Satisfiable Fairness in Cooperative Games with Transferable Utility

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Abstract

The driving question behind cooperative game theory is how to divide a cooperatively generated value among the players. The dominant solution concept is presently the core. Other solution concepts include the τ-value, the Shapley value and the egalitarian core.
Wherever cooperative game theory is used to model human behavior, the question arises as to whether the modelled solutions can be considered *fair*. Now, while some solution concepts are motivated by certain notions of fairness, the term itself cannot be accurately defined. The word carries a range of semantics as diverse as *equity of needs*, *performance fairness* and *equal opportunities*. In addition, the degree of personal inequity aversion varies between cultures.
This paper provides a sanity condition for different fairness notions called *satisfiability*. Furthermore, different fairness predicates on the imputation space are defined and their satisfiability is discussed.
The proposed fairness concepts include respecting a pre-order of relative value on the player set as given by the game's payoff function, compatibility with splitting the game into a purely cooperative and a trivial component and respecting a pre-order of relative value on the lattice of coalitions, which can be thought of as the formation of labor unions. A discussion whether the solution concepts mentioned above meet these fairness predicates is also included.

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1 Introduction

Research involving the Ultimatum Game, the Dictator Game and the Trust Game show that fairness concepts have to be taken into consideration if game theory is to be a realistic model for real life human behavior (see for example [Hein 04], [Oost 04], [Fehr 99]).

The approaches of inequity aversion [Fehr 99] and envy-freeness [Arag 92], [Fole 67] adhere to an equal entitlement of all players to the total generated value and, thus, encounter difficulties when the game's characteristic function imposes different strategic worth to different players. The Shapley value is anonymous, additive and satisfies the dummy player axiom. However, the Shapley value is not necessarily an element of the core of a superadditive game. Several authors have decided to use the τ-value as their solution concept when addressing matters of fairness in the past (see for example [Zele 08] or [Bran 02]). But, like the Shapley value, it need not be an element of the core for all games. The egalitarian core (see [Arin 01] and [Arin 08]) is a strong and very interesting approach to fairness. We will later see why this solution is less convincing for games where the value of singleton coalitions is non-zero.

The paper is organized as follows: Section 2 introduces basic notation and definitions. Section 3 proposes a way to split games into a trivial component and a purely cooperative component. Section 4 then introduces a rationality-of-fairness reasoning that is build around the concept of *satisfiable predicates* on the imputation space.

Section 5 contains the main results of our efforts. We introduce different fairness concepts and discuss their satisfiability. Also we will discuss which solution concepts meet these fairness criteria and which ones do not. First we will introduce the pre-order of cooperational value on the player set as given by a game's payoff function. We can then introduce *relative player value fairness*, a relatively weak but unconditionally satisfiable concept that labels as fair those imputations that respect this pre-order. Subsequently, we introduce a split version of this fairness condition as a stronger concept that remains satisfiable. Next, we introduce a very strong fairness concept where solutions need to respect all relative-value relations between coalitions and show that, while this *set value fairness* is not satisfiable for super-additive games, a weaker notion, called *labor union fairness*, is satisfiable for convex games.

2 Definitions

2.1 Cooperative Games and Imputations

We generally follow the notation given in [Krab 05].

2.1. Definition. A **cooperative game** is a tuple (N, v), where N is a set of players and $v : \mathfrak{P}(N) \longrightarrow \mathbb{R}$, $v(\emptyset) = 0$ is a function that we call **payoff function** for coalitions.

For $|N| = n$ we call (N, v) a **cooperative** n**-person game**.

We expect the players of an n-person game to be numbered 1, ..., n and write $N = \{1, \dots n\}$. The function v gives the value of a coalition in the sense that for $A \subseteq N$, the players in coalition A can obtain a total payoff of $v(A)$ through cooperation (regardless of how other players cooperate)

This is obviously a definition for transferable utility games. When we say *game*, we mean *TU game*.

A game is called **superadditive** if $v(A) + v(B) \leq v(A \cup B)$ holds for all coalitions A, B with $A \cap B = \emptyset$ (see [Krab 05]). It is called **average convex**, if the inequality $\sum_{i \in A} [v(A) - v(A \setminus \{i\})] \leq \sum_{i \in A} [v(B) - v(B \setminus \{i\})]$ holds for all coalitions $A \subseteq B \subseteq N$ (see [Iñar 93]) and it is called **convex**, if the inequality $v(A) + v(B) \leq v(A \cup B) + v(A \cap B)$ holds for all coalitions $A, B \subseteq N$ (see again [Krab 05]).

Convexity implies both superadditivity and average-convexity.

We write $MC_i(K)$ for the **marginal contribution** of player i to coalition $K \subset N \setminus \{i\}$ that is: $MC_i(K) := v(K \cup \{i\}) - v(K)$.

The central question in cooperative game theory is how the total payoff can be distributed in a way that satisfies all players.

An imputation is a distribution of payoff that grants each player at least the amount that he can gain by playing solo.

2.2. Definition. Let (N, v) be a cooperative n-person game. An **imputation** of v is a vector $x \in \mathbb{R}^n$ with $\sum_{i=1}^{n} x_i = v(N)$ and $x_i \geq v(\{i\})$ for all $i \in \{1, \dots, n\}$.

A vector that only satisfies the first condition is called **pre-imputation**.

The set of imputations in (N, v) is denoted by $I(v)$.

2.2 Solution Concepts

2.2.1 The Core

A cooperative n-person game (N, v) has imputations whenever $\sum_{i=1}^{n} v(\{i\}) \leq v(N)$, and it has infinitely many imputations when the inequality is strict.

In the latter case it is necessary for the players to agree on specific imputations. Several criteria for the feasibility of imputations have been introduced, the most prominent one being the *core property* first defined in modern form by [Gill 59].

2.3. Definition. The **core** $C(v)$ of the cooperative n-person game (N, v) is the set of those imputations x that satisfy the following **core property**: $v(K) \leq \sum_{i \in K} x_i$ for all $K \subseteq N$.

Note that by definition any game's core is a convex and bounded polytope.

Since the core of a game will often also have infinitely many elements, the question of choice of a single imputation remains. In the rest of this section we will introduce several solution concepts for this problem. We will use the term **one-point solution** concept for mappings that take each game (N, v) to a singleton subset of $I(v)$ and **partial one-point solution**, when the image of this mapping is either singleton or empty.

Where no confusion can arise, we identify singleton sets with their only element.

2.2.2 Strong ϵ-Core and Least-Core

If for any coalition K we have $v(K) = \sum_{i \in K} x_k$, that is if x lies on the bounding hyperplane of the core, there is no real incentive for that coalition to participate in the grand coalition. Therefore it can be usefull to study the interior of the core:

2.4. Definition. For some number $\epsilon \in \mathbb{R}$ The **strong ϵ-core** of the cooperative n-person game (N, v) is the set of those imputations x that satisfy the following core property: $v(K) - \epsilon \leq \sum_{i \in K} x_i$ for all $K \subseteq N$.

Clearly, regardless of whether the core is empty, the strong ϵ-core will be non-empty for a large enough value of ϵ and empty for small enough (large negative) values of ϵ.

Following this line of reasoning, the **least-core**, introduced in [Masc 78], is the intersection of all non-empty strong ϵ-cores. It can also be viewed as the strong ϵ-core for the smallest value of that makes the set non-empty [Bilb 00].

2.2.3 The Shapley Value

The **Shapley value** of a cooperative n-person game is a one-point solution concept defined on the class of super-additive games introduced in [Shap 53].

The most common formula for the Shapley value $Sh(v) \in \mathbb{R}^n_+$ of a game (N, v) is as follows: $Sh_i(v) := \sum_{K \subseteq N \setminus \{i\}} \gamma_n(K) \cdot MC_i(K)$ for all $i = 1, \ldots, n$, where $\gamma_n(K) := (n!)^{-1} \cdot |K|! \cdot (n - |K| - 1)!$ for all $K \subseteq N,\ K \neq N$.

The Shapley value is a possible distribution of the total payoff, that is: $\sum_{j=1}^{n} Sh_j(v) = v(N)$. Proof for this property can be found in [Drie 88]. If we assume the game (N, v) to be superadditive, we have $v(K \cup \{i\}) - v(K) \geq v(\{i\})$, hence the Shapley value is an imputation.

2.2.4 The τ-Value

We now define the τ-value of a cooperative n-person game, a one-point solution concept introduced by Stef Tijs in [Tijs 81].

For $i = 1 \ldots n$ we call $u_i = v(N) - v(N \setminus \{i\}) = MC_i(N \setminus \{i\})$ the **utopia value** of player i. In a core imputation no player can ever get a payoff that exceeds his utopia value. Therefore the **utopia vector** u is an upper bound for core imputations.

Generally every player will end up getting less than his utopia value, because for all interesting games $v(N) \leq u_1 + \ldots + u_n$. The **g**ap function $g : \mathfrak{P}(N) \longrightarrow \mathbb{R}$ is thus defined as $g(K) = (\sum_{j \in K} u_j) - v(K)$.

For every player i the **concession value** λ_i is then defined to be the minimum of $g(K)$, where K ranges over all coalitions that contain the player i.

For core elements $x \in C(v)$ we find $\left(\forall i \in N\right)\ u_i - \lambda_i \leq x_i \leq u_i$, a proof for this Proposition is given, for example, in [Krab 05]

If $\sum_{i \in N} \lambda_i \geq g(N)$ holds and $g(K)$ is never negative, the game (N, v) is called **quasi-balanced**. Every game with non-empty core is indeed quasi-balanced (see for example [Krab 05]). The **τ-value** can then be defined.

2.5. Definition. The τ-value of a quasi-balanced cooperative n-person game (N, v) is given by: $\tau(v) := u - \frac{g(N)}{\sum_{i \in N} \lambda_i} \cdot \lambda$.

A *iff* criterion for $\tau(v) \in C(v)$ is that the following implication holds for all coalitions K:

$$g(K) \geq 0 \land 2 \leq |K| \leq n - 2 \Longrightarrow \frac{\lambda(N)}{g(N)} \geq \frac{\lambda(K)}{g(K)}$$

where $\lambda(K)$ is the sum over the λ_i with $i \in K$.

Again, proof for the Proposition can be found in [Krab 05]

2.2.5 The Egalitarian Core and Egalitarian Values

Egalitarianism is the strife of a community to spread the total wealth of the community as equally as possible among its members, while satisfying certain stability requirements of the allocation. The notion of egalitarianism is frequently used outside the theory of transferable utility games. See for example [Thom 94] for applications in bargaining theory and Moulin Dutta and Ray first introduced the egalitarian allocation in [Dutt 89], as a solution concept that combines (recursively defined) stability and egalitarianism. For convex games the egalitarian allocation always exists and it is an element of the core.

Arin and Iñarra use another definition of egalitarian allocations ([Arin 01]). They use the same notion of egalitarianism—the widely accepted Lorenz criterion—but as a notion of stability they use the game's core. As a consequence, the latter type of egalitarian allocation exists for a given game precisely when the core of the game is not empty.

In the class of convex games both notions coincide (see again [Arin 01]). This, together with the guarantee of existence for a relatively large and manageable class of games, makes the notion of Arin and Iñarra an interesting alternative. The most prominent drawback of their definition is that more than one allocation may be egalitarian.

This motivates Arin and Kuipers to introduce several solution concepts that assign exactly one egalitarian allocation to each game with non-empty core in [Arin 08].

From the latter article we take the following definitions:

2.6. Definition. For two players i and j in a game (N, v), an allocation $x \in I(v)$, and a real number $\alpha > 0$, we say that (i, j, x, α) is an **equalizing bilateral transfer** (of size α from i to j with respect to x) if $x_i - \alpha \geq x_j + \alpha$.

Now an imputation $x \in C(v)$ is called **egalitarian** if no core allocation y is the result of an equalizing bilateral transfer with respect to x. A core allocation x is **strongly egalitarian** if no core allocation y is the result of a finite sequence of equalizing bilateral transfers starting from x.

We write $C_e(v)$ for the set of egalitarian core allocations and $C_{se}(v)$ for strongly egalitarian core allocations.

2.7. Definition. For a balanced game (N, v), the least squares Solution $\mathrm{LS}(v)$ is defined as the unique allocation x in $C(v)$ for which $\|x\| < \|y\|$ for all $y \in C(v)$.

Here, $\|x\|$ denotes the Euclidean length $\sqrt{\sum_{i \in N} x_i^2}$ of x.

The existence of the unique allocation minimizing the Euclidean length is obvious, since it is the solution of an optimization problem with continuous (even quadratic) objective on a compact set (with linear constraints).

Arin et al. prove, that LS is indeed strongly egalitarian for all games with non-empty core.

Now that we have recalled the basic definitions, the next chapter will explain a certain way to think about games, that we will use later for different fairness considerations.

3 Splitting Cooperative n-Person Games

Let (N, v) be a cooperative n-person game. Note that every player $i \in N$ of the game will get a payoff of at least $v(\{i\})$. This payoff is guaranteed to the player even if cooperation fails and, in case of cooperative play, regardless of the particular imputation chosen.

3.1. Definition. The vector $s(v) = (v(\{1\}), \dots, v(\{n\}))^T$ of individual payoffs is called **non-cooperative component** or **trivial component** of the game (N, v).

Where no confusion arises as to which game function v is meant, we also write s for $s(v)$.

When the non-cooperative payoffs are given to the players beforehand, a purely cooperative game remains, that is a game in which no payoffs can be generated without cooperation.

3.2. Definition. Let (N, v) be a cooperative n-person game. Then we define

$$\begin{aligned} v_0 \,:\, \mathfrak{P}(N) &\longrightarrow \mathbb{R} \\ K &\longmapsto v_0(K) := v(K) - \sum_{k \in K} v(\{k\}). \end{aligned}$$

We call the game (N, v_0) the **purely cooperative component** of the game (N, v).

3.3. Proposition.

(i) *We have both* $I(v_0) + s = I(v)$ *and* $C(v_0) + s = C(v)$.

(ii) *The game* (N, v_0) *is superadditive (average-convex, convex) if and only if* (N, v) *is superadditive (average-convex, convex).*

Proof.

(i) $I(v_0)$ is the set of positive vectors x with $\|x\|_1 = v_0(N)$, while the imputations of v are those vectors of norm $v(N)$ that have $x_i \geq s_i$ for all i.

Since $v(N) - v_0(N) = \|s\|_1$ we have $I(v_0) + s = I(v)$.

The core $C(v)$ is defined via $\sum_{k \in K} x_k \geq v(K)$, which can be written as $\sum_{k_i n K} x_k - s_k \geq v(K) - s(K)$, which is the core property of the game v_0 for the imputation $y = x - s$ and coalition K.

(ii) The transition of superadditivity and convexity from v to v_0 and vice versa follows with $\sum_{i \in K} s_i + \sum_{i \in L} s_i = \sum_{i \in K \cup L} s_i + \sum_{i \in K \cap L} s_i$.

The transition of average-convexity follows with $MC_i(v) = v(K \cup \{i\}) - v(K) = v_0(K \cup \{i\}) + \sum_{k \in K \cup \{i\}} v(\{k\}) - v_0(K) - \sum_{k \in K} v(\{k\}) = MC_i(v_0) + v(\{i\})$.

□

We will now see, that the Shapley value and the τ-value both split, whenever they exist.

3.4. Proposition. *The Shapley value of a superadditive game* (N, v) *splits, i.e. for an additive game* (N, v) *with non-cooperative component* s *and purely cooperative component* v_0 *we have* $Sh(v) = Sh_(v_0) + s$.

Proof. Since the Shapley value is additive, it suffices to check that the Shapley value of the purely non-cooperative game $v_s(A) = \sum_{a \in A} s_a$ is again s. $\square$

3.5. Proposition. *Consider a quasi-balanced game (N, v) with non-cooperative component s and purely cooperative component (N, v_0).*
Then $\tau(v) = \tau(v_0) + s$.

Proof. Consider the utopia vector of the purely cooperative component

$$u_0 = \left(MC_1^{v_0}(N \setminus \{1\}), \ldots, MC_n^{v_0}(N \setminus \{n\})\right)^T,$$

the cooperative component's gap function g_0 and the concession vector λ_0.
Recalling Proposition 3.3 it is obvious that $u_0 + s = u$, $g_0 = g$ and $\lambda_0 = \lambda$.
This all lets us conclude that

$$\begin{aligned}\tau(v_0) \coloneqq u_0 - \frac{g_0(N)}{\lambda_0(N)} \cdot \lambda_0 &= u - s - \frac{g(N)}{\lambda(N)} \cdot \lambda \\ &= \tau(v) - s.\end{aligned}$$

$\square$

3.6. Proposition. *For general cooperative n-person games, neither $C_e(v) = C_e(v_0) + s$ nor $C_{se}(v) = C_{se}(v_0) + s$ hold.*

Proof. Consider a two player game with non-cooperative component $s = (1, 0)^T$ and purely cooperative component $v(N) = 2$, thus $v_0(N) = 1$.
Then $C_e(v_0) = C_{se}(v_0) = \{(0.5, 0.5)^T\}$, while $C_e(v) - s = C_{se}(v) - s = \{(0, 1)^T\}$. $\square$

4 Predicates on the Imputation Space

We will introduce different fairness concepts as *predicates* on the imputation space and then study their *satisfiability*, so let us start out by defining these two words.

4.1. Definition. A **predicate** on the Imputation Space of a cooperative n-person game is a mapping P that assigns every game (N, v) a subset $P(v) \subseteq I(v)$.

We will be interested in whether or not certain one-point solution concepts satisfy certain predicates.

Some predicates that commonly appear in game theory are the following

- The **dummy player predicate** DP rules out those imputations with positive payoffs for players that contribute nothing.
- All set valued or one-point, total or partial solution concepts are predicates.
- A (partial) one-point solution concept satisfies **anonymity** if for any permutation σ of the player set N we have $P(v)_i = P(\sigma(v))_{\sigma}(i)$.
- A (partial) one-point solution concept is **additive** if for two cooperative n-person games (N, v) and (N, w) (where P(v) and P(w) are both non-empty) the equation $P(v + w) = P(v) + P(w)$ holds.

Concerning the splitting of games from the previous Section, we can also define predicates that split.

4.2. Definition. A predicate P on the Imputation Space of cooperative n-person games is said to **split** if for all (N, v) we have $P(v_0) + s(v) = P(v)$.

4.1 Rationality of Fairness - Satisfiable Predicates

It is not practical to work with (or propagate) fairness concepts that are too strong. The one-point solution according to a "Every Player Must Get The Same"-fairness principle, for example, would (for many games) produce vectors that are not even imputations.

A sanity condition for fairness predicates is whether they are satisfiable within the core—ideally for any game, possibly for a class of games.

4.3. Definition. A predicate P is **satisfiable** in a class $\mathcal{G}$ of games if for each game $(N, v) \in \mathcal{G}$ the implication $C(v) \neq \emptyset \Longrightarrow P(v) \cap C(v) \neq \emptyset$ holds.

One can argue that it is rational for a player to demand a *fair* share with respect to some satisfiable fairness predicate, because *he knows he can*. If the players can agree on a notion of fairness that is satisfiable in all games that they are about to play, they can agree to choose fair imputations even before the game rules are known to them, that is while they are still equals. And once the game rules become apparent, they would still stick to their fairness notion, since it produces core imputations and there is no sound reason to quit cooperation under these circumstances.

5 Several Fairness Predicates

Let us now propose some definitions of what might be considered a *fair* imputation and study their satisfiability.

5.1 Relative Player Value Fairness

The amount of payoff that is considered *fair* in this first notion of fairness must reflect on the *relative value* of the players.

5.1. Definition. We define the following relation on the set of players:

$$i \succ j \quad :\Longleftrightarrow \quad \big(\forall A \subseteq N \setminus \{i, j\}\big)\, v(A \cup \{i\}) \geq v(A \cup \{j\}) \tag{1}$$

In that case we say that player i **rivals** player j in the sense that he is *at least as good as* player j. That means that coalitions with i, but without j, are at least as successful as the coalitions obtained by replacing i with j. Two players that rival each other, are **coequal players**. We write $i \sim j$ for $i \succ j \succ i$.

5.2. Proposition. *This relation $\succ$ is a pre-order on the set of players. That is, it is reflexive and transitive.*

The proof of this Proposition is done using simple arithmetic.

By subtracting $v(A)$ on both sides of the equation (1) we find that a player i rivals player j *iff* his possible contribution to a coalition A that initially contains neither of the two is always greater than or equal to the contribution that player j could make.

5.3. Definition. The **relative player value fairness** concept is given by the predicate $\mathcal{F}_{\succ}(v) \subseteq I(v)$ where an imputation $x \in \mathcal{F}_{\succ}(v)$, if we find that $x_i \geq x_j$ holds for each $i \succ j$, that is, if the distribution of the payoff reflects the relation of rivalry.

We will now check that $\mathcal{F}_{\succ}$ is satisfiable for all games.

We start out by introducing some helpful notation and then prove two lemmata.

5.4. Definition. For $N = \{1, \ldots, n\}$ and $i, j \in N$ we define the permutation $\sigma_{\{i,j\}} : N \longrightarrow N$ to map j to i, i to j and k to k for all $k \notin \{i, j\}$.

5.5. Lemma. *Let $c \in C(v)$ be a core imputation that is unfair in the sense that $i \succ j$, but $c_i < c_j$. Then $c_{\sigma_{\{i,j\}}(k)}$ is a core imputation as well.*

Proof. With $v(\{i\}) \geq v(\{j\})$ it is obvious that $c_{\sigma_{\{i,j\}}(k)}$ is still an imputation.
It remains to show, that all inequalities of the form $v(K) \leq \sum_{k \in K} c_{\sigma_{\{i,j\}}(k)}$ hold. For $i, j \notin K$ and for $i, j \in K$ we still find that $v(K) \leq \sum_{k \in K} c_k = \sum_{k \in K} c_{\sigma_{\{i,j\}}(k)}$.
For $i \in K, j \notin K$ we have $v(K) \leq \sum_{k \in K} c_k = c_i + \sum_{k \in K \setminus \{i\}} c_k < c_j + \sum_{k \in K \setminus \{i\}} c_k = \sum_{k \in K} c_{\sigma_{\{i,j\}}(k)}$
and for $i \notin K, j \in K$ we check that $v(K) \leq v((K \setminus \{j\}) \cup \{i\}) \leq \sum_{k \in K \setminus \{j\} \cup \{i\}} c_k = \sum_{k \in K} c_{\sigma_{\{i,j\}}(k)}$, which concludes the proof of the lemma. □

5.6. Lemma. *Let $c \in C(v)$ be a core imputation and $s_1 \sim s_2 \sim \ldots \sim s_k$, that is the k players s_i are coequals.*
Assume that there are $i, j \in \{1, \ldots k\}$ with $c_{s_i} \neq c_{s_j}$, i.e. the imputation c does not satisfy the relative player value fairness criterion for this coequality.
Then $d \in C(v)$ with $d_k := c_k$ for $k \notin \left\{s_i \,\middle|\, i = 1, \ldots, k\right\}$ and $d_{s_i} := \frac{1}{k} \sum_{i=1}^{k} c_{s_i}$ for all $i = 1, \ldots, k$ is also a core imputation.

Proof. The vector d is an imputation since $\sum_{i \in N} d_i = \sum_{i \in N} c_i$ and for all s_i the value d_{s_i} is at least as big as $\min c_{s_i} > v(\{s_i\}) = v(\{s_1\})$.
Let $A \subset \{1, \ldots, k\}$ be a subset of the coequal players with $\frac{1}{|A|} \sum_{i \in A} c_{s_i} > \frac{1}{k} \sum_{i=1}^{k} c_{s_i}$, that is the players from A get an above-average payoff in imputation c relative to their coequals in $\left\{s_i \,\middle|\, i \notin A\right\}$.
Then there exists a subset $\mathcal{A} \subset \{1, \ldots, k\}$ with $|\mathcal{A}| = |A|$ and $\frac{1}{|\mathcal{A}|} \sum_{i \in \mathcal{A}} c_{s_i} < \frac{1}{k} \sum_{i=1}^{k} c_{s_i}$, i.e. the c payoff for the players from $\mathcal{A}$ is below the average (simply let $\mathcal{A}$ consist of the indices of the $|A|$ players with the smallest payoff in c).
With that in mind, we prove for all $K \subseteq N$ that the inequality $v(K) \leq \sum_{x \in K} d_x$ holds.
For $K \subseteq N$ we have either $\sum_{x \in K} c_x \leq \sum_{x \in K} d_x$ or $\sum_{x \in K} c_x > \sum_{x \in K} d_x$.
The first case yields $v(K) \leq \sum_{i \in K} c_i \leq \sum_{i \in K} d_i$. In the second case we obviously have $\{1, \ldots k\} = A \cup B$ with $A \subset K$ and $B \subset K^{\mathsf{C}}$ and A satisfying $\frac{1}{|A|} \sum_{i \in A} c_{s_i} > \frac{1}{k} \sum_{i=1}^{k} c_{s_i}$.
Now let $\mathcal{A}$ be as above and consider $K' := K \setminus \left\{s_i \,\middle|\, i \in A\right\} \cup \left\{s_i \,\middle|\, i \in \mathcal{A}\right\}$. Then we have $v(K) = v(K') \leq \sum_{i \in K'} d_i = \sum_{i \in K \setminus \{s_i \mid i \in A\}} d_i + |\mathcal{A}| \cdot \frac{1}{k} \sum_{i=1}^{k} c_{s_i} = \sum_{i \in K} d_i$. □

With these two lemmata in place we can prove the following theorem:

5.7. Theorem. *Relative player value fairness is satisfiable for all cooperative n-person games. That is to say: For a cooperative n person game (N, v) with nonempty core $C(v) \neq \emptyset$ the intersection $C(v) \cap \mathcal{F}_{>}(v)$ is nonempty as well.*

Proof. Starting with any core element x we can construct a relative player value fair core element in a few steps by solving all the violations of fairness, for which we use the two lemmata.

Our strategy for this is:

First of all we look at the pairs of players a and b where a rivals b but they are not coequal, that is $a > b$, but not $b > a$. For these pairs we ensure that $x_a \geq x_b$ holds by sorting individual payoffs according to lemma 5.5. This is possible and also easy to do; every computer programmer knows a whole range of algorithms which do just that.

For chains of coequal players we now use Lemma 5.6 to reach fairness. The order we created relative to other non-coequal players in the first step is not destroyed in the process, because convex combinations of several real numbers are always below the largest and above the smallest of these numbers. □

5.8. Example. *We consider an example of an exchange economy (see [Drie 88] or [Krab 05]). In this example the payoff function is given by $v(K) = \min\{|K \cap P|, \alpha|K \cap Q|$ for all $K \subseteq N$ where $P \cup Q = N, P \cap Q = \emptyset, \frac{1}{2} \leq \alpha \leq 1$ are given. An interpretation of the payoff function is the following: the players in P and Q respectively own two different types of goods which can be used to generate value, when brought together in the relation $\alpha^{-1}\ :\ 1$.*

We assume $n = 3, P = \{1\}, Q = \{2, 3\}$.and obtain:

$$v(N) = 1, v(\{1,2\}) = v(\{1,3\}) = \alpha, v(\{2,3\}) = 0,$$
$$v(\{1\}) = v(\{2\}) = v(\{3\}) = 0.$$

Let $\alpha = \frac{1}{2}$, then the greatest payoff is obtained for the grand coalition N. Core imputations are all those positive vectors $x = (a, b, c)^T$ that satisfy $a + b + c = 1, a + b \geq \frac{1}{2}$ and $a + c \geq \frac{1}{2}$.

The Shapley value of the game and its τ-value are both given by $(\frac{1}{2}, \frac{1}{4}, \frac{1}{4})^T$, which is also the only element of the least-core.

$LS(v)$ is given by $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})^T$, which is the only strongly egalitarian imputation, while the egalitarian core also includes $(\frac{1}{2}, \frac{1}{4}, \frac{1}{4})^T$ and the convex hull of these two points.

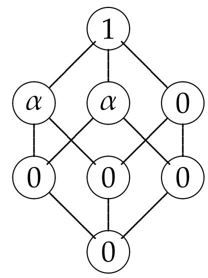

Figure 1: The 3 Person Exchange Economy

Relative player value fair solutions of the game are given by

$$C(v) \cap \mathcal{F}_{\succ}(v) = \left\{ (a,b,b)^T \in \mathbb{R}^3 \;\middle|\; a + 2b = 1, \tfrac{1}{3} \leq a \leq 1 \right\},$$

since we have $1 \succ 2 \sim 3$. *Examples for relative player value unfair core solutions are* $(0, 0.5, 0.5)^T$ *and* $(0.5, 0.5, 0)^T$.

5.9. Theorem. *The* τ*-value, the Shapley-value and the elements of* $C_e(v)$ *are elements of* $\mathcal{F}_{\succ}(v)$*, whenever they exist.*

Proof. For $i \succ j$ we have $u_i \geq u_j$ and $\lambda_i \leq \lambda_j$, hence $\tau(v)_i \geq \tau(v)_j$.

Also for $i \succ j$ player i has larger marginal contributions and thus his Shapley value is not smaller than player j's Shapley value.

Now let $x \in C_e(v) \setminus \mathcal{F}_{\succ}(v)$. So there are two players i and j with $i \succ j$ and $x_j = x_i + 2 * \alpha$ for some positive α.

Let x' be the result of the equalizing bilateral transfer of size α from player j to player i.

Then, since $x \in C(v)$ and (with Lemma 5.5) $c_{\sigma_{\{i,j\}}(k)} \in C(v)$, with the convexity of the core it follows that $x' \in C(v)$.

For any $x \in C_e(v)$ however there are no core imputations that are the result of equalizing bilateral transfers, thus we indeed have $C_e(v) \subseteq \mathcal{F}_{\succ}(v)$. □

5.2 Split Relative Player Value Fairness

One shortcoming of the relative player value fairness concept is that it is not compatible with splitting games. The intuitive equation $\mathcal{F}_{\succ}(v_0) + s = \mathcal{F}_{\succ}(v)$ does not necessarily hold, as the following example shows.

5.10. Example. *Let $n = 2, v(\{1\}) = 0, v(\{2\}) = 1, v(\{1,2\}) = 2$. Then $v_0(\{1,2\}) = 1$ and $v_0(\{i\}) = 0$ for $i = 1,2$. Thus $\mathcal{F}_{>}(v_0)$ is the singleton set containing only the equal distribution $(0.5, 0.5)^T$, while $\mathcal{F}_{>}(v)$ contains not only $(0.5, 1.5)^T$, but also $(0,2)^T$ and the egalitarian value $(1,1)^T$.*

Taking the Exchange Economy game from example 5.8 as v_0 and adding $s = (0, 0.5, 0.5)^T$ one arrives at an example where neither $\mathcal{F}_{>}(v_0) + s \subseteq \mathcal{F}_{>}(v)$ nor $\mathcal{F}_1(v_0) + s \supseteq \mathcal{F}_1(v)$ hold.

We can however define a fairness predicate that splits in the following way.

5.11. Definition. We define the set of **split relative player value fair** imputations of a game $(N.v)$ to be the intersection of $\mathcal{F}_{>}(v)$ and $\mathcal{F}_{>}(v_0) + s$, where $v = v_0 + s$, i.e. v_0 is the purely cooperative component of v and s is its trivial component.

$$\mathcal{F}^s_{>}(v) = \mathcal{F}_{>}(v) \cap \mathcal{F}_{>}(v_0) + s.$$

In order to prove that this predicate is satisfiable for all games, we first prove the following Lemma.

5.12. Lemma. *Let (N, v) be a cooperative n-person game and ϵ be a positive real number.*

Suppose that $i+\epsilon > j$, in the sense that for every $K \subseteq N \backslash \{i, j\}$ we have $v(K \cup \{i\}) + \epsilon \geq v(K \cup \{j\})$.

Let further $x \in C(v)$ with $x_j > x_i + \epsilon$.

We define $\alpha := x_j - x_i - \epsilon$ and the imputation x' through

$$(x')_j = x_j - \frac{\alpha}{2}$$
$$(x')_i = x_i + \frac{\alpha}{2}$$
$$(x')_k = x_k \text{ for all } k \notin \{i, j\}.$$

In these circumstances we have $x' \in C(v)$.

Proof. For $K \subseteq N, \{i, j\} \subseteq K$ we have $x_K = (x')_K$ and $x \in C(v)$, so the vector x' does not violate the core condition for coalition K. The same holds for $K \subseteq N \setminus \{i, j\}$.

For $i \in K, j \notin K$ we have $(x')_K \geq x_K \geq v(K)$.

Now for $i \notin K, j \in K$ we have $(x')_K = (x')_{K\setminus\{j\}} + (x')_j \geq (x')_{K\setminus\{j\}} + \epsilon + (x')_i \geq \epsilon + v(K \setminus \{j\} \cup \{i\}) \geq v(K)$.

The last inequality holds since $i + \epsilon \succ j$ and the one before that is true since the core condition holds for $j \notin K$, as we have seen. $\square$

Now we can prove the Theorem.

5.13. Theorem. *The predicate $\mathcal{F}^s_\succ$ is satisfiable for all games, since $C_e(v_0) + s \subseteq \mathcal{F}^s_\succ(v)$.*

Proof. We already know that $C_e(v_0) \subseteq \mathcal{F}_\succ(v_0)$ (see Theorem 5.9). Thus it remains to show that for every $x \in C_e(v_0)$ the core imputation $x+s$ is always a member of $F_\succ(v)$.

Now let $i \succ j$ and assume that we have $(x+s)_i < (x+s)j$. It follows from $i \succ j$ that $s_i \geq s_j$. Define $\sigma := s_i - s_j$.

Let us look at the game (N, v_0) and note that in this game we have $i + \sigma \succ j$ (using the notation we introduced in Lemma 5.12).

Since $x_i \leq x_j + \sigma$, it follows from that Lemma that the result of an equalizing bilateral transfer of size $0.5 * (x_j - x_i - \sigma)$ from player j to player i would be element of the core $C(v_0)$, which obviously contradicts $x \in C_e(v_0)$.

Thus the assumption $(x+s)_i < (x+s)j$ was wrong and we have $i \succ j \Longrightarrow (x+s)_i \geq (x+s)_j$. $\square$

5.14. Theorem. *The τ-value and the Shapley-value are elements of $\mathcal{F}^s_\succ(v)$, whenever they exist.*

The elements of $C_e(v)$ are, in general, not.

Proof. Both the τ-value and the Shapley value split. In addition they are both relative player value fair. Hence they are both split relative player value fair.

Consider $n = 2$ with $v(\{1\}) = 1, v(\{2\}) = 0, v(\{1,2\}) = 2$, then the only egalitarian value is $(1,1)^T$, while $\mathcal{F}^s_\succ(v) = \{(1.5, 0.5)^T\}$, which shows that, in general, $C_e(v)$ is not included in $\mathcal{F}^s_\succ(v)$. $\square$

5.3 Set Value Fairness

Note that in the Exchange Economy game from example 5.8 the imputation $(1,0,0)^T$ is a relative player value fair core imputation. Player 1 is the dominant player of the game, thus the concept of relative player value fairness does not prevent him from taking all the profit.

Players 2 and 3 of the game might reason that *together* they are worth no less than player 1 and demand equal shares between him and the two of them. This fairness consideration can be modelled in the following way

5.15. Definition. We define the following relation on the set of coalitions:

$$K \sqsupseteq L \quad :\Longleftrightarrow \quad \Big(\forall A \subseteq N \setminus (L \cup K)\Big)\, v(A \cup K) \geq v(A \cup L) \tag{2}$$

In that case we say that coalition K **rivals** coalition L in the sense that it is *at least as good as* coalition L.

This relation $\sqsupseteq$ is a pre-order on the powerset lattice of N.

5.16. Definition. The **set value fairness** concept is given by the predicate $\subseteq I(v)$ where $x \in \mathcal{F}_{\sqsupseteq}(v)$, if we find that $\sum_{k \in K} x_k \geq \sum_{l \in L} x_l$ holds for each $K \sqsupseteq L$, that is, if the distribution of the payoff reflects the relation of coalitional rivalry.

It turns out, that set value fairness is not a satisfiable predicate on the class of super-additive games, as the following superadditive counterexample shows:

5.17. Example. *Consider the three player Exchange Economy once again, but this time take* $\alpha = 0.9$*. We call this game* **Overemployment Game** *since it can be interpreted in the following way:*

Player 1 is an employer who can make good profit (total payoff=0.9) by employing an employee. There are however two equivalent employees that offer to work at his firm. If he employs both, the result is even better (total payoff of 1), but the marginal contribution of the second worker employed is small.

Any imputation that pays more than 0.1 to either worker is not element of this game's core, since $u_2 = u_3 = 0.1$*. However we have* $\{1\} \sqsupseteq \{2,3\} \sqsupseteq \{1\}$*, so the only imputation in* $\mathcal{F}_{\sqsupseteq}(v)$ *is* $(0.5, 0.25, 0.25)^T$.

The same example shows that the τ-value, the Shapley-value and egalitarian solutions are not always set value fair.

5.4 Labor Union Fairness

We have not yet been able to prove or disprove satisfiability of set value fairness for convex games. However a weaker form of fairness, called labor union fairness, is satisfiable for convex games.

5.18. Definition. The **set value fairness** concept is given by the predicate $\subseteq I(v)$ where $x \in \mathcal{F}_{lu}(v)$, if we find that $\sum_{k \in K} x_k \geq x_l$ holds for each $K \sqsupseteq \{l\}$, where K is a class of coequal players.

Obviously we have $\mathcal{F}_{\sqsupseteq}(v) \subseteq \mathcal{F}_{lu}(v) \subseteq \mathcal{F}_{>}(v)$. While labor union fairness does not compare any set of likely or unlikely coalitions it ensures that a labor union of coequals do not receive less in sum than any individual player *(manager)*, whom they, as a labor union, rival, does.

If players observe that they are coequals and agree to share a fairness notion that guarantees them equal payoffs, it is quite natural to assume that they would act as a sort of labor union, trying to maximize their individual outcome by increasing the outcome of this labor union. This is why labor unions of coequal players might seem more natural than mixed labor unions.

In order to show that $\mathcal{F}_{lu}$ is satisfiable for all convex games, we first show a Lemma which is pretty intuitive geometrically:

5.19. Lemma. *Let (N, v) be a convex cooperative n-person game and $I = \{i_1, i_2, ..., i_r\}$ be a set of coequal players. Let $x \in \mathcal{F}_{>}(v) \cap C(v)$ be a relative player value fair core element.*

Let further $K_0 \subseteq N \setminus I, K_1 = K_0 \cup \{i_1\}, K_2 = K_1 \cup \{i_2\}, ..., K_r = K_0 \cup I$.

For a coalition K let us write $x(K)$ for the sum $\sum_{k \in K} x_k$.

If there exists an $a \in \{1, ..., r-1\}$ with $v(K_a) = x(K_a)$, it follows that $v(K_s) = x(K_s)$ for all $s \in \{0, ..., r\}$.

Proof. We will first see that for all $s > a$ the equation holds.

Assume that there is some $s \geq a$ with $v(K_s) < x(K_s)$. In that case let $l := \min\left\{l' \in \{1, ...r-a\} \,\middle|\, v(K_{a+l'}) < x(K_{a+l'})\right\}$.

Then obviously $MC_{i_{a+l}}(K_{a+l-1}) < x_{i_{a+l}} = x_{i_a}$.

But also, since $x \in C(v)$, we have $v(K_{a-1}) \leq x(K_{a-1})$, thus $MC_{i_a}(K_{a-1}) \geq x_{i_a}$.

Convexity ensures that $MC_{i_a}(K_{a-1}) \leq MC_{i_a}(K_{a+l} \setminus \{i_a\}) = MC_{i_{a+l}}(K_{a+l-1})$.

Concluding we have $x_{i_a} \leq MC_{i_a}(K_{a-1}) = MC_{i_{a+l}}(K_{a+l-1}) < x_{i_a}$, which is a contradiction.

The proof for all $s < a$ is the same. □

We now implicitly prove satisfiability for $\mathcal{F}_{lu}$ in the following Theorem.

5.20. Theorem. *$C_e(v) \subseteq \mathcal{F}_{lu}(v)$ for convex games (N, v).*

Proof. Let $[i]_\sim = \{i_1, ..., i_r\}$ and suppose $[i]_\sim \sqsupseteq \{j\}$.

Let further $x \in C_e(v)$.

Suppose that $r \cdot x_i \leq x_j$. Then for sure $x_i < x_j$ and thus (since x is egalitarian) there exists no egalizing bilateral transfer of any size from j to i without leaving the core. In other words, every transfer leaves the core and thus there is a coalition K with $i \notin K$, $j \in K$ and $v(K) = x(K)$.

Then, with the Lemma 5.19, we can assume that K does not meet $[i]_\sim$.

Now we have $v(K \cup [i]_\sim \setminus \{j\}) \geq v(K) = x(K)$ and since $rx_i \leq x_j$ we also have $x(K) \geq x(K \cup [i]_\sim \setminus \{j\})$.

On the other hand $x \in C(v)$, thus $v(K \cup [i]_\sim \setminus \{j\}) \leq x(K \cup [i]_\sim \setminus \{j\})$, thus $v(K \cup [i]_\sim \setminus \{j\}) = x(K) = x(K \cup [i]_\sim \setminus \{j\})$.

It follows that $rx_i = x_j$. $\square$

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