

Cooperation, Imitation and Correlated Matching

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April 22, 2009

Abstract

A setting where players are matched into pairs to play a Prisoners' Dilemma game is studied. Players are not rational in that they simply imitate the more successful actions they observe. Furthermore, a certain correlation is added to the matching process: players belonging to a pair where both parties cooperate repeat partner next period while all other players are randomly matched into pairs. While under complete random matching cooperation vanishes for any initial interior condition, the correlation in the matching process considered in this paper makes a significant amount of cooperation the unique outcome under mild conditions. Furthermore, it is shown that no separating equilibrium, i.e. a situation where cooperators and defectors are not matched together, exists.

JEL Classification: C71, C73, C78.

Keywords: Cooperation, Correlated Matching, Imitation, Prisoners' Dilemma.

*I would like to thank Subir Bose, Gianni de Fraja, Friederike Mengel, Mark Le Quement, Carlos Oyarzun and Ludovic Renou for comments and suggestions. I would also like to thank the attendants to the Internal Seminar series at the University of Leicester for useful comments.

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1 INTRODUCTION

Individuals learn by imitation when their choices are based on the success of the alternatives they observe others choose. Under mild conditions a population whose behavior follows imitation learns not to play dominated actions (see, for instance, Schlag (1998) Remark 6). Thus, if every period imitative players are randomly matched to play a Prisoner's Dilemma game cooperation vanishes in the long run. Given the importance of cooperation and its constant presence in societies (see Axelrod (1984)) and the relevance of imitation for modeling bounded rational behavior (see, for example, Banerjee (1992), Eshel et al (1998) or Ellison and Fudenberg (1995)) the question we raise is: can cooperation survive when players learn by to imitation?

We answer this question by exploring the mechanism by which players in a population are matched to play a Prisoners' Dilemma game. In particular, the novelty of this paper is that a certain correlation is introduced to the matching process: players who cooperated with each other last period meet again in the next period whilst the rest of players are randomly matched into pairs. This matching mechanism captures the simple idea that a player should have no incentives of repeating partner unless the partner played cooperatively last period.

The correlated matching process considered in this paper adds a positive externality to playing cooperatively: in a situation where two players are playing cooperatively, switching action has the disadvantage that next period a new opponent, who might not be so keen on playing cooperatively, is faced. However, this argument is rational, do the previous statement holds when players simply imitate each other? The answer is positive. To understand this consider the simple situation where the population is divided into two groups: players who play cooperatively and are matched with another one also playing cooperatively and players who do not play cooperatively and are matched with a player playing likewise. The way matching works implies that pairs where both players cooperate repeat partner. Therefore, in this case, players that play cooperatively always enjoy more payoff than these not playing cooperatively. Hence, non-cooperative players may imitate a cooperative ones, making the survival of cooperation possible.

In the results of this paper three main conclusions are achieved: First, under mild conditions and for any interior initial condition the survival of a non-negligible amount of cooperation is guaranteed. That is, the situation where no player cooperates is not stable if mild conditions on the payoff matrix and/or the specific imitative rule employed are satisfied. Second, no separating equilibrium exists. This means that, apart from the equilibria on the boundaries, a situation where cooperative players do not face non-cooperative ones is not an equilibrium. Finally, we show that all players cooperating is an stable outcome under certain

conditions.

To our knowledge, only Levine and Pesendorfer (2007) and Bergstrom (2003) study similar settings to the one considered in this paper. Levine and Pesendorfer (2007) show that cooperation can survive within a group of players who learn by imitation if the encounters are non-anonymous, meaning that each player holds some information about the strategy of the player with whom she is matched. Bergstrom (2003) proves conditions under which cooperation survives in an evolutionary model where players are of two types, cooperator or defector, and are more likely to be matched with another player of the same type.

The difference between this paper and Levine and Pesendorfer (2007) lies in that in our model there is a set of matches that are anonymous whilst in Levine and Pesendorfer (2007) all matches are non-anonymous to a certain degree. The present paper differs from Bergstrom (2003) in that players can change their actions from one period to another. Thus, playing cooperatively in the present period is no guarantee of exhibiting a cooperative behavior in the next period.

The issue of partner selection in cooperative games has recently attracted attention from experimental economists. Duffy and Ochs (2009) conduct an experiment where a Prisoners Dilemma game with two treatments is considered. In the first treatment matching is completely random whereas in the second one each player always repeats partner. The authors find that cooperation does not emerge in the random matching setting while it does in the fixed pairs treatment. Yang et al (2007) present an experiment where a Prisoner Dilemma game is played and individuals with similar histories are more likely to be matched together. Results show that cooperation does have higher chances of survival when this history-dependent correlation is added to the matching process. Grimm and Mengel (2009) develop an experiment where players choose between two Prisoner's Dilemma games that differ in the gains from defection. Choosing the game with lower gains signals the player's willingness to cooperate. Grimm and Mengel find that this self selection significantly increases the amount of cooperation.

The rest of the paper is organized as follows. In Section 2 we develop the model. Section 3 presents the main analysis and the results. In Section 4 we discuss the robustness of the results and the assumptions as well as present some extensions. Finally, Section 5 concludes.

2 THE MODEL

Consider a continuum of identical players uniformly distributed on the interval $[0, 1]$ with the standard Borel-Lebesgue measure. At the beginning of each period $t = 0, 1, 2, \dots$ every

player in the population is paired with another one and plays the following symmetric stage game against her partner:

Table 1: The Stage Game

	C	D
C	a, a	b, c
D	c, b	d, d

Where C stands for cooperate and D stands for defect. The stage game above has the standard Prisoners' Dilemma structure: $c > a > d > b$ with $a, b, c, d \in \mathbb{R}$.

After the stage game is played all the pairs where at least one player chose D are broken while the rest of pairs, i.e. where both players chose C , are maintained. After that, unpaired players are randomly matched into pairs. The distribution of pairs at the beginning of period $t = 0$ is given.

Given the description above, at the beginning of each period the population is divided into three sets: players who played C last period and were matched with an opponent who also played C last period, γ , players who played C but were facing an opponent who played D last period, σ , and the rest, denoted by $1 - \gamma - \sigma$. We use γ , σ and $1 - \gamma - \sigma$ to denote exchangeably both the sets and their respective measure. Thus, for instance, γ is both the set of players who played C last period and were matched with an opponent who also played C and the measure (fraction) of players who played C last period and were matched with an opponent who also played C . Therefore, we have that $\gamma \in [0, 1]$, $\sigma \in [0, 1]$ and $1 - \gamma - \sigma \in [0, 1]$. Evidently, $\sigma + \gamma \leq 1$ with equality only in the case when $\gamma = 1$ (if $\sigma + \gamma = 1$ then all players played C and thus all players were matched with another one playing C). Note that the fraction of population who maintain partner equals γ . Furthermore, note that all players in σ are matched with a player in $1 - \gamma - \sigma$ and, thus, $\sigma < 1 - \gamma - \sigma$.

Players in the population follow very simple decision rules. In particular, they observe the action and payoff of a random individual from the population and base their choice of action for the stage game on this information plus the information from own action and payoff. All players in the population are equally likely to be observed. Results presented in this paper do not depend on how many players are observed since we are dealing with a continuous population.

Let $A \in \{C, D\}$ be the action set and let $P(\{i, a_i, \pi_i\} \{j, a_j, \pi_j\}) \in [0, 1]$ be the probability of changing action for player $i \in [0, 1]$ if she, who played action $a_i \in A$ and obtained payoff $\pi_i \in \mathbb{R}$, observes player $j \in [0, 1]$ who chose action $a_j \in A$ and achieved payoff $\pi_j \in \mathbb{R}$. Some

assumptions on the probabilities of changing action are needed for the analysis.

Assumptions.

1. If $a_i = a_j$ then $P(\{i, a_i, \pi_i\}\{j, a_j, \pi_j\}) = 0$,
2. $P(\{i, a_i, \pi_i\}\{j, a_j, \pi_j\}) > 0$ if and only if $\pi_i < \pi_j$ and,
3. for all $i, j \in [0, 1]$ and all $a_i, a_j \in A$:
 - if $\pi_j > \pi'_j$ then $P(\{i, a_i, \pi_i\}\{j, a_j, \pi_j\}) \geq P(\{i, a_i, \pi_i\}\{j, a_j, \pi'_j\})$,
 - if $\pi_i < \pi'_i$ then $P(\{i, a_i, \pi_i\}\{j, a_j, \pi_j\}) \geq P(\{i, a_i, \pi'_i\}\{j, a_j, \pi_j\})$.

The first two assumptions are standard in imitation models (see, for instance, Schlag (1998)). Assumption 1 implies that players update their choice of action only if the player they observe played a different action than the one they chose. Assumption 2 means that there is a positive probability of adapting a different action if and only if the observed action yielded more payoff than own action. The third assumption is a monotonicity condition that relates to reinforcement learning models (see, for example, Börgers et al (2004) and Rustichini (1999)). It means that the probability of changing action is weakly increasing in observed payoff and weakly decreasing in own payoff.

We simplify notation when using the function $P(\{i, a_i, \pi_i\}\{j, a_j, \pi_j\})$ as follows: Denote by $P_D^\gamma : A^2 \times \mathbb{R}^2 \rightarrow [0, 1]$ the probability with which a player in γ changes to playing D . Let $P_D^\sigma : A^2 \times \mathbb{R}^2 \rightarrow [0, 1]$ be the probability with which a player who belongs to σ , changes to playing D . Finally, denote by $P_C^{1-\gamma-\sigma} : A^2 \times \mathbb{R}^2 \rightarrow [0, 1]$ the probability with which a player in $1 - \gamma - \sigma$ changes to playing C .

Assumptions 1–3 impose some restrictions on the functional forms of P_D^γ , P_D^σ and $P_C^{1-\gamma-\sigma}$. Define Ω as $\Omega = \{(\gamma, \sigma) \in \mathbb{R}_+^2 : \gamma + \sigma < 1 \cup (\gamma, \sigma) = (1, 0)\}$ and consider that the system is at any point in Ω excluding $(1, 0)$. The function P_D^γ is only positive if the player in γ observes a player in $1 - \gamma - \sigma$ (Assumption 1) that was matched with a player in σ (Assumption 2). In this case, the payoff of the observed player equals c while own payoff equals a . Thus, we can write P_D^γ as

$$P_D^\gamma = (1 - \gamma - \sigma) \frac{\sigma}{1 - \gamma} f(c, a) \quad (1)$$

for some function $f : \mathbb{R}^2 \rightarrow [0, 1]$. The two arguments in f are observed payoff and own payoff respectively. The function f is weakly increasing in its first argument and weakly decreasing in its second argument by Assumption 3 and, by Assumption 2, has $f(\pi', \pi) = 0$ for any $\pi' > \pi$.

The function P_D^σ is only positive if the player in σ observes a player in $1 - \gamma - \sigma$. In this case, two different situations arise: if the player observed was matched with a player in σ then the observed payoff equals c and own payoff equals b , on the other hand, if the observed player faced a player in $1 - \gamma - \sigma$ then observed payoff equals d and own payoff equals b . Therefore, we have that

$$P_D^\sigma = (1 - \gamma - \sigma) \left[\frac{\sigma}{1 - \gamma} f(c, b) + \frac{1 - \sigma - \gamma}{1 - \gamma} f(d, b) \right]. \quad (2)$$

Finally, $P_C^{1-\gamma-\sigma}$ is only positive if the player in $1 - \gamma - \sigma$ was matched with a player also in $1 - \gamma - \sigma$ and the observed player belongs to γ . In this case observed payoff equals a while own payoff equals d . Hence, we have that

$$P_C^{1-\gamma-\sigma} = \gamma \frac{(1 - \gamma - \sigma)}{1 - \gamma} f(a, d). \quad (3)$$

Let γ^t and σ^t denote the values of γ and σ respectively at each point in time $t = 0, 1, 2, \dots$ before the stage game is played with $(\gamma^0, \sigma^0) \in \Omega$ given. At $t = 0$ and prior to the starting of the game all players not in γ^0 are randomly and uniformly matched into pairs. For notational convenience the argument t in the functions P_D^γ, P_D^σ and $P_C^{1-\gamma-\sigma}$ is omitted.

The model just described gives rise to the following system of difference equations when $\gamma < 1$:

$$\begin{aligned} \sigma^{t+1} = & \gamma^t (1 - P_D^\gamma) + \sigma^t (1 - P_D^\sigma) + (1 - \gamma^t - \sigma^t) P_C^{1-\gamma-\sigma} \\ & - \gamma^t (1 - P_D^\gamma)^2 - \frac{(\sigma^t (1 - P_D^\sigma) + (1 - \gamma^t - \sigma^t) P_C^{1-\gamma-\sigma})^2}{1 - \gamma^t}, \end{aligned} \quad (4)$$

$$\begin{aligned} \gamma^{t+1} = & \gamma^t (1 - P_D^\gamma)^2 \\ & + \frac{(\sigma^t (1 - P_D^\sigma) + (1 - \gamma^t - \sigma^t) P_C^{1-\gamma-\sigma})^2}{1 - \gamma^t}. \end{aligned} \quad (5)$$

Equation (4) tells us the measure of players who played C in period t and were matched with another one playing D in t . The value of σ^{t+1} is computed as follows: The first three terms represent all players who played C in t (note that players in γ^t and σ^t played C in $t - 1$ but may have played D in t). The fourth term subtracts the pairs in γ^t where both players played C again in t . Finally, the fifth term subtracts the players not in γ^t who played C in t and were matched with another one who also played C in t .

Equation (5) is the measure of players who played C in period t and that were matched with another player playing C in t . The value of γ^{t+1} is determined as follows: The first term adds the pairs in γ^t where both players played C in t as well. The second term adds the

players not in γ^t who played C in t and were matched with another player who also played C in t .

Next, we define what an equilibrium of the model at hands is. Intuitively, an equilibrium is a situation where the measure of players belonging to each of the three sets γ , σ and $1 - \gamma - \sigma$ does not change. Formally,

Definition 1. *An equilibrium is a point $(\gamma, \sigma) \in \Omega$ such that $\gamma^{t+1} = \gamma^t$ and $\sigma^{t+1} = \sigma^t$ whenever $\gamma^t = \gamma$ and $\sigma^t = \sigma$.*

Among all equilibria it is useful to single out the separating equilibria. A separating equilibrium is an equilibrium where there is a fraction of the population playing C against themselves, $\gamma \in (0, 1)$, while all the other players choose D , $1 - \gamma - \sigma = 1 - \gamma$. That is, in a separating equilibrium $\sigma = 0$ and the population is completely separated between cooperators and defectors.

Definition 2. *A separating equilibrium is an equilibrium where $\gamma \in (0, 1)$ and $\sigma = 0$.*

Further to the definitions of equilibrium, it is necessary for the analysis to distinguish between the different notions of stability. The following definitions are based on Khalil (1995). Let $B_r(\gamma, \sigma)$ be the ball of radius $r > 0$ around the point (γ, σ) .

Definition 3. *The equilibrium $(\gamma, \sigma) \in \Omega$ is*

- *stable if for any $\varepsilon > 0$ there exists $\delta > 0$ such that if $(\gamma^0, \sigma^0) \in \Omega \cap B_\delta(\gamma, \sigma)$ then $(\gamma^t, \sigma^t) \in \Omega \cap B_\varepsilon(\gamma, \sigma)$ for all $t \geq 0$,*
- *unstable if it is not stable,*
- *asymptotically stable if it is stable and $\delta > 0$ can be chosen such that for any $\kappa < \varepsilon$ if $(\gamma^0, \sigma^0) \in \Omega \cap B_\delta(\gamma, \sigma)$ then*

$$\| \lim_{t \rightarrow \infty} (\gamma^t, \sigma^t) - (\gamma, \sigma) \| < \kappa,$$

- *a repeller if there exists a $\delta > 0$ such that if $(\gamma^0, \sigma^0) \in \Omega \cap B_\varepsilon(\gamma, \sigma)$ for all $\varepsilon \in (0, \delta)$ then $(\gamma^t, \sigma^t) \notin \Omega \cap B_\delta(\gamma, \sigma)$ for some $t \geq 0$,*

Whenever results from simulations are presented the specific imitative behavior players use follows what is known as the Proportional Imitation Rule (PIR henceforth) due to Schlag (1998). According to the PIR an action is adopted with a probability equal to the normalized difference between observed and own payoff. Given that c is the maximum payoff achievable

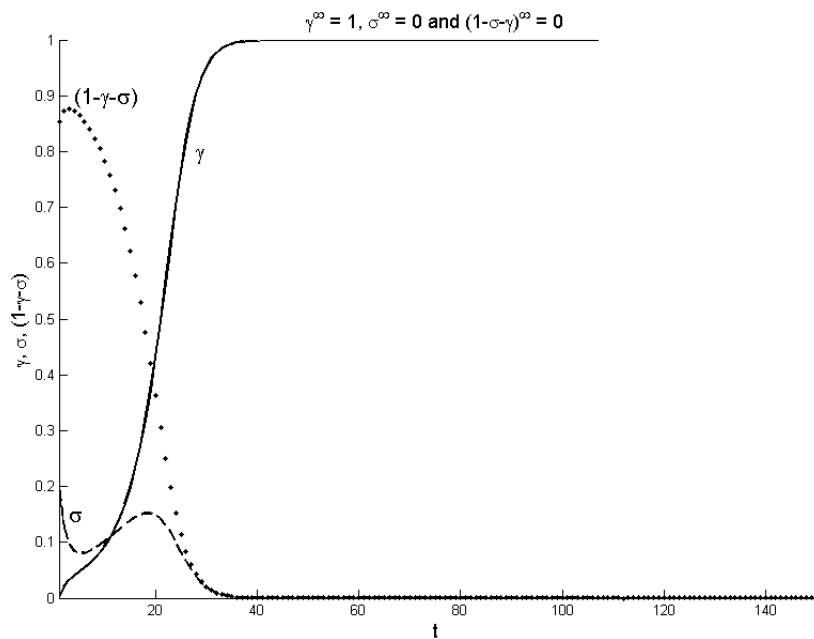
and b is the minimum one, the function f for the PIR when observed action is different from own action and observed payoff, π' , is higher than own payoff, π , is given by

$$f(\pi', \pi) = \frac{1}{c-b} (\pi' - \pi).$$

If $\pi' \leq \pi$ or observed action is equal to own action then $f(\pi', \pi) = 0$.

In order to illustrate the behavior of the model Figure 1 shows the result of a simulation. As it can be seen during the first periods the amount cooperative players matched with non-cooperative ones, σ , decreases. This is due to the fact that during these first stages most cooperative players enjoy less period than cooperative ones. However, as times evolves, more and more cooperative players meet each other. After this grouping stage is over, the payoff from cooperating is on average greater than that from not cooperating. This is due to the fact that most players playing cooperatively are matched with a player also playing cooperatively. The level of cooperation increases from there until all players have adapted the cooperative action.

Figure 1: Simulation: PIR with $a = 0.4$, $b = -0.1$, $c = 0.5$, $d = 0$ and $(\gamma_0, \sigma_0) = (0, 0.2)$



3 RESULTS

3.1 RANDOM MATCHING

In this Subsection the benchmark case of random matching is considered. Under random matching all pairs are broken after the stage game is played. We show that under random matching cooperation vanishes for any interior initial condition. The full analysis of the random matching case is presented in the Appendix, here we restrict our attention to the main result from this analysis.

Proposition 1. *Under random matching, for any initial interior condition*

$$\lim_{t \rightarrow \infty} \gamma^t + \sigma^t = 0.$$

Proof. See Lemma 1 in the Appendix. □

As Proposition 1 shows, under random matching cooperation does not survive in the population. This is the known result that under mild monotonicity conditions (Assumption 3) imitation rules out dominated actions (Schlag (1998)).

With random matching, playing cooperatively is always dominated by the non-cooperative behavior. However, once a correlation is added to the matching process this is no longer the case. If the pairs where both players were cooperative are preserved, then a positive externality to being cooperative is added. When two players play cooperatively they achieve the second highest payoff and, thus, are not too likely to imitate other players. If no player in the pair changes action they keep on achieving the second highest payoff and thus their repeated interaction makes cooperation possible. This is true even though players base their actions for the present period on the information obtained in the last period and not on the information from any previous period.

3.2 CORRELATED MATCHING

We now revert back to the case with correlated matching, i.e. pairs where both players cooperated are maintained in the next period. A first result is that there exist no separating equilibria.

Proposition 2. *No separating equilibrium exists.*

Proof. First, note that from (4) and (5) we have that

$$(1 - \gamma - \sigma) P_C^{1-\gamma-\sigma} - \gamma P_D^\gamma - \sigma P_D^\sigma = 0. \tag{6}$$

The next step is showing that in an equilibrium with $\sigma = 0$ no pairs in γ are ever broken. Assume the contrary, if some pairs are broken that means that some set of players from γ play D . If we are at time t this implies that $\gamma^{t+1} < \gamma^t$ unless some set of players in $1 - \sigma^t - \gamma^t$ switch to C . If this happens, however, we will have that some of them will be matched against players who played D in t . Hence, if a pair is broken either $\gamma^{t+1} < \gamma^t$ or $\sigma^{t+1} > 0$, a contradiction to the definition of separating equilibrium.

Given that in a separating equilibrium no pairs are ever broken and that $\gamma \in (0, 1)$ it follows that all players always play the same action in the stage game. This implies that players in γ^t obtain a payoff of a while players in $1 - \sigma^t - \gamma^t$ obtain a payoff of d . Thus, from Assumptions 2 it follows that $P_D^\gamma = 0$ and $P_C^{1-\gamma-\sigma} > 0$. However, when $P_D^\gamma = 0$ equation (6), implies that

$$(1 - \gamma - \sigma) P_C^{1-\gamma-\sigma} = 0.$$

Since $\gamma \in (0, 1)$, $\sigma = 0$ and $P_C^{1-\gamma-\sigma} > 0$, we have that $(1 - \gamma - \sigma) P_C^{1-\gamma-\sigma} > 0$, a contradiction to the condition above. \square

The intuition behind the result above is straightforward: In a separating equilibrium cooperative players, γ , obtain a payoff of a whilst all the other players, $1 - \gamma$, obtain a payoff of $d < a$. Thus, non-cooperative players imitate cooperative ones but cooperative players do not imitate non-cooperative ones. Therefore, the situation with complete separation between cooperators and defectors is not an equilibrium.

The system of difference equations (4) and (5) is highly nonlinear. In particular, the two expressions on right hand side of both (4) and (5) are polynomials of degree 6 in σ and γ . For studying the behavior of the model at hands we employ what is known as the continuous time approximation (see, for instance, Benaïm and Weibull (2003)). This approximation consists of taking the time step between interactions to zero while the response to each interaction is also taken to zero and at the same rate. Below the continuous time approximation is more deeply explained. Numerical and analytical comparisons between the behavior of the original discrete time model and the behavior of the continuous time approximated model can be found in Section 4.1.

The continuous time approximation is a procedure widely employed in the learning literature (Benaïm and Weibull (2003)). Although this technique is usually applied to eliminate the randomness of a certain model, we apply it here to decrease the order of the difference equations as well as to move from difference equations to differential equations where more tools for the analysis exist.

In the continuous time approximation of the model, each time interval is divided into $1/\delta$ subintervals with $\delta \in (0, 1]$. Thus, the time scale is then $t = 0, \delta, 2\delta, \dots, 1, 1 + \delta, 1 + 2\delta, \dots$. Within each of this subintervals only a fraction δ of the original response to each interaction occurs. That is, if the probability of changing action in the original model is P then the probability of changing action in the continuous time version is given by δP . Furthermore, in the continuous time version of the model only a fraction δ of the couples where both players played cooperatively are maintained. Evidently, if $\delta = 1$ the two models are equivalent. The continuous time version of the model is obtained by taking the limit when δ tends to 0.

The usefulness of the continuous time approximation lies in the fact that given that δ is made arbitrarily small, all the terms of order δ and higher, denoted by $o(\delta)$, are negligible and, hence, can be ignored.

When $\gamma < 1$ the continuous time model is given by

$$\begin{aligned}\sigma^{t+\delta} &= \gamma^t (1 - \delta P_D^\gamma) \delta P_D^\gamma + \sigma^t (1 - \delta P_D^\sigma) + (1 - \gamma^t - \sigma^t) \delta P_C^{1-\gamma-\sigma} \\ &\quad - \delta \frac{\left(\sigma^t (1 - \delta P_D^\sigma) + (1 - \gamma^t - \sigma^t) \delta P_C^{1-\gamma-\sigma} \right)^2}{1 - \gamma^t}, \\ \gamma^{t+\delta} &= \gamma^t (1 - \delta P_D^\gamma)^2 \\ &\quad + \delta \frac{\left(\sigma^t (1 - \delta P_D^\sigma) + (1 - \gamma^t - \sigma^t) \delta P_C^{1-\gamma-\sigma} \right)^2}{1 - \gamma^t}.\end{aligned}$$

Which can be rewritten as

$$\begin{aligned}\frac{\sigma^{t+\delta} - \sigma^t}{\delta} &= \gamma^t P_D^\gamma - \sigma^t P_D^\sigma + (1 - \gamma^t - \sigma^t) P_C^{1-\gamma-\sigma} - \frac{(\sigma^t)^2}{1 - \gamma^t} + o(\delta), \\ \frac{\gamma^{t+\delta} - \gamma^t}{\delta} &= -2\gamma^t P_D^\gamma + \frac{(\sigma^t)^2}{1 - \gamma^t} + o(\delta).\end{aligned}$$

Thus, in the continuous time approximation of the model, where δ is taken to zero, if we write $\dot{x} = \frac{x^{t+\delta} - x^t}{\delta}$ with $x = \{\gamma, \sigma\}$ and drop the superscript t for notational convenience we can rewrite the equations determining the evolution of the population as follows:

$$\begin{aligned}\dot{\sigma} &= \gamma P_D^\gamma - \sigma P_D^\sigma + (1 - \gamma - \sigma) P_C^{1-\gamma-\sigma} - \frac{\sigma^2}{1 - \gamma}, \\ \dot{\gamma} &= -2\gamma P_D^\gamma + \frac{\sigma^2}{1 - \gamma}.\end{aligned}$$

Finally, if we substitute the value of P_D^γ , P_D^σ and $P_C^{1-\gamma-\sigma}$ from equations (1), (2) and (3) into the two equations above and consider points with $\gamma < 1$ we obtain

$$\dot{\sigma} = \frac{1}{1 - \gamma} \left[(1 - \gamma - \sigma) \gamma \sigma f(c, a) - (1 - \gamma - \sigma) \sigma^2 f(c, b) - (1 - \gamma - \sigma)^2 \sigma f(d, b) + (1 - \gamma - \sigma)^2 \gamma f(a, d) - \sigma^2 \right], \quad (7)$$

$$\dot{\gamma} = \frac{1}{1 - \gamma} \left[-2(1 - \gamma - \sigma) \gamma \sigma f(c, a) + \sigma^2 \right]. \quad (8)$$

The convenience of working with the continuous time approximation is clear: we are left with an homogeneous system of two differential equations of order three in two unknowns that is more tractable than the original system of difference equations of order six.

The main result of this paper is stated below. If a certain condition on the payoff matrix and/or the learning rule is satisfied, the existence of a significant amount of cooperation is the only stable equilibrium. Even if the initial amount of cooperators is small, the cooperative behavior will grow popular.

Proposition 3. *From any interior initial condition, if $f(a, d) > 2f(c, a)f(d, b)$ then the equilibrium $(0, 0) \in \Omega$ is a repeller and the system converges to an equilibrium with $\gamma + \sigma > 0$. Furthermore, if $f(a, d) < 2f(c, a)f(d, b)$ then the equilibrium $(0, 0) \in \Omega$ is asymptotically stable.*

Proof. See Lemma 2 in the Appendix. □

For a better understating of the condition in Proposition 3 consider the following example:

Example. *Assume the payoff matrix of the stage game is given by*

Table 2: The Stage Game - Example

	C	D
C	$\pi_b - \pi_c, \pi_b - \pi_c$	$-\pi_c, \pi_b$
D	$\pi_b, -\pi_c$	$0, 0$

with $1 > \pi_b > \pi_c > 0$. In this case we can interpret π_b as the benefit a player receives when her pair cooperates and π_c as the cost of cooperating.

Corollary of Proposition 3. *Assume the stage game is the one given in Table 2 and that players employ the PIR. From any interior initial condition, if $\pi_b > \pi_c\sqrt{3}$ then a significant amount of cooperation is present in the long run.*

The idea behind the survival of cooperation is the following: Imagine a situation where only a small fraction of players cooperate. Some of these players will be matched together so they repeat partner next period. This set of players playing cooperatively and that are matched together obtain the second-maximum payoff, a . Since only very few players cooperate, there is almost no player obtaining the maximum payoff, c . Thus, under certain conditions more non-cooperative players imitate cooperative ones than cooperative players imitate non-cooperative ones.

Even if the condition in Proposition 3 is not satisfied, it may still happen that cooperation survives. As the following result shows, if $f(a, d) > f(c, a)$ then all cooperate is asymptotically stable meaning that if initially a sufficient amount of people cooperate then all the population ends up cooperating.

Proposition 4. *The equilibrium $(1, 0) \in \Omega$ is asymptotically stable if $f(a, d) > f(c, a)$.*

Proof. See Lemma 3 □

To better understand the result in Proposition 4 imagine a situation where almost all players cooperate. In this case, if most defectors are match against other defectors then cooperative players achieve higher payoff than non-cooperative ones. Thus, under certain conditions the amount of cooperators will increase until all players cooperate. Assume, on the other hand, than most defectors are matched with cooperators. In this situation defectors achieve higher payoff than cooperators and, thus, the total amount of cooperation in the population decreases. However, the correlation in the matching process favors matches between cooperators and tends to leave defectors matched with another defectors. If conditions in Proposition 4 are satisfied, the payoff from cooperating eventually surpasses that of non-cooperating and the amount of cooperation increases in the population until all players cooperate.

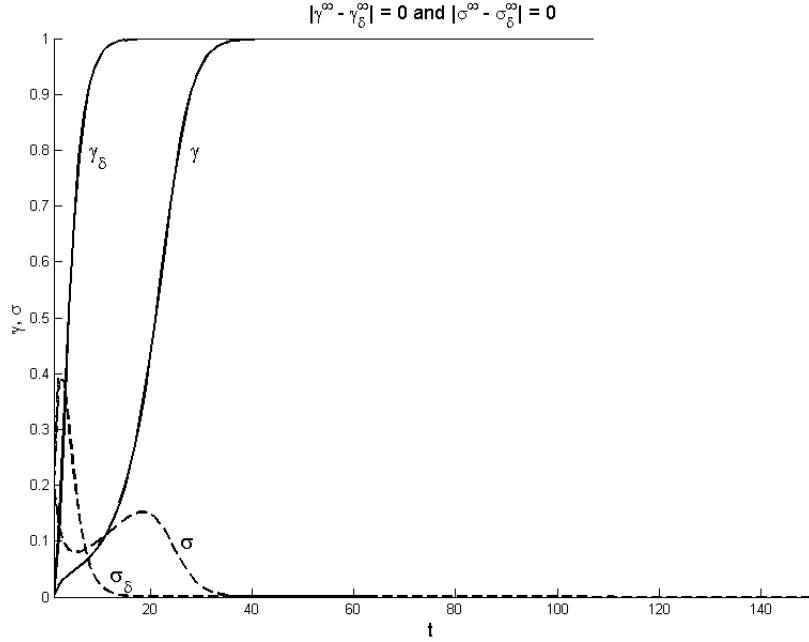
4 ROBUSTNESS CHECKS AND EXTENSIONS

4.1 COMPARISON BETWEEN DISCRETE TIME AND CONTINUOUS TIME

The continuous time version of the model has been used to obtain our main result. Next we examine to which extent this approximation is accurate. Before we present a formal result the two versions of the model are compared using simulations. Figure 2 shows the simulation of both models for the same set of parameters as those employed in Figure 1. The value of δ is set to 0.001.

As one can see in Figure 2, the continuous time model behaves similar to the original discrete time model, specially when close to the boundaries. The fact that the behavior of the two models is almost identical when close to the boundaries is proven below. Since the aim of this paper is to investigate when cooperation can be sustained the fact that both models converge when (γ, σ) is close to $(0, 0)$ allows us to conclude that if cooperation survives in the continuous time version of the model then so it does in the original discrete time model and vice versa.

Figure 2: Simulation: PIR with $a = 0.4$, $b = -0.1$, $c = 0.5$, $d = 0$, $(\gamma_0, \sigma_0) = (0, 0.2)$ and $\delta = 0.001$



Proposition 5. Define $\Delta x = x^{t+1} - x^t$ with $x \in \{\gamma, \sigma\}$ and let $B_r(\gamma, \sigma)$ be the ball of radius $r > 0$ around the point (γ, σ) . For any $\varepsilon > 0$ and any given $(\gamma, \sigma) \in \Omega \cap B_\varepsilon(0, 0)$ we have that if $\gamma^t = \gamma$ and $\sigma^t = \sigma$ then

$$\begin{aligned} |\dot{\gamma} - \Delta\gamma| &\leq o(\varepsilon^2), \\ |\dot{\sigma} - \Delta\sigma| &\leq o(\varepsilon^2). \end{aligned}$$

Furthermore, for any $\kappa > 0$ and any given $(\gamma, \sigma) \in \Omega \cap B_\kappa(1, 0)$ we have that if $\gamma^t = \gamma$ and $\sigma^t = \sigma$ then

$$\begin{aligned} |\dot{\gamma} - \Delta\gamma| &\leq o(\kappa^2), \\ |\dot{\sigma} - \Delta\sigma| &\leq o(\kappa^2). \end{aligned}$$

Proof. If $(\gamma, \sigma) \in \Omega \cap B_\varepsilon(0, 0)$ then $\gamma, \sigma \leq \varepsilon$. Take equations (5) and (7) and substitute P_D^γ, P_D^σ and $P_C^{1-\gamma-\sigma}$ from equations (1), (2) and (3). After some algebra we have that

$$\begin{aligned} |\dot{\gamma} - \Delta\gamma| &\leq -2\gamma P_D^\gamma + 2\gamma P_D^\gamma + o(\varepsilon^2) \\ &= o(\varepsilon^2), \\ |\dot{\sigma} - \Delta\sigma| &\leq -\gamma P_D^\gamma + \gamma P_D^\gamma + o(\varepsilon^2) \\ &= o(\varepsilon^2). \end{aligned}$$

Similarly, when $(\gamma, \sigma) \in \Omega \cap B_\kappa(1, 0)$ we have that $\sigma \leq \kappa$ and that $1 - \gamma - \sigma \leq \kappa$. Proceeding as above we simplify to obtain

$$\begin{aligned} |\dot{\gamma} - \Delta\gamma| &\leq -2\gamma P_D^\gamma + 2\gamma P_D^\gamma + o(\kappa^2) \\ &= o(\kappa^2), \\ |\dot{\sigma} - \Delta\sigma| &\leq \gamma P_D^\gamma + \gamma P_D^\gamma + o(\kappa^2) \\ &= o(\kappa^2). \end{aligned}$$

□

4.2 EXTENSIONS

The long run behavior of the population in our model is determined to a certain extent by the initial condition, i.e. if no player cooperates initially then no player will ever cooperate. This fact disappears as long as mutations or mistakes are introduced in the model. Given that we are dealing with a continuum of population, introducing mistakes is a straightforward task.

Assume that at any given period with a given small probability $\varepsilon > 0$ a player makes a mistake and chooses the action she intended not to. In this case and given that a continuum of population exists, each period exactly a fraction ε of players will make mistakes. More specifically, a fraction $\varepsilon(\gamma + \sigma)$ of players that intended to play C will play D and a fraction $\varepsilon(1 - \gamma - \sigma)$ of players that intended to play D will play C .

In this case, the results presented will still be valid so long as an equilibrium is defined as the situation where for any probability of making mistakes ε , the change in γ and σ is always smaller or equal than $\varepsilon\gamma$ and $\varepsilon\sigma$ respectively. The convenience of adding mistakes is that unstable equilibria are eliminated. That is, in the model with mistakes, if $f(a, d) > 2f(c, a)f(d, b)$ then cooperation emerges independently of the initial conditions.

5 CONCLUSIONS

The present paper investigated cooperation in a setting where players who learn by imitating more successful strategies are matched to play a Prisoner's Dilemma game. The contribution of the present paper to the literature lies in the way matching takes place: only these players belonging to a pair were both cooperated repeat partner while the rest of players are randomly matched into pairs.

In the benchmark case with random matching we showed that cooperation vanishes for any interior initial condition. When moving to the correlated matching setting we proved

that if mild conditions on the payoff matrix and/or the specific way imitation takes place are satisfied then a significant amount of cooperation appears from any initial interior condition. Furthermore, we found that no separating equilibrium exists.

We believe our work is also novel in the way the continuous time approximation is employed. This approximation simplifies calculations in models where randomness plays an important role. In our setting we employed the continuous time approximation to decrease the complexity of the system of difference equations that governs the evolution of the population. We proved that both the original discrete time model and the continuous time approximated model behave similarly when the system is close to the boundaries. Thus, if cooperation survives in the continuous time model then it also survives in the original model.

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APPENDIX

RANDOM MATCHING

With random matching there is no need to distinguish between players who cooperated and were paired with a player who also cooperated, γ , and players who cooperated and faced a player who did not cooperate, σ . Thus, these two sets of players are grouped to the same set $\omega = \gamma + \sigma$.

Let ω^{t+1} the fraction of players who played C at time t with $\omega^0 \in [0, 1]$ given. Let $1 - \omega^{t+1}$ be the fraction of players who played D at time t . Furthermore, let $P_D^\omega : A^2 \times \mathbb{R}^2 \rightarrow [0, 1]$ be the probability by which a player who played C switches to playing D and let $P_C^{1-\omega} : A^2 \times \mathbb{R}^2 \rightarrow [0, 1]$ be the probability by which a player who played D switches to playing C . Assume P_D^ω and $P_C^{1-\omega}$ satisfy assumptions 1 – 3. The evolution of the fraction of population playing C is given by

$$\omega^{t+1} = \omega^t(1 - P_D^\omega) + (1 - \omega^t)P_C^{1-\omega} \quad (9)$$

where P_D^ω and $P_C^{1-\omega}$ are derived below.

Proceeding in a similar fashion as in equation (2) we have that $P_C^{1-\omega}$, which is the probability that a player in $1 - \omega$ changes to action C , is only positive if the player in $1 - \omega$ is matched with another one playing D and observes an individual playing C that is matched with a player who also chose action C . In this case the observed payoff equals a while own payoff equals d . Hence, we can write $P_C^{1-\omega}$ as follows:

$$P_C^{1-\omega} = (1 - \omega)\omega^2 f(a, d). \quad (10)$$

On the other hand, P_D^ω , which is the probability that a player in ω changes to playing D , is only positive if the player in ω observes a player in $1 - \omega$. Three different situations can occur now: First, if the player in ω is paired with a player in ω and the player she observes is also paired with a player in ω then its payoff equals a while observed payoff equals c . Second, if the player in ω is paired with a player in $1 - \omega$ and the player she observes is paired with a player in ω then own payoff equals b while observed payoff equals c . Finally, if the player in ω is paired with a player in $1 - \omega$ and the player she observes is also paired with a player in $1 - \omega$ then own payoff equals b while observed payoff equals d . Therefore, we can write

$$P_D^\omega = (1 - \omega) [\omega^2 f(c, a) + (1 - \omega)\omega f(c, b) + (1 - \omega)^2 f(d, b)]. \quad (11)$$

Lemma 1. *With random matching only $\omega = 1$ and $\omega = 0$ are equilibria. Furthermore, for any interior initial condition*

$$\lim_{t \rightarrow \infty} \omega^t = 0.$$

Proof. We can see from equation (9) that both $\omega = 1$ and $\omega = 0$ are equilibria. The proof is completed by showing that from any point with $\omega \in (0, 1)$ the system converges in the limit to $\omega = 0$.

Since $\omega \in (0, 1)$, using assumptions 2 and 3 we obtain

$$\begin{aligned} \omega^2 f(c, a) + (1 - \omega)\omega f(c, b) + (1 - \omega)^2 f(d, b) &> (1 - \omega)\omega f(c, b) \\ &\geq (1 - \omega)\omega f(c, d) \\ &\geq (1 - \omega)\omega f(a, d). \end{aligned}$$

Thus, we have that

$$\begin{aligned} \omega f(a, d) &< \omega^2 f(a, d) + \omega^2 f(c, a) + \\ &\quad (1 - \omega)\omega f(c, b) + (1 - \omega)^2 f(d, b). \end{aligned}$$

Multiplying both sides by $\omega(1 - \omega)$ and use equations (10) and (11) to obtain

$$P_C^{1-\omega} < \omega (P_D^\omega + P_C^{1-\omega}). \quad (12)$$

From (9) we have that $\Delta\omega = P_C^{1-\omega} - \omega(P_D^\omega + P_C^{1-\omega})$. Hence, by equation (12) we know that whenever $\omega \in (0, 1)$, $\Delta\omega < 0$. Thus, no point $\omega \in (0, 1)$ can be an equilibrium and the system cannot converge to $\omega = 1$ from any initial condition $\omega \in (0, 1)$.

We still have to show that the system cannot converge to a cycle nor to a point that is not an equilibrium. The fact that the system does not converge to a cycle follows from the observation that for all $\omega \in (0, 1)$, $\Delta\omega < 0$. To show that the system cannot converge to a non-equilibrium point note first that the function $\Delta\omega$ is a polynomial in ω and, hence, continuous in ω .

Assume the system converges to a non-equilibrium point $\hat{\omega} \in (0, 1)$. Given that for all $\omega \in (0, 1)$ we have that $\Delta\omega < 0$, immediately to the right of the point $\hat{\omega}$ it holds that $\Delta\omega \rightarrow 0$ while at $\hat{\omega}$ it holds that $\Delta\omega < 0$. That is, $\lim_{\delta \rightarrow 0^+} \Delta\omega|_{\hat{\omega}+\delta} = 0$ while $\Delta\omega|_{\hat{\omega}} < 0$. Thus, $\Delta\omega$ is not continuous in $\hat{\omega}$, a contradiction. \square

LEMMATA

Lemma 2. *The equilibrium $(0, 0) \in \Omega$ is a repeller if $f(a, d) > 2f(c, a)f(d, b)$ and is asymptotically stable if $f(a, d) < 2f(c, a)f(d, b)$.*

Proof. Define the set $\Sigma_r = (\gamma, \sigma) \in \Omega \cap B_r(0, 0)$. If we take $\varepsilon > 0$ to be small we can disregard terms of order $o(\varepsilon^2)$ and write the system (7) and (8) when $(\gamma, \sigma) \in \Sigma_\varepsilon$ as

$$\begin{aligned}\dot{\sigma} &= -f(d, b)\sigma + f(a, d)\gamma, \\ \dot{\gamma} &= 0.\end{aligned}$$

The approximation above is correct up to a term of order ε^2 . Thus, when the process is arbitrarily close to $(0, 0)$ the change in γ with respect to the change in σ is negligible. The system above converges to $\sigma = \gamma \frac{f(a, d)}{f(d, b)}$. Hence, if we start in Σ_ε with ε small the process first converges to a situation where $\sigma = \gamma \frac{f(a, d)}{f(d, b)}$. The system may hit the path $\sigma = \gamma \frac{f(a, d)}{f(d, b)}$ outside the set Σ_ε . This poses no problem as the further away from $(0, 0)$ the system can be in this case is in the set $\Sigma_\varepsilon \frac{f(a, d)}{f(d, b)}$, which is also arbitrarily close to $(0, 0)$ when ε is small.

After starting in Σ_ε and once the system reaches $\sigma = \gamma \frac{f(a, d)}{f(d, b)}$ we can rewrite (8) as

$$\dot{\gamma} = \left(1 - 2\frac{f(d, b)f(c, a)}{f(a, d)}\right)\sigma^2.$$

The equation of the motion of σ is irrelevant because in the neighborhood of $(0, 0)$ the system moves along the path $\sigma = \gamma \frac{f(a, d)}{f(d, b)}$ as we just proved. To be more precise, the Center

Manifold Theorem is being used here (see Sastry (1999) Section 7.8 or Khalil (1995) Section 8.1).

Given that at most four equilibrium points exists we fix $\varepsilon > 0$ such that no equilibrium points exists in $\Sigma_\varepsilon \setminus (0, 0)$. Assume now that $f(a, d) > 2f(c, a)f(d, b)$ so $\dot{\gamma} > 0$ in Σ_ε . Local stability implies that for any $\varepsilon > 0$ we can find a $\kappa < \varepsilon$ such that if the system starts in Σ_κ then it never leaves Σ_ε . Assume this is the case.

For any $\kappa < \varepsilon$ if $f(a, d) > 2f(c, a)f(d, b)$ then $\dot{\gamma} > 0$. Thus, since $\dot{\gamma} > 0$ and $\sigma = \gamma \frac{f(a, d)}{f(d, b)}$ if the system starts in the boundary of Σ_κ then it will leave that set. Assume that the system, after leaving Σ_κ , does not hit the boundary of the other bigger set Σ_ε . Since for any point in Σ_ε we have that $\dot{\gamma} > 0$ and by continuity of (7) and (8) everywhere except in $\gamma = 1$, if the process does not hit the boundary of Σ_ε then we must have that there exists a point $(\gamma, \sigma) \in \Sigma_\varepsilon \setminus (0, 0)$ such that $\dot{\gamma} = 0$ and thus $\dot{\sigma} = 0$. That is, there must exists at least one equilibrium point in $\Sigma_\varepsilon \setminus (0, 0)$ which is a contradiction.

Thus, if the process starts in Σ_κ it must hit the boundary of Σ_ε . We know that for any point in Σ_ε if $f(a, d) > 2f(c, a)f(d, b)$ then $\dot{\gamma} > 0$ and $\sigma = \gamma \frac{f(a, d)}{f(d, b)}$. Thus, starting in boundary of Σ_κ the process leaves Σ_ε , which is the condition for the point $(0, 0) \in \Omega$ to be a repeller.

Assume now that $f(a, d) < 2f(c, a)f(d, b)$. Since $\dot{\gamma} < 0$, $\sigma = \gamma \frac{f(a, d)}{f(d, b)}$, continuity and the fact that no equilibrium point exists in $\Sigma_\varepsilon \setminus (0, 0)$, for any $\kappa < \rho$ with $\rho < \frac{f(d, b)}{f(a, d)}\varepsilon$ if the system starts in $\Sigma_\varepsilon \setminus \Sigma_\kappa$ then it eventually enters the set Σ_κ . This is the condition for asymptotic stability. \square

Lemma 3. *The equilibrium $(1, 0) \in \Omega$ is asymptotically stable if $f(a, d) > f(c, a)$.*

Proof. For proving the lemma we employ Lyapunov's method:

Proposition 6 (Theorem ?? in Khalil (1995)). *Let $x = 0$ be an equilibrium point for a system described by*

$$\dot{x} = f(x)$$

where $f : U \rightarrow \mathbb{R}^n$ is a locally Lipschitz and $U \subset \mathbb{R}^n$ a domain that contains the origin. Let $V : U \rightarrow \mathbb{R}$ be a continuously differentiable, positive definite function in U with $V(0) = 0$.

- *If $\dot{V}x = \partial V / \partial x$ is negative semidefinite, then $x = 0$ is a stable equilibrium point.*
- *If \dot{V} is negative definite, then $x = 0$ is an asymptotically stable equilibrium point.*

Consider the domain Ω . Take the system given (7) and (8) when $\gamma < 1$ and $\dot{\gamma} = 0$ and $\dot{\sigma} = 0$. Since $\sigma < 1 - \gamma$ this system is continuously differentiable and, thus, Lipschitz continuous.

The proposition easily applies to the equilibrium $(1, 0)$ by just requiring that the function V is such that $V(1, 0) = 0$. Take $V(\gamma, \sigma) = 1 - \gamma - \sigma$.

Define the set $\Theta_r = \{(\gamma, \sigma) \in \Omega \cap B_r(1, 0)\}$. It is easy to see that $V(1, 0) = 0$ and $V(\gamma, \sigma) > 0$ for points in Ω around $(1, 0)$. We still have to show that there exists a $r > 0$ such that $\dot{V}(\gamma, \sigma) < 0$ for $(\gamma, \sigma) \in \Theta_r$. From (7) and (8) the function \dot{V} is given by

$$\dot{V}(\gamma, \sigma) = -\frac{1 - \gamma - \sigma}{1 - \gamma} [-\gamma\sigma f(c, a) - \sigma^2 f(c, b) - (1 - \gamma - \sigma)\sigma f(d, b) + (1 - \gamma\sigma)\gamma f(a, d)].$$

When the system is close to $(1, 0)$ we can rewrite the expression above as

$$\dot{V}(\gamma, \sigma) = \frac{1 - \gamma - \sigma}{1 - \gamma} [\sigma f(c, a) - (1 - \gamma - \sigma)f(a, d)].$$

Thus, given that $\sigma < 1 - \gamma - \sigma$ if $f(a, d) > f(c, a)$ then $\dot{V}(\gamma, \sigma) < 0$ around $(1, 0)$ and, hence, by Proposition 6 the point $(1, 0)$ is asymptotically stable. \square

Lemma 4. *No periodic solutions (cycles) exist.*

To show that no periodic solutions exist Theorem 3.1 in Hethcote (1976) is employed.

Proposition 7 (Theorem 3.1 Hethcote (1976)). *Assume that f and g are continuously differentiable in an open connected region D , that no solution path of*

$$\begin{aligned} \dot{x} &= f(x, y), \\ \dot{y} &= g(x, y) \end{aligned}$$

leaves D , and that D contains at least one equilibrium point. If there exists a function $\varphi : D \rightarrow \mathbb{R}$ which is continuously differentiable in D and such that

$$\frac{\partial \varphi f}{\partial x} + \frac{\partial \varphi g}{\partial y}$$

has the same sign throughout D , then there are no closed paths (periodic solutions) in D .

Proof of Lemma 4. Take $\varphi = \frac{1 - \gamma}{(1 - \sigma - \gamma)\sigma}$ and consider the set $D = \Omega \setminus \{(x, 0) \text{ with } x \in [0, 1]\}$. It is easy to notice that function φ is continuous and differentiable in D . The fact that the system does not leave D follows from the continuity of (7) and (8).

As in the proof of Theorem 3.1 Hethcote (1976), if there are no equilibrium points in D then no closed path can exist in D . Assume, thus, that at least one equilibrium point exists in D . From (7) and (8) we have that

$$\begin{aligned} \frac{\partial \varphi \dot{\sigma}}{\partial \sigma} + \frac{\partial \varphi \dot{\gamma}}{\partial \gamma} &= -f(c, b) + f(d, b) - \frac{\gamma}{\sigma} f(a, d) - \\ &\quad (1 - \gamma - \sigma) \frac{\gamma}{\sigma^2} f(a, d) - \frac{1}{1 - \gamma - \sigma} - 2f(c, a). \end{aligned}$$

By assumptions 2 and 3, $f(c, b) \geq f(d, b)$ and $f(a, d), f(c, a) > 0$. Thus, $\frac{\partial \varphi \dot{\sigma}}{\partial \sigma} + \frac{\partial \varphi \dot{\gamma}}{\partial \gamma} < 0$ for all $(\gamma, \sigma) \in D$.

Thus, using Proposition 7 the system (7) and (8) has no periodic solutions in D . We still have to show that the system does not have periodic solutions in Ω .

As already mentioned, if the system starts in D then it never leaves that set. Furthermore, if the system starts in $(x, 0)$ with $x \in (0, 1)$ then by equation (8) it enters the set D . Thus, starting from any point other than $(0, 0)$ and $(1, 0)$ the system enters and/or stays in D and never leaves that set. Given that both $(0, 0)$ and $(1, 0)$ are equilibria it can be concluded that no periodic solutions exists in Ω . \square