Revenue Comparison in Common-Value Auctions: two examples*

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Abstract

Milgrom and Weber (1982) established that for symmetric auction environments in which players' (affiliated) values are symmetrically distributed, expected revenue in the second-price sealed-bid auction is at least as large as in the first-price sealed-bid auction. We provide two simple examples of a common-value environment showing this ranking can fail when players are asymmetrically informed.

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1 Introduction

Two archetypal auction frameworks often studied are the independent-private-values model and the common-value model. Celebrated results of the theory include the Revenue Equivalence Theorem for the symmetric independent private-values model, and its extension to symmetric affiliated-values models: the expected revenue in the second-price auction is at least as large as that in the first-price auction.[1] In this note we show, in a discrete common-value setting, that this ranking of revenues is not generally maintained when players' signals are not symmetrically distributed.

The standard approach models an auction as a Bayesian game and then derives properties of the Bayesian equilibrium. In a private-values auction, each player knows his own value for the object being sold but knows only the distribution of his rivals' values, not their actual realizations. The benchmark private-values model incorporates the following assumptions: (i) bidders are risk-neutral; (ii) bidders do not face budget constraints; (iii) bidders' values are independently distributed; and (iv) bidders' values are symmetrically distributed. A central finding under these assumptions is the Revenue Equivalence Theorem: the first-price sealed-bid auction and the second-price sealed-bid auction yield the same expected revenue for the seller.[2]

If any one of assumptions (i)–(iv) is dropped then the revenue equivalence of the first-price and second-price auctions is generally not valid. We are particularly interested in the consequences of relaxing assumptions (iii) and (iv).[3] Milgrom and Weber (1982) replace assumption (iii) with the following: (iii′) the "state" variable and players' signals are affiliated. This affiliated-values model includes as special cases the independent–private-values environment and the common-value environment (Milgrom and Weber, 1982, p. 1095). Milgrom and Weber then show that the expected revenue in the second-price auction is at least as great as in the first-price auction. In the *private*-values setting Maskin and Riley (2000) provided examples showing this ranking may fail if players' values are not symmetrically distributed. Complementing their analysis, we examine a *common*-value environment in which players' signals are not symmetrically distributed. In particular, we suppose each player's information is represented by a finite partition of the set of possible common values, and these partitions differ between players. We show that Milgrom and Weber's ranking of expected revenues can fail if players' information is not symmetric.

Hausch (1987) and Parreiras (2006) have also compared revenues in asymmetric common-value auctions. Hausch (1987) supposes the value of the object and players' signals are discrete random variables. While he allows players' signals not to have the same distribution, there is a strong assumption of symmetry

[1]See Krishna (2002) and Milgrom (2004) for excellent presentations of the state of the art of auction theory as well as its development.

[2]This equivalence of expected revenues for the first-price and second-price auctions is a special case of the more general equivalence results of Myerson (1981) and Riley and Samuelson (1981).

[3]For the effects of relaxing (i), see Holt (1980) and Matthews (1987); for (ii), see Che and Gale (1998, 2006); for (iii), see Milgrom and Weber (1982); and for (iv), see Maskin and Riley (2000).

about conditional probabilities of one player's signal given the other player's realized signal; Hausch derives symmetric equilibria to the first-price and second-price auctions. For these symmetric equilibria, Hausch extends the Milgrom-Weber revenue ranking, showing expected revenue in the second-price auction is at least as large as in the first-price auction. Hausch then provides an auction environment in which players' signals violate his assumption of symmetry on the conditional distributions of players' signals. In this example he compares an asymmetric equilibrium in the first-price auction (he finds there are no symmetric equilibria) to a *symmetric* equilibrium in the second-price auction, finding the revenue in the first-price auction is greater. While acknowledging that equilibrium selection is key, Hausch provides no justification for selecting the symmetric equilibrium of the second-price auction beyond that this strategy was also an equilibrium in the environment where signals satisfied his symmetry condition.

Parreiras (2006) considers asymmetric hybrid auctions in which the price paid by the winner is a weighted average of the winning price in the first-price auction and the second-price auction. Signals are continuously distributed. He shows that when there is any positive (but not full) weight on the second-price auction, the hybrid auction yields at least the expected revenue of the (stand-alone) first-price auction, which is in line with the Milgrom-Weber ranking. The difference between Parreiras' result and the reversals Hausch and we find, however, is not due to the modelling of signals as discrete versus continuous—Hausch's model admits the case in which signals are discrete and have identical distributions conditional on the state; in this case he finds the second-price auction yields at least the expected revenue of the first-price auction. Rather, it appears the finding in Parreiras' hybrid auction is due to each player's having a significant chance of winning (approximately 1/2) the auction, for any realized common value, and so they bid relatively aggressively. In contrast, we will see in our second-price auction equilibrium (as in Einy *et al.*, 2002), in each state a player expects to win the auction only when the other player's signal is worse, so he (the first player) bids conservatively.

When comparing a seller's expected revenues in first-price and second-price auctions, equilibrium selection is critical. Generally the second-price auction yields a vast set of equilibria, so it may not be surprising that *some* equilibrium provides lower expected revenue than does the first-price auction. One should ask, too, whether such a ranking reversal holds for "sensible" equilibria. In private-value models, one might view weakly-dominant strategies as most appealing and so select this equilibrium; in symmetric affiliated-values models, a symmetric equilibrium is the natural selection. In an asymmetric common-value neither of these avenues is open. However, in a discrete auction framework more general than ours, Einy *et al.* (2002) use dominance-solvability to make a unique selection among the set of equilibria in the second-price auction. Their selection is the equilibrium we consider. Among all equilibria in undominated strategies, the selection of Einy *et al.* is the only one that guarantees all bidders nonnegative expected payoffs, and it is the only sophisticated equilibrium (as in Moulin, 1986) if uninformed bidders are added to the auction.

In our examples with a finite set of possible values for the object, each player's information is represented by a partition of this state space. Each player learns which element of his partition contains the realized common value. Asymmetry is captured by players having different partitions. As players' signals are finite, equilibrium involves randomized bids.

The examples of the next section show the dominance-solvable equilibrium of the second-price auction may yield strictly less expected revenue than the equilibrium of the first-price auction.

2 Two Examples

Denote by u the random common value of the object and by Ω the set of possible common values for the object.

Example 1. $\Omega = \{1, 2\}$, $\Pr(u = 1) = \Pr(u = 2) = 1/2$, $\Pi^1 = \{\{1\}, \{2\}\}$, *and* $\Pi^2 = \{\{1, 2\}\}$

Player 1 is perfectly informed about the state while player 2 knows only that the states $u = 1$ and $u = 2$ are equally likely.[4] It is easily verified that first-price auction has no equilibrium in pure strategies. The unique equilibrium has the following form: when observing $\{1\}$ player 1 bids $b_1 = 1$ for sure, and when observing $\{2\}$ player 1 randomizes over a nondegenerate interval $[1, \bar{x}]$; player 2 randomizes over this same interval.

In the second-price auction, the equilibrium selection offered by Einy *et al.* (2002) has player 1 bid $b_1 = 1$ when observing $\{1\}$ and $b_1 = 2$ when observing $\{2\}$, and it has player 2 bid $b_2 = 1$. Einy *et al.* explain this equilibrium selection by pointing out that player 1's strategy is weakly dominant; and, among undominated strategies, the one selected for player 2 is the only one that ensures player 2's expected payoff is nonnegative, regardless of player 1's strategy (this selection is discussed further in Section 3 below).

It is now clear that the first-price auction yields greater expected revenue than the second-price auction: revenue in the second-price equilibrium is sure to equal 1, while it the first-price auction it is never less than 1 and, conditional on the state being $u = 2$, it is greater than 1 with positive probability. Hence, expected revenue in the first-price auction strictly exceeds 1. □

The reader may feel cheated by Example 1 in that the revenue comparison did not even require an explicit calculation of expected revenue in the first-price auction. We hope the next example is more satisfying. Here, with three possible states, neither player is always perfectly informed or perfectly uninformed.

[4] This example is a discrete version of Wilson (1967) (also see Weverbergh, 1979, and Engelbrecht-Wiggans *et al.*, 1983).

Example 2. $\Omega = \{1,2,3\}$, $\Pr(u=1) = \Pr(u=2) = \Pr(u=3) = 1/3$, $\Pi^1 = \{\{1\},\{2,3\}\}$, *and* $\Pi^2 = \{\{1,2\},\{3\}\}$.

If $u = 1$ then player 1 knows the state, and if $u = 3$ then player 2 knows the state. If $u = 2$ then both players are unsure of the state: player 1, observing $\{2,3\}$, assigns equal probability to states $u = 2$ and $u = 3$; player 2, observing $\{1,2\}$, assigns equal probability to the states $u = 1$ and $u = 2$.

We let F^1_L and V^1_L denote the cumulative distribution function (cdf) of the bid and the conditional expected payoff, respectively, for player 1 when he observes the "low" signal, $\{1\}$; and we let F^1_H and V^1_H denote his cdf and conditional expected payoff when he observes his "high" signal, $\{2,3\}$. Similarly, we let F^2_L and V^2_L denote the cdf and conditional expected payoff, respectively, for player 2 when he observes his "low" signal, $\{1,2\}$; and we let F^1_H and V^1_H denote his cdf and conditional expected payoff when he observes the "high" signal, $\{3\}$. We look for an equilibrium in which player 1 bids 1 for sure if he observes $\{1\}$; if he observes $\{2,3\}$, then he bids according to the cdf F^1_H over the interval $[x_1, x_3]$. Additionally, if player 2 observes $\{1,2\}$, then he bids according to the cdf F^2_L over the interval $[x_1, x_2]$; if he observes $\{3\}$, then he bids according to the cdf F^2_H over the interval $[x_2, x_3]$.

We conjecture $V^1_L = V^2_L = 0$ and $x_1 = 1$. We have the following: indifference among possible bids requires, for player 1 observing $\{2,3\}$,

$$V^1_H = \frac{1}{2}(2-b_1)F^2_L(b_1) + \frac{1}{2}(3-b_1)F^2_H(b_1) \qquad \forall b_1 \in (x_1, x_3]; \tag{1}$$

for player 2, observing $\{1,2\}$,

$$V^2_L = \frac{1}{2}(1-b_2) + \frac{1}{2}(2-b_2)F^1_H(b_2) \qquad \forall b_2 \in [x_1, x_2] \tag{2}$$

and, observing $\{3\}$,

$$V^2_H = (3-b_2)F^1_H(b_2) \qquad \forall b_2 \in [x_2, x_3]. \tag{3}$$

Evaluating (1) at $b_1 = x_2$ and $b_1 = x_3$ we find

$$V^1_H = \frac{1}{2}(2-x_2) = \frac{1}{2}(2-x_3) + \frac{1}{2}(3-x_3), \tag{4}$$

where the first equality follows because $F^2_L(x_2) = 1$ and $F^2_H(x_2) = 0$, the second because $F^2_L(x_3) = 1$ and

$F_H^2(x_3) = 1$. Similarly, for player 2 we recall $V_L^2 = 0$ and evaluate (2) at $b_2 = x_2$ to obtain

$$0 = V_L^2 = \frac{1}{2}(1 - x_2) + \frac{1}{2}(2 - x_2)F_H^1(x_2); \tag{5}$$

evaluating (3) at $b_2 = x_2$ and $b_2 = x_3$ we find

$$V_H^2 = (3 - x_2)F_H^1(x_2) = 3 - x_3, \tag{6}$$

where the second equality follows because $F_H^1(x_3) = 1$. Equations (4)–(6) provide three equations in the three unknowns x_2, x_3, and $F_H^1(x_2)$, for which the unique solution is

$$x_2 = 4/3 \qquad x_3 = 13/6 \qquad F_H^1(x_2) = 1/2. \tag{7}$$

Using the values from (7) in equations (1)–(3), we find

$$V_H^1 = 1/3 \qquad \text{and} \qquad V_H^2 = 5/6. \tag{8}$$

With our conjectures $V_L^1 = V_L^2 = 0$ and $x_1 = 1$, substituting values from (8) into (1)–(3) yields the following: for player 1,

$$F_L^1(b_1) = \begin{cases} 1 & \text{if } b_1 \geq 1 \\ 0 & \text{if } b_1 < 1 \end{cases} \qquad \text{and} \qquad F_H^1(b_1) = \begin{cases} 1 & \text{if } 13/6 \leq b_1 \\ \dfrac{5}{6(3-b_1)} & \text{if } 4/3 \leq b_1 < 13/6 \\ \dfrac{b_1 - 1}{2 - b_1} & \text{if } 1 \leq b_1 < 4/3 \\ 0 & \text{if } b_1 < 1; \end{cases}$$

for player 2,

$$F_L^2(b_2) = \begin{cases} 1 & \text{if } 4/3 \leq b_2 \\ \dfrac{2}{3(2-b_2)} & \text{if } 1 \leq b_2 < 4/3 \\ 0 & \text{if } b_2 < 1 \end{cases} \qquad \text{and} \qquad F_H^2(b_2) = \begin{cases} 1 & \text{if } 13/6 \leq b_2 \\ \dfrac{3b_2 - 4}{3(3-b_2)} & \text{if } 4/3 \leq b_2 < 13/6 \\ 0 & \text{if } b_2 < 4/3. \end{cases}$$

Figure 1 depicts these cdfs. It can be proved that the these strategies constitute the unique equilibrium to the first-price auction.

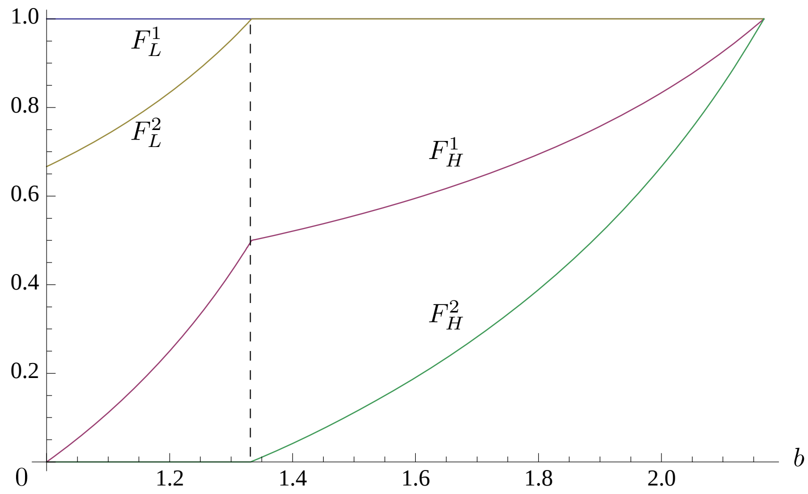

Figure 1: Equilibrium cdfs in Example 2

Because the good is sold with probability one, we can follow Hausch (1987) in calculating the seller's expected revenue in the first-price auction as

$$\begin{aligned}\begin{pmatrix}\text{Expected}\\ \text{Revenue}\end{pmatrix} &= \begin{pmatrix}\text{Expected } u\end{pmatrix} - \begin{pmatrix}\text{Expected Payoff}\\ \text{to Player 1}\end{pmatrix} - \begin{pmatrix}\text{Expected Payoff}\\ \text{to Player 2}\end{pmatrix}\\ &= 2 - \frac{2}{3} \times V_H^1 - \frac{1}{3} \times V_H^2\\ &= \frac{3}{2}. \end{aligned} \tag{9}$$

Now consider the equilibrium of Einy *et al.* in the second-price auction: player 1 bids 1 when observing $\{1\}$ and bids 2 when observing $\{2,3\}$; player 2 bids 1 when observing $\{1,2\}$ and bids 3 when observing $\{3\}$. Correspondingly, the sales price is 1 when $u \in \{1,2\}$ and is 2 when $u = 3$; the *ex ante* expected revenue is, therefore,

$$\frac{2}{3} \times 1 + \frac{1}{3} \times 2 = \frac{4}{3}. \tag{10}$$

Comparing (9) with (10) we see the first-price auction yields greater expected revenue than the second-price auction. □

3 Equilibrium Selection in the Second-Price Sealed-Bid Auction

Here we discuss further the selection of equilibrium in the second-price common-value auction. For Example 2, the dominance-solvability intuition provided by Einy *et al.* is as follows. Player 2 observing the "high" signal $\{3\}$ bids $b = 3$ as this is his weakly dominant strategy. Now consider player 1 observing $\{2, 3\}$; if the true state is 3, then player 2 will bid 3, so the weakly dominant strategy for player 1 is to bid 2. Next consider player 2 observing $\{1, 2\}$. If player 1 observing $\{2, 3\}$ bids $b_1 = 2$, and if player 1 observing $\{1\}$ bids $b_1 = 1$ (which is his weakly dominant strategy), then any bid $b_2 \in [1, 2]$ is an optimal bid for player 2.[5] Observe that for player 2 any bid above 1 will give him exactly the same payoff as bidding $b_2 = 1$. Therefore, player 2 might sensibly choose to be cautious, bidding $b_2 = 1$. While we find this intuition reasonable, it does not nail down a strategy for player 2 observing $\{1, 2\}$, and other orders of elimination can give different equilibrium strategies for player 1 observing $\{2, 3\}$.

Einy *et al.* (2002) recognize this limitation of the above argument, and they use the "sophisticated" equilibria of Moulin (1986) to address the weakness that equilibrium selection may depend on the order in which weakly dominated strategies are eliminated. But even the sophisticated equilibria admit some flexibility in the specification of some player's bid at some particular information set. Nevertheless, in all sophisticated equilibria except one, in at least one information set the *ex post* weak player bids incautiously in the sense that a strictly negative payoff would be earned for some conceivable bids of the rival bidder. Among others, one justification Einy *et al.* provide for their selection is that this is the only sophisticated equilibrium that assures each bidder a nonnegative payoff, regardless of the other bidder's actions (as noted below, Larson makes a similar assumption). Thus, in the Einy *et al.* selection each player's choice of strategy is "immune" to small trembles by the other player.[6]

Larson (forthcoming) considers a perturbation of common value auctions. As a special case of his model, suppose that player i's value for the object is $v + \varepsilon_i$, where v represents the common-value component, and ε_i the (independent) random private-value component with support $[-\bar{\varepsilon}, \bar{\varepsilon}]$, where $0 < \bar{\varepsilon} < 0.25$, $i = 1, 2$. Larson finds that adding this small element of private value determines a unique equilibrium in the perturbed game. The equilibrium of the basic common-value auction is now found as the distributions of the ε_i converge to 0. We next explain how applying Larson's methods in our setting yields the Einy *et al.* equilibrium selection in Example 2. Consider player 2 observing $\{1, 2\}$. Larson assumes a player will never bid more than the expected value of the object conditional on his observed set. By this assumption, player 2 observing $\{1, 2\}$ and the private component ε_2 will not bid more than $1.5 + \varepsilon_2$. If the true state is 2, player 1 will observe $\{2, 3\}$ and not bid less than $2 - \bar{\varepsilon}$, which is sure to exceed $1.5 + \bar{\varepsilon}$, an upper bound on player 2's bid. Given that player 2 is sure not to win when the common value is 2, he recognizes that if he wins when observing $\{1, 2\}$,

[5] Larson (forthcoming) has noted the same difficulty in isolating the "right" equilibrium. He writes that "...when a bidder, in formulating a best response to her rival's strategy, considers the event of winning [the second-price auction] with a particular price, she is effectively conditioning on full information about the common value. Given this, there is no particular reason that a poorly informed bidder needs to bid cautiously." This is the situation when player 2 observes $\{1, 2\}$.

[6] One of the other justifications that Einy *et al.* provide is that the addition of an uninformed player will yield for the other players the strategies of the equilibrium selection.

then his value must be $1+\varepsilon_2$—consequently, he bids $1+\varepsilon_2$ at information set $\{1,2\}$.[7] As $\bar{\varepsilon} \to 0$, this strategy for player 2 converges to that specified by Einy *et al.* A similar analysis applies to player 1 observing $\{2,3\}$. Thus, Larson's perturbation technique also selects in our second-price common-value auction the equilibrium of Einy *et al.*

Parreiras (2006) also selects an equilibrium by perturbing the second-price auction slightly. He supposes that, given $\varepsilon > 0$, players submit bids; the winner has the high bid and pays a price that is a weighted sum of his actual bid and the lower bid: it is $1-\varepsilon$ times the second-highest bid plus ε times the winning bid. Because the winning bidder pays a price that increases with his actual bid, the usual multiplicity in second-price auctions is eliminated. Parreiras selects as the equilibrium in the second-price auction the limit of equilibrium strategies of the hybrid auction as $\varepsilon \to 0$. Applying hybrid auctions of Parreiras to our setting is a hard task. For example, when observing $\{3\}$, player 2's unique weakly dominant strategy in the second-price auction is to bid $b = 3$. But in the hybrid auction, for every $\varepsilon > 0$ player 2 observing $\{3\}$ uses a mixed strategy over the same interval as player 1 who observes $\{2,3\}$. In this sense, the hybrid auction of Parreiras retains a strong similarity with the first-price action, for all $\varepsilon > 0$. This may be the reason Parreiras finds that each player has a significant chance of winning (approximately $1/2$) the auction.

For completeness we conclude by considering in our Example 2 the meaning of a *symmetric* equilibrium in the second-price auction, as in Hausch (1987). Each player may observe a "high" or a "low" signal. For player 2 observing the "high" signal $\{3\}$, his weakly dominant strategy is to bid $b = 3$. For player 1 observing the "high" signal $\{2,3\}$, symmetry will have him bid $b = 3$. On the other side, for player 1 observing the "low" signal $\{1\}$, his weakly dominant strategy is to bid $b = 1$. For player 2 observing the "low" signal $\{1,2\}$, symmetry will have him bid $b = 1$. The symmetric strategies having players bid 1 when observing their low signal and 3 when observing their high signal indeed constitutes an equilibrium. This equilibrium has the imperfectly player bid conservatively when observing a low signal and aggressively when observing a high signal.[8]

In summary, our equilibrium selection in the second-price auction coincides with that of Einy *et al.* (2002) and Larson (forthcoming).

[7] Bidding more than $1+\varepsilon_2$ is incautious as it exposes player 2 to the possibility of winning the auction and paying a price $1+\varepsilon_1$ that strictly exceeds his own value of $1+\varepsilon_2$, thus leaving player 2 with negative surplus.

[8] It can be checked that this "symmetric" equilibrium in the second-price auction indeed yields greater expected revenue than does the first-price auction.

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