\Gamma \backslash \{f(S,d)\}$ is non-empty. It can be shown that either $\Gamma \cap \{x \in IR(S,d) | x_1 = d_1\} \neq \emptyset$ or $\Gamma \cap \{x \in IR(S,d) | x_2 = d_2\} \neq \emptyset$. There are two cases to be considered:

(i) If $d \in \Gamma$, then $f(S,d) = l[d, b(S,d)] \cap PO(S) = KS$.

(ii) If $d \notin \Gamma$, then either $(\alpha, d_2) \in \Gamma$ for some $\alpha \in (d_1, f_1(S,d))$ or (d_1, β) for some $\beta \in (d_2, f_2(S,d))$. Without loss of generality assume $(\alpha, d_2) \in \Gamma$ for some $\alpha \in (d_1, f_1(S,d))$. It is straightforward to show that $\Gamma \backslash (f(S,d) \cup (\alpha, d_2)) \subset D(S,d,f)$ by MD, DCONT and RIA-Concessions. $(\alpha, d_2) \in D(S,d,f)$ by DCONT. But it violates SDM as $\alpha > d_1$. Therefore d must be in Γ and $f = KS$. ■

Proof of Proposition 6. It can be easily seen that P satisfies these five axioms. Suppose f satisfies DCONT, PCONT, SIR, WPO and RIA-Inclusion; then we show that $f = P$. It is sufficient to show that DCONT, PCONT, SIR, WPO and RIA-Inclusion imply SSN. Denote by $\Sigma^{nl} \subset \Sigma$ the class of all bargaining problems that are non-level (see de Clippel, 2007). Since every $(S,d) \in \Sigma$ can be approximated by a sequence of bargaining problems in Σ^{nl}. By PCONT, it is sufficient to show the claim is true in Σ^{nl}. Pick any (S,d) and (T,d) in Σ^{nl} such that $S \subset T$ and $(T, f(S,d)) \in \Sigma^{nl}$. First we show that there is a sequence $\{x^i\}$ with $x^i \in D(S,d,f)$ such that $\lim_{i\to\infty} x^i = f(S,d)$. Denote by $B_\epsilon(y) = \{x \in \mathbb{R}^2 |\ \|x - y\| < \epsilon\}$ the open ball of radius $\epsilon > 0$ centered at y. For any given $\epsilon > 0$, we show that $B_\epsilon(f(S,d)) \cap D(S,d,f) \neq \emptyset$. Pick any $z \in B_\epsilon(f(S,d)) \cap l(d, f(S,d))$. $z < f(S,d)$ by SIR. If $f(S,z) = f(S,d)$, the claim is established. Suppose now $f(S,z) \neq f(S,d)$. Assume without loss of generality $f_1(S,z) > f_1(S,d)$; then $f_2(S,z) \leq f_2(S,d)$ by WPO. Define $\zeta \equiv (z_1, f_2(S,d)) \in B_\epsilon(f(S,d))$. Since S is non-level, $(S,\zeta) \in \Sigma$. $f_2(S,\zeta) > f_2(S,d)$ by SIR, and hence $f_1(S,\zeta) \leq f_1(S,d)$ by WPO. $f(S, l[z,\zeta]) \subset WPO(S)$. Since $l[z,\zeta]$ is connected (in the Hausdorff topology) and $f(S,\cdot)$

is continuous by DCONT, $f(S, l[z, \zeta])$ is connected. Consequently there exists some $\gamma \in l[z, \zeta]$ such that $f(S, \gamma) = f(S, d)$. As ϵ is arbitrary, we can always find a sequence $\{x^i\}$ such that $x^i \in D(S, d, f)$ for all i and $\lim_{i \to \infty} x^i = f(S, d)$. Then $x^i \in D(T, d, f)$ for all i by RIA-Inclusion and DCONT completes the proof. ■

7.2 Part B: $\alpha-$RIA Results

$\alpha-$Robustness of Intermediate Agreements with Identical Relative Gains ($\alpha-$RIA-Gains) Suppose $(S, d), (T, d) \in \Sigma$ and pick any $\alpha \in [0, 1)$. A solution f satisfies **$\alpha-$RIA-Gains** if (i) $f(S, d) \in IR(S, d)\backslash\{d\}$ and $f(T, d) \in IR(T, d)\backslash\{d\}$ and (ii) $\theta(d, f(S, d)) = \theta(d, f(T, d))$ implies that there exists $x \in D(S, d, f) \cap D(T, d, f)$ with $x \geq \alpha d + (1 - \alpha) \min\{f(S, d), f(T, d)\}$.

$\alpha-$RIA-Gains strengthens RIA-Gains by requiring that there exists at least one common intermediate agreement which dominates $\alpha d + (1 - \alpha) \min\{f(S, d), f(T, d)\}$. From the negotiation process point of view, it can be seen as a condition on the speed of convergence. This common intermediate agreement can be arbitrarily close to d if we pick α sufficiently close to 1. Note that DPC implies $\alpha-$RIA-Gains as well. Therefore, the following straightforward extension of Proposition 2 improves Theorem 1 of de Clippel (2007).

Proposition 7 *N is the unique solution satisfying MD and $\alpha-$RIA-Gains for all $\alpha \in (0, 1)$.*

$\alpha-$Robustness of Intermediate Agreements with Identical Relative Concessions ($\alpha-$RIA-Concessions) Suppose $(S, d), (T, d) \in \Sigma$ and $\alpha \in [0, 1)$. If (i) $f(S, d) \in IR(S, d)\backslash\{d\}$ and $f(T, d) \in IR(T, d)\backslash\{d\}$ and (ii) $\theta(b(T, f(S, d)), f(T, d)) = \theta(b(T, f(S, d)), f(S, d))$, then there exists $x \in D(S, d, f) \cap D(T, d, f)$ with $x \geq \alpha d + (1 - \alpha) \min\{f(S, d), f(T, d)\}$; moreover, $f_i(T, d) = b_i(T, d)$ for some i only if $b(T, d) \in T$.

It is straightforward to show the following:

Proposition 8 *KS is the unique solution satisfying SDM, MD and $\alpha-$RIA-Concessions.*

Proposition 9 *KS is the unique solution satisfying SDM, PO and $\alpha-$RIA-Concessions.*

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