Von Neumann-Morgenstern Farsightedly Stable Sets in Two-Sided Matching †

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Abstract

We adopt the notion of von Neumann-Morgenstern (vNM) farsightedly stable sets to predict which matchings are possibly stable when agents are farsighted in one-to-one matching problems. We provide the characterization of vNM farsightedly stable sets: a set of matchings is a vNM farsightedly stable set if and only if it is a singleton set and its element is a corewise stable matching. Thus, contrary to the vNM (myopically) stable sets [Ehlers, *J. of Econ. Theory* 134 (2007), 537-547], vNM farsightedly stable sets cannot include matchings that are not corewise stable. Moreover, we show that our main result is robust to many-to-one matching problems with responsive preferences.

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1 Introduction

Gale and Shapley (1962) have proposed the simple two-sided matching model, known as the marriage problem, in which matchings are one-to-one. There are two disjoint sets of agents, men and women, and the problem is to match agents from one side of the market with agents from the other side where each agent has the possibility of remaining single. They have shown that the set of corewise stable matchings is nonempty. A matching is corewise stable if there is no subset of agents who by forming all their partnerships only among themselves (and having the possibility of becoming single), can all obtain a strictly preferred set of partners.[1] Recently, Ehlers (2007) has characterized the von Neumann-Morgenstern stable sets (hereafter, vNM stable sets) in one-to-one matching problems. A set of matchings is a vNM stable set if this set satisfies two robustness conditions: (internal stability) no matching inside the set is dominated by a matching belonging to the set; (external stability) any matching outside the set is dominated by some matching belonging to the set. Ehlers has shown that the set of corewise stable matchings is a subset of any vNM stable set.

The notions of corewise stability and of vNM stable set are myopic notions since the agents cannot be farsighted in the sense that individual and coalitional deviations cannot be countered by subsequent deviations. An interesting contribution is Diamantoudi and Xue (2003) who have investigated farsighted stability in hedonic games (of which one-to-one matching problems are a special case) by introducing the notion of the coalitional largest farsighted conservative stable set which coincides with the largest consistent set due to Chwe (1994).[2] The largest consistent set is based on the indirect dominance relation which captures the fact that farsighted agents consider the end matching that their move(s) may lead to. Diamantoudi and Xue (2003) have shown that in hedonic games with strict preferences core partitions are always contained in the largest consistent set.

In this paper, we adopt the notion of von Neumann-Morgenstern farsightedly stable sets (hereafter, vNM farsightedly stable sets) to predict which matchings are possibly stable when agents are farsighted. This concept has been studied by Chwe

[1] We refer to Roth and Sotomayor (1990) for a comprehensive overview on two-sided matching problems (marriage problems and college admissions or hospital-intern problems).

[2] Other approaches to farsightedness in coalition and/or network formation are suggested by the work of Xue (1998), or Herings, Mauleon, and Vannetelbosch (2004, 2009).

(1994) who introduced the notion of indirect dominance into the standard definition of vNM stable sets. Thus, a set of matchings is a vNM farsightedly stable set if no matching inside the set is *indirectly* dominated by a matching belonging to the set (internal stability), and any matching outside the set is *indirectly* dominated by some matching belonging to the set (external stability).

Our main result is the characterization of vNM farsightedly stable sets in one-to-one matching problems. A set of matchings is a vNM farsightedly stable set if and only if it is a singleton set and its element is a corewise stable matching. Thus, contrary to the vNM (myopically) stable sets, vNM farsightedly stable sets cannot include matchings that are not corewise stable ones. In other words, we provide an alternative characterization of the core in one-to-one matching problems. Finally, we show that our main result is robust to many-to-one matching problems with responsive preferences: a set of matchings is a vNM farsightedly stable set if and only if it is a singleton set and its element is a setwise stable matching.

With respect to other farsighted concepts of stability, we show that matchings that do not belong to any vNM farsightedly stable sets (hence, that are not corewise stable matchings) may belong to the largest consistent set.

The paper is organized as follows. Section 2 introduces one-to-one matching problems and standard notions of stability. Section 3 defines vNM farsightedly stable sets. Section 4 provides the characterization of vNM farsightedly stable sets in one-to-one matching problems. Section 5 deals with many-to-one matching problems. Section 6 concludes.

2 One-to-one matching problems

A one-to-one matching problem consists of a set of N agents divided into a set of men, $M = \{m_1, ..., m_r\}$, and a set of women, $W = \{w_1, ..., w_s\}$, where possibly $r \neq s$. We sometimes denote a generic agent by i and a generic man and a generic woman by m and w, respectively. Each agent has a complete and transitive preference ordering over the agents on the other side of the market and the prospect of being alone. Preferences are assumed to be strict. Let P be a preference profile specifying for each man $m \in M$ a strict preference ordering over $W \cup \{m\}$ and for each woman $w \in W$ a strict preference ordering over $M \cup \{w\}$: $P = \{P(m_1), ..., P(m_r), P(w_1), ..., P(w_s)\}$, where $P(i)$ is agent i's strict preference ordering over the agents on the other side of the market and himself (or herself). For instance, $P(w) = m_4, m_1, w, m_2, m_3, .., m_r$ indicates that woman w prefers m_4 to m_1 and she prefers to remain single rather than to marry anyone else. We write $m \succ_w m'$ if woman w strictly prefers m to m', $m \sim_w m'$ if w is indifferent between m and m', and $m \succeq_w m'$ if $m \succ_w m'$ or $m \sim_w m'$. Similarly, we write $w \succ_m w'$, $w \sim_m w'$, and $w \succeq_m w'$. A one-to-one matching market is simply a triple (M, W, P).

A matching is a function $\mu : N \to N$ satisfying the following properties: (i) $\forall m \in M$, $\mu(m) \in W \cup \{m\}$; (ii) $\forall w \in W$, $\mu(w) \in M \cup \{w\}$; and (iii) $\forall i \in N$, $\mu(\mu(i)) = i$. We denote by $\mathcal{M}$ the set of all matchings. Given matching μ, an agent i is said to be unmatched or single if $\mu(i) = i$. A matching μ is *individually rational* if each agent is acceptable to his or her mate, i.e. $\mu(i) \succeq_i i$ for all $i \in N$. For a given matching μ, a pair $\{m, w\}$ is said to form a blocking pair if they are not matched to one another but prefer one another to their mates at μ, i.e. $w \succ_m \mu(m)$ and $m \succ_w \mu(w)$. A coalition S is a subset of the set of agents N.[3]

Definition 1 (corewise enforceability) *Given a matching μ, a coalition $S \subseteq N$ is said to be able to enforce a matching μ' over μ if the following conditions hold:*

(i) *$\mu'(i) \notin \{\mu(i), i\}$ implies $\{i, \mu'(i)\} \subseteq S$ and*

(ii) *$\mu'(i) = i \neq \mu(i)$ implies $\{i, \mu(i)\} \cap S \neq \varnothing$.*

Condition (i) says that any new match in μ' that does not exist in μ should be between players in S, and condition (ii) states that in order to destroy an existing match in μ, one of the two players involved in that match should belong to coalition S. Notice that the concept of enforceability is independent of preferences.

Definition 2 *A matching μ is directly dominated by μ', or $\mu < \mu'$, if there exists a coalition $S \subseteq N$ of agents such that $\mu' \succ_i \mu$ $\forall i \in S$ and S can enforce μ' over μ.*

Definition 2 gives us the definition of direct dominance. The direct dominance relation is denoted by $<$. A matching μ is *corewise stable* if there is no subset of agents who by forming all their partnerships only among themselves, possibly dissolving some partnerships of μ, can all obtain a strictly preferred set of partners. Formally, a matching μ is corewise stable if μ is not directly dominated by any other

[3]Throughout the paper we use the notation $\subseteq$ for weak inclusion and $\subsetneq$ for strict inclusion.

matching $\mu' \in \mathcal{M}$.[4] Let $C(<)$ denote the set of corewise stable matchings. Gale and Shapley (1962) have proved that the set of corewise stable matchings is non-empty. Sotomayor (1996) has provided a non-constructive elementary proof of the existence of stable marriages.[5]

Another concept used to study one-to-one matching problems is the vNM stable set (von Neumann and Morgenstern, 1944), a set-valued concept, that imposes both internal and external stability. A set of matchings is a vNM stable set if (internal stability) no matching inside the set is directly dominated by a matching belonging to the set, and (external stability) any matching outside the set is directly dominated by some matching belonging to the set.

Definition 3 *A set of matchings* $V \subseteq \mathcal{M}$ *is a vNM stable set if*

(i) *for all* $\mu \in V$*, there does not exist* $\mu' \in V$ *such that* $\mu' > \mu$*;*

(ii) *for all* $\mu' \notin V$ *there exists* $\mu \in V$ *such that* $\mu > \mu'$*.*

Definition 3 gives us the definition of a vNM stable set $V(<)$. Let $\mathcal{V}(<)$ be the set of all vNM stable sets. Ehlers (2007) has studied the properties of the vNM stable sets in one-to-one matching problems. He has shown that the set of corewise stable matchings is a subset of any vNM stable set. Example 1 illustrates his main result.

Example 1 (Ehlers, 2005) *Let* $M = \{m_1, m_2, m_3\}$ *and* $W = \{w_1, w_2, w_3\}$*. Let* P *be such that:*

$P(m_1)$	$P(m_2)$	$P(m_3)$	$P(w_1)$	$P(w_2)$	$P(w_3)$
w_1	w_2	w_3	m_2	m_3	m_1
w_2	w_3	w_1	m_3	m_1	m_2
m_1	m_2	m_3	w_1	w_2	w_3
w_3	w_1	w_2	m_1	m_2	m_3

[4] Setting $|S| \leq 2$ in the definition of corewise stability, we obtain Gale and Shapley's (1962) concept of pairwise stability that is equivalent to corewise stability in one-to-one matchings with strict preferences.

[5] Roth and Vande Vate (1990) have demonstrated that, starting from an arbitrary matching, the process of allowing randomly chosen blocking pairs to match will converge to a corewise stable matching with probability one in the marriage problem. In relation, Jackson and Watts (2002) have shown that if preferences are strict, then the set of stochastically stable matchings coincides with the set of corewise stable matchings.

Let

$$\mu = \begin{pmatrix} m_1 & m_2 & m_3 \\ w_1 & w_2 & w_3 \end{pmatrix},\ \mu' = \begin{pmatrix} m_1 & m_2 & m_3 \\ w_2 & w_3 & w_1 \end{pmatrix},\ \mu'' = \begin{pmatrix} m_1 & m_2 & m_3 \\ w_3 & w_1 & w_2 \end{pmatrix}.$$

It can be shown that the set of corewise stable matchings is $C = \{\mu'\}$ *and* $V = \{\mu, \mu', \mu''\}$ *is the unique vNM stable set.*□

In Example 1 the matchings μ and μ'' belong to the unique vNM stable set because μ' does not directly dominate neither μ nor μ'' even though μ and μ'' are not individually rational matchings (either all women or all men prefer to become single). However, farsighted women may decide first to become single in the expectation that further marriages will be formed leading to μ'. The women prefer μ' to μ and once everybody is divorced, men and women prefer μ' to the situation where everybody is single. A similar reasoning can be made for μ'' with the roles of men and women reversed. Then, we may say that (i) μ' farsightedly dominates μ, (ii) μ' farsightedly dominates μ'', and (iii) $V = \{\mu, \mu', \mu''\}$ is not a reasonable candidate for being a vNM farsightedly stable set.

3 Von Neumann-Morgenstern farsighted stability

The indirect dominance relation was first introduced by Harsanyi (1974) but was later formalized by Chwe (1994). It captures the idea that coalitions of agents can anticipate the actions of other coalitions. In other words, the indirect dominance relation captures the fact that farsighted coalitions consider the end matching that their matching(s) may lead to. A matching μ' indirectly dominates μ if μ' can replace μ in a sequence of matchings, such that at each matching along the sequence all deviators are strictly better off at the end matching μ' compared to the status-quo they face. Formally, indirect dominance is defined as follows.

Definition 4 *A matching* μ *is indirectly dominated by* μ'*, or* $\mu \ll \mu'$*, if there exists a sequence of matchings* $\mu^0, \mu^1, ..., \mu^K$ *(where* $\mu^0 = \mu$ *and* $\mu^K = \mu'$*) and a sequence of coalitions* $S^0, S^1, ..., S^{K-1}$ *such that for any* $k \in \{1, ..., K\}$*,*

(i) $\mu^K \succ_i \mu^{k-1}\ \forall i \in S^{k-1}$*, and*

(ii) *coalition* S^{k-1} *can enforce the matching* μ^k *over* μ^{k-1}*.*

Definition 4 gives us the definition of indirect dominance. The indirect dominance relation is denoted by $\ll$. Direct dominance is obtained by setting $K = 1$ in Definition 4. Obviously, if $\mu < \mu'$, then $\mu \ll \mu'$.

Diamantoudi and Xue (2003) have investigated farsighted stability in hedonic games (of which one-to-one matching problems are a special case) introducing the notion of the coalitional largest farsighted conservative stable set which coincides with the largest consistent set due to Chwe (1994).[6]

Definition 5 *$Z(\ll) \subseteq \mathcal{M}$ is a consistent set if $\mu \in Z(\ll)$ if and only if $\forall \mu'$, S such that S can enforce μ' over μ, $\exists \mu'' \in Z(\ll)$, where $\mu' = \mu''$ or $\mu' \ll \mu''$, such that $\mu(i) \not<_i \mu''(i)$ for some $i \in S$. The largest consistent set $\Gamma(\ll)$ is the consistent set that contains any consistent set.*

Interestingly, Diamantoudi and Xue (2003) have proved that in hedonic games with strict preferences (i) any partition belonging to the core indirectly dominates any other partition, and (ii) core partitions are always contained in the largest consistent set. Thus, in one-to-one matching markets, for all $\mu' \neq \mu$ with $\mu \in C(<)$ we have that $\mu \gg \mu'$, and $C(<) \subseteq \Gamma(\ll)$. However, the largest consistent set may contain more matchings than those matchings that are corewise stable as shown in Example 2.

Example 2 (Knuth, 1976) *Let $M = \{m_1, m_2, m_3, m_4\}$ and $W = \{w_1, w_2, w_3, w_4\}$. Let P be such that:*

$P(m_1)$	$P(m_2)$	$P(m_3)$	$P(m_4)$	$P(w_1)$	$P(w_2)$	$P(w_3)$	$P(w_4)$
w_1	w_2	w_3	w_4	m_4	m_3	m_2	m_1
w_2	w_1	w_4	w_3	m_3	m_4	m_1	m_2
w_3	w_4	w_1	w_2	m_2	m_1	m_4	m_3
w_4	w_3	w_2	w_1	m_1	m_2	m_3	m_4
m_1	m_2	m_3	m_4	w_1	w_2	w_3	w_4

Let

$$\mu' = \begin{pmatrix} m_1 & m_2 & m_3 & m_4 \\ w_1 & w_3 & w_2 & w_4 \end{pmatrix}, \mu'' = \begin{pmatrix} m_1 & m_2 & m_3 & m_4 \\ w_4 & w_2 & w_3 & w_1 \end{pmatrix}.$$

[6] The largest consistent set always exists, is non-empty, and satisfies external stability (i.e. any matching outside the set is indirectly dominated by some matching belonging to the set). But a consistent set does not necessarily satisfy the external stability condition. Only the largest consistent set is guaranteed to satisfy external stability.

There are ten corewise stable matchings where men (m_1, m_2, m_3, m_4) *are matched to women* (w_1, w_2, w_3, w_4), (w_2, w_1, w_3, w_4), (w_1, w_2, w_4, w_3), (w_2, w_1, w_4, w_3), (w_2, w_4, w_1, w_3), (w_3, w_1, w_4, w_2), (w_3, w_4, w_1, w_2), (w_3, w_4, w_2, w_1), (w_4, w_3, w_1, w_2), *and* (w_4, w_3, w_2, w_1), *respectively. It can be shown that* μ' *and* μ'' *are not corewise stable matchings. We already know that if* $\mu \in C(<)$ *then* $\mu \in \Gamma(\ll)$. *We will show that* μ' *and* μ'' *belong to the largest consistent set,* $\Gamma(\ll)$. *We know that if* $\mu \in C(<)$ *then for all* $\widehat{\mu} \neq \mu$ *we have that* $\mu \gg \widehat{\mu}$. *Moreover, we have that (i) for each* $i \in N$ *there is* $\mu \in C(<)$ *such that* $\mu(i) = \mu'(i)$, *and (ii) for each* $i \in N$ *there is* $\mu \in C(<)$ *such that* $\mu(i) = \mu''(i)$. *Hence, for all* μ''', S *such that* S *can enforce* μ''' *over* μ', $\exists \mu \in C(<) \subseteq \Gamma(\ll)$, *where* $\mu''' \ll \mu$, *such that* $\mu'(i) \not<_i \mu(i)$ *for some* $i \in S$. *Thus,* $\mu' \in \Gamma(\ll)$. *Similarly, for all* μ''', S *such that* S *can enforce* μ''' *over* μ'', $\exists \mu \in C(<) \subseteq \Gamma(\ll)$, *where* $\mu''' \ll \mu$, *such that* $\mu''(i) \not<_i \mu(i)$ *for some* $i \in S$*; and,* $\mu'' \in \Gamma(\ll)$. *So, the largest consistent set may contain more matchings than those matchings that are corewise stable.*$\square$

Now we give the definition of a vNM farsightedly stable set due to Chwe (1994).

Definition 6 *A set of matchings* $V \subseteq \mathcal{M}$ *is a vNM farsightedly stable set with respect to* P *if*

(i) *for all* $\mu \in V$, *there does not exist* $\mu' \in V$ *such that* $\mu' \gg \mu$;

(ii) *for all* $\mu' \notin V$ *there exists* $\mu \in V$ *such that* $\mu \gg \mu'$.

Definition 6 gives us the definition of a vNM farsightedly stable set $V(\ll)$. Let $\mathcal{V}(\ll)$ be the set of all vNM farsightedly stable sets. Part **(i)** in Definition 6 is the internal stability condition: no matching inside the set is indirectly dominated by a matching belonging to the set. Part **(ii)** is the external stability condition: any matching outside the set is indirectly dominated by some matching belonging to the set.[7] Chwe (1994) has shown that the largest consistent set always contains the

[7]Diamantoudi and Xue (2007) have extended the notion of the Equilibrium Binding Agreement or EBA (see Ray and Vohra, 1997) with unrestricted coalitional deviations by using the vNM stable set with the indirect dominance relationship. They have studied whether the agents will reach or not efficient agreements when they can negotiate openly and form coalitions. They have shown that, while the extended notion of the EBA facilitates the attainment of efficient agreements, inefficient agreements can arise, even if utility transfers are possible. However, no characterization of the vNM stable set with the indirect dominance relationship is provided.

vNM farsightedly stable sets. That is, if $V(\ll)$ is a vNM farsightedly stable set, then $V(\ll) \subseteq \Gamma(\ll)$.

We next reconsider the above examples to show that non corewise stable matchings, that belong either to the vNM stable set or to the largest consistent set, do not survive the stability requirements imposed by introducing farsightedness into the concept of vNM stable sets.

Example 1 (*continue*) Remember that $C(<) = \{\mu'\}$ is the set of corewise stable matchings and $V(<) = \{\mu, \mu', \mu''\}$ is the unique vNM stable set. It is easy to verify that $\mu' \gg \mu$ and $\mu' \gg \mu''$. Let $\mu^0 = \mu$, $\mu^1 = \varnothing$ (all agents are single), $\mu^2 = \mu'$, $S^0 = \{w_1, w_2, w_3\}$, and $S^1 = N$. We have (i) $\mu^2 \succ \mu^0$ $\forall i \in S^0$ and S^0 can enforce μ^1 over μ^0, (ii) $\mu^2 \succ \mu^1$ $\forall i \in S^1$ and S^1 can enforce μ^2 over μ^1. Thus, $\mu^2 \gg \mu^0$ or $\mu' \gg \mu$. Similarly, it is easy to verify that $\mu' \gg \mu''$. Hence, $\{\mu, \mu', \mu''\}$ cannot be a vNM farsightedly stable set, nor can $\{\mu, \mu'\}$ or $\{\mu', \mu''\}$ be a vNM farsightedly stable set since internal stability is violated. Moreover, μ does not indirectly dominate μ' and μ'' does not indirectly dominate μ'. It implies that the sets $\{\mu, \mu''\}$, $\{\mu\}$ or $\{\mu''\}$ cannot be vNM farsightedly stable sets as they violate the external stability condition. In fact, $V(\ll) = \{\mu'\}$ is the unique vNM farsightedly stable set.□

Example 2 (*continue*) Remember that μ' and μ'' belong to the largest consistent set but are not corewise stable. First, we show that $\{\mu'\}$ cannot be a vNM farsightedly stable set since the external stability condition would be violated. Indeed, for instance, μ' does not indirectly dominate the matching where men (m_1, m_2, m_3, m_4) are matched to women (w_2, w_1, w_3, w_4). Second, we show that a set composed of μ' and other matching(s) cannot be a vNM farsightedly stable set since the internal stability condition would be violated. Indeed, (i) $\mu \gg \mu'$ if $\mu \in C(<)$; (ii) $\mu' \gg \mu''$ and $\mu' \ll \mu''$; (iii) the matchings where men (m_1, m_2, m_3, m_4) are matched to women (w_1, w_3, w_4, w_2), (w_3, w_1, w_2, w_4), (w_1, w_4, w_2, w_3), (w_1, w_4, w_3, w_2), (w_4, w_1, w_2, w_3), (w_4, w_1, w_3, w_2), (w_2, w_3, w_1, w_4), (w_2, w_3, w_4, w_1), (w_3, w_2, w_1, w_4), (w_3, w_2, w_4, w_1), (w_2, w_4, w_3, w_1), and (w_4, w_2, w_1, w_3), respectively, indirectly dominate μ', but μ' does not indirectly dominate any of these matchings. Thus, the largest consistent set may contain more matchings than those matchings that belong to the vNM farsightedly stable sets.□

4 Main results

From Definition 6, we have that, for $V(\ll)$ to be a singleton vNM farsightedly stable set, only external stability needs to be verified. That is, the set $\{\mu\}$ is a vNM farsightedly stable set if and only if for all $\mu' \neq \mu$ we have that $\mu \gg \mu'$.

In order to show our main results we use Lemma 1 that shows that μ indirectly dominates μ' if and only if there does not exist a pair $\{i, \mu'(i)\}$ that blocks μ. In other words, an individually rational matching μ does not indirectly dominate another matching μ' if and only if there exists a pair $\{i, \mu'(i)\}$ that blocks μ.

Lemma 1 *Consider any two matchings $\mu', \mu \in \mathcal{M}$ such that μ is individually rational. Then, $\mu \gg \mu'$ if and only if there does not exist a pair $\{i, \mu'(i)\}$ such that both i and $\mu'(i)$ prefer μ' to μ.*

Hence, if μ is individually rational and there does not exist a pair $\{i, \mu'(i)\}$ that blocks μ, then $\mu \gg \mu'$. The proof of this lemma, as well as all other proofs, may be found in the appendix.

Lemma 2 *Consider any two matchings $\mu', \mu \in \mathcal{M}$ such that μ' is individually rational. Then $\mu \gg \mu'$ implies that μ is also individually rational.*

The next theorem shows that every corewise stable matching is a vNM farsightedly stable set.

Theorem 1 *If μ is a corewise stable matching, $\mu \in C(<)$, then $\{\mu\}$ is a vNM farsightedly stable set, $\{\mu\} = V(\ll)$.*

Since μ is a corewise stable matching, there is no pair of players matched at any other matching μ' and such that they both prefer μ' to μ. Then, Lemma 1 applies and μ indirectly dominates any other matching μ'. Thus, it follows that if $\mu \in C(<)$ then $\{\mu\}$ is a vNM farsightedly stable set.[8] But, a priori there may be other vNM farsightedly stable sets of matchings. We now show that the only possible vNM farsightedly stable sets are singleton sets whose elements are the corewise stable matchings.

[8]Diamantoudi and Xue (2003) were first to show that in hedonic games (of which marriage problems are a special case) with strict preferences, any partition belonging to the core indirectly dominates any other partition. Here, we provide an alternative proof of their result for one-to-one matching problems with strict preferences, and in Section 5 we show that this result also holds for many-to-one matching problems with responsive preferences.

Theorem 2 *If* $V(\ll) \subseteq \mathcal{M}$ *is a vNM farsightedly stable set of matchings then* $V(\ll) = \{\mu\}$ *with* $\mu \in C(<)$*.*

We here provide a sketch of the proof. Because of Theorem 1, it is clear that if $V(\ll)$ has more than one element and has a nonempty intersection with the core $C(<)$, then internal stability is violated since any element of the core indirectly dominates any other matching. Secondly if $V(\ll) = \{\mu\}$ with $\mu \notin C(<)$, then there exists a deviating coalition that can enforce a new matching in which all coalition members are better off. Then this new matching cannot be indirectly dominated by μ and hence external stability is violated. A third possibility is that $V(\ll)$ has more than one element and has an empty intersection with $C(<)$. In this case, we pick any element μ_1 of $V(\ll)$ and construct a deviation to a matching μ_1' such that no blocking pair of μ_1 blocks μ_1'. In order to satisfy external stability there must be a $\mu_2 \in V(\ll)$ such that $\mu_2 \gg \mu_1'$. We then show that internal stability cannot be satisfied: either $\mu_2 \gg \mu_1$ or $\mu_1 \gg \mu_2$. In particular, we prove that $V(\ll)$ cannot contain only matchings that are not corewise stable since the three different types of possible deviations from any non corewise stable matching in $V(\ll)$ are not compatible with the conditions of internal and external stability. First, we show that $V(\ll)$ cannot contain non-individually rational matchings. Second, we show that for any $\mu_1 \in V(\ll)$ there are not two single agents that block μ_1. Finally, we show that for any $\mu_1 \in V(\ll)$ there is no married agent i such that $\{i, j\}$ blocks μ_1 with $\mu_1(i) \neq j$. Therefore, we can conclude that there does not exist a vNM farsightedly stable set $V(\ll)$ containing at least two different matchings and satisfying $V(\ll) \cap C(<) = \varnothing$. We refer the reader to the appendix for a detailed proof. Hence, while Ehlers (2007) has shown that the set of corewise stable matchings is a subset of vNM (myopically) stable sets, vNM farsightedly stable sets cannot include matchings that are not corewise stable ones.[9]

[9] The notion of bargaining set for one-to-one matching problems defined by Klijn and Masso (2003) was a first attempt to reflect the idea that agents are not myopic. The bargaining set is the set of matchings that have no justified objection. They have shown that the set of corewise stable matchings is a subset of the bargaining set. So, contrary to the vNM farsightedly stable set; the bargaining set can contain matchings outside the core.

5 Many-to-one matching problems

A many-to-one matching problem consists of a set of N agents divided into a set of hospitals, $H = \{h_1, ..., h_r\}$, and a set of medical students, $I = \{i_1, ..., i_s\}$, where possibly $r \neq s$. For each hospital $h \in H$ there is a positive integer q_h called the quota of hospital h, which indicates the maximum number of positions to be filled. Let $Q = \{q_h\}_{h \in H}$. Each hospital $h \in H$ has a strict, transitive, and complete preference relation over the set of medical students I and the prospect of having its position unfilled, denoted h. Hospital h's preferences can be represented by a strict ordering of the elements in $I \cup \{h\}$; for instance, $P(h) = i_1, i_2, h, i_3, ...$ denotes that hospital h prefers to enroll i_1 rather than i_2, that it prefers to enroll either one of them rather than leave a position unfilled, and that all other medical students are unacceptable. Each medical student $i \in I$ has a strict, transitive, and complete preference relation over the set of hospitals H and the prospect of being unemployed. Student i's preferences can be represented by a strict ordering of the elements in $H \cup \{i\}$; for instance, $P(i) = h_2, h_1, h_3, i, ...$ denotes that the only positions student i would accept are those offered by h_2, h_1, and h_3, in that order. Let $P = (\{P(h)\}_{h \in H}, \{P(i)\}_{i \in I})$. A many-to-one matching market is simply (H, I, P, Q).

Definition 7 *A matching μ is a mapping from the set $H \cup I$ into the set of all subsets of $H \cup I$ such that for all $i \in I$ and $h \in H$:*

(a) either $|\mu(i)| = 1$ and $\mu(i) \subseteq H$ or else $\mu(i) = i$.

(b) $\mu(h) \in 2^I$ and $|\mu(h)| \leq q_h$.

(c) $\mu(i) = \{h\}$ if and only if $i \in \mu(h)$.

We denote by $P^*(h)$ the preference relation of hospital h over sets of students. We assume that $P^*(h)$ is responsive to $P(h)$. That is, whenever $\mu'(h) = \mu(h) \cup \{i\} \setminus \{i'\}$ for $i' \in \mu(h)$ and $i \notin \mu(h)$, then h prefers $\mu'(h)$ to $\mu(h)$ (under $P^*(h)$) if and only if h prefers i to i' (under $P(h)$). Under this condition, as in Roth and Sotomayor (1990), we can associate to the many-to-one matching problem a one-to-one matching problem in which we replace hospital h by q_h positions of h denoted by $h^1, h^2, ..., h^{q_h}$. Each of these positions has preferences over individuals that are identical with those of h. Each student's preference list is modified by replacing h, wherever it appears on his or her list, by the string $h^1, h^2, ..., h^{q_h}$ in that order. That is, if i prefers h_1 to h_2, then i prefers all positions of h_1 to all positions of h_2, and i prefers h_1^1, to all the other positions of h_1.

In many-to-one matching problems, it makes sense to distinguish between setwise enforceability and corewise enforceability. Let $P^{\mu(i)}$ denote the power set of the set $\mu(i)$.

Definition 8 (setwise enforceability) *Given a matching μ, a coalition $S \subseteq N$ is said to be able to enforce a matching μ' over μ if the following conditions hold:*

(i) $\mu'(h) \notin P^{\mu(h)} \cup \{h\}$ *implies* $\mu'(h) \setminus \mu(h) \cup \{h\} \subset S$ *and*

(ii) $\mu'(h) \in P^{\mu(h)} \cup \{h\}$, $\mu'(h) \neq \mu(h)$, *implies either h or $\mu(h) \setminus \mu'(h)$ or h together with a strict nonempty subset of $\mu(h) \setminus \mu'(h)$ should be in S.*

Condition (i) says that any new match in μ' that contains different partners than in μ should be such that h and the different partners of h belong to S. Condition (ii) states that in order to leave some (or all) positions unfilled of one existing match in μ, either h or the students leaving such positions or h and some strict subset of such students should be in S. Depending on the notion of enforceability used (setwise or corewise), we obtain the setwise direct dominance relation ($\overset{s}{<}$) or the corewise direct dominance relation ($\overset{c}{<}$), the set of setwise stable matchings ($C(\overset{s}{<})$) or the set of corewise stable matchings ($C(\overset{c}{<})$), the vNM setwise stable sets ($V(\overset{s}{<})$) or the vNM corewise stable sets ($V(\overset{c}{<})$). A matching μ is setwise stable if there is no subset of agents who by forming new partnerships only among themselves, possibly dissolving some partnerships of μ to remain within their quotas and possibly keeping other ones, can all obtain a strictly preferred set of partners.[10] A matching of the many-to-one matching problem is setwise stable if and only if the corresponding matchings of the associated one-to-one matching problem are (setwise) stable (see Roth and Sotomayor, 1990). However, this result does not hold for corewise stability.

In many-to-one matching problems with responsive preferences, the indirect dominance relation is invariant to the notion of enforceability in use. Indeed, if μ is indirectly dominated by μ' under corewise enforceability, it is obvious that μ is indirectly

[10] On the contrary, a matching μ is corewise stable if there is no subset of agents who by forming all their partnerships only among themselves, can all obtain a strictly preferred set of partners. Obviously, setwise enforceability and corewise enforceability are equivalent in one-to-one but not in many-to-one matching problems.

dominated by μ' under setwise enforceability. In the other direction, if μ is indirectly dominated by μ' under setwise enforceability, then μ is indirectly dominated by μ' under corewise enforceability because if for some $i \in S$ we have that $\mu'(i) \cap \mu(i) \neq \varnothing$, then i could first become "single" and then match with $\mu'(i)$, instead of matching directly with $\mu'(i) \backslash \mu(i)$.

Lemma 3 *A matching μ is indirectly dominated by μ' in a many-to-one matching problem if μ is indirectly dominated by μ' in the associated one-to-one matching problem.*

However, one can have a situation in which μ is indirectly dominated by μ' in a many-to-one matching problem but not in the associated one-to-one matching problem. Let us now write a condition that characterizes indirect dominance in the many-to-one case. Given two matchings $\mu', \mu \in \mathcal{M}$ and any hospital h, let $\mu'_W(h, \mu)$ be the set of students matched to h in μ' that are worse off in μ (compared to be matched to h). Likewise let $\mu'_B(h, \mu)$ and $\mu'_I(h, \mu)$ be the set of students matched to h in μ' that are respectively better off and equally well off in μ. Let $\mu'_{IW}(h, \mu) = \mu'_I(h, \mu) \cup \mu'_W(h, \mu)$.

Lemma 4 *Consider any two matchings $\mu', \mu \in \mathcal{M}$ such that μ and μ' are individually rational. Then, $\mu \gg \mu'$ if and only if there does not exist a hospital h such that $\mu'_{IW}(h, \mu) \succcurlyeq_h \mu(h)$ and $\mu'_W(h, \mu)$ is non-empty.*

From Theorem 1 and Theorem 2, using Lemma 3 and Lemma 4, we obtain the following corollary.

Corollary 1 *In a many-to-one matching problem with responsive preferences, a set of matchings is a vNM farsightedly stable set if and only if it is a singleton set and its element is a setwise stable matching.*

Thus, our characterization of the vNM farsightedly stable set for one-to-one matching problems extends to many-to-one matching problems with responsive preferences. This result contrasts with Ehlers (2007) who has shown that there need not be any relationship between the vNM corewise stable sets of a many-to-one matching problem and its associated one-to-one matching problem. Example 4 illustrates our main result for many-to-one matching problems with responsive preferences: vNM farsightedly stable sets only contain setwise stable matchings. Thus, if there is a matching of the many-to-one matching problem that is corewise stable but not setwise stable, then this matching is never a vNM farsightedly stable set.

Example 3 (Ehlers, 2005) *Consider a many-to-one matching problem and its associated one-to-one matching problem with* $H = \{h_1, h_2\}$, $I = \{i_1, i_2, i_3, i_4\}$, $q_{h_1} = 2$, $q_{h_2} = 1$, *and* P *such that:*

$P(h_1^1)$	$P(h_1^2)$	$P(h_2)$	$P(i_1)$	$P(i_2)$	$P(i_3)$	$P(i_4)$
i_1	i_1	i_2	h_2	h_1^1	h_1^1	h_1^1
i_2	i_2	i_1	h_1^1	h_1^2	h_1^2	h_1^2
i_3	i_3	i_3	h_1^2	h_2	h_2	h_2
i_4	i_4	i_4	i_1	i_2	i_3	i_4
h_1^1	h_1^2	h_2				

Let

$$\mu = \begin{pmatrix} h_1^1 & h_1^2 & h_2 & i_4 \\ i_2 & i_3 & i_1 & i_4 \end{pmatrix}, \mu' = \begin{pmatrix} h_1^1 & h_1^2 & h_2 & i_4 \\ i_1 & i_3 & i_2 & i_4 \end{pmatrix}, \widetilde{\mu} = \begin{pmatrix} h_1^1 & h_1^2 & h_2 & i_3 \\ i_2 & i_4 & i_1 & i_3 \end{pmatrix},$$

$$\widetilde{\mu}' = \begin{pmatrix} h_1^1 & h_1^2 & h_2 & i_3 \\ i_1 & i_4 & i_2 & i_3 \end{pmatrix}, \widetilde{\mu}'' = \begin{pmatrix} h_1^1 & h_1^2 & h_2 & i_1 \\ i_3 & i_4 & i_2 & i_1 \end{pmatrix}, \widehat{\mu} = \begin{pmatrix} h_1^1 & h_1^2 & h_2 & i_4 \\ i_1 & i_2 & i_3 & i_4 \end{pmatrix}$$

Ehlers (2005) has shown that $V(\overset{c}{<}) = \{\mu, \mu', \widetilde{\mu}, \widetilde{\mu}', \widetilde{\mu}''\}$ *in this many-to-one matching problem, while* $V(\overset{c}{<}) = \{\mu, \mu'\}$ *in its associated one-to-one matching problem. Notice that* $V(\overset{s}{<}) = \{\mu, \mu'\}$ *in both the many-to-one matching problem and its associated one-to-one matching problem. The set of setwise stable matchings is* $C(\overset{s}{<}) = \{\mu\}$. *However,* $\mu' \in C(\overset{c}{<})$ *in the many-to-one matching problem, since* μ' *is not directly dominated via any coalition (for instance, the coalition* $\{h_1, i_2\}$ *that prefers* $\widehat{\mu}$ *to* μ' *cannot enforce* $\widehat{\mu}$ *over* μ'*, and the members of the coalition* $\{h_1, i_1, i_2\}$ *that can enforce* $\widehat{\mu}$ *over* μ' *do not all prefer* $\widehat{\mu}$ *to* μ'*). Now, applying our results we have that* $V(\ll) = \{\mu\}$ *in both the many-to-one matching problem and its associated one-to-one matching problem. Contrary to the direct dominance relation,* μ *indirectly dominates* μ'*. Indeed, the sequence of deviations is as follows. First,* i_2 *leaves hospital* h_2*; second, hospital* h_2 *hires student* i_1*; third, hospital* $h_1 = \{h_1^1, h_1^2\}$ *hires student* i_2 *(either directly when using setwise enforceability, or first leaving student* i_3 *and then hiring both students* i_2 *and* i_3 *when using corewise enforceability).*□

6 Conclusion

We have characterized the vNM farsightedly stable sets in one-to-one matching problems: a set of matchings is a vNM farsightedly stable set if and only if it is a singleton set and its element is a corewise (hence, setwise) stable matching. Thus, we have provided an alternative characterization of the core in one-to-one matching problems. Finally, we have shown that our main result is robust to many-to-one matching problems with responsive preferences: a set of matchings is a vNM farsightedly stable set if and only if it is a singleton set and its element is a setwise stable matching.

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Appendix

Proof of Lemma 1. Let $B(\mu', \mu)$ be the set of men and women who are strictly better off in μ than in μ'. Accordingly, let $I(\mu', \mu)$ and $W(\mu', \mu)$ be the set of men and women who are indifferent between μ and μ' and worse off in μ than in μ', respectively.

($\Rightarrow$) Assume on the contrary that $\mu \gg \mu'$ and that there is a pair $\{i, \mu'(i)\}$ such that both prefer μ' to μ. For μ to indirectly dominate μ' it must be that i or $\mu'(i)$ get divorced along the path from μ' to μ. But both i and $\mu'(i)$ belong to $W(\mu', \mu)$, and then, they will never divorce. Hence $\mu \not\gg \mu'$, a contradiction.

($\Leftarrow$) We will prove it by showing that $\mu \gg \mu'$ if the above condition is satisfied. Assume that for all pairs $\{i, \mu'(i)\}$ such that $\mu'(i) \neq \mu(i)$, either i or $\mu'(i)$ or both belong to $B(\mu', \mu)$. Notice that every agent i single in μ' that accepts a match with someone else in μ also belongs to $B(\mu',\mu)$ since μ is individually rational. Next construct the following sequence of matchings from μ' to μ: μ^0, μ^1, μ^2 (where $\mu^0 = \mu'$, $\mu^1 = \{\mu^1(i) = i,\ \mu^1(\mu'(i)) = \mu'(i)$ for all $i \in B(\mu',\mu)$, and $\mu^1(j) = \mu'(j)$ otherwise$\}$, and $\mu^2 = \mu$), and the following sequence of coalitions S^0, S^1 with $S^0 = B(\mu',\mu)$ and $S^1 = B(\mu',\mu) \cup \{\mu(i)$ for $i \in B(\mu',\mu)\}$. Then, coalition S^0 can enforce μ^1 over μ^0 and coalition S^1 can enforce μ^2 over μ^1. Moreover, $\mu^2 \succ \mu^0$ for S^0, and $\mu^2 \succ \mu^1$ for S^1 because every mate of $i \in B(\mu',\mu)$ in μ^2 (in μ) also prefers his or her mate in μ^2 to being single in μ^1. Indeed, for every $i \in B(\mu',\mu)$, either $\mu^2(i) \in B(\mu',\mu)$ and hence both prefer μ^2 to μ^1, or $\mu^2(i) \in W(\mu',\mu)$. In this last case, $\mu^2(i)$ must have lost his or her mate in μ^0 and $\mu^0(\mu^2(i))$ must belong to $B(\mu',\mu)$ since otherwise $\mu^0(\mu^2(i))$ and $\mu^2(i)$ would form a blocking pair of μ^2, and this by assumption is not possible. Hence $\mu^2(i)$ must be single in μ^1. Then, since μ^2 is individually rational, $\mu^2(i)$ must prefer accepting his or her mate in μ^2 than remaining single at μ^1. So, we have that $\mu \gg \mu'$. ■

Proof of Lemma 2. Suppose not. Then, there exists $i \in N$ that prefers to be single than to be married to $\mu(i)$ in μ. Since $\mu \gg \mu'$ and μ' is individually rational, we have that i was either single at μ' or matched to $\mu'(i) \succ_i i$. But then in the sequence of moves between μ' and μ, the first time i has to move she/he was either matched with $\mu'(i)$ or single and, hence, i cannot belong to a coalition S^{k-1} that can enforce the matching μ^k over μ^{k-1} and such that all members of S^{k-1} prefer μ to μ^{k-1}, contradicting the fact that $\mu \gg \mu'$. ■

Proof of Theorem 1. We only need to verify condition (ii) in Definition 6: for all $\mu' \neq \mu$ we have that $\mu \gg \mu'$. Since $\mu \in C(<)$, we know that $\forall \mu' \neq \mu$, $\nexists\, i \in M$ and $j \in W$ such that $\mu'(i) = j$ and $\mu' \succ \mu$ for both i and j. Since μ is individually rational, we have from Lemma 1 that $\mu \gg \mu'$. ■

Proof of Theorem 2. Notice that if $V(\ll) \subseteq C(<)$, then $V(\ll)$ is a vNM farsightedly stable set only if $V(\ll)$ is a singleton set $\{\mu\}$ with $\mu \in C(<)$. >From Theorem 1, we know that for all $\mu' \neq \mu$, $\mu \gg \mu'$. Suppose now that $V(\ll) \subsetneq C(<)$. Then, either $V(\ll) \cap C(<) = \varnothing$ or $V(\ll) \cap C(<) \neq \varnothing$.

Suppose first that $V(\ll) \cap C(<) \neq \varnothing$. Then, there exists a matching $\mu \in V(\ll$

$) \cap C(<)$, and we know that for all $\mu' \neq \mu$, $\mu \gg \mu'$ since $\{\mu\}$ is a vNM farsightedly stable set. But, then there exists a matching $\mu' \neq \mu \in V(\ll)$ such that $\mu \gg \mu'$, violating the internal stability condition.

Suppose now that $V(\ll) \cap C(<) = \varnothing$. Then, we will show that $V(\ll)$ is not a vNM farsightedly stable set because either the internal stability condition (condition **(i)** in Definition 6) or the external stability condition (condition **(ii)** in Definition 6) is violated.

Assume $V(\ll) = \{\mu\}$ is a singleton. Since $\mu \notin C(<)$ there exists a deviating coalition S in μ and a matching $\mu' \in \mathcal{M}$ such that $\mu' \succ_i \mu$ for all $i \in S$ and S can enforce μ' over μ. Then, $\mu \not\gg \mu'$ and the external stability condition is violated.

Assume now that $V(\ll)$ contains more than one matching that do not belong to $C(<)$. Take any matching $\mu_1 \in V(\ll)$. Since $\mu_1 \notin C(<)$, there exists at least a pair of agents $\{i,j\}$ such that $\mu_1(j) \neq i$ (or a single agent $\{i\}$) and a matching $\mu_1' \in \mathcal{M}$ such that $\mu_1' \succ \mu_1$ for both i and j (or $\mu_1' \succ \mu_1$ for i), and $\{i,j\}$ (or i) can enforce μ_1' over μ_1, i.e. such that $\mu_1'(j) = i$ (or $\mu_1'(i) = i$). Let $S(\mu_1)$ be the set of blocking pairs of μ_1. Consider the deviation from μ_1 to μ_1' of the subset of blocking pairs $S'(\mu_1) \subseteq S(\mu_1)$, where $S'(\mu_1)$ contains the maximum number of blocking pairs and is such that the subset $S(\mu_1) \diagdown S'(\mu_1)$ does not contain any blocking pair of μ_1'.[11]

[11] We now establish formally that μ_1' exists. Define M' to be the set of men that belong to some blocking pair of μ_1 : $M' = \{i \in M : \exists j \in W \cup \{i\} \text{ such that } ij \in S(\mu_1)\}$. Equally, define W' to be the set of women that belong to some blocking pair of μ_1 : $W' = \{j \in W : \exists i \in M \cup \{j\} \text{ such that } ij \in S(\mu_1)\}$.

Let us restrict the preferences for all $i \in M'$ and $j \in W'$ such that they only rank the people they can form a deviating coalition with. To be clear, the preferences of $i \in M'$ are only over the set $W_i' = \{j \in W' \cup \{i\} \text{ such that } ij \in S(\mu_1)\}$. For each $j \in W'$, her preferences are resricted to the set $M_j' = \{i \in M' \cup \{j\} \text{ such that } ij \in S(\mu_1)\}$. Let $P'(\mu_1)$ denote these restricted preference.

By making use for instance of the 'restricted' deferred acceptance algorithm, we know that the matching problem $\{M', W', P'(\mu_1)\}$ has at least one stable matching, call it μ'. Then define μ_1' as follows. All agents that do not belong to either M' or W' do not belong to $S'(\mu_1)$ (they do not move themselves although they can lose their match in the move from μ_1 to μ_1' if they were initially matched to some of the deviating players in $S'(\mu_1)$). Now consider the people in M' and W'. Every pair $\{i, \mu'(i)\}$ of μ' belongs to $S'(\mu_1)$ and, hence, both i and $\mu'(i)$ move from μ_1 to μ_1' to their match in μ'. Every single agent at μ' prefering being single at μ' rather than being married at μ_1 belongs to $S'(\mu_1)$ and we let them becoming single in the move from μ_1 to μ_1'. Every single agent at μ' prefering being married at μ_1 rather than being single at μ' does not belong to $S'(\mu_1)$ but to $S(\mu_1) \diagdown S'(\mu_1)$. Every pair $\{i, \mu_1(i)\}$ of μ_1 with $\mu_1(i) = \mu'(i)$ is such that i (and/or $\mu_1(i)$) belongs to $S(\mu_1) \diagdown S'(\mu_1)$ when i (and/or $\mu_1(i)$) belongs to M' (belongs to W'). Clearly, In order for $V(\ll)$ being a vNM farsightedly stable set we need that the following conditions are satisfied:

(i) for any other matching $\mu_2 \in V(\ll)$, $\mu_2 \neq \mu_1$, it should be that $\mu_1 \not\gg \mu_2$ and $\mu_2 \not\gg \mu_1$

(ii) for all $\mu' \notin V(\ll)$ there should exist $\mu \in V(\ll)$ such that $\mu \gg \mu'$ (in particular, we need that there exists a matching $\mu_2 \in V(\ll)$ such that $\mu_2 \gg \mu'_1$ for each matching, like μ'_1, that can be enforced by any subset of blocking pairs of any matching in $V(\ll)$).

We will show that $V(\ll)$ is not a vNM farsightedly stable set because one of the above conditions is not satisfied. In particular, we will prove that $V(\ll)$ cannot contain only matchings that are not corewise stable since the three different types of possible deviations from any non corewise stable matching in $V(\ll)$ are not compatible with the above two conditions. First, we show that $V(\ll)$ cannot contain non-individually rational matchings. Second, we show that for any $\mu_1 \in V(\ll)$ there are not two single agents i and j such that $\{i, j\}$ blocks μ_1. Finally, we show that for any $\mu_1 \in V(\ll)$ there is no married agent i such that $\{i, j\}$ blocks μ_1. Therefore, we can conclude that there does not exist a vNM farsightedly stable set $V(\ll)$ containing at least two different matchings and satisfying $V(\ll) \cap C(<) = \varnothing$.

1. Assume that $S'(\mu_1)$ contains a blocking pair $\{i\}$ that divorces $\mu_1(i)$. Consider the deviation of $\{i\}$ from μ_1 to μ''_1 where he or she divorces $\mu_1(i)$, while the other blocking pairs do not move. Then if $\mu_2 \gg \mu''_1$ (in order for $V(\ll)$ satisfying external stability), we also have that $\mu_2 \gg \mu_1$ since i will never marry someone else and becoming worse off than being single. Hence, the internal stability condition is violated and $V(\ll)$ is not a vNM farsightedly stable set. So $V(\ll)$ cannot contain non-individually rational matchings.

2. Assume that $S'(\mu_1)$ contains at least a blocking pair $\{i, j\}$ that are single at μ_1 but married at μ'_1, i.e., $\mu'_1(i) = j$. Consider the deviation of $\{i, j\}$ from μ_1 to μ''_1 where they get married, while the other blocking pairs do not move. Then if $\mu_2 \gg \mu''_1$ (in order for $V(\ll)$ satisfying external stability), we will

the subset $S(\mu_1) \setminus S'(\mu_1)$ does not contain any blocking pair of μ'_1 because otherwise μ' would not be a stable matching for the matching problem $\{M', W', P'(\mu_1)\}$. Since in any stable matching the set of single agents is always the same, then $S'(\mu_1)$ contains the maximum possible number of blocking pairs such that $S(\mu_1) \setminus S'(\mu_1)$ does not contain any blocking pair of μ'_1.

show that the internal stability condition is violated. Two sub-cases have to be considered.

2.1. If i and j are still married in μ_2 (or they are married to someone else preferred to j and i respectively), then $\mu_2 \gg \mu_1$ and the internal stability condition is violated.

2.2. On the contrary, assume that one of them, i, has divorced from j (leaving j single in μ_2 like she was in μ_1) to marry to another woman $\mu_2(i)$ preferred to j. But notice that the position (matching) of any other agent different than i and j in μ_1'' is the same than in μ_1, since only i and j married at μ_1'' while they were single at μ_1, and then since $\mu_2 \gg \mu_1''$, we should have that $\{i, \mu_2(i)\} \in S(\mu_1)$. But then, the pair $\{i, j\}$ cannot belong to $S'(\mu_1)$, since j is not the best partner for i. Thus, consider the deviation of $\{i, \mu_2(i)\}$ from μ_1 to μ_1''' where they get married, while the other blocking pairs do not move. Then, if we have that $\mu_2 \gg \mu_1'''$ (in order for $V(\ll)$ satisfying external stability), we also have $\mu_2 \gg \mu_1$ and the internal stability condition is violated.

3. Assume that all blocking pairs $\{i, j\} \in S'(\mu_1)$ are such that at least one of the blocking partners (or both of them) is married at μ_1 with someone else, $\mu_1(j) \neq i, j$ and now at μ_1' they get married $\mu_1'(j) = i$. Assume that in the deviation from μ_1 to μ_1' all blocking pairs in $S'(\mu_1)$ get married so that at μ_1' no other blocking pair exists ($S(\mu_1) \setminus S'(\mu_1)$ does not contain any blocking pair of μ_1'). Three sub-cases have to be considered.

3.1. If at μ_2 we have that every blocking pair $\{i, j\} \in S'(\mu_1)$ is such that i and j are still married or they are married to someone else but preferred to j and to i, respectively, then $\mu_2 \gg \mu_1$ and the internal stability condition is violated.

3.2. Assume that at μ_2 no initial blocking pair $\{i, j\} \in S'(\mu_1)$ is still married, and that in each blocking pair $\{i, j\} \in S'(\mu_1)$ we have that i is marrying $\mu_2(i) \neq j$ divorcing from j. Hence, in order to have that $\mu_2 \gg \mu_1'$ but $\mu_2 \not\gg \mu_1$ and given Lemma 1 and Lemma 2, we need that there does not exist a pair $\{k, \mu_1'(k)\}$ married at μ_1' (or a single agent k) that blocks μ_2, and that there exists a pair $\{k, \mu_1(k)\}$ married (or a single agent k)

at μ_1 that blocks μ_2, with μ_2 individually rational. Notice that the only change between μ_1 and μ_1' is that each blocking pair $\{i,j\} \in S'(\mu_1)$ gets married leaving $\mu_1(j)$ (and possibly $\mu_1(i)$) single. Then we will prove that whenever $\mu_2 \gg \mu_1'$ but $\mu_2 \not\gg \mu_1$, we have that $\mu_1 \gg \mu_2$, violating the internal stability condition. By Lemma 1, we only need to show that there does not exist a pair $\{k, \mu_2(k)\}$ married at μ_2 (or a single agent k) that blocks μ_1.

Since $\mu_2 \gg \mu_1'$ but $\mu_2 \not\gg \mu_1$, we have that:

(i) For each pair $\{i,j\} \in S'(\mu_1)$, whenever i is better off at μ_2 than at μ_1' (and then better than at μ_1), $\mu_2(i)$ is worse off at μ_2 than at μ_1', because no other blocking pair different from the ones contained in $S'(\mu_1)$ exists at μ_1 (and hence $\mu_2(i)$ is worse off at μ_2 than at μ_1).

(ii) Also all the remaining initial partners $\mu_1(i)$ (and $\mu_1(j)$) that have been left by i (and by j) with $\{i,j\} \in S'(\mu_1)$, when single at μ_2, prefer μ_1 to μ_2 because otherwise μ_1 would not be individually rational. Whenever some remaining initial partner $\mu_1(i)$ (and $\mu_1(j)$) that has been left by i (and by j) with $\{i,j\} \in S'(\mu_1)$, is not single at μ_2 but married to someone else $\mu_2(\mu_1(i))$ (married to $\mu_2(\mu_1(j))$), either $\mu_1(i)$ or $\mu_2(\mu_1(i))$ (either $\mu_1(j)$ or $\mu_2(\mu_1(j))$) or both should prefer μ_1 to μ_2 because otherwise the pair $\{\mu_1(i), \mu_2(\mu_1(i))\}$ (the pair $\{\mu_1(j), \mu_2(\mu_1(j))\}$) would have been also a blocking pair at μ_1.

(iii) Moreover, since the pairs $\{i,j\} \in S'(\mu_1)$ are the only blocking pairs at μ_1, every pair of agents $\{k,l\}$ such that $\mu_2(l) = k$ and $\mu_1'(k) \neq l$ with $\mu_1'(k) = \mu_1(k)$ and $\mu_1'(l) = \mu_1(l)$, is such that if one of the mates prefers μ_2 to μ_1' (and hence, prefers μ_2 to μ_1) then the other prefers μ_1' to μ_2 (and hence, prefers μ_1 to μ_2) because otherwise they would have been a blocking pair at μ_1, and all single agent k at μ_2 that is married at μ_1', with $\mu_1'(k) = \mu_1(k)$, prefer μ_1' to μ_2 (and hence, prefer μ_1 to μ_2) because otherwise μ_1' (and hence, μ_1) would not be individually rational.

So, (i), (ii) and (iii) show that every pair of agents matched (every single agent) at μ_2 and not matched (not single) at μ_1 that contains one of the initial deviating players in $S'(\mu_1)$, or one of the players initially matched at μ_1 to some player i with $\{i,j\} \in S'(\mu_1)$, or one of the players in some of the initial pairs that do not change in the move from μ_1 to μ_1', is such that one of the mates (or both) prefers μ_2 to μ_1 while the other prefers μ_1 to μ_2.

Thus, we have proved that when $\mu_2 \gg \mu_1'$ but $\mu_2 \not\gg \mu_1$, we have that there is a pair $\{k, \mu_1(k)\}$ married (or a single agent k) at μ_1 that blocks μ_2 and that every other pair of agents matched at μ_2 and not matched at μ_1 is such that one of the mates prefers μ_2 to μ_1 while the other prefers μ_1 to μ_2. So, there does not exist a pair $\{i, \mu_2(i)\}$ matched at μ_2 that blocks μ_1, and then by Lemma 1 we have $\mu_1 \gg \mu_2$, violating the internal stability condition.

3.3. Finally, consider the case in which μ_2 contains some but not all initial blocking pairs from μ_1 contained in $S'(\mu_1)$. Then, consider the deviation from μ_1 to μ_1'' such that $\mu_1'' \succ \mu_1$ by all the initial blocking pairs belonging to $S''(\mu_1) \subsetneq S'(\mu_1)$ that are still married at μ_2 and that can be enforced by such blocking pairs from μ_1. Since $\mu_2 \gg \mu_1''$ (in order for $V(\ll)$ satisfying external stability), we will have that $\mu_2 \gg \mu_1$ violating the internal stability condition. ■

Proof of Lemma 3. If μ is indirectly dominated by μ' in the associated one-to-one matching problem, then there exists a sequence of matchings $\mu^0, \mu^1, ..., \mu^K$ (where $\mu^0 = \mu$ and $\mu^K = \mu'$) and a sequence of coalitions $S^0, S^1, ..., S^{K-1}$ consisting only of individual students or hospitals, or of student-hospital pairs and such that for any $k \in \{1, ..., K\}$, $\mu^K \succ_i \mu^{k-1}$ $\forall i \in S^{k-1}$, and coalition S^{k-1} can enforce the matching μ^k over μ^{k-1}. But then, μ is indirectly dominated by μ' in the many-to-one matching problem by the deviations of the sequence of coalitions consisting of the same singletons or pairs. ■

Proof of Lemma 4.

($\Rightarrow$) Assume on the contrary that $\mu \gg \mu'$ and that there exists a hospital h such that $\mu'_{IW}(h, \mu) \succ_h \mu(h)$ and $\mu'_W(h, \mu)$ is non-empty. At no step along the path between μ' and μ will any $i \in \mu'_{IW}(h, \mu)$ leave h. So, along the path between μ' and μ, hospital h must at some point get rid of any $i \in \mu'_W(h, \mu)$. Since μ' is individually rational, then $\mu'(h) \succ_h \mu(h)$ and h will never initiate a move at μ' in order to go to μ. Hence, some or all the students in $\mu'_B(h, \mu)$ that prefer μ to

μ' would leave h. Since μ' is individually rational, any intermediate matching obtained once some students in $\mu'_B(h,\mu)$ leave h between μ' and the matching in which h is only matched to $\mu'_{IW}(h,\mu)$, are all preferred by h to this last matching in which h is matched to $\mu'_{IW}(h,\mu)$. So, at any step along the path between μ' and the matching in which h is only matched to $\mu'_{IW}(h,\mu)$, h is in a better position compared to μ. But then h never has an incentive to get rid of any $i \in \mu'_W(h,\mu)$. Hence $\mu \not\gg \mu'$, a contradiction.

($\Longleftarrow$) We will prove it by construction. In step one let anyone (student or hospital) get rid of all her matches in μ' if they are better off at μ. After this step, only hospitals who are (weakly) worse off in μ compared to μ' may still have some students they are matched to (those who belong to $\mu'_{IW}(h,\mu)$). In step two, let these hospitals get rid of all their matches (all $i \in \mu'_{IW}(h,\mu)$). They will want to do so, since, by assumption, they are better off at μ compared to being matched only to $\mu'_{IW}(h,\mu)$. After step two everyone is alone. In step three, allow all matches necessary to obtain μ. This is possible since μ is individually rational. ■

Proof of Corollary 1.

($\Longrightarrow$) First we prove that if μ is a setwise stable matching, $\mu \in C(\overset{s}{<})$, then $\{\mu\}$ is a vNM farsightedly stable set, $\{\mu\} = V(\ll)$. This follows immediately from Lemma 3. If μ is setwise (pairwise) stable, then it indirectly dominates any other matching (from Lemma 1). Hence external stability is satisfied and using Lemma 3 we have that if μ is a setwise stable matching, $\mu \in S(<)$, then $\{\mu\}$ is a vNM farsightedly stable set of the many-to-one matching problem.

($\Longleftarrow$) The proof runs exactly along the same lines as the proof of Theorem 2 by simply proving that $V(\ll)$ cannot contain only matchings that are not setwise stable since the different types of possible deviations from any non setwise stable matching in $V(\ll)$ are not compatible with the conditions of internal and external stability that $V(\ll)$ must satisfy.

Indeed, assume now that $V(\ll)$ contains more than one matching that do not belong to the set of setwise stable matchings $C(\overset{s}{<})$. Take any matching $\mu_1 \in V(\ll)$ where $\mu_1 \notin C(\overset{s}{<})$. Let $S(\mu_1)$ be the set of blocking coalitions (in the sense of setwise stability) of μ_1. Consider the deviation from μ_1 to μ'_1 of the subset of blocking coalitions $S'(\mu_1) \subseteq S(\mu_1)$, where $S'(\mu_1)$ contains the maximum number of blocking coalitions and is such that the subset $S(\mu_1) \setminus S'(\mu_1)$ does not contain any blocking coalition of μ'_1. We now establish formally that μ'_1 exists. We do so by making use of the property of responsive preferences which allow us to make use of the equivalence between setwise stability in the many-to-one case and core stability in the one-to-one counterpart of the matching problem.

Define H^* to be the set of (copies) of hospitals : $H^* = (h_1^1, h_1^2, ..., h_1^{q_{1h}}, ..., h_r^1, h_r^2, ..., h_r^{q_{rh}})$. Denote $S^*(\mu_1^*)$ to be the set of blocking pairs of the one-to-one version of $\mu_1 : \mu_1^*$. Define $H^{*\prime}$ to be the set of (copies) of hospitals that belong to some blocking pair of $\mu_1^* : H^{*\prime} = \{h \in H^* : \exists i \in I \cup \{h\}$ such that $hi \in S^*(\mu_1^*)\}$. Define $I^{*\prime}$ to be the set of students that belong to some blocking pair of $\mu_1^* : I^{*\prime} = \{i \in I : \exists h \in H^* \cup \{i\}$ such that $hi \in S^*(\mu_1^*)\}$. Consider the deviation from μ_1^* to $\mu_1^{*\prime}$ of the subset of blocking coalitions $S^{*\prime}(\mu_1^*) \subseteq S^*(\mu_1^*)$, where $S^{*\prime}(\mu_1^*)$ contains the maximum number of blocking coalitions and is such that the subset $S^*(\mu_1^*) \setminus S^{*\prime}(\mu_1^*)$ does not contain any blocking coalition of $\mu_1^{*\prime}$. Let $S'(\mu_1)$ be defined as follows:
$S'_{h_j}(\mu_1) = \left\{h_j\widetilde{\mu}(h_j), \text{ where } \widetilde{\mu}(h_j) = \{i_1, ..., i_k\} \text{ is such that } h_j^1 i_1, ..., h_j^k i_k \in S^{*\prime}(\mu_1^*)\right\}$.
Let $S'(\mu_1) = \bigcup_{h \in H^{*\prime}} S'_{h_i}(\mu_1)$. Notice now that if there exists a coalition $S \in S(\mu_1) \setminus S'(\mu_1)$ that is a blocking coalition of μ'_1, then there exists a subcoalition of S containing just one hospital and a student that also blocks μ'_1. But then this hospital-student pair must also belong to $S^*(\mu_1^*) \setminus S^{*\prime}(\mu_1^*)$, a contradiction. This implies that, by responsiveness of preferences, in the many-to-one case the deviation from μ_1 to μ'_1 of the subset of blocking coalitions $S'(\mu_1) \subseteq S(\mu_1)$, where $S'(\mu_1)$ contains the maximum number of blocking coalitions and is such that the subset $S(\mu_1) \setminus S'(\mu_1)$ does not contain any blocking coalition of μ'_1, exists. Once we have shown the existence of μ'_1, the proof follows the proof of Theorem 2 but replacing now Lemma 1 by Lemma 4. ■

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