# Second Place is Good Enough: On the Optimality of Larger 2nd Prizes in Contests

Dylan B. Minor*
Haas School of Business
UC Berkeley

July 2009

**Abstract**

Contests are ubiquitous; from a true tournament to auctions, from competing employees to competing firms, much can be cast as a contest. Consequently, there has been much literature examining the optimal design of contests. Interestingly, few studies have considered the role of convex costs or designer concave benefits, which is how we model many real world applications found in a contest setting. We show under such a setting, given sufficient number of participants, it is best to offer a larger second over first prize. However, in such a contest, non-monotonicities emerge, for which we propose a mechanism dubbed the generalized second prize contest. We then examine indivisible prizes, finding again with sufficient number of participants, it is best to offer the sole prize to second place instead of first. Finally, we provide some applications of our findings ranging from regulation to innovation.

*The author would like to thank John Morgan for his extensive comments and support.

# 1 Introduction

Many economic settings can be cast as a contest. Whether firms are competing for business or to avoid a regulator, whether sales people are competing for prizes, nonprofits are vying for donors' dollars or even politicians seeking election, we have multiple agents seeking after prizes. In most every setting it is simply assumed prizes receive, or should receive, a monotonic ordering: first prize is the greatest, second the next greatest, and so forth.

However, we ask the question is this ever suboptimal? That is, for example, should we ever offer a larger second than first prize? What about if there is only one prize to offer? Surprisingly, we find in a large class of realistic settings it is optimal to offer a larger second (or even exclusive second) over first prize. The intuition is in offering a larger second prize we lose some effort from the most able, which are most likely to receive first prize; however, we receive increased effort from the less able as a result of their more likely earned second prize being larger. If this increased effort overcomes the most able's lesser effort, total effort is increased.

However, in this setting of larger second prizes, we also encounter non-monotonic bidding structures. Thus, we find a mechanism we dub the generalized second prize contest (GSPC hereafter) to solve this non-monotonicity problem. This solution also works in the case of indivisible prizes.

The paper is organized as follows. Our next section provides a brief overview of extant contest literature. We then delve into our bidding function and generalized second prize contest mechanism. Next, we show when this mechanism provides more total revenue, or effort, compared with the classical contest structure of (weakly) monotonic prize ordering. Finally, we provide some applications and a concluding discussion.

# 2 Related Literature

The contest literature can be divided into three main strands. One of the first, and earliest strands was initiated by Tullock (1980). He set the problem up as players having a chance of winning a contest as a function of a particular contestant's effort vis-a-vis all other contestants' effort level. Much of the focus of this literature is the degree of rent dissipation through rent seeking. That is, determining what percent of the prize is exerted in effort to obtain such prize. Here the analysis is often concerned with how efficient a contest is- e.g., politicians competing for election.

Another strand has to do with casting a contest as an all-pay auction, with the war of attrition as a leading characterization (see Bulow & Klemperer (1999) for a generalization and survey). Here we find we can analyze contest outcomes and participant behavior by drawing on the rich auction literature. However, of note, is it has almost been universally assumed effort costs are linear, as in auctions a bidder's cost of a bid is most often linear. Nonetheless, in practice, and in most economic applications the firm or individual's cost function it is most often assumed to be convex.

The third strand has to do with designing optimal contests, whether it is deciding how much to allocate between multiple prizes or deciding between single or multiple stage contests. Moldovanu and Sela (2001) provide a survey of this work. Moldovanu and Sela make the important contribution in this literature of allowing participant costs to be convex. In doing so they show whereas with linear (participant) contests a contestant designer would optimally have a contest as a winner-takes-all structure, with convex costs now providing a (weakly) less second prize is usually superior.

Meanwhile, still none of the literature, to our knowledge, has considered the contestant designer's benefit function being concave as opposed to linear. This can be thought of as either a risk averse designer or simply a designer that values each contestant's level of effort in a diminishing manner. For example, a professor might prefer increasing a lesser able student's grade from 65 to 70% versus the most able student's performance 95% to 100%.

In this paper, we show having concave designer benefits is essentially equivalent to assuming convex or linear contestant costs, having the same general effect of increasingly the likelihood of optimally offering a larger second prize. Further, as intuition would suggest, concave benefits amplify the effect of convex costs.

Finally, all the above literature has generally simply assumed prizes have a monotonic ordering with first prize being the greatest. We now relax this assumption, as well as relax the linearity of costs and benefits.

# 3 The Bidding Function

We begin by examining our bidding function and consider its behavior as we offer a larger second than first prize in a contest with a maximum of two prizes. We will mostly use the notation found in Moldovanu and Sela (2001). They define the bidding, or effort, function in the face of two prizes, with the first prize being weakly greater than the second prize, thus:

$$b(c) = g(A(c)(1-\alpha) + B(c)(\alpha))$$

We again generalize their setting by having $\alpha \in [0,1]$ - as opposed to $\alpha \in [0,.5]-$ denoting the value of the 2nd prize as $\alpha$ and $1-\alpha$ the value of the first prize, giving a total prize mass of 1. The inverse of our cost (of effort) function $\gamma(x)$ of participants is $g(x)$. We assume the cost of effort is convex. We further assume $c$ represents the privately known cost type of each participant. The *total* cost of effort $x$ is then $c\gamma(x)$. We further assume $c$ is bounded above zero to assume away costless effort. This cost type $c$ is drawn from some cdf $F(\cdot)$, which then allows us to define $A(c)$ and $B(c)$ found in the above bidding function as:

$$A(c) \equiv (k-1)\int_{c}^{\bar{c}} \frac{1}{a}(1-F(a))^{k-2} \times F'(a)da$$

$$B(c) \equiv (k-1)\int_{c}^{\bar{c}} \frac{1}{a}(1-F(a))^{k-3} \times [(k-1)F(a)-1] \times F'(a)da$$

This bidding function is simply the result of contestants each solving their first order condition of expected benefit vis-a-vis their given cost. All is well until we allow $\alpha > .5$ (i.e., second prize is larger than first). In this case, the bidding function then becomes non-monotonic as we prove in our next lemma.

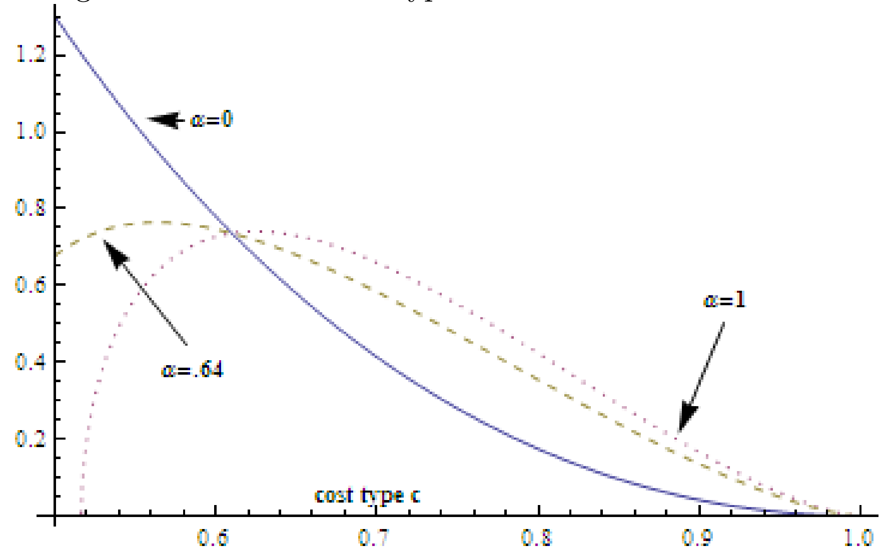
**Lemma 1** *If $\alpha > .5$, the contestant bidding function becomes single peaked with a maximum at $\widehat{c}$ such that $F(\widehat{c}) = \frac{2\alpha-1}{k\alpha-1}$*

Proof: see appendix.

We now turn to a graphical example of the bidding function for some intuition on the effect of increasing the value of the second prize compared with the first prize. Here we assume $\gamma(x) = x^2$, $c \in U[.5,1]$,and $k = 5$ participants:

Our horizontal axis represents cost type $c$. The vertical axis is optimal effort for a given type. The blue (solid) line is the bidding function assuming $\alpha = 0$ (i.e., only a 1st prize is offered) and the red (dotted) shows bidding under $\alpha = 1$ (i.e., only a second prize is offered). These two lines show the trade off between offering more of a first versus second prize. The 1st prize always increases the effort of the lowest cost types until about type .61. However as the cost becomes greater for a given type, then it is the second prize that creates more effort. Hence, offering more of a second prize increases the effort of the roughly top 80% of cost types, but reduces

Figure 1: Effort of Each Type for Different Prize Divisions



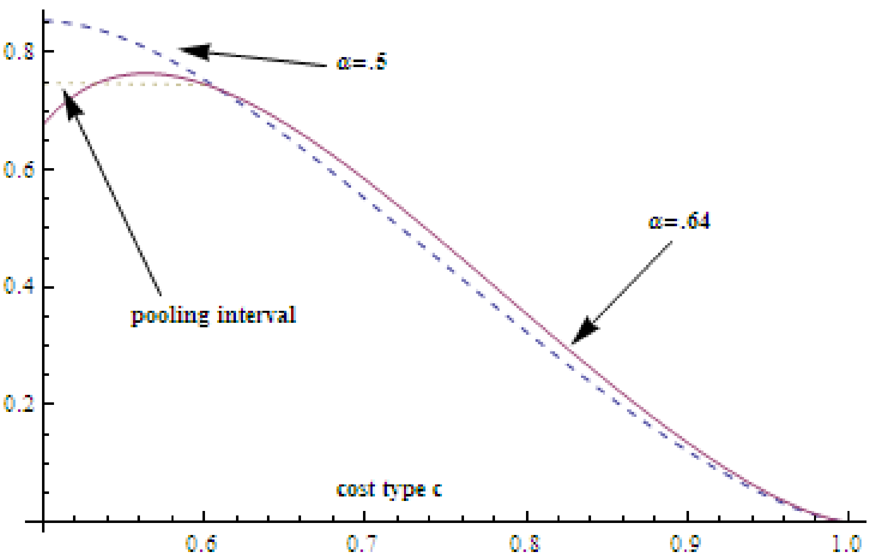

the effort of the bottom 20% of cost types. Thus, where the marginal increase of increasing the 2nd prize equals the marginal loss in reducing the 1st prize, we find our optimal $\alpha^*$.

Finally, the yellow (dashed) line traces the bidding function of $\alpha \approx .64$, which is

the optimal $\alpha$ for this example. Of course, even though $\alpha \approx .64$ yields the highest total expected effort, it is not feasible due to the non-monotonicity. We now turn to making such payoff feasible.

# 4 Generalized 2nd Prize Contest

We propose the generalized second prize contest (GSPC), which fixes the non-monotonicity of the bidding function as $\alpha > .5$. We "iron" out the non-monotonic part of the bidding function by creating a pooling interval. In particular, we find some maximal effort level $e^*$ at which pooling will occur. Any exerting effort below this level will be ranked by effort, as before, to determine prize allocation. However, any contestants at $e^*$ will be pooled. If there is only one such contestant, they receive 1st prize. The next highest effort contestant with effort below $e^*$ will get second prize. If there are two or more contestants in the effort pooling interval, first and second prize will be randomly allocated with equal chance among contestants along




the pooling interval. For example, if 3 people pool, each of them has a separate 1/3 chance of getting 1st or 2nd prize. Hence, there is a 1/9 chance a contestant receives both first and second prize. This allocation is then the same as the auction literature that typically assumes a "tie" is broken by equal random allocation among tied bidders.

To see an example of GSPC mechanism, we continue our last bidding function example and add the location of the pooling interval:

Here the blue (dashed) line represents effort if we instead set $\alpha = .5$,whereas the red (solid) line shows $\alpha \approx .64$. The yellow (dotted) line then shows the effort level of the pooling interval. Note if the area between the blue and red line but below the yellow line is greater than the area above the yellow line and below the blue line, then the GSPC generates more total effort than a contest constraining $\alpha = .5$.

It turns out we can always use a generalized second prize contest mechanism, finding a unique symmetric equilibrium, as our next proposition gives:

**Proposition 2** *The generalized 2nd prize contest mechanism always exists, meets all incentive compatibility constraints, induces a (weakly) monotonic bidding function, and is unique given a prize allocation*

**Proof:**

We first show the mechanism meets all incentive computability constraints. For contestants who would optimally provide effort below $e^*$, this problem is just as before so their bidding function remains as under a contest with no pooling, which we will call *no pool* bidding or *no pool* contest, depending on the context. Call this cutoff $c^*$ such that for all $c \in [c^*, \overline{c}]$ these participants provide their effort below $e^*$ as under *no pool*. Now all that have costs of effort $c \in [\underline{c}, c^*]$ are to be in the pooling effort interval. For this pooling group we now must make sure such effort interval is incentive compatible against the deviation of exerting more or less effort than $e^*$, make sure such $c^*$ is beyond $\widehat{c}$ (i.e., the peak of the non-modified bidding function) to induce a (weakly) monotonic bidding function, and also make sure at $c^*$ such participant is indifferent between optimal effort solved under *no pool* and the pooling interval payoff. We check each of these necessary conditions in turn.

Let the total prize be worth 1, as before. Let $\alpha > \frac{1}{2}$ be the second place share with the remainder being the first place share. Suppose there are $k$ players. Let $p$ be the measure of types in the pooling interval. Then, the expected payoff from bidding in the pooling interval is

$$\pi_{pool} = \underbrace{\sum_{i=1}^{k-1} \binom{k-1}{i} \frac{1}{i+1} p^i (1-p)^{k-1-i}}_{\text{1 or more other contestants pool}} + \underbrace{(1-p)^{k-1}(1-\alpha)}_{\text{No other contestant pool}}$$

We first need to verify the contestant $c^*$ at the end of the pooling interval (i.e., type $c^*$ where $p = F(c^*)$) is indifferent between pooling or exerting the identical effort $e^*$ under *no pool* bidding. When we set $p \equiv F(c^*)$, the payoff for $c^*$ under a *no pool* contest is as follows:

$$\pi_{no\ pool} = (k-1)\alpha p (1-p)^{k-2} + (1-\alpha)(1-p)^{k-1}$$

Hence, we require $\pi_{pool} = \pi_{no\ pool}$, thus we solve for the indifferent value of $p$

$$\begin{aligned}
\sum_{i=1}^{k-1} \binom{k-1}{i} \frac{1}{i+1} p^i (1-p)^{k-1-i} + (1-\alpha)(1-p)^{k-1} &= (k-1)\alpha p (1-p)^{k-2} + (1-\alpha)(1-p)^{k-1} \\
\sum_{i=1}^{k-1} \binom{k-1}{i} \frac{1}{i+1} p^i (1-p)^{k-1-i} &= (k-1)\alpha p (1-p)^{k-2}
\end{aligned}$$

Now divide by $(1-p)^{k-1}$ to obtain

$$\sum_{i=1}^{k-1} \binom{k-1}{i} \frac{1}{i+1} \left(\frac{p}{1-p}\right)^{i} = (k-1)\alpha \frac{p}{1-p}$$

Now, with a change of variable, let $z = \frac{p}{1-p}$ to obtain

$$\sum_{i=1}^{k-1} \binom{k-1}{i} \frac{1}{i+1} (z)^{i-1} = (k-1)\alpha$$

Now fix $\alpha$ and $k$. Then the LHS of the equality is strictly increasing in $z$,which is strictly increasing in $p$. Further at the limit as $p \to 0 \Rightarrow z \to 0$, the LHS converges to $\frac{k-1}{2}$, only the $i = 1$ term remains[1]. The RHS is then a greater finite number $(k-1)\alpha > \frac{k-1}{2}$ with $\alpha > .5$ as $p \to 0$.

Oppositely, as $p \to 1$ the LHS approaches $+\infty$,whereas the RHS is again a finite number. Hence, there exists a *unique* $p^* \in (0, 1)$ that solves the above equation. Solving for $p^*$ then determines both $e^*$ and $c^*$. Hence, once we fix $\alpha$ and $k$, we can always find our needed $c^*$ *uniquely.* Further, by meeting the above equality we have actually also met the $IC$ constraint, which we call $IC_{down}$, for preventing pooling types from deviating down; thus, we see $IC_{down}$ binds. Note also $p^*$ is increasing in $\alpha$. However, $p^*$ can be either increasing or decreasing in $k$ depending on the parameterization, as $k$ affects both $\alpha$ and $p$ (holding $\alpha$ constant) in a complex way.

Once we have $e^*$ and $c^*$ we already know any $c \in [c^*, \overline{c}]$ does not want to deviate, as they are already choosing their optimal effort per the *no pool* bidding structure. Meanwhile, any $c \in [\underline{c}, c^*)$ will not want to deviate by providing less effort than the pooling effort level because if it was not worth it for the $c^*$ type to do so, then it certainly is not worth it for the lower cost types. That is, in considering whether to exert less effort, the $c^*$ type trades off the saved cost of less effort with a reduced expected gross benefit. Thus, if the $c^*$ type's cost savings did not justify less effort, it certainly will not be justified for those with lower cost (savings) facing the same reduced expected benefit.

Now we need to check that a participant in the pooling interval does not want to deviate up, as doing so would guarantee a first prize. The payoff from deviating up is thus:

$$\pi_{up} = 1 - \alpha$$

[1] This can also be seen by taking the limit of the generating function of $\sum_{i=1}^{k-1} \binom{k-1}{i} \frac{1}{i+1} (z)^{i-1} \equiv \frac{(1+z)^k - (k)z - 1}{(k)z^2}$ as $z \to 0$.

Hence, we require that

$$\pi_{pool} - \pi_{up} \geq 0$$

Substituting.

$$\pi_{in} - \pi_{up} = \sum_{i=1}^{k-1} \binom{k-1}{i} \frac{1}{i+1} p^i (1-p)^{k-1-i} - \left(1 - (1-p)^{k-1}\right)(1-\alpha)$$

Now, recall by the binomial theorem:

$$1 - (1-p)^{k-1} = \sum_{i=1}^{k-1} \binom{k-1}{i} p^i (1-p)^{k-1-i}$$

Hence

$$\pi_{in} - \pi_{up} = \sum_{i=1}^{k-1} \binom{k-1}{i} \left(\frac{1}{i+1} - (1-\alpha)\right) p^i (1-p)^{k-1-i}$$

Clearly, if $\frac{1}{k} > (1-\alpha) \iff \alpha \geq 1 - \frac{1}{k}$, $IC_{up}$ is met, as it means all sums being added above are positive. This requirement simply says the second prize share $\alpha$ needs to be weakly greater than 1 minus the inverse the number of participants. Thus, this requirement increases in $k$; however, the optimal $\alpha^*$ is also increasing in $k$. It is then trivial if $\alpha = 1$ (i.e., there is only a second prize), $IC_{up}$ is met. However, this sufficient condition is obviously more than needed.

The precise requirement is readily found by solving the generating function of $\sum_{i=1}^{k-1} \binom{k-1}{i} \left(\frac{1}{i+1} - (1-\alpha)\right) p^i (1-p)^{k-1-i}$ :

$$\frac{(1-p)^{k-1}(p - 1 - kp\alpha + (\frac{1}{1-p})^{k-1}(1 + kp(\alpha - 1))}{kp}$$

The term of interest is $(p - 1 - kp\alpha + (\frac{1}{1-p})^{k-1}(1 + kp(\alpha-1))$, as this determines if the entire equation is (weakly) positive and thus $IC_{up}$ is met. We can then solve for when this term is (weakly) greater than zero:

$(p - 1 - kp\alpha + (\frac{1}{1-p})^{k-1}(1 + kp(\alpha-1)) \geq 0 \Rightarrow kp\alpha((\frac{1}{1-p})^{k-1} - 1) \geq 1 - p + (kp - 1)(\frac{1}{1-p})^{k-1} \Rightarrow \alpha \geq \frac{1-p}{kp((\frac{1}{1-p})^{k-1}-1)} + \frac{(kp-1)(\frac{1}{1-p})^{k-1}}{kp((\frac{1}{1-p})^{k-1}-1)} > 1 - \frac{1}{k}$, our first sufficient condition. Since the designer gets to choose $\alpha$, this condition can regardless always be met since any $\alpha \in [\frac{1-p}{kp((\frac{1}{1-p})^{k-1}-1)} + \frac{(kp-1)(\frac{1}{1-p})^{k-1}}{kp((\frac{1}{1-p})^{k-1}-1)}, 1]$ will satisfy $IC_{up}$. We later find a sufficient condition that will not only guarantee the GSPC provides more total revenue but also that $\alpha = 1$ is optimal (see appendix). Additionally, we will consider indivisible prizes, which means we again have $\alpha = 1$ or $\alpha = 0$. Finally, when we do allow for divisible prizes, we could allow that the designer to simply declare any observed effort greater than $e^*$ is still counted as $e^*$. Since effort is costly, no player would ever exert greater than $e^*$.

Lastly, we also need to check the type $c^* \geq \widehat{c}$. That is, we need to make sure the indifference point from where we end the pooling interval is *after* the single peak of the *no pool* bidding function; otherwise, we still have not solved the non-monotonicity problem. Recall our $IC_{down}$ condition was the following being (weakly) positive:

$$\sum_{i=1}^{k-1} \binom{k-1}{i} \frac{1}{i+1} (z)^{i-1} - (k-1)\alpha$$

We then substitute in $z = \frac{F(\widehat{c})}{1-F(\widehat{c})}$, where $F(\widehat{c}) = \frac{2\alpha-1}{k\alpha-1}$ as found in our first Lemma. If the expression is negative, it means $c^* > \widehat{c}$, since $IC_{down}$ is not yet met at $\widehat{c}$. That is, we need to choose a larger $p^* > F(\widehat{c})$ (since the above is strictly increasing in $z$, which is strictly increasing in $p$) to meet $IC_{down}$. But this then means we get $c^* > \widehat{c}$.

To see we always have $c^* > \widehat{c}$, first note $\frac{dF(\widehat{c})}{d\alpha} = \frac{\partial}{\partial\alpha} \frac{2\alpha-1}{k\alpha-1} = \frac{k(1-2\alpha)-1}{(k\alpha-1)^2} < 0$ (for $\alpha \geq .5$). This then means $\frac{\partial z}{\partial \alpha} < 0$ when evaluated at $c = \widehat{c}$ since $z$ is strictly increasing in $p \equiv F(c)$. Now we take the derivative of our $IC_{down}$ condition with respect to $\alpha$ and consider its value when evaluated at $c = \widehat{c}$:

$$\frac{d}{d\alpha}\left(\sum_{i=1}^{k-1} \binom{k-1}{i} \frac{1}{i+1} (z)^{i-1} - (k-1)\alpha\right) = \underbrace{\sum_{i=1}^{k-1} \binom{k-1}{i} \frac{1}{i+1}(i-1)(z)^{i-2} \underbrace{\frac{\partial z}{\partial \alpha}}_{<0}}_{\leq 0} + \underbrace{-(k-1)}_{<0} < 0$$

Hence, our $IC_{down}$ condition is strictly decreasing in $\alpha$. This means if we can show such expression is non positive at $\alpha = .5$, we are done. Recall, as we already showed, when $\alpha = .5$, we get $z = 0 \Rightarrow \sum_{i=1}^{k-1} \binom{k-1}{i} \frac{1}{i+1} (z)^{i-1} \to \frac{k-1}{2}$. But this then means $\sum_{i=1}^{k-1} \binom{k-1}{i} \frac{1}{i+1} (z)^{i-1} - (k-1)\frac{1}{2} = \frac{k-1}{2} - \frac{k}{2} + \frac{1}{2} = 0$ with $\alpha = .5$. Hence, since $\sum_{i=1}^{k-1} \binom{k-1}{i} \frac{1}{i+1} (z)^{i-1} - (k-1)\alpha$ is strictly decreasing in $\alpha$, it has to be the case for any $\alpha > .5$ we get $c^* > \widehat{c}$.$\square$

Also note if $\alpha \leq .5$, then the GSPC collapses to *no pool* since the bidding function is then monotonic and thus the pooling interval is zero. Therefore, we now see our generalization not only allows any allocation of second and first prize, but, in particular, it also allows us to offer *only* a second prize. Having proven the existence of the GSPC, we now show when it is superior to limiting a contest to have a second prize no greater than a first prize.

# 5 GSPC With Divisible Prizes

In the case of divisible prizes it is then natural to solve for an optimal $\alpha^*$ such that we maximize revenue. That is, we solve:

$$\max_{\alpha \in [0,1]} R(\alpha) = k \int_{\underline{c}}^{\overline{c}} g(A(c) + \alpha(B(c) - A(c))) \times F'(c) dc$$

If we extend the designer's benefit function to be concave, we then have:

$$\max_{\alpha \in [0,1]} R(\alpha) = k \int_{\underline{c}}^{\overline{c}} h(g(A(c) + \alpha(B(c) - A(c)))) \times F'(c) dc$$

Here we introduce $h(\cdot)$ to be the designer's benefit function, whereas in the linear designer benefit case we simply get $h(x) = x$. Note having $h(\cdot)$ concave and participant effort costs linear is mathematically equivalent to having convex participant costs and linear designer benefits. Since having both convex costs and concave benefits complicates the analysis without much added insight, we will focus on linear designer benefit and convex participant costs for the balance of the paper. However, we will provide some examples of concave benefits.

We can find $\alpha^*$ by taking the first order condition and solving for $\alpha^*$:

$$R'(\alpha) = k \int_{\underline{c}}^{\overline{c}} g'(A(c) + \alpha(B(c) - A(c)))(B(c) - A(c)) \times F'(c) dc \equiv 0$$

Unfortunately, $\alpha^*$ must be solved for numerically, thus requiring a given set of parameters for a given problem. Here are some examples of the optimal $\alpha$ given the total participants $k$ and $< w, z >$, where the contest designer's benefit function is $y^w$

and the contestant's *total* cost function is $c \cdot x^z$. Throughout our examples we again assume $c$ is distributed uniform such that $c \in U[.5, 1]$:

| Optimal Second Prize $\alpha^* \in [0,1]$ | | | | |
|---|---|---|---|---|
| benefit/cost | $k$ participants | | | |
| $< w, z >$ | 3 | 4 | 5 | 10 |
| $(1,2)$ | .38 | .54 | .64 | .84 |
| $(1,3)$ | .50 | .65 | .74 | .89 |
| $(.5,2)$ | .54 | .69 | .77 | .91 |

For example, with $k = 3$ participants, functional form of participant cost $c{\cdot}\gamma(x) = c{\cdot}x^2$ and designer concave benefit of $y^{.5}$, the optimal prize allocation is 54% to second and 46% to first. This gives us a ratio of prizes of $\frac{.54}{.46}$,or about a 17% greater second prize.

Thus, with a bit more convexity of cost, concavity of designer benefit or more participants, we can quickly get the optimal allocation being great than 50% to second prize. However, these optimal $\alpha^*$ are based on an unfeasible bidding function due to its non-monotonicity. We now consider when can we know the feasible contest with a larger 2nd prize, the GSPC, does indeed provide greater revenue than restricting $\alpha \leq .5$. First we need a Lemma for our proposition of when this is so.

**Lemma 3** *A sufficient and necessary condition for $\widetilde{R}(\alpha) > R(\frac{1}{2})$ for some $\alpha \in (\frac{1}{2}, 1]$ is $R'(\frac{1}{2}) > 0$.*

Proof: First note if $R'(\frac{1}{2}) > 0$, then $\widetilde{R}'(\frac{1}{2}) > 0$ since $\widetilde{R}(\alpha)$ and $R(\alpha)$ are the same function for all $\alpha \in [0, \frac{1}{2}]$. Also, as shown in the appendix, the GSPC revenue function $\widetilde{R}(\alpha)$ is strictly concave in $\alpha$. This then means, due to concavity, if $\widetilde{R}'(\frac{1}{2}) > 0$, it has to be the case $\widetilde{R}(\alpha) > R(\frac{1}{2}) = \widetilde{R}(\frac{1}{2})$ for some $\alpha \in (\frac{1}{2}, 1]$. To see necessity of this result, note if $\widetilde{R}'(\frac{1}{2}) \leq 0$ it cannot be the case $\widetilde{R}(\alpha) > R(\frac{1}{2}) = \widetilde{R}(\frac{1}{2})$, again due to the concavity of $\widetilde{R}(\alpha)$ in $\alpha$. □

**Corollary 4** *When $\alpha^* \leq \frac{1}{2}$ binds under a classical contest (i.e., $R'(\frac{1}{2}) > 0$), offering a larger second prize through the GSPC always yields more total revenue than restricting $\alpha = .5$.*

This Corollary then gives the desired result that whenever we would optimally offer a larger second prize under the classical contest (but we cannot because it is not feasible), the GSPC, which is always feasible, will provide more total effort through using this mechanism with a larger second prize. In other words, it is never optimal to force $\alpha \leq .5$. We are using the term revenue over effort to accommodate that under concave designer benefit functions it is total revenue and not effort *per se* that we are maximizing.

**Proposition 5** *With sufficient number of participants, offering a larger second prize through the GSPC yields more total revenue than offering a (weakly) larger first prize.*

Proof: see appendix.

Though this is a limit result, it need not take many participants to optimally choose a larger second prize under the GSPC; as the table above shows, having as few as $k = 3$ participants yields a larger optimal second prize (i.e., $\alpha^* > .5 \iff R'\left(\frac{1}{2}\right) > 0$).

Now many prizes in practice aren't readily divided up into equal (or even multiple prizes). For example, take the position of CEO. A firm would not (likely) want to divide this into 10 smaller equal positions due to (presumed) synergy of the CEO multi-tasking. Also, we could think of certain prizes costing the designer in terms of both a fixed and variable cost for each prize unit offered. With sufficient fixed costs, the designer will want to limit the number of prizes, maybe even only offering a single prize. We now seek to answer the question *when is it better under an indivisible prize to offer it to second place over first place.*

# 6 GSPC With an Indivisible Prize

From our previous analysis of divisible prizes, we have an obvious condition for offering only a second prize versus first prize being optimal if $\widetilde{R}(1) > 0$. However, this is more than is needed. There may be some $0 < \alpha < 1$ such that $\widetilde{R}(\alpha) > \widetilde{R}(0)$ and yet will still have $\widetilde{R}(1) > \widetilde{R}(0)$. Indeed, as long as $\alpha < 1$ is great enough, we will still have $\widetilde{R}(1) > \widetilde{R}(0)$ (i.e., even though $\widetilde{R}'(1) < 0$).

We now consider a variety of examples comparing the revenue of offering only a first prize versus only a second prize. We as before assume total cost is $cx^z$ for effort $x$ and cost type $c \in U[.5, 1]$. The designer's benefit function is simply $y^w$, where $y$ is a contestant's total effort. We report the increased revenue in the table below.

Strikingly, total revenue is increased some 10 to 20% once we have four or five contestants by offering a prize only to second place over first place. The intuition is although shifting the prize from first to second causes us to lose some effort from the lowest cost types, we enjoy increased effort form all the "average" and "high" cost types, which more than offset the reduced effort of the low types. These "average" types exert more effort simply because they now have a better chance of winning a now larger second prize.

| $\frac{\text{Revenue of } \alpha=1}{\text{Revenue of } \alpha=0} - 1$ | | | | |
|---|---|---|---|---|
| benefit/cost | $k$ participants | | | |
| $< w, z >$ | 3 | 4 | 5 | 10 |
| $(1, 2)$ | 1.9% | 6.4% | 9.4% | 16.4% |
| $(1, 3)$ | 6.4% | 11.3% | 14.5% | 21.2% |
| $(.5, 2)$ | 7.2% | 11.9% | 14.9% | 21.3% |

Also crucial is convexity of costs over linear costs (or, equivalently, concavity of designer benefit over linear benefit in the face of linear participant costs). Moldovanu and Sela (2001) show with linear costs (and only linear benefits) offering a winner-takes-all (i.e., $\alpha = 0$) is optimal. However, under convex costs (or concave designer benefits), the importance of further incentivizing the "average" and "high" cost types is increased since effort costs are now increasingly costly; it is providing the prize now to second over first that increases such incentives. To the extent such increased effort from such increased incentivizing offsets the reduced effort from the low cost types, it is optimal to then only offer the second prize.

It may seem a bit strange to think of only offering a second prize and no first prize. However, recall with the GSPC our pooling interval: as long as two or participants pool, these lowest (cost) types will have a chance of receiving the sole second prize.

Now that we have developed some intuition of our GSPC, as well as numerical examples showing its potential to increase revenue over a classical contests that limits $\alpha \leq .5$ (or requires $\alpha = 0$ or $\alpha = 1$ due to indivisibility), we turn to some applications-both in the case of divisible and indivisible prizes.

# 7 Applications

We consider three classes of examples: positive divisible prizes, negative divisible prizes, and (positive) indivisible prizes. For all these examples we simply assume contestant cost is convex (or equivalently designer benefit function is negative).

A quintessential example of a positive divisible prize is the organization setting of incentivizing sales people. If we assume a world where fellow sales people do not know each other's differential cost of effort, and do not observe each others level of actual effort until results are made known (e.g., geographically separated sales people), we have the setting of a divisible second prize contestant. That is, with sufficient number of sales people, sufficient convex sales person cost of effort, a sufficient concave firm (or sales manager) benefit function, we will want to reward a larger second than first prize. Recall, however, with our GSPC, the mechanism calls for a maximal effort pooling interval where if two or more provide such an effort level, they have equal chance of garnering a first or second prize. Thus, if two or more sales people have a great deal of sales, it will be a push as to which gets which sales award. However, if all provide lower effort, then the second most effort should get more of a prize than first.

For the second class of examples, consider firms competing on lowering their toxic emissions to limit regulation penalties. That is, assume even the firm with the greatest reduction will still face some cost or tax from the regulator. If firms have convex costs (or the regulator has a concave benefit function over total effort) and the regulator wants to lower total emissions, letting the second most toxin reducing firm off a bit lighter (i.e., assessing a lesser penalty for the second most toxin reducing firm) will actually create more overall reduction of emissions.

Similarly, we can consider an NGO making demands of a firm for their level of corporate social responsibility (CSR) commitment. If the NGO demands a greater level (and thus cost) from the most able provider of CSR than from the second most able (i.e., the NGO is toughest against the most "responsible" firm), the greater will be overall firm level CSR activity, again assuming privately known costs.

Finally, consider the provision of a single prize: the awarding of a patent. If the patent office values total innovative activity, by awarding the patent to the inventor providing the second greatest effort, more total effort will be exerted by inventors. If the order of beginning effort on innovation is not observable, we could think of this as the second mover getting the patent over the first mover. That is, this scheme of the second mover getting the patent creates more overall innovative activity.

# 8 Conclusion

Whether it be business, politics, or life, much is actually a contest. As such, it makes sense to consider how to best design a contest when we face contestants with realistic convex costs and/ or the designer having concave benefit over effort. When we do so, we find we often will actually want to offer a greater second over first prize, if not a sole second prize. Indeed, as long as we have sufficient number of participants, we are assured offering a greater second prize is optimal.

Unfortunately, bidding under such incentivizing becomes non-monotonic. For such problem we found a solution we dubbed the generalized second prize contest (GSPC) mechanism. We then saw how we can create greater revenue through this GSPC.

We also considered just a few of many examples where the GSPC could be applied to create more total revenue. One challenge, however, is in practice, we rarely see non-monotonic prize ordering. Consequently, to further explore its potential, we are currently planning a series of experiments on the effects of larger second prizes.

Another consideration is we did not consider dynamic contests. So too, we are beginning some work to discover how robust offering larger second prizes is when we have contestants interacting repeatedly over time. Both reputation building and monitoring begin to be a factor.

Finally, we only considered the case of two prizes. It would be interesting to expand our exploration and consider the question when we can offer $k$ prizes with $n \geq k$ contestants, which prize should be largest? What about if prizes are indivisible- which place should receive the sole prize? We suspect we will find a $k$ prize analog of our results.

Whatever the case, we hope this notion of second best often being better than first best spurs further research and exploration.

# 9 Appendix

**Lemma 1:** *If* $\alpha > .5$, *the contestant bidding function becomes single peaked with a maximum at* $\widehat{c}$ *such that* $F(\widehat{c}) = \frac{2\alpha-1}{k\alpha-1}$

Proof: We first write the bidding function as $b(\alpha, c) = g((1-\alpha)A(c) + \alpha B(c))$, where $g(\cdot)^{-1} = \gamma(\cdot)$ (i.e., $g(\cdot)$ is the inverse of the cost function). Now note $\frac{d}{dc}g((1-\alpha)A(c)+\alpha B(c)) = \underbrace{g'((1-\alpha)A(c)+\alpha B(c))}_{>0}[(1-\alpha)A'(c)+\alpha B'(c)]$. The former term is always positive for $c \in [\underline{c}, \overline{c})$ since $g(\cdot)$ is strictly increasing and $(1-\alpha)A(c)+\alpha B(c)$ is always positive. The latter term, we will see, is single peaked, thus making our entire expression single peaked. Expanding $(1-\alpha)A'(c)+\alpha B'(c)$, we get:

$$(1-\alpha)(-(k-1)\frac{1}{c}(1-F(c))^{k-2}\times F'(c)+\alpha((k-1)\frac{1}{c}(1-F(c))^{k-3}\times[(1-(k-1)F(c)]\times F'(c)$$

Rearranging terms then gives$\underbrace{(k-1)\frac{1}{c}(1-F(c))^{k-3}\times F'(c)\times}_{>0}[(F(c)-1)(1-\alpha)+\alpha(1-(k-1)F(c)]$. The former term is always positive so we only focus on the latter term, which further rearranging gives: $F(c)-\alpha F(c)-1+\alpha+\alpha-k\alpha F(c)+\alpha F(c) = -k\alpha F(c)-(1-F(c))+2\alpha$

First note at the lowest cost type $\underline{c}$,we get simply $2\alpha-1$, which is always positive for $\alpha > .5$ at $c = \underline{c}$. That is, our bidding function is increasing at the lowest cost type. Similarly, with the highest cost type, we get $-k\alpha+2\alpha$, which is always negative for $k \geq 3$. Thus, our bidding function is decreasing at the highest cost type.

Now solving the above for a unique zero gives: $-k\alpha F(c)-(1-F(c))+2\alpha \equiv 0 \Rightarrow (k\alpha-1)F(c) = 2\alpha-1 \Rightarrow F(c) = \frac{2\alpha-1}{k\alpha-1}$. Define then $\widehat{c}$ such that $F(\widehat{c}) = \frac{2\alpha-1}{k\alpha-1}$.

Now when $c \in [\underline{c}, \widehat{c})$ we have $a \equiv F(c)$:

$$\underbrace{(1-k\alpha)}_{<0}a+2\alpha-1$$

Once we fix $\alpha$ and $k$, we see the above expression, which then determines the sign of the derivative of the bidding function, is strictly *decreasing* in $a$. At $a = \frac{2\alpha-1}{k\alpha-1}$, the above expression equals zero. Meanwhile, with $a \in [\underline{c}, \widehat{c})$ the expression is positive and with $a \in (\widehat{c}, \overline{c}]$ the expression is negative. Hence, the bidding function is single peaked at $\widehat{c}$. Thus, our type $\underline{c} < \widehat{c} < \overline{c}$ provides the highest effort over all types. $\square$.

**Lemma 6** *$R$ and $\widetilde{R}$ are concave in $\alpha$*

Proof: Recall $R(\alpha) = k\int_{\underline{c}}^{\bar{c}} g(A(c) + \alpha(B(c) - A(c))) \times F'(c)dc.$

Taking the first derivative with respect to $\alpha$ yields:

$R'(\alpha) = k\int_{\underline{c}}^{\bar{c}} g'(A(c) + \alpha(B(c) - A(c))) \times (B(c) - A(c))) \times F'(c)dc.$

Taking the derivative again with respect to $\alpha$ yields:

$R''(\alpha) = k\int_{\underline{c}}^{\bar{c}} g''(A(c) + \alpha(B(c) - A(c))) \times (B(c) - A(c)))^2 \times F'(c)dc.$

Since $g''(\cdot) < 0$ (i.e., because $g(\cdot)$ is concave), we get $R''(\alpha) < 0$, as desired.

Extending this to the GSPC, note $\widetilde{R}(\alpha)$ is the same as $R(\alpha)$ is the same as above except rather than the integral having the lower bound of $\underline{c}$ it is $c^*$, the marginal type to pool. In addition, our $F(c)$ has an atom at $c^*$. In particular, $F'(c^*) \equiv f(c^*) = p^*$. Thus, our term $g''(A(c) + \alpha(B(c) - A(c))) \times (B(c) - A(c)))^2$ is weighted by $p^*$ at $c^*$. Hence, all terms remain negative, which means, $R''(\alpha) < 0$,as desired. □

**Lemma 7** *$F(c^{**})$ is bounded above by $\frac{2}{k}$*

Proof: First recall by definition $B(c^{**}) - A(c^{**}) = 0$. We write $B(c) - A(c) = (k-1)\int_c^{\bar{c}} \frac{(1-F(a))^{k-3}}{a} \times (k \cdot F(a) - 2)\, F'(a)da$. Note when $F(c) = \frac{2}{k}$, we have $\frac{(1-F(a))^{k-3}}{a} \times (k \cdot F(a) - 2)\, F'(a) = 0$, which means $B(c) - A(c) = (k-1)\int_c^{\bar{c}} \frac{(1-F(a))^{k-3}}{a} \times (k \cdot F(a) - 2)\, F'(a)da > 0$, since then all the terms under the integral are positive. But this then means it must be $F(c^{**}) < \frac{2}{k}$ since we require $B(c^{**}) - A(c^{**}) = (k-1)\int_{c^{**}}^{\bar{c}} \frac{(1-F(a))^{k-3}}{a} \times (k \cdot F(a) - 2)\, F'(a)da = 0$. Thus, $F(c^{**})$ is bounded above by $\frac{2}{k}$ and $c^{**}$ is approaching $\underline{c}$ in the limit. □

**Proposition 5** *With sufficient number of participants, offering a larger second prize through the GSPC yields more total revenue than offering a (weakly) larger first prize.*

Proof Sketch: We want to show there exists some $k^*$ such that we are assured $R'(\frac{1}{2}) > 0$. First we write $R'(\frac{1}{2}) = k\int_{\underline{c}}^{\bar{c}} g'(\frac{1}{2}(A(c) + B(c))) \times (B(c) - A(c))) \times$

$F'(c)dc$. An increase in $k$ increases the derivative of the revenue function in two ways (beyond simply making the integral result larger). First, it reduces the bid, which is $\frac{1}{2}(A(c)+B(c))$, for all but the lowest of cost types, which means $g'(\cdot)$ then becomes larger (since $g(\cdot)$ is concave). To see this, first note $A(c) + B(c) = (k-1)(k-2)\int_c^{\overline{c}} \frac{(1-F(a))^{k-3}}{a} F(a)F'(a)da$. For large $k$, this becomes approximately $k^2 \int_c^{\overline{c}} \frac{(1-F(a))^k}{a} F(a)F'(a)da$. We then have $\frac{\partial}{\partial k}\left[k^2 \int_c^{\overline{c}} \frac{(1-F(a))^k}{a} F(a)F'(a)da\right] = k^2 \int_c^{\overline{c}} \frac{(1-F(a))^k}{a} F(a)Log(1-F(a))F'(a)da + 2k\int_c^{\overline{c}} \frac{(1-F(a))^k}{a} F(a)F'(a)da$,which is increasingly negative in $c$ and negative for all but the lowest cost of types (i.e., $Log[1-F(a)]$ ranges from 0 to $-\infty$ from $\underline{c}$ to $\overline{c}$, respectively). Second, as shown in the previous Lemma, an increase in $k$ reduces $F(c^{**})$. Note we then have $B(c)-A(c)<0$ when $c<c^{**}$ and $B(c)-A(c)>0$ with $c>c^{**}$. Therefore, increasing $k$ *decreases* the region for which $g'(\cdot)$ is multiplied by a negative term (i.e., by $B(c)-A(c)$ when $c<c^{**}$), and in the limit $B(c)-A(c)\geq 0$ almost everywhere.

That is, we have as $k\to\infty$, $(B(c)-A(c)))\geq 0$ almost everywhere and we have $g'(\cdot)$ is increasing in $k$, which means there must exist some $k$ such that $R'(\frac{1}{2})>0$. □

Now we consider a sufficient, but not necessary condition, that yields the GSPC always provides more total revenue than offering a weakly larger first prize. Additionally, it provides the surprising case that even in the case of divisible prizes it is optimal to offer only a second prize under GSPC, and thus also gives under indivisible prizes, offering only second place a prize dominates offering only first place a prize.

**Sufficient Condition**: If $k$ is adequately large, or (participant) costs are convex enough, or (designer) benefits are concave enough, such that $F(c^{**})<p^*$,where $c^{**}$ is defined as the type that bids equally under a second only or first only prize (i.e., $A(c^{**})=B(c^{**})$), our sufficient condition is met. That is, if the type indifferent between bidding under the pooling interval and *no pool* contest is greater (i.e., higher cost) than the type indifferent between bidding under a first only versus second only prize, the condition is met. Note also we may have linear costs and concave benefits or convex costs and linear benefits as opposed to both concavity and convexity, respectively.

**Proposition 8** *If our sufficient condition is met, we have* $\widetilde{R}(\alpha^*)>R(\frac{1}{2})$, *whenever* $\alpha^*>\frac{1}{2}$. *That is, the GSPC provides more total revenue than limiting the contest to having no greater second than first prize.*

**Proof:**

By assumption $R(\alpha^*) > R(\frac{1}{2}) = \widetilde{R}(\frac{1}{2})$. (recall whenever $\alpha^* \leq \frac{1}{2}$ the GSPC collapses to the traditional contest with no pooling interval).

Now we have $\widetilde{R}'(\alpha) = k \int\limits_{c*}^{\bar{c}} g'(A(c) + \alpha(B(c) - A(c))) \times (B(c) - A(c)) \times F'(c)dc$, where there is an atom at $c^*$ such that $F'(c^*) = p^*$ . Since $F(c^{**}) < p^* = F(c^*)$, we know $c^{**} < c^*$. But this then means each term under the integral $\int\limits_{c*}^{\bar{c}} g'(A(c) + \alpha(B(c) - A(c))) \times (B(c) - A(c)) \times F'(c)$ is positive since with $c^* > c^{**}$ we have $(B(c) - A(c)) > 0$ and $g'(A(c) + \alpha(B(c) - A(c))) > 0$, since all bids are weakly positive and $g(\cdot)$ is increasing. The lower bound of the integral's term is also then a positive term multiplied by a positive measure. Hence, by definition this then means $\widetilde{R}'(\alpha^*) > 0$. Since $\widetilde{R}(\alpha)$ is concave in $\alpha$ (see above) it must then be $\widetilde{R}(\alpha^*) > \widetilde{R}(\frac{1}{2}) = R(\frac{1}{2})$. □

This proposition then gives another important implication: when our condition is met it is best to offer $\alpha = 1$, as we show in our corollary.

**Corollary 9** *If our sufficient condition is met, $\widetilde{R}(1) \geq \widetilde{R}(\alpha)$ for all $\alpha \in [0, 1]$. That is, the optimal prize allocation is to give a sole second prize and no first prize. Consequently, in this setting, under indivisible contests, it is better to only offer a second over first prize.*

**Proof:**

From the argument in our previous Proposition, we then see $\widetilde{R}(\alpha) > 0$ since all the terms under the integral of the derivative of the revenue function are positive for all $\alpha \in [0, 1]$ (with our sufficient condition being met). But then since $\widetilde{R}(\alpha)$ is concave in $\alpha$ (see appendix), it has to be that $\widetilde{R}(1) \geq \widetilde{R}(\alpha)$ for all $\alpha \in [0, 1]$. □

# References

[1] Bulow, Jeremey and Paul Klemperer. 1999. "The Generalized War of Attrition." The American Economic Review, Vol. 89, No. 1 , pp. 175-189

[2] Moldovanu, Benny and Aner Sela. 2001. "The Optimal Allocation of Prizes in Contests." The American Economic Review, Vol. 91, No. 3, pp. 542-558

[3] Tullock, Gordon, "Efficient Rent Seeking," in J. Buchanan, R. Tollison, and G. Tullock, eds., Toward a Theory of the Rent-Seeking Society, College Station: Texas A&M University Press, 1980.