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Distributed Welfare Games

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We consider a variation of the resource allocation problem. In the traditional problem, there is a global planner who would like to assign a set of players to a set of resources so as to maximize welfare. We consider the situation where the global planner does not have the authority to assign players to resources; rather, players are self-interested. The question that emerges is how can the global planner entice the players to settle on a desirable allocation with respect to the global welfare? To study this question, we focus on a class of games that we refer to as distributed welfare games. Within this context, we investigate how the global planner should distribute the welfare to the players. We measure the efficacy of a distribution rule in two ways: (i) Does a pure Nash equilibrium exist? (ii) How does the welfare associated with a pure Nash equilibrium compare to the global welfare associated with the optimal allocation? In this paper we explore the applicability of cost sharing methodologies for distributing welfare in such resource allocation problems. We demonstrate that obtaining desirable distribution rules, such as distribution rules that are budget balanced and guarantee the existence of a pure Nash equilibrium, often comes at a significant informational and computational cost. In light of this, we derive a systematic procedure for designing desirable distribution rules with a minimal informational and computational cost for a special class of distributed welfare games. Furthermore, we derive a bound on the price of anarchy for distributed welfare games in a variety of settings. Lastly, we highlight the implications of these results using the problem of sensor coverage.

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1. Introduction

Resource allocation problems are of fundamental importance in many applications. Traditionally, researchers have aimed at developing centralized algorithms to determine efficient allocations (Feige and Vondrak (2006), Ageev and Sviridenko (2004), Ahuja et al. (2003)). However, in many modern applications, these centralized algorithms are not applicable and/or desirable. One concrete example which we will focus on in this paper is the problem of sensor coverage. In sensor coverage, the goal is to allocate a fixed number of sensors across a given mission space so as to maximize the probability of detecting a particular event (Li and Cassandras (2005)). In this case, a centralized algorithm requires that a central authority maintains complete knowledge of the environment and can communicate directly with each sensor during the entire mission. Both requirements may be unrealistic in large and/or hostile environments. The same issues arise in many computer network resource allocation problems, e.g., wireless access point assignment (Kaumann et al. (2007)), congestion control (Garcia et al. (2000), Akella et al. (2002)), and wireless power management (Enrique Campos-Nañez (2008), Li and Cassandras (2005)). There are also many examples outside of computer systems, e.g., minimizing aggregate congestion in a transportation system. In this setting, a global planner does not have the authority to assign drivers to roads; rather, a global planner must entice drivers appropriately, possibly through taxation, to settle on a desirable allocation (Sandholm (2002)).

Recently, there has been surge of research aimed at understanding the possibility of decentralizing (localizing) decisions in resource allocation problems (Kaumann et al. (2007), Mhatre et al. (2007), Komali and MacKenzie (2007), Zou and Chakrabarty (2004), Srivastava et al. (2005), Enrique Campos-Nañez (2008)). One approach for accomplishing this is to model the resource allocation problem as a non-cooperative game where the players, e.g, the sensors in the sensor coverage problem or the drivers in the transportation systems, selfishly pursue their own independent objectives which may or may not be in conflict with other players. In the context of a non-cooperative game, each player is assigned an action set in addition to a utility function which depends not only on the player's action, but also the actions of all other players. For example, in the sensor coverage problem, each sensor's action set could represent possible search locations and each sensor's utility function could assess the quality of a particular search location given the search locations of all other sensors. The goal is to design utility functions appropriately so that the resulting game has desirable characteristics.

There are wide-ranging advantages to this game-theoretic form of a distributed architecture, including robustness to failures and environmental disturbances, reducing communication requirements, improving scalability, etc. However, several challenges arise when seeking to design and implement such a distributed system (Arslan et al. (2007), Marden et al. (2007a)). The primary challenge, and the focus of this paper, is the following: how can a global planner entice the players to settle on something desirable with regards to the global objective? Equivalently, in the case of engineered systems such as sensor coverage, how can a global planner *design utility functions* for the players so that they will settle on a desirable allocation?

Non-cooperative resource allocation has recently been applied to a number of computer network applications, e.g., Akella et al. (2002), Kaumann et al. (2007), Mhatre et al. (2007), Komali and MacKenzie (2007), Zou and Chakrabarty (2004), Enrique Campos-Nañez (2008). However, in the current literature, the design of utility functions is highly dependent on the application at hand. For example, Enrique Campos-Nañez (2008) focuses on efficiently managing a tradeoff between energy usage and sensing capability in sensor networks while Komali and MacKenzie (2007) focuses on topology control in ad-hoc networks. The respective utility designs are as independent as the problem domains. However, the analysis of the utility designs in both cases are very similar. Among other things, they investigate the existence and efficiency of (pure) Nash equilibria. These are the primary two questions because, in a non-cooperative setting where players are self-interested, a Nash equilibrium represents an individually agreeable allocation. Therefore, the existence of efficient equilibria for a utility design is of the utmost importance.

The goal of this paper is to establish a general framework, independent of any specific application domain, for (i) investigating the feasibility of non-cooperative resource allocation and (ii) designing desirable utility functions. To that end, in this paper we will consider a class of resource allocation games that we refer to as *distributed welfare games*. A distributed welfare game is a resource allocation game where each player's utility is defined as some fraction of the total welfare garnered (see Section 3). Therefore, designing a utility function is equivalent to defining a distribution rule that depicts how the welfare garnered from a specific allocation is distributed to the players. The primary goal is to design distribution rules for distributed welfare games that guarantee the following two properties.

(i) *Existence:* A distribution rule should guarantee that a Nash equilibrium exists.
(ii) *Efficiency:* A distribution rule should guarantee that all Nash equilibria are efficient with respect to the global objective.

In addition to the two properties above, which are of primary importance, in many applications there are additional requirements that distribution should satisfy. We will investigate four such features:

(iii) *Tractability:* Computing the distribution rule should be tractable in games with a large number of players.
(iv) *Low informational requirement:* The distribution rule should only use limited information. Further, it should be as independent of the structure of of the global welfare function as possible.
(v) *Budget-balance:* In many problems that involve costs, the cost must be completely absorbed by the players. For example, in a network formation problem the cost associated with building and maintaining a network must be completely distributed to the players.
(vi) *Potential game:* If the distribution rule results in the formulation of a potential game, then one can appeal to a variety of distributed algorithms to ensure that the players converge to a Nash equilibrium.

With the goal of developing distribution rules that satisfy the above properties, we first focus on developing distribution rules that guarantee the existence of a Nash equilibrium (property i) while satisfying as many of (iii)-(vi) as possible. To accomplish this, we begin by investigating the applicability of cost-sharing methodologies (Young (1994), Shapley (1953), Hart and Mas-Colell (1989)). In particular, the first set of results (Section 4) illustrate that cost sharing methodologies can be used effectively as distribution rules in distributed welfare games. In a cost sharing problem, there are a group of players that are allowed to form coalitions with one another. Each coalition is accompanied by an associated worth (or cost) that must be distributed completely to the members of the coalition. These methodologies seek to understand how the worth of a coalition should be dispersed to its members to satisfy various objectives. While reminiscent of a distribution rule for distributed welfare games, there is one key difference. In a distributed welfare game, the welfare garnered by a particular coalition is dependent on the resource the coalition is utilizing. Therefore, there is an extra degree of flexibility in distributed welfare games when compared to cost sharing problems as each player must select not only a coalition, but also a set of resources.

Building upon results from Hart and Mas-Colell (1989), we show that one can use cost sharing methodologies as distribution rules in distributed welfare games. However, we demonstrate that these approaches do not satisfy all of the desirable properties above. For example, rules that are budget balanced and guarantee the existence of a Nash equilibrium, often come with significant informational and computational costs. This fact has also been noted by recent result of Chen et al. (2008), which studies a special class of distributed welfare games where the welfare (or cost) garnered at a particular resource is independent of the coalition of players utilizing that resource. The authors show that a budget balanced distribution rule guarantees the existence of a Nash equilibrium if and only if the welfare is distributed in accordance with each player's weighted Shapley value (Hart and Mas-Colell (1989)), which is intractable in general.

The results of Chen et al. (2008) ensure that attaining a systematic procedure for defining distribution rules satisfying all the desirable properties for arbitrary distributed welfare games is unattainable. Thus, the natural question that emerges is whether there exists special classes of distributed welfare games where one can establish distribution rules satisfying the above properties. Our second set of results (Section 5) investigates this question. We identify three easily verifiable properties of distribution rules, see Conditions 5.1–5.3, that guarantee the existence of a Nash equilibrium in any distributed welfare game where players are restricted to selecting a single resource. Furthermore, we derive a systematic procedure for designing desirable distribution rules with minimal informational and computational costs in these single selection distributed welfare games when the welfare function takes on a specified form.

Having provided a general class of distribution rules that guarantee the existence of a Nash equilibrium, the next task (Section 6) is to understand whether the Nash equilibria are efficient (property ii). We measure the efficiency of a Nash equilibrium using the well known measures: *price of anarchy* and *price of stability*. The price of anarchy (stability) is defined as the worst-case ratio between the global welfare evaluated at the worst (best) Nash equilibrium and the optimal welfare. These measures have been studied extensively in several application domains (Vetta (2002), Johari and Tsitsiklis (2004), Nissan et al. (2007)). In general, the price anarchy in distributed welfare games can be arbitrarily close to 0; however, when we restrict our attention to submodular welfare functions, which is a common assumption in many resource allocation problems (Vetta (2002), Krause and Guestrin (2007)), we can develop distribution rules that obtain a welfare within 1/2 of that of the optimal assignment. Furthermore, we tighten this price of anarchy bound in a variety of settings. This compares favorably with the best known results of centralized approximations for resource allocation problems with submodular welfare functions, which guarantee welfare within $1-1/e \approx 0.6321$ of the optimal (Feige and Vondrak (2006), Ageev and Sviridenko (2004), Ahuja et al. (2003)). Surprisingly, this comparison demonstrates that the inefficiency resulting from localizing decisions in resource allocation problems is relatively small when the welfare functions are submodular.

To ground the discussion of distributed welfare games, we end the paper in Section 7 with an illustration of the theory developed for the particular case of the sensor coverage problem. We model the sensor coverage problem as a non-cooperative game and we directly apply the results of this paper to develop tractable, budget balanced distribution rules that guarantee the existence of efficient Nash equilibria. Furthermore, in the case of sensor coverage, we demonstrate how to exploit the structure of the welfare function to tighten the price of anarchy results.

It should be noted that this paper predominantly focuses on equilibrium behavior in distributed welfare games. An alternative question that is of equal importance is understanding how players reach an equilibrium in a distributed fashion. While not focusing on this question in detail, we illustrate the applicability of the theory of *learning in games* (Young (1998), Monderer and Shapley (1996a), Marden et al. (2008, 2007b,a)) as a distributed control mechanism for coordinating group behavior. For example, if a distributed welfare game constitutes a potential game, then a global planner can appeal to a variety of distributed learning algorithms of varying complexity to guarantee that the group behavior converges to a Nash equilibrium Young (1998), Monderer and Shapley (1996a), Marden et al. (2008, 2007b,a,c) . There are fewer distributed learning algorithms available for distributed welfare games that possess at least one Nash equilibrium but do not constitute potential games (Young (2008)). An important future direction is developing suitable learning algorithms that exploit the structure inherent within distributed welfare games.

2. Background

In this paper we consider *resource allocation games* that consist of a set of players $N := \{1, ..., n\}$ and a finite set of resources $\mathcal{R}$ that are to be shared by the players. Each player $i \in N$ is assigned an action set $\mathcal{A}_i \subseteq 2^{\mathcal{R}}$ and a utility function of the form $U_i : \mathcal{A} \to \mathbb{R}$ where $2^{\mathcal{R}}$ denotes the power sets of $\mathcal{R}$ and $\mathcal{A} := \mathcal{A}_1 \times \cdots \times \mathcal{A}_n$ is referred to as the set of joint actions. Therefore, a player may have the option of selecting multiple resources and the player's utility may be influenced by the actions of other players.

For an action profile $a = (a_1, a_2, ..., a_n) \in \mathcal{A}$, let a_{-i} denote the profile of player actions *other than* player i, i.e., $a_{-i} = (a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n)$. With this notation, we will sometimes write a profile a of actions as (a_i, a_{-i}). Similarly, we may write $U_i(a)$ as $U_i(a_i, a_{-i})$. Furthermore, let $\mathcal{A}_{-i} := \prod_{j \neq i} \mathcal{A}_j$ denote the set of possible collective actions of all players other than player i.

We will focus on analyzing equilibrium behavior in such games. A well-known equilibrium concept that emerges in non-cooperative games is that of a *pure Nash equilibrium*. An action profile $a^* \in \mathcal{A}$ is called a *pure Nash equilibrium* if for all players $i \in N$,

$$U_i(a_i^*, a_{-i}^*) = \max_{a_i \in \mathcal{A}_i} U_i(a_i, a_{-i}^*). \tag{1}$$

A pure Nash equilibrium represents a scenario for which no player has an incentive to unilaterally deviate. We will henceforth refer to a pure Nash equilibrium as simply an equilibrium.

One class of games discussed in this paper is *potential games* (Monderer and Shapley (1996b)). In a potential game, the change in a player's utility that results from a unilateral change in strategy equals the change in some global potential function. Specifically, there is a function $\phi : \mathcal{A} \to \mathbb{R}$ such that for every player $i \in N$, for every $a_{-i} \in \mathcal{A}_{-i}$, and for every $a_i', a_i'' \in \mathcal{A}_i$,

$$U_i(a_i', a_{-i}) - U_i(a_i'', a_{-i}) = \phi(a_i', a_{-i}) - \phi(a_i'', a_{-i}). \tag{2}$$

When this condition is satisfied, the game is called a potential game with the potential function ϕ. In potential games, any action profile maximizing the potential function is an equilibrium, hence every potential game possesses at least one such equilibrium.

For a more comprehensive review of the game-theoretic concepts introduced in this section, we refer the readers to Fudenberg and Tirole (1991), Young (1998, 2005), Nissan et al. (2007).

3. Distributed Welfare Games

A *distributed welfare game* is a non-cooperative formalization of a resource allocation game with a specific structure enforced on player utility functions. The formalization includes a wide variety of resource allocation problems including network routing, wireless power management, sensor coverage, and others. To illustrate the applicability of distributed welfare games, we will focus in detail on the case of the sensor coverage problem in Section 7.

To define distributed welfare games, consider a resource allocation game with a global welfare function, $W : \mathcal{A} \to \mathbb{R}_+$. Each player's utility is defined as some portion of the welfare and must satisfy the following properties: for any player $i \in N$ and action profile (allocation) $a \in \mathcal{A}$

(i) $U_i(a) \geq 0$,
(ii) $\sum_{i \in N} U_i(a) \leq W(a)$.

This structure permits the total global welfare to be distributed arbitrarily to the players.

An example of a utility function that satisfies these conditions is equally distributing the global welfare

$$U_i(a) = \frac{1}{n} W(a). \tag{3}$$

This utility design guarantees that any allocation that maximizes the welfare is an equilibrium. (This could also be observed by noticing that we have a potential game.) However, a player needs to know the selections of all other players in addition to the structural form of the global welfare function in order to evaluate his utility, which is typically impractical.

Our focus in this paper is on understanding the degree to which players' utility functions can be *localized* while at the same time guaranteeing both existence and efficiency of equilibria. To this end, we will restrict our attention to *separable* welfare functions of the form

$$W(a) = \sum_{r \in \mathcal{R}} W^r(a^r),$$

where $W^r : 2^N \to \mathbb{R}_+$ is the welfare function for resource r and a^r denotes the subset of players that selected resource r in the joint allocation a, i.e., $a^r := \{i \in N : r \in a_i\}$. To simplify notation, we will commonly write $W^r(a^r)$ as just $W^r(a)$. Because the welfare function is assumed to be separable, we will also restrict player utility functions to be *local* and *separable*, i.e., of the form

$$U_i(a_i, a_{-i}) = \sum_{r \in a_i} f_i(r, a)\, W^r(a), \tag{4}$$

where $\{f_1(r,a), ..., f_n(r,a)\}$ defines how the global reward garnered from resource r is distributed across the players. We will refer to $\{f_1(r,a), ..., f_n(r,a)\}_{r \in \mathcal{R}, a \in \mathcal{A}}$ as the *distribution rule*. A distribution rule must satisfy the following properties: for any player $i \in N$, resource $r \in \mathcal{R}$, and action profile $a \in \mathcal{A}$

(i) $f_i(r,a) \geq 0$,
(ii) $r \notin a_i \Rightarrow f_i(r,a) = 0$,
(iii) $\sum_{i \in N} f_i(r,a) \leq 1$.

We will refer to distribution rules that satisfy (iii) with equality as *budget balanced distribution rules*.

There are a number of desirable properties of distribution rules, which we discussed in Section 1. Of primary importance is that the distribution rule guarantees existence and efficiency of equilibria. Furthermore, it is often desirable that distribution rules are tractable, budget-balanced, have a low informational requirement, and lead to a potential game formulation. The focus of this paper is designing distribution rules that satisfy these objectives.

4. Methods for Distributing Welfare

In this section, we explore several natural approaches for designing distribution rules in distributed welfare games. These approaches are derived from methodologies in the cost sharing literature (Young (1994)), such as the Shapley value (Shapley (1953), Hart and Mas-Colell (1989)). In particular, distributing the welfare garnered at each resource in a distributed welfare game can be viewed as a cost sharing problem. We will see that cost sharing methodologies can be effective as distribution rules in distributed welfare games. However, we will also see that there are several issues that limit their applicability. Resultantly, we will discuss alternative approaches to designing distribution rules for distributed welfare games in the Section 5.

4.1. Equally Shared Utilities

The utility design in (3) disseminates the total welfare equally to all players and guarantees the existence of efficient equilibria. Unfortunately, it requires players to use global (rather than local) information and thus it is not a local distribution rule of the form (4).

Suppose the welfare from each resource is divided equally amongst the players that selected the resource, i.e.,

$$U_i(a_i, a_{-i}) = \sum_{r \in a_i} \left(\frac{1}{\sum_j I\{r \in a_j\}} \right) W^r(a), \tag{5}$$

where $I\{\cdot\}$ is the usual indicator function. In general, such a design cannot guarantee the existence of an equilibrium as the following example illustrates. For alternative examples, see Arslan et al. (2007).

EXAMPLE 1 (EQUALLY SHARED UTILITIES). *Consider a two player distributed welfare game with player set $N = \{1, 2\}$, resources $\mathcal{R} = \{r^1, r^2\}$, actions set $\mathcal{A}_1 = \mathcal{A}_2 = \mathcal{R}$, utility functions of the form (5), and a separable welfare function as illustrated below. Note that the game does not possess an equilibrium.*

Welfare (Player 1 rows, Player 2 columns):

	$\emptyset$	r^1	r^2
$\emptyset$	0	4	1
r^1	6	6	7
r^2	5	9	10

Payoffs (Player 1 rows, Player 2 columns):

	$\emptyset$	r^1	r^2
$\emptyset$	$0,0$	$0,4$	$0,1$
r^1	$6,0$	$3,3$	$6,1$
r^2	$5,0$	$5,4$	$5,5$

One problem with equally shared utilities is that players' utility functions are not *aligned* with their contribution to the global welfare. However, if players are anonymous with regards to their impact on the global welfare function then the equally shared utilities in (5) guarantee the existence of an equilibrium. Players are anonymous with regards to their impact on the global welfare function if for any action profiles $a, a' \in \mathcal{A}$, $\sigma^r(a) = \sigma^r(a') \Rightarrow W^r(a) = W^r(a')$, where $\sigma^r(a) := |\{i \in N : r \in a_i\}|$ denotes the number of players utilizing resource r given the allocation a. Hence, the welfare generated by a particular resource depends only on the number of players utilizing that resource, not the specific players utilizing the resource.

PROPOSITION 1. *If a distributed welfare game has anonymous players then an equilibrium exists under the equal share utility design (5).*

Proof: It is straightforward to show that any distributed welfare game with anonymous players is a congestion game (Rosenthal (1973), Monderer and Shapley (1996b)), with the following specification:
(a) resources: $\mathcal{R}$,
(b) cost functions: $c^r(k) = \frac{W^r(k)}{k},\ k > 0$, where k is the number of players utilizing resource r, and
(c) utility functions: $U_i(a) = \sum_{r \in a_i} c^r(\sigma^r(a))$.
Any congestion game is a potential game with potential function

$$\phi(a) = \sum_{r \in \mathcal{R}} \sum_{k=1}^{\sigma^r(a)} c^r(k).$$

Therefore, an equilibrium is guaranteed to exist in such a game.
□

4.2. Marginal Contribution Utilities

The structural requirement of equally shared utilities requires that players need to know the total welfare garnered from a particular resource in addition to the number of players utilizing that resource in order to evaluate their utilities. While this requirement is desirable from an implementation stand point, in general this design does not guarantee the existence of an equilibrium. However, by conditioning a distribution rule not only on the welfare garnered, but also on each player's marginal contribution to the existing welfare, we can guarantee the existence of an equilibrium even when players are not anonymous.

Specifically, suppose each player's utility is set as his marginal contribution to the global welfare, i.e.,

$$U_i(a_i, a_{-i}) = W(a_i, a_{-i}) - W(a_i^0, a_{-i}), \tag{6}$$

where a_i^0 designate the null action for player i. We will refer to the utility design in (6) as the *wonderful life utility* (WLU) (Wolpert and Tumor (1999)). We will assume throughout this paper that $a_i^0 = \emptyset$; however, in general a_i^0 could be set as any fixed action in the player's action set. This translates to a distribution rule of the form

$$f_i(r, a) = \frac{W^r(a) - W^r(a_i^0, a_{-i})}{W^r(a)}, \tag{7}$$

hence the distribution rule is dependent on both the current welfare and each player's marginal contribution to the current welfare. It is well known that assigning each player a utility as in (6) results in a potential game with potential function W; hence any action profile that maximizes the global welfare is an equilibrium. However, other equilibria may also exist under the wonderful life utility design.

There are two limitations of the the marginal contribution utility design. First, each player needs to be able to compute his marginal contribution to the welfare in order to evaluate his utility. Second, the wonderful life utility may distribute more (or less) welfare than is gathered; hence, it may not satisfy condition (iii) of distributed welfare games. While the first limitation cannot be relaxed in this setting, it remains an open question as to whether the second limitation can be addressed utilizing a similar informational requirement.

One natural approach for extending (7) is to disperse welfare proportionally to each player's marginal contribution

$$U_i(a_i, a_{-i}) = \sum_{r \in a_i} \left(\frac{W^r(a) - W^r(a_i^0, a_{-i})}{\sum_j W^r(a) - W^r(a_j^0, a_{-j})} \right) W^r(a). \tag{8}$$

We will refer to (8) as the *normalized wonderful life utility* (NWLU). However, the following example illustrates that such a design does not always guarantee the existence of an equilibrium.

EXAMPLE 2 (WLU AND NWLU). *Consider the setup and separable welfare function in Example 1. The wonderful life utility and normalized wonderful life utility for each player are illustrated below. Note that while the WLU ensures that the optimal allocation is indeed an equilibrium, the players payoffs may be more or less than the welfare garnered. Furthermore note that no equilibrium exists in the case of NWLU.*

Player 1 \ Player 2	$\emptyset$	r^1	r^2
$\emptyset$	$0, 0$	$0, 4$	$0, 1$
r^1	$6, 0$	$2, 0$	$6, 1$
r^2	$5, 0$	$5, 4$	$9, 5$

Payoffs – WLU

Player 1 \ Player 2	$\emptyset$	r^1	r^2
$\emptyset$	$0, 0$	$0, 4$	$0, 1$
r^1	$6, 0$	$6, 0$	$6, 1$
r^2	$5, 0$	$5, 4$	$6.4, 3.6$

Payoffs – NWLU

4.3. The Shapley Value

While WLU guarantees the existence of an equilibrium in all settings, it may distribute more or less reward than the welfare garnered. It turns out that we can rectify this problem by using a common cost sharing methodology known as the Shapley value (Shapley (1953)). For any subset of players $\widetilde{N} \subseteq N$, resource $r \in \mathcal{R}$, and player $i \in \widetilde{N}$, the Shapley value of player i is defined as (Shapley (1953), Hart and Mas-Colell (1989), Haeringer (2006))

$$Sh_i^r(\widetilde{N}) := \sum_{S \subseteq \widetilde{N}: i \in S} \frac{(|\widetilde{N}|-2)!(|S|-1)!}{|\widetilde{N}|!} \left(W^r(S) - W^r(S \setminus \{i\})\right). \tag{9}$$

Suppose each player's utility function is defined as

$$U_i(a_i, a_{-i}) := \sum_{r \in a_i} Sh_i^r(a^r), \tag{10}$$

where $Sh_i^r(a^r)$ is player i's Shapley value at resource r given the allocation of players a^r. Unfortunately, utilizing (9) comes with a significant informational and computational cost. Each player needs to know not only his marginal contribution to the existing welfare but also his perceived marginal contribution to the welfare garnered from any subset of players in order to evaluate his utility. Roughly speaking, utilizing (9) requires each player to have complete knowledge of the structural form of the welfare function in addition to the capabilities of all players.

PROPOSITION 2. *Consider any resource allocation game. If each player is assigned a utility of the form (10), then the ensuing game is a budget balanced distributed welfare game. Furthermore, it is a potential game with the following potential function* $\phi : \mathcal{A} \to \mathbb{R}$

$$\phi(a) := \sum_{r \in \mathcal{R}} \sum_{S \subseteq a^r} \frac{1}{|S|} \left(\sum_{T \subseteq S} (-1)^{|S|-|T|} W^r(T) \right). \tag{11}$$

Proof: We will prove this proposition by using the potential function derived in Hart and Mas-Colell (1989). First, we will express the Shapley value of player i as a weighted sum of unanimity games (Hart and Mas-Colell (1989), Haeringer (2006)) which takes on the form

$$Sh_i^r(\widetilde{N}) = \sum_{S \subseteq \widetilde{N}: i \in S} \frac{1}{|S|} \left(\sum_{T \subseteq S} (-1)^{|S|-|T|} W^r(T) \right). \tag{12}$$

Let $\alpha_S^r := \sum_{T \subseteq S} (-1)^{|S|-|T|} W^r(T)$ and $\phi^r : 2^N \to \mathbb{R}$ be the resource specific potential function (Hart and Mas-Colell (1989))

$$\phi^r(a) := \sum_{S \subseteq a^r} \frac{\alpha_S^r}{|S|}.$$

Let $a \in \mathcal{A}$ be any allocation. Player i's marginal utility is

$$\begin{aligned} U_i(a) - U_i(a_i^0, a_{-i}) &= \sum_{r \in a_i} Sh_i^r(a^r), \\ &= \sum_{r \in a_i} \left(\sum_{S \subseteq a^r : i \in S} \frac{\alpha_S^r}{|S|} \right), \\ &= \sum_{r \in a_i} \left(\sum_{S \subseteq a^r} \frac{\alpha_S^r}{|S|} - \sum_{S \subseteq a^r \setminus i} \frac{\alpha_S^r}{|S|} \right), \end{aligned}$$

$$= \sum_{r \in a_i} \left(\phi^r(a) - \phi^r(a_i^0, a_{-i}) \right),$$
$$= \phi(a) - \phi(a_i^0, a_{-i}).$$

Therefore, for any player i, actions $a_i', a_i'' \in \mathcal{A}_i$, and allocation $a_{-i} \in \mathcal{A}_{-i}$

$$U_i(a_i', a_{-i}) - U_i(a_i'', a_{-i}) = \phi(a_i', a_{-i}) - \phi(a_i'', a_{-i}).$$

□

It is also worth noting that this potential function could be computed recursively. For any resources $r \in \mathcal{R}$ define $\phi^r(\emptyset) := 0$. One can recursively evaluate ϕ^r according to the following equation: for any $S \subseteq N$,

$$\phi^r(S) = \frac{1}{|S|}\left[W^r(S) + \sum_{i \in S} \phi^r(S \setminus \{i\}) \right].$$

There are two limitations of the Shapley value utility design that may prevent it from being applicable. First, there is a high informational requirement as each player must be able to compute his marginal contribution to all action profiles in order to evaluate his utility. Second, in general computing a Shapley value is intractable in games with a large number of players. This is highlighted explicitly in either (9) or (12) where computation of the Shapley value requires a weighted summation over all subsets of players. However, it should be noted that this computational cost is lessened dramatically if there are a limited number of distinct "classes" of players, see Conitzer and Sandholm (2004). For example, if players are anonymous then the Shapley value is equivalent to the equal share distribution rule in (5).

4.4. The Weighted Shapley Value

A generalization to the Shapley Value that is often studied in the cost sharing literature is the weighted Shapley value (Shapley (1953), Hart and Mas-Colell (1989), Haeringer (2006)). Define $\omega_i \in \mathbb{R}_+$ as the weight of player i. Let $\omega := \{\omega_i\}_{i \in N}$ be the associated weight weight vector. For any weight vector $\omega \in \mathbb{R}_+^n$, subset of players $\widetilde{N} \subseteq N$, resource $r \in \mathcal{R}$, and player $i \in \widetilde{N}$, the weighted Shapley value is defined as (Haeringer (2006))

$$\widetilde{Sh}_i^r(\widetilde{N}) := \sum_{S \subseteq \widetilde{N} : i \in S} \frac{\omega_i}{\sum_{j \in S} \omega_j} \left(\sum_{T \subseteq S} (-1)^{|S|-|T|} W(T) \right).$$

Note that the Shapley value is recaptured if $\omega_i = 1$ for all players $i \in N$.

Suppose each player is assigned the following utility

$$U_i(a_i, a_{-i}) := \sum_{r \in a_i} \widetilde{Sh}_i^r(a^r). \tag{13}$$

where $\widetilde{Sh}_i^r(a^r)$ is player i's weighted Shapley value at resource r given the allocation of players a^r. Note that if players are anonymous, the weighted Shapley distributes the welfare according to $w_i / \sum_{j \in N} w_j$. We will state the following proposition without proof to avoid redundancy.

PROPOSITION 3. *Consider any resource allocation game. If each player is assigned a utility of the form (13), then the ensuing game is a budget balanced distributed welfare game. Furthermore, the resulting game is a potential game.*

The weighted Shapley value does not result in as clean a closed form expression for the potential function as the Shapley value in (11). However, as with the Shapley value, the potential function can be computed recursively and is of the form

$$\widetilde{\phi}(a) := \sum_{r \in \mathcal{R}} \widetilde{\phi}^r(a),$$

where $\widetilde{\phi}^r(a)$ is the resource specific potential function, $\widetilde{\phi}^r(\emptyset) := 0$, and for any subset $S \subseteq N$ (Hart and Mas-Colell (1989))

$$\widetilde{\phi}^r(S) = \frac{1}{\sum_{i \in S} \omega_i} \left[W^r(S) + \sum_{i \in S} \omega_i \widetilde{\phi}^r(S \setminus \{i\}) \right].$$

Finally, note that the weighted Shapley utility design suffers from the same drawbacks as the Shapley value utility design.

4.5. Comparison of distribution rules

To this point we have surveyed five distribution rules, each of which was motivated by methodologies from the cost sharing literature. We will end this section by summarizing the advantages and disadvantages of these rules. Table 1 illustrates the tradeoff between desirable features of a distribution rule and the computational and informational requirement needed to obtain such a rule. The only setting where a distribution rule achieves all of the desired properties is the case where players are anonymous. In this setting, the Shapley value is equivalent to equally share distribution rule in (5). In the next section, our goal will be to develop other distribution rules that attain (nearly) all of our desired properties.

Distribution Rule	Existence of Equilibrium	Potential Game	Budget Balanced	Tractable	Informational Requirement
Equally Shared (anonymous)	yes	yes	yes	yes	Low
Equally Shared	no	no	yes	yes	Low
WLU	yes	yes	no	yes	Medium
NWLU	no	no	yes	yes	Medium
Shapley	yes	yes	yes	no	High
Weighted Shapley	yes	yes	yes	no	High

Table 1 Summary of Distribution Rules for Distributed Welfare Games

5. Single Selection Distributed Welfare Games

The results of the previous section suggest that, in general it is not possible to establish distribution rules that guarantee all of our desired properties. As a result, in this section we focus on a simplified setting where players are only allowed to select a single resource, $\mathcal{A}_i = \mathcal{R}$, as opposed to multiple resources, $\mathcal{A}_i \subseteq 2^{\mathcal{R}}$. In this restricted setting, the goal will be to develop distribution rules that guarantee several of our desired properties.

To this end, we identify three sufficient conditions that determine whether a distribution rule guarantees the existence of an equilibrium for a particular game. Furthermore, we develop a systematic procedure for designing desirable distribution rules in single selection distributed welfare game with a particular class of welfare functions.

5.1. Sufficient Conditions for Existence of an Equilibrium

In this section, we identify three sufficient conditions on players' utility functions that determine whether a distribution rule guarantees the existence of an equilibrium in any single selection resource allocation game. These sufficient conditions translate to pairwise comparisons of players' utility functions. Before stating the conditions, we will introduce the following notation. The allocation $(r^{(i)}, r^{(j)}, a_{-ij})$, denotes the situation where player i selects resource $r^{(i)}$, player j selects resource $r^{(j)}$, and all other player select resources according to $a_{-ij} \in \mathcal{A}_{-ij} := \prod_{k \neq i,j} \mathcal{A}_k$. Likewise, let $U_i(r^{(i)}, r^{(j)}, a_{-ij})$ and $U_j(r^{(i)}, r^{(j)}, a_{-ij})$ denote player i and j's respective utility for the given allocation.

CONDITION 5.1. Let i and j be any two players. If for any resource $r \in \mathcal{R}$ and any allocation $a_{-ij} \in \mathcal{A}_{-ij}$

$$U_i(r, r, a_{-ij}) > U_j(r, r, a_{-ij}),$$

then for all resources $r' \in \mathcal{R}$ and all allocations $a'_{-ij} \in \mathcal{A}_{-ij}$

$$U_i(r', r', a'_{-ij}) \geq U_j(r', r', a'_{-ij}).$$

In this situation, we will say that player i is stronger than player j. Furthermore, we will require that player strengths are a transitive property, that is if player i is stronger than player j who is stronger than player k, then player i is also stronger than player k.

CONDITION 5.2. Suppose player i is stronger than player j. For any resource $r \in \mathcal{R}$ and any action profile $a_{-ij} \in \mathcal{A}_{-ij}$, the following holds

$$U_i(r, a_j^0, a_{-ij}) \geq U_i(r, r, a_{-ij}).$$

CONDITION 5.3. Suppose player i is stronger than player j. For any resources $r \in \mathcal{R}$ and action profile $a_{-ij} \in \mathcal{A}_{-ij}$, the following holds

$$\frac{U_j(a_i^0, r, a_{-ij})}{U_i(r, a_j^0, a_{-ij})} \geq \max_{r^* \in \mathcal{R}} \frac{U_j(r^*, r^*, a_{-ij})}{U_i(r^*, r^*, a_{-ij})}.$$

THEOREM 1. *Consider any single selection resource allocation game. If each player's utility function satisfies Conditions 5.1, 5.2, and 5.3 then an equilibrium exists.*

Proof: We begin by renumbering the players in order of strengths with player 1 being the strongest player. This is possible because of Condition 5.1.

We will construct an equilibrium by letting each player select his action one at a time in order of strength. The general idea of the proof is that once a player selects an action, the player will never seek to deviate regardless of the other player's action. First, player 1 selects the resource $r^{(1)}$ according to

$$r^{(1)} \in \arg\max_{r \in \mathcal{R}} U_1(r, a_{-1}^0) \tag{14}$$

Next, player 2 selects action $r^{(2)}$ according to

$$r^{(2)} \in \arg\max_{r \in \mathcal{R}} U_2(r^{(1)}, r, a_{-12}^0).$$

If $r^{(1)} \neq r^{(2)}$, then by (14) and Condition 5.2 we know that

$$U_1(r^{(1)}, a_{-1}^0) \geq U_1(r^{(2)}, a_{-1}^0) \geq U_1(r^{(2)}, r^{(2)}, a_{-12}^0).$$

Therefore, player 1 can not improve his utility by switching his strategy, i.e.,

$$U_1(r^{(1)}, r^{(2)}, a_{-12}^0) \geq U_1(r, r^{(2)}, a_{-12}^0),\ \forall r \in \mathcal{R}.$$

If $r^{(1)} = r^{(2)} = r$, then by Condition 5.3, we know that for any resource $\widetilde{r} \in \mathcal{R}$, $\widetilde{r} \neq r$,

$$\begin{aligned} \frac{U_2(\widetilde{r}, r, a^0_{-12})}{U_1(r, \widetilde{r}, a^0_{-12})} &= \frac{U_2(a^0_1, r, a^0_{-12})}{U_1(r, a^0_2, a^0_{-12})}, \\ &\geq \frac{U_2(r, r, a^0_{-12})}{U_1(r, r, a^0_{-ij})}. \end{aligned}$$

Using the above inequality, we can conclude that for any resource $\widetilde{r} \in \mathcal{R}$

$$U_2(r, r, a^0_{-12}) \geq U_2(r, \widetilde{r}, a^0_{-12}) \Rightarrow U_1(r, r, a^0_{-12}) \geq U_1(\widetilde{r}, r, a^0_{-12}).$$

Therefore, player 1 cannot improve his utility by switching his strategy.

If $n = 2$, then $a = (r^{(1)}, r^{(2)})$ would be an equilibrium. Otherwise this argument could be repeated n times to construct an equilibrium.

□

It remains an open question as to whether Conditions 5.1 – 5.3 guarantee additional properties pertaining to the structure of the game besides existence of an equilibrium. For example, if each player's utility function satisfies Conditions 5.1 – 5.3, is the game a potential game or some variant?

5.2. Generalized Method for Distributing Welfare

The previous section illustrates three sufficient conditions on distribution rules that guarantee the existence of an equilibrium. These conditions provide a useful tool, but ideally it would be desirable to provide a systematic procedure for designing such a rule. In this section we provide such a procedure.

Inspired by the three conditions of the previous section, we introduce a procedure for defining each player's strength, denoted as $(s_1, ..., s_n) \in R^n_+$, such that if each player is assigned a utility function that distributes welfare proportionally according to the players' strengths, i.e.,

$$U_i(r, a_{-i}) = \frac{s_i}{\sum\limits_{j \in N: a_j = r} s_j} W^r(a) \tag{15}$$

then the resulting game is guaranteed to possess at least one equilibrium. However, in order to attain our results we must limit the analysis to a particular class of welfare functions that we introduce in the next section. Note that the utility design in (15) has a similar informational and computational requirement to the equally shared utility in (5).

5.2.1. Admissible Welfare Functions. We limit ourselves to welfare functions that satisfy the following condition: for any resource $r \in R$ there exists a constant $\bar{W}^r > 0$ such that for any player $i \in N$, resource $r \in R$, and action profile $a_{-i} \in \mathcal{A}_{-i}$

$$W^r(r, a_{-i}) - W^r(a^0_i, a_{-i}) = f_i \left(\bar{W}^r - W^r(a^0_i, a_i) \right), \tag{16}$$

for some player dependent constant $f_i \in (0, 1]$. We will define the *leftover* at resource r given the allocation a as $L^r(a) := \bar{W}^r - W^r(a^0_i, a_i)$. Therefore a player's marginal contribution to the welfare is linear in the size of the leftover.

This form of welfare function is common in many applications. For example, the sensor coverage problem that we consider in Section 7 has a welfare function of this form.

5.2.2. Definition of Strength. In order to guarantee the existence of an equilibrium, it is enough to define strengths as follows. For some $k \in [0,1]$, define the strength of each player i, s_i, as the solution to

$$f_i = (1-k)s_i + k\frac{s_i}{1+s_i}. \tag{17}$$

It is easy to see that the strength of any player i is increasing in both k and f_i. According to (17), the strength of any player $i \in N$ for the special case of $k=0$ or $k=1$ is precisely

$$s_i = \begin{cases} f_i, & k=0, \\ \frac{f_i}{1-f_i}, & k=1. \end{cases}$$

By referring back to (16), these extreme points correspond to

$$\begin{aligned} f_i &= \frac{W^r(r,a_{-i}) - W^r(a_i^0,a_{-i})}{L^r(a_i^0,a_{-i})}, \\ \frac{f_i}{1-f_i} &= \frac{W^r(r,a_{-i}) - W^r(a_i^0,a_{-i})}{L^r(r,a_{-i})}. \end{aligned}$$

Note that $s_i \geq 0$ for all players $i \in N$.

THEOREM 2. *Consider any single selection distributed welfare game with a welfare function of the form (16). If each player's strength is defined according to (17) then distributing welfare proportionally according to the players' strengths as in (15) guarantees the existence of an equilibrium.*

We defer the proof of this result to appendix A.

6. Efficiency of Equilibria in Distributed Welfare Games

In addition to guaranteeing the existence of an equilibrium, it is important for a distribution rule to guarantee that the equilibria are efficient. In this section, we focus on bounding the efficiency of equilibria in distributed welfare games. We gauge the efficiency of equilibria using use the well known measures of *price of anarchy (PoA)* and *price of stability (PoS)* (Nissan et al. (2007)). In terms of distributed welfare games, the PoA gives a lower bound on the global welfare achieved by any equilibrium while the PoS gives a lower bound on the global welfare associated with the best equilibrium for any distributed welfare game. Specifically, let $\mathcal{G}$ denote a set of distributed welfare games. For any particular game $G \in \mathcal{G}$ let $\mathcal{E}(G)$ denote the set of equilibria, $PoA(G)$ denote the price of anarchy, and $PoS(G)$ denote the price of stability for the game G where

$$PoA(G) := \min_{a^{ne} \in \mathcal{E}(G)} \frac{W(a^{ne})}{W(a^{opt})} \tag{18}$$

$$PoS(G) := \max_{a^{ne} \in \mathcal{E}(G)} \frac{W(a^{ne})}{W(a^{opt})}, \tag{19}$$

where $a^{opt} \in \arg\max_{a^* \in \mathcal{A}} W(a^*)$. We define the PoA and PoS for the set of distributed welfare games $\mathcal{G}$ as

$$PoA(\mathcal{G}) := \inf_{G \in \mathcal{G}} PoA(G), \tag{20}$$

$$PoS(\mathcal{G}) := \inf_{G \in \mathcal{G}} PoS(G). \tag{21}$$

In general, the price of anarchy can be arbitrarily close to 0 in distributed welfare games. However, when the welfare function is submodular it is possible to attain a much better price of anarchy. A set valued function $W : 2^{\mathcal{A}} \to \mathbb{R}$ is submodular if $W(X) + W(Y) \geq W(X \cap Y) + W(X \cup Y)$ for all $X, Y \subseteq 2^{\mathcal{A}}$. Submodularity corresponds to the notion of a decreasing marginal contribution and is a common in many resource allocation problems, e.g., Vetta (2002), Krause and Guestrin (2007). Further, it is a key property underlying the design of many centralized algorithms for these problems.

We can interpret Theorem 3.4 in Vetta (2002) in the context of distributed welfare games to provide a fairly weak condition on the interaction between the welfare function W and the utility functions which guarantees that the price of anarchy is at least 1/2.

PROPOSITION 4. *(Vetta (2002)) Consider any distributed welfare game where*
(i) the welfare function W is submodular and
(ii) each player's utility is at least equal to his marginal contribution to the global welfare

$$U_i(a) \geq W(a) - W(a_i^0, a_{-i}), \;\; \forall i \in N, \; a \in \mathcal{A}.$$

If an equilibrium exists, then the price of anarchy is 1/2.

To provide a basis for comparison, computing the optimal assignment for a general distributed welfare game is NP-complete (Murphey (1999)). Further, the best known approximation algorithms guarantee only to provide a solution that is within $1 - 1/e \approx 0.6321$ of the optimal (Feige and Vondrak (2006), Ageev and Sviridenko (2004), Ahuja et al. (2003)). Thus, the 1/2 price of anarchy in this scenario is comparable to the best centralized solution.

While the generality of Proposition 4 is useful, the applicability is limited because it does not guarantee the existence of an equilibrium. Hence, its applicability depends on the results we have proven in Section 4.

COROLLARY 1. *Consider any distributed welfare game with a submodular global welfare function. If*
(i) players are anonymous and assigned an equally shared utility as in (5), or
(ii) players are assigned a wonderful life utility as in (6), or
(iii) players are assigned a (weighted) Shapley value utility as in (10) or (13),
then an equilibrium exists and the price of anarchy is 1/2.

The four distribution rules depicted in Corollary 1 all guarantee the existence of an equilibrium. Note that the wonderful life utility design satisfies condition (ii) of Proposition 4 with equality in addition to condition (iii) of distributed welfare games since the welfare function is submodular. Additionally, the Shapley and weighted Shapley values can easily be seen to satisfy condition (ii) of Proposition 4 when the welfare function is submodular. Finally, note that the price of anarchy is tight for these rules. This is illustrated in Appendix B.

To this point we have focussed exclusively on bounding the price of anarchy. Interestingly, when we focus on the price of stability there is a distinction between rules that are budget balanced and those that are not. In particular, the price of stability under any rule that is equivalent to equal share in the case of anonymous players is 1/2 (see Appendix B for an illustrative example). Hence, the price of stability of the Shapley value distributed rule is 1/2. Thus, any *fair* budget balanced rule has both a price of anarchy and a price of stability of 1/2.

In contrast, if the distribution rule is allowed to violate the budget balance constraint, then the price of stability can be 1. For example, under the wonderful life utility design, the optimal assignment is an equilibrium, thus the price of stability is 1. However, like the budget balance rules, the price of anarchy bound of 1/2 can be shown to be tight.

Distribution Rule	Budget Balanced	Price of Anarchy	Price of Stability
Equally Shared (anonymous)	yes	1/2	1/2
WLU	no	1/2	1
Shapley Value	yes	1/2	1/2

Table 2 Price of Anarchy and Price of Stability Comparison

6.1. Single Selection Distributed Welfare Games

We now move from general distributed welfare games to single selection games. In this case, we can strengthen the results of Proposition 4 utilizing Conditions 5.1–5.3 which guarantee the existence of an equilibrium.

PROPOSITION 5. *Consider a single selection distributed welfare game with a submodular welfare function and a budget balanced distribution rule. If Conditions 5.1–5.3 are satisfied, then an equilibrium exists and the price of anarchy is 1/2.*

Proof: The result follows from the fact that Conditions 5.1 – 5.3 combine to ensure that Condition (ii) of Proposition 4 is satisfied.

For any allocation $a \in \mathcal{A}$, the marginal contribution of any player $i \in N$ to the global welfare is

$$W(a_i, a_{-i}) - W(a_i^0, a_{-i}) = U_i(a_i, a_{-i}) + \sum_{j \neq i} \big(U_j(a_i, a_{-i}) - U_j(a_i^0, a_{-i})\big)$$

by noting that $U_i(a_i^0, a_{-i}) = 0$. To complete the proof, it is enough to show that for any player $j \neq i$,

$$U_j(a_i, a_{-i}) \leq U_j(a_i^0, a_{-i}),$$

since it follows that

$$U_i(a_i, a_{-i}) + \sum_{j \neq i} \big(U_j(a_i, a_{-i}) - U_j(a_i^0, a_{-i})\big) \leq U_i(a_i, a_{-i}).$$

Therefore, we will complete the proof by proving that Conditions 5.2 and 5.3 imply that for any players $i, j \in N$ and any allocation $a_{-ij} \in \mathcal{A}_{-ij}$

$$U_i(r, a_j^0, a_{-ij}) \geq U_i(r, r, a_{-ij}).$$

Assume, without loss of generality, that player i is stronger than player j. We can rewrite Condition 5.3 as

$$\begin{aligned} U_j(a_i^0, r, a_{-ij}) &\geq \frac{U_i(r, a_j^0, a_{-ij})}{U_i(r, r, a_{-ij})} U_j(r, r, a_{-ij}), \\ &\geq U_j(r, r, a_{-ij}), \end{aligned}$$

where the second inequality comes from Condition 5.2 which ensures that $\frac{U_i(r, a_j^0, a_{-ij})}{U_i(r, r, a_{-ij})} \geq 1$. This completes the proof.
□

6.2. Anonymous Distributed Welfare Games

Our bounds on the price of anarchy to this point have been independent of the number of players. In this section, we investigate the relationship between the price of anarchy and the number of players, albeit in the limited case where players are anonymous with regard to their impact on the global welfare. Furthermore, we will analyze the price of anarchy when the number of players at the equilibrium and optimal allocations differ. Specifically, let $W(a^{ne}; n + \delta)$ be the total welfare garnered by an equilibrium consisting of $n + \delta$ players. Likewise, let $W(a^{opt}; n)$ be the total welfare garnered by an optimal allocation consisting of n players.

THEOREM 3. *Consider any distributed welfare game with anonymous players where*
(i) the welfare function W is submodular,
(ii) the action set of player i is $\mathcal{A}_i \subseteq 2^{\mathcal{R}}$ and $\mathcal{A}_i = \mathcal{A}_j$ for all $i, j \in N$, and

(iii) each player's utility is at least equal to his marginal contribution to the global welfare

$$U_i(a) \geq W(a) - W(a_i^0, a_{-i}), \ \ \forall i \in N, \ a \in \mathcal{A}.$$

If an equilibrium exists, then the relative price of anarchy satisfies

$$\frac{W(a^{ne}; n+\delta)}{W(a^{opt}; n)} \geq \frac{n+\delta}{2n+\delta-1}.$$

Proof: We will prove the result by bounding $W(a^{opt}; n)$ in terms of $W(a^{ne}; n+\delta)$. First, notice that an upper bound on the $W(a^{opt}; n)$ is if one player in the optimal allocation can attain the entire welfare garnered at the equilibrium, $W(a^{ne}; n+\delta)$, and all other players attain $\min_{i\in N} U_i(a^{ne}; n+\delta)$ where $U_i(a^{ne}; n+\delta)$ represents the utility player i receives for the allocation a^{ne} consisting of $n+\delta$ players. To see that this upper bound holds, note first that (iii) guarantees that each player's utility is an upper bound on the player's contribution to the global welfare. Further, by combining the definition of an equilibrium with the fact that the welfare function is submodular, we see that no additional player can attain a utility higher than $\min_{i\in N} U_i(a^{ne}; n+\delta)$ once $W(a^{ne}; n+\delta)$ is covered. Thus, we have

$$W(a^{opt}; n) \leq W(a^{ne}; n+\delta) + (n-1) \min_{i\in N} U_i(a^{ne}; n+\delta).$$

Now, noting that

$$\min_{i\in N} U_i(a^{ne}; n+\delta) \leq \frac{W(a^{ne}; n+\delta)}{n+\delta}$$

gives

$$W(a^{ne}; n+\delta) + (n-1) \min_{i\in N} U_i(a^{ne}; n+\delta) \leq W(a^{ne}; n+\delta) \left(1 + \frac{n-1}{n+\delta}\right)$$

which easily gives the bound in the theorem

$$\begin{aligned} \frac{W(a^{ne}; n+\delta)}{W(a^{opt}; n)} &\geq \frac{1}{1+\frac{n-1}{n+\delta}} \\ &= \frac{n+\delta}{2n+\delta-1}. \end{aligned}$$

□

Notice that Theorem 3 shows that the worst-case price of anarchy is increasing as the number of players increases and that as $n \to \infty$ the price of anarchy approaches $1/2$, which matches Proposition 4. See Appendix B for an example which illustrates that this bound is tight. Furthermore, note that all the utility design methods previously studied, i.e., equally shared, wonderful life, and (weighted) Shapley value utility, satisfy the three conditions of Theorem 3. Hence, if the welfare function is submodular, then an equilibrium is guaranteed to exist and the bound on the relative price of anarchy holds. Lastly, note that the price of anarchy, $\delta = 0$, is bounded by

$$\frac{W(a^{ne}; n)}{W(a^{opt}; n)} \geq \frac{n}{2n-1}.$$

7. Motivational Example: The Sensor Coverage Problem

To ground the discussion in this paper, we will now highlight many of the issues we have discussed using a specific resource allocation problem: the sensor coverage problem. This problem is of particular interest given the growing deployment of sensor networks in a wide range of applications including surveillance, military, environmental monitoring, and beyond. The goal of the sensor coverage problem is to allocate a fixed number of sensors across a given "mission space" so as to maximize the probability of detecting a particular event. For a more detailed introduction to the problem, refer to Iyengar and Brooks (2005), Li and Cassandras (2005).

In modeling this problem, we divide the mission space into a finite set of sectors denoted as $\mathcal{R}$ and define an events density function, or relative reward/value function, $V(r)$, over $\mathcal{R}$, where $V(r) \geq 0, \forall r \in \mathcal{R}$. This formulation of the problem is common (Dhillon et al. (2004), Iyengar and Brooks (2005), Zou and Chakrabarty (2004)). Note that $V(r)$ often has a very intuitive meaning, e.g., in the case of enemy submarine tracking, $V(r)$ represents the a priori probability that an enemy submarine is situated in sector r.

There are a finite number of autonomous sensors (or players) denoted as $N = \{1,...,n\}$ allocated to the mission space. Each sensor i is capable of sensing activity in (monitoring) possibly multiple sectors simultaneously based on its chosen location. The set of possible monitoring choices for sensor i is denoted as $\mathcal{A}_i \subseteq 2^{\mathcal{R}}$. Let $\mathcal{A} = \prod_{i \in N} \mathcal{A}_i$ be the set of joint actions, or monitoring choices, for all players. The probability that sensor i detects an event in sector r given his current monitoring choice a_i is denoted as $p_i(r, a_i)$. We will assume that the detection probabilities satisfy:

$$r \in a_i \Leftrightarrow p_i(r, a_i) > 0,$$
$$r \notin a_i \Leftrightarrow p_i(r, a_i) = 0,$$

This sensor model is quite general and can accommodate a variety of settings, such as a sensing capability that degrades exponentially in distance, obstacles, etc.

For a given joint action profile $a := \{a_1, ..., a_n\}$, the joint probability of detecting an event in sector r is

$$P(r, a) = 1 - \prod_{i \in N} [1 - p_i(r, a_i)].$$

The goal of the global planner in this scenario is to allocate the sensors in a way that maximizes the probability of detecting an event, which is characterized by the following global welfare function (Li and Cassandras (2005))

$$W(a) = \sum_{r \in \mathcal{R}} V(r) P(r, a). \tag{22}$$

Computing the optimal sensor allocation is an NP-hard combinatorial optimization problem – Murphey (1999) shows this for a structurally equivalent version of the weapon targeting problem. Resultantly, research has traditionally centered around developing heuristic methods to quickly obtain near optimal allocations, where the degree of suboptimality is dependent on the structure of the global objective, e.g., Ahuja et al. (2003).

7.1. The Sensor Coverage Game

Rather than view the sensor coverage problem as a centralized optimization problem, our focus is on the design of autonomous sensors that are individually capable of making their own independent decision in response to local information. To model this situation, we will view the interactions of the sensors as a non-cooperative resource allocation game where each sensor i is assigned a utility function $U_i : \mathcal{A} \to \mathbb{R}$ that defines his payoff (utility) for each monitoring profile. Note that a sensor's utility could be adversely affected by the monitoring choices of other sensors.

We will refer to the non-cooperative game theoretic formulation of the sensor coverage problem as the *sensor coverage game*. The engineering question in this setting is how to design the sensor utility functions, or equivalently, how should a global planner distribute the global welfare to the sensors. In the case of large sensors networks local utility functions are a necessity.

7.2. Local Utility Designs for the Sensor Coverage Game

The sensor coverage game is a simple example of a distributed welfare game. Thus, we can apply our results in order to design distribution rules that guarantee the existence of an equilibrium. In particular, we can immediately conclude that the wonderful life and the (weighted) Shapley value utility designs will guarantee the existence of an equilibrium.

In the case of the sensor coverage problem, the wonderful life utility design can be written as

$$U_i(a_i, a_{-i}) = \sum_{r \in a_i} V(r) p_i(r, a_i) \prod_{j \in N,\, j \neq i} (1 - p_j(r, a_j)), \tag{23}$$

In addition, the weighted Shapley value can be written as

$$U_i(a_i, a_{-i}) = w_i \sum_{r \in a_i} V(r) \sum_{S \subseteq N,\, S \in r} \frac{\prod_{j \in S} p_j(r, a_j)}{\sum_{j \in S} w_j} (-1)^{|S|-1}, \tag{24}$$

where w_j is the weight of sensor j. Recall that the Shapley value can be attained with $w_j = 1$. We know from our previous discussions that both (23) and (24) guarantee the existence of an equilibrium. Further, since the welfare function is submodular, both rules yield a price of anarchy of 1/2. Further, the price of stability of the wonderful life design is 1 and the price of stability of the weighted Shapley design is 1/2.

As we discussed previously, there are many issues that limit the practical applicability of the wonderful life and (weighted) Shapley designs. The most important issues are that the wonderful life design is not budget balanced and that, in general, computing the (weighted) Shapley values is not tractable.

7.3. Single Selection Sensor Coverage Games

In this section we will focus on the sensor coverage game in which each sensor is only capable of selecting a single sector, i.e., $\mathcal{A}_i = \mathcal{R}$. Furthermore, we will assume that each sensor has an invariant detection probability, i.e., for each sensor $i \in N$ there exists an invariant detection probability $p_i > 0$ such that

$$p_i(r, a_i) = \begin{cases} p_i, & \text{if } r = a_i, \\ 0, & \text{if } r \neq a_i. \end{cases}$$

Let $W^r(a) := P(r,a)V(r)$ be the welfare generated from sector r given the allocation a. The marginal contribution of player i to the global welfare provided that the $a_i = r$ is

$$W^r(r, a_{-i}) - W^r(a_i^0, a_{-i}) = p_i \left(V(r) - W^r(a_i^0, a_{-i}) \right), \tag{25}$$

which fits into the class of admissible welfare functions developed in Section 5.2. Therefore we can appeal to Theorem 2 to design distribution rules that yield the existence of an equilibrium and Theorem 5 to bound the price of anarchy. This is summarized in the following corollary.

COROLLARY 2. *Consider any single selection sensor coverage game where each sensor has an invariant detection probability. Fix $k \in [0,1]$. Define the strength of player i as the solution to the equation*

$$p_i = (1-k)s_i + k \frac{s_i}{1 + s_i}.$$

If the utility of sensor i is defined as

$$U_i(r, a_{-i}) = \frac{s_i}{\sum_{j \in N : a_j = r} s_j} V(r) P(r, a). \tag{26}$$

then the resulting game is a budget balanced distributed welfare game and possesses at least one equilibrium. Furthermore, the price of anarchy is 1/2.

An example of a distribution rule that guarantees the existence of an equilibrium is dividing the welfare proportionally according to players' detection probabilities

$$U_i(r, a_{-i}) = \frac{p_i}{\sum_{j \in N: a_j = r} p_j} P(r, a) V(r). \tag{27}$$

This corresponds to the special case when $k = 0$, and we refer to this as the *proportional share utility design*. An alternative distribution rule that guarantees the existence of an equilibrium is the *normalized wonderful life utility design* which takes on the form

$$\begin{aligned} U_i(r, a_{-i}) &= \left(\frac{W(r, a_{-i}) - W(a_i^0, a_{-i})}{\sum_{j \in N: a_j = r} W(r, a_{-j}) - W(a_j^0, a_{-j})} \right) P(r, a) V(r), \\ &= \left(\frac{\frac{p_i}{1-p_i}}{\sum_{j \in N: a_j = r} \frac{p_j}{1-p_j}} \right) P(r, a) V(r). \end{aligned} \tag{28}$$

This corresponds to the special case when $k = 1$. Both of these distribution rules have a price of anarchy of 1/2 because the welfare function is submodular.

It is interesting to note that the distribution rules captured in (27) and (28) are extremely sensitive to perturbations. For example, consider a slight variant of (27)

$$U_i(r, a_{-i}) = \frac{p_i^\beta}{\sum_{j \in N: a_j = r} p_j^\beta} P(r, a) V(r),$$

for some $\beta > 0$. An equilibrium is only guaranteed to exist for the special case when $\beta = 1$. (See Appendix C.)

It is also important to point out that neither of the distribution rules captured in (27) and (28) correspond to particular weights for the weighted Shapley value. (See Appendix D.) Thus, a priori, one does not expect them to guarantee the existence of an equilibrium. This is especially true in light of the results of Chen et al. (2008), who prove that all distribution rules that guarantee the existence of an equilibrium must correspond to a weighted Shapley distribution rule (13) in a special case of distributed welfare games.

7.4. Anonymous Sensor Coverage Games

In the case of anonymous sensors, that is all sensors have the same detection probability p, we can obtain tighter bounds on the price of anarchy by appealing to Theorem 3.

COROLLARY 3. *Consider any anonymous sensor coverage game where the action set of sensor i is $\mathcal{A}_i \subseteq 2^{\mathcal{R}}$ and $\mathcal{A}_i = \mathcal{A}_j$ for all $i, j \in N$. If each sensor is assigned an equally shared utility*

$$U_i(a) = \sum_{r \in a_i} \frac{1}{\sigma^r(a)} V(r) P(r, a), \tag{29}$$

then the resulting game possesses at least one equilibrium and the relative price of anarchy satisfies

$$\frac{W(a^{ne}; n + \delta)}{W(a^{opt}; n)} \geq \frac{n + \delta}{2n + \delta - 1}.$$

We can specialize these bounds further to attain the following bound, which illustrates the impact of the detection probability on the price of anarchy.

THEOREM 4. *Consider a single-sector anonymous sensor coverage game with n sensors each having invariant detection probability p. Under the equal share utility design (29) the price of anarchy is bounded by*

$$\frac{W(a^{ne})}{W(a^{opt})} \geq \left(\frac{a^*}{n} + \frac{1-(1-p)^{n-a^*}}{1-(1-p)^n}\right)^{-1}$$

$$\text{where} \quad a^* = \begin{cases} n-1, & p=1; \\ n - \frac{\log\left(n\frac{\log(1/(1-p))}{1-(1-p)^n}\right)}{\log(1/(1-p))}, & p<1. \end{cases}$$

We defer the proof of Theorem 4 to the appendix. We illustrate the price of anarchy in Figure 1. Note that this bound is a decreasing function of the detection probability of the sensors, which is intuitive.

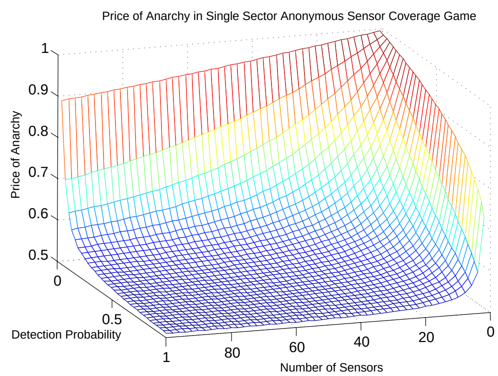


Figure 1 Price of Anarchy in Single Sector Anonymous Sensor Coverage Game

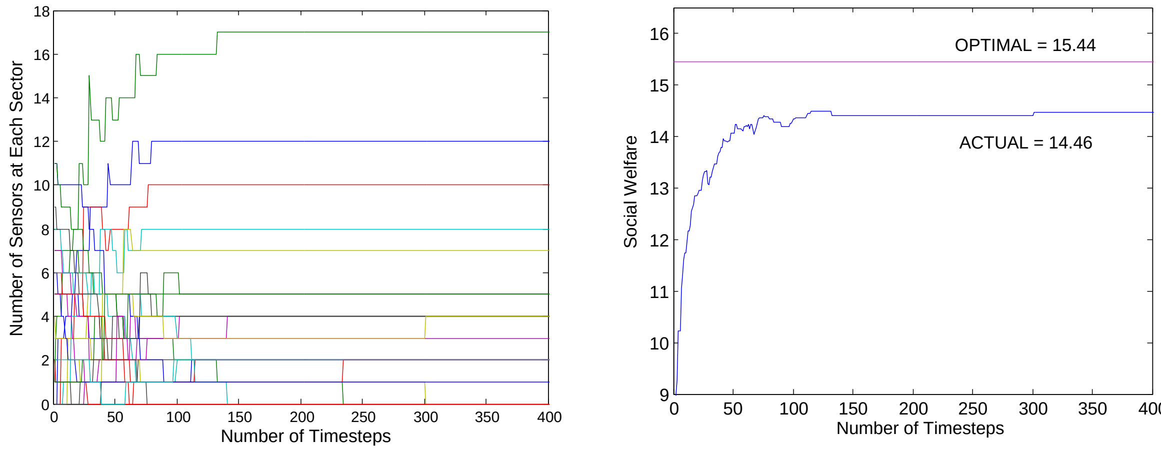
7.5. Simulation Experiments

To this point, we have explored equilibrium behavior in the sensor coverage game. The question that remains is how can autonomous sensors reach an equilibrium in a distributed fashion. While not focusing on this question in detail, we illustrate the applicability of the theory of learning in games as a local control mechanism for coordinating group behavior.

We consider a sensor coverage game with 100 single-sector homogenous sensors with invariant detection probability $p = 0.25$. The mission space is $\mathcal{R} = \{r_1, ..., r_{25}\}$. The reward for each sector is randomly assigned from a uniform distribution; two sectors according to $U[0,6]$, four sectors according to $U[0,3]$, and the remaining according to $U[0,1]$. Each sensor is capable of monitoring any of the 25 sectors, i.e., $\mathcal{A}_i = \mathcal{R}$ and is assigned an equally shared utility design (29).

There is a large body of literature analyzing distributed learning algorithms in potential games (Young (1998), Monderer and Shapley (1996a), Marden et al. (2008, 2007b,a)). We will apply *fading memory joint strategy fictitious play with inertia*, which guarantees convergence to an equilibrium in any potential game while maintaining computational tractability even in large-scale games (Marden et al. (2008)). Fading memory joint strategy fictitious play with inertia can be described as follows:

1. **Initialization:** Each sensor i is assigned a perceived utility $V_i^{a_i}(1) \in \mathbb{R}$ for each action $a_i \in \mathcal{A}_i$. One can think of $V_i^{a_i}(1)$ as sensor i's initial belief about the utility he would receive for playing action a_i in the ensuing time step.
2. **Action Selection:** At time $t \geq 1$, each sensor i plays the following strategy:
   - $a_i(t) \in \arg\max_{a_i \in \mathcal{A}_i} V_i^{a_i}(t)$ with probability $(1-\epsilon)$,
   - $a_i(t) = a_i(t-1)$ with probability (ϵ),



(a) Evolution of Number of Sensors in Each Sector over Mission (b) Evolution of Welfare Function over Mission

Figure 2 Simulation results for sensor coverage game.

where $\epsilon \in (0,1)$ is referred to as the sensor's inertia, i.e., a probabilistic reluctance to changing strategies.

3. **Belief Propagation:** Each sensor i updates his beliefs as

$$V_i^{a_i}(t+1) = (1-\lambda)V_i^{a_i}(t) + \lambda\, U_i(a_i, a_{-i}(t)), \;\; \forall a_i \in \mathcal{A}_i,$$

where $\lambda \in (0,1]$ is the discount factor.

4. Return to Step 2 and repeat.

It is worth noting that fading memory joint strategy fictitious play with inertia only requires each sensor to observe the number of players at each sector. The identity of the observed player is unimportant. If all players adhere to the prescribed learning rule, then the action profile, $a(t)$, will converge almost surely to an equilibrium.

We use the following discount factor and inertia: $\lambda = 0.5$ and $\epsilon = 0.02$. Figure 2(a) illustrates the evolution of the number of sensors at each sector. The identity of the sectors is unimportant as the key observation is that behavior settles down at an equilibrium. Figure 2(b) illustrates the evolution of the global welfare in addition to the efficiency gap between the equilibrium and the optimal. From Theorem 4, we know that the price of anarchy must be greater than 0.541. Our simulation illustrates that Theorem 4 provides a very conservative estimate of the efficiency since the price of anarchy we observe is 0.936.

8. Concluding Remarks

In this paper, we focus on a class of games that we refer to as distributed welfare games. These games are formulated to study the feasibility of approaching resource allocation problems via distributed, non-cooperative techniques. In particular, these games allows us to study how the method used to divide the global welfare among participating (selfish) players impacts the existence and efficiency of equilibria in resource allocation games.

The paper illustrates that cost sharing methodologies are applicable to distributed welfare games, but suffer from a number of drawbacks, such as leading to intractable distribution rules with high informational requirements. This leads us to investigate the possibility of developing alternative distribution rules. To this end, we derive three sufficient conditions on distribution rules that guarantee the existence of an equilibrium in the setting where players are only allowed to select a single resource. Further, we use these sufficient conditions to develop a systematic procedure for designing desirable distribution rules with minimal informational and computational costs in these single selection distributed welfare games when the welfare function takes on a specified form. We illustrate the applicability of this procedure using the sensor coverage problem. We also derive general bounds on the price of anarchy in distributed welfare games and application specific bounds on the price of anarchy for the sensor coverage problem. It should be noted that the structure of the global welfare function W for sensor coverage parallels those for many other problems, e.g., weapon target assignment, fault detection, and surveillance.

A number of interesting research questions related to distributed welfare games remain. Of primary importance is the use of learning rules to ensure fast convergence to an equilibrium. When players are anonymous, we demonstrated that there are several distributed learning algorithms that guarantee players will reach an equilibrium. However, it remains to apply learning algorithms for the distributed welfare games with players that are not anonymous. In the case of the Shapley value and wonderful life designs, the same learning algorithms can be applied, since both designs lead to potential games. However, it is unclear whether the other distribution rules introduced in the paper, e.g. the proportional share rule, give rise to a potential game.

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Appendix A: Proof of Theorem 2

We will prove the result by verifying Conditions 5.1-5.3.

Step 1: Verification of Condition 5.1

According to (17), for any $k \in [0,1]$ we have

$$f_i \geq f_j \;\Rightarrow\; s_i \geq s_j.$$

Therefore, Condition 5.1 is satisfied.

Step 2: Verification of Condition 5.2

Let $a := (r, a_j^0, a_{-ij})$ and $a' := (r, r, a_{-ij})$ be any two action profile where $s_i \geq s_j$. According to (15), we have

$$\begin{aligned} U_i(a) &= \frac{s_i}{\sum_{j \in N: a_j = r} s_j} W^r(a), \\ &= \frac{s_i}{S} W^r(a), \end{aligned}$$

where $S := \sum_{j \in N: a_j = r} s_j$. Similarly, we have

$$\begin{aligned} U_i(a') &= \frac{s_i}{\sum_{j \in N: a'_j = r} s_j} W^r(a'), \\ &= \frac{s_i}{S + s_j} \left(W^r(a) + f_j L^r(a) \right). \end{aligned}$$

Verifying Condition 5.2 involves proving that

$$\frac{s_i}{S} W^r(a) \geq \frac{s_i}{S + s_j} \left(W^r(a) + f_j L^r(a) \right),$$

which is equivalent to proving that

$$W^r(a) \geq S \left(\frac{f_j}{s_j} \right) L^r(a).$$

Letting $f_j = (1-k)s_j + k\frac{s_j}{1+s_j}$ and expanding the right hand side we obtain

$$\begin{aligned} S \left(\frac{f_j}{s_j} \right) L^r(a) &= S \left(\frac{(1-k)s_j + k\frac{s_j}{1+s_j}}{s_j} \right) L^r(a), \\ &= S \left((1-k) + k\frac{1}{1+s_j} \right) L^r(a), \\ &\leq S L^r(a), \\ &= \sum_{i \in N: a_i = r} s_i L^r(a). \end{aligned}$$

Since $s_i \leq \frac{f_i}{1-f_i}$, we have

$$\begin{aligned} \sum_{i \in N: a_i = r} s_i L^r(a) &\leq \sum_{i \in N: a_i = r} \frac{f_i}{1-f_i} L^r(a), \\ &= \sum_{i \in N: a_i = r} W^r(a) - W^r(a_i^0, a_{-i}), \\ &\leq W^r(a), \end{aligned}$$

where the last inequality results from the welfare structure in (16).

Step 3: Verification of Condition 5.3

Let $a := (a_i^0, a_j^0, a_{-ij})$ and $a' := (r, r, a_{-ij})$ where $s_i \geq s_j$. The utility of player i for the allocation (r, a_j^0, a_{-ij}) is

$$\begin{aligned} U_i(r, a_j^0, a_{-ij}) &= \frac{s_i}{\sum_{j \in N: a_j = r} s_j} \left(W^r(a) + f_i L^r(a) \right), \\ &= \frac{s_i}{S + s_i} \left(W^r(a) + f_i L^r(a) \right). \end{aligned}$$

where $S := \sum_{j \neq i : a_j = r} s_j$. The utility of player j for the allocation (a_i^0, r, a_{-ij}) is

$$U_j(a_j^0, r, a_{-ij}) = \frac{s_j}{S + s_j} \left(W^r(a) + f_j L^r(a) \right).$$

Verifying Condition 5.3 involves proving that

$$\frac{\frac{s_j}{S+s_j} \left(W^r(a) + f_j L^r(a) \right)}{\frac{s_i}{S+s_i} \left(W^r(a) + f_i L^r(a) \right)} \geq \frac{s_j}{s_i}.$$

Using simple algebra, this is equivalent to showing that

$$(S + s_i) \left(W^r(a) + f_j L^r(a) \right) \geq (S + s_j) \left(W^r(a) + f_i L^r(a) \right).$$

Rewriting the above equation, we obtain

$$\begin{aligned} W^r(a)(s_i - s_j) &\geq (S + s_j) f_i L^r(a) - (S + s_i) f_j L^r(a), \\ &= S(f_i - f_j) L^r(a) + (s_j f_i - s_i f_j) L^r(a). \end{aligned}$$

Dividing through by $(s_i - s_j)$, we are left to show that

$$W^r(a) \geq \left(\frac{f_i - f_j}{s_i - s_j} \right) S L^r(a) + \left(\frac{s_j f_i - s_i f_j}{s_i - s_j} \right) L^r(a). \tag{30}$$

We will first focus on the first term of (30)

$$\begin{aligned} \frac{f_i - f_j}{s_i - s_j} &= \frac{(1-k)s_i + k\frac{s_i}{1+s_i} - (1-k)s_j - k\frac{s_j}{1+s_j}}{s_i - s_j}, \\ &= (1-k) + k\frac{1}{(1+s_i)(1+s_j)}, \\ &\leq 1. \end{aligned}$$

Focusing on the numerator of the second term of (30), we have

$$\begin{aligned} s_j f_i - s_i f_j &= (1-k) \left(s_j \frac{s_i}{1+s_i} - s_i \frac{s_j}{1+s_j} \right), \\ &= k \left(\frac{s_i s_j (1+s_j) - s_i s_j (1+s_i)}{(1+s_i)(1+s_j)} \right), \\ &= k \left(\frac{s_i s_j (s_j - s_i)}{(1+s_i)(1+s_j)} \right), \\ &\leq 0. \end{aligned}$$

In conclusion, we have

$$\begin{aligned} \left(\frac{f_i - f_j}{s_i - s_j} \right) S L^r(a) + \left(\frac{s_j f_i - s_i f_j}{s_i - s_j} \right) L^r(a) &\leq S L^r(a), \\ &\leq W^r(a). \end{aligned}$$

This completes the proof.

Appendix B: Tightness of Theorem 3 and Price of Stability in Fair Budget Balanced Rules

To illustrate the tightness of Theorem 3, consider a anonymous single-selection sensor coverage game with the equal share utility design (29). There are $n+\delta$ sensors each having invariant detection probability $p=1$ and $n+\delta$ sectors. One sector, r, has reward 1 and the other sectors have reward $1/(n+\delta)$.

Notice that if all $n+\delta$ sensors cover r, then the total reward garnered is 1 and the assignment is an equilibrium. However, the optimal assignment with n sensors has one sensor in each sector and achieves reward $1+(n-1)/(n+\delta)$. Thus, we achieve the bound in Theorem 3.

Appendix C: Sensitivity of Distribution Rules

To illustrate the sensitivity of the distribution rules presented in Theorem 2 we consider the following variation of the proportional share utility design:

$$g_i(r,a)=\frac{p_i^\beta}{\sum_{j:a_j=r}p_j^\beta}.$$

Note that when, $\beta=1$, this is equivalent to the proportional share utility design. When $\beta\rightarrow 0$, this rule becomes the equal share utility design, and when $\beta\rightarrow\infty$, this rule gives the entire reward to the sensor with the highest detection probability. So, we have already seen that an equilibrium exists when $\beta=1$ and does not exist when $\beta=0$. We will now see that for any $\beta\neq 1$, an equilibrium is not guaranteed to exist.

Consider 2 sectors x and y and 2 sensors 1 and 2. The rewards are $V(x)=V(y)=1$ and the detection probabilities are $p_1>p_2$. Now, we are going to set a scenario where if $\beta>1$ sensor 1 chases sensor 2 and if $\beta<1$ sensor 2 chases sensor 1.

The utilities to each sensor when both are in the same sector are

$$U_i(x,x)=U_i(y,y)=\frac{p_i^\beta}{p_1^\beta+p_2^\beta}(1-(1-p_1)(1-p_2))$$

If the sensors are in different sectors, the utilities are

$$U_i(x,y)=U_i(x,y)=p_i$$

Now, consider player 1. The difference in utility between being together and being apart is equal to:

$$\begin{aligned}U_1(x,x)-U_1(y,x) &= \frac{p_1^\beta}{p_1^\beta+p_2^\beta}(1-(1-p_1)(1-p_2))-p_1\\ &= \frac{p_2p_1^\beta}{p_1^\beta+p_2^\beta}\left(1-p_1-\left(\frac{p_1}{p_2}\right)^{1-\beta}\right)\end{aligned}$$

Similarly, for player 2, we have the difference in utility between being together and being apart is equal to:

$$U_2(x,x)-U_2(y,x) = \frac{p_1p_2^\beta}{p_1^\beta+p_2^\beta}\left(1-p_2-\left(\frac{p_2}{p_1}\right)^{1-\beta}\right)$$

If $\beta=1$ both players prefer to be apart regardless of p_1,p_2. However, if $\beta\neq 1$, the sensors exhibit chasing behavior. In particular, set $p_1=0.5$ and $p_2=\epsilon>0$. Then, by letting $\epsilon\rightarrow 0$, we can observe chasing behavior. In particular, as $\epsilon\rightarrow 0$, when $\beta<1$ we see that sensor 1 prefers to be away from sensor 2 and sensor 2 prefers to be with sensor 1, and when $\beta>1$ the sensors have the opposite preferences.

Appendix D: Relationship of Distribution Rules to the Weighted Shapley Value

One might expect a priori that all distribution rules that guarantee the existence of a Nash equilibrium can be viewed as a version of the weighted Shapley. This is especially true in light of recent results (Chen et al. (2008)) which prove such a characterization in the case of a related model.

However, our results in the example of the sensor coverage game illustrate that this is not the case. In particular, we give examples of two simple rules that guarantee the existence of an equilibrium which are not weighted Shapley values.

D.1. The Proportional Share Utility Design

eq:dist-ratio-marginal

We will start with the proportional share design in (27). Consider a sensor coverage game with 2 sensors where each sensor i has an invariant detection probability p_i. In such a game, we can write the proportional share rule as (after some algebra)

$$U_1(r,r) = \left(p_1 - \frac{p_1}{p_1+p_2} p_1 p_2\right) V(r)$$

and we can write the weighted Shapley value as

$$U_1(r,r) = \left(p_1 - \frac{w_1}{w_1+w_2} p_1 p_2\right) V(r)$$

Thus, for the proportional share rule to correspond to a weighted Shapley value, it must be that $p_i/p_j = w_i/w_j$ for all i,j. So, $w_i = cp_i$ for some constant c for all i.

Now, consider a sensor coverage game with 3 sensors. In this case, we can write the proportional share rule as

$$U_1(r,r,r) = p_1 \left(1 + \left(\frac{p_1p_2 + p_2p_3 + p_3p_1}{p_1+p_2+p_3}\right) + \frac{p_1p_2p_3}{p_1+p_2+p_3}\right) V(r)$$

and we can write the weighted Shapley value as

$$U_1(r,r,r) = w_1 \left(\frac{p_1}{w_1} + \left(\frac{p_1p_2}{w_1+w_2} + \frac{p_1p_3}{w_1+w_3}\right) + \frac{p_1p_2p_3}{w_1+w_2+w_3}\right) V(r)$$

But, it is easy to see that no choice of w_i can both satisfy $p_i/p_j = w_i/w_j$ and make the above two equations equal.

Notice that the reason proportional share utilities cannot be viewed as weighted Shapley values is not that for a particular number of players there are not weights that could be chosen so that the utilities can be viewed as a weighted Shapley value. Rather, the issue is that these weights are required to be the same across all sets of possible players. It is this extra constraint that leads to the existence of rules that guarantee the existence of an equilibrium but are not weighted Shapley values.

D.2. The Normalized Wonderful Life Utility Design

We will consider a 2 sensor game where sensor i has an invariant detection probability p_i. Then, we have that the normalized wonderful life utility design can be written as (after some algebra)

$$U_1(r,r) = \left(p_1 - \frac{\frac{p_2}{1-p_2}}{\frac{p_1}{1-p_1} + \frac{p_2}{1-p_2}} p_1 p_2\right) V(r)$$

and we can write the weighted Shapley value as

$$U_1(r,r) = \left(p_1 - \frac{w_i}{w_1+w_2} p_1 p_2\right) V(r)$$

So, the only way to satisfy these equations is to have $\frac{w_i}{w_j} = \frac{p_j(1-p_i)}{p_i(1-p_j)}$. So, $w_i = c(1-p_i)/p_i$ for some constant c for all i.

Once again, it can be verified that such a weight scheme will not work for a three player game. The calculation is tedious, so we do not include it.

Appendix E: Proof of Theorem 4

Theorem 4 follows from the following sequence of lemmas.

LEMMA 1. *Consider a single-sector sensor coverage game with n sensors each having fixed detection probability p. The price of anarchy is bounded by*

$$\left(\max_{a+b\leq n, a\geq 0, b\geq 1} \left\{ \frac{a}{a+b} + \frac{1-(1-p)^b}{1-(1-p)^{a+b}} \right\} \right)^{-1}$$

where the maximum is taken over integer a, b.

Proof: We will again describe the optimal placement in terms of the Nash placement. For each sector r covered by the Nash, either every sensor is also present at that sector in the optimal placement, or some number m_r "move" in the optimal. The sensors that move, can provide an additional reward that is bounded by their contribution in the Nash, but will drop the reward gathered from sector r. Note that $0 \leq m_r \leq \sigma^r(a^{ne}) - 1$.

This gives that

$$\begin{aligned} W(a^{opt}) &\leq \sum_{r\in\mathcal{R}:\sigma^r(a^{ne})>0} \max_{m_r\in[0,\sigma^r(a^{ne})-1]} \left\{ m_r U_r(a^{ne}) + (1-(1-p)^{\sigma^r - m_r}) V(r) \right\} \\ &= \sum_{r\in\mathcal{R}:\sigma^r(a^{ne})>0} \max_{m_r\in[0,\sigma^r(a^{ne})-1]} \left\{ \frac{m_r}{\sigma^r(a^{ne})} + \frac{1-(1-p)^{\sigma^r(a^{ne})-m_r}}{1-(1-p)^{\sigma^r(a^{ne})}} \right\} (1-(1-p)^{\sigma^r(a^{ne})}) V(r) \end{aligned}$$

Letting $a_r = m_r$ and $b_r = \sigma^r(a^{ne}) - m_r$, we have

$$\begin{aligned} W(a^{opt}) &\leq \sum_{r\in\mathcal{R}:\sigma^r(a^{ne})>0} \max_{a_r+b_r=\sigma^r(a^{ne}), a_r\geq 0, b_r\geq 1} \left\{ \frac{a_r}{a_r+b_r} + \frac{1-(1-p)^{b_r}}{1-(1-p)^{a_r+b_r}} \right\} (1-(1-p)^{\sigma^r(a^{ne})}) V(r) \\ &\leq \sum_{r\in\mathcal{R}:\sigma^r(a^{ne})>0} \max_{a+b\leq n, a\geq 0, b\geq 1} \left\{ \frac{a}{a+b} + \frac{1-(1-p)^b}{1-(1-p)^{a+b}} \right\} (1-(1-p)^{\sigma^r(a^{ne})}) V(r) \\ &= \max_{a+b\leq n, a\geq 0, b\geq 1} \left\{ \frac{a}{a+b} + \frac{1-(1-p)^b}{1-(1-p)^{a+b}} \right\} \sum_{r\in\mathcal{R}:\sigma^r(a^{ne})>0} (1-(1-p)^{\sigma^r(a^{ne})}) V(r) \\ &= \max_{a+b\leq n, a\geq 0, b\geq 1} \left\{ \frac{a}{a+b} + \frac{1-(1-p)^b}{1-(1-p)^{a+b}} \right\} W(a^{ne}) \end{aligned}$$

which completes the proof.

□

To obtain a more explicit form of the price of anarchy, we will first relax the constraints and then we will characterize the maximal a, b.

LEMMA 2.

$$\max_{a+b\leq n, a\geq 0, b\geq 1} \left\{ \frac{a}{a+b} + \frac{1-(1-p)^b}{1-(1-p)^{a+b}} \right\} \leq \max_{a+b=n, a\geq 0, b\geq 1} \left\{ \frac{a}{a+b} + \frac{1-(1-p)^b}{1-(1-p)^{a+b}} \right\}$$

where the LHS is taken over integer a, b and the RHS is taken over real-valued a, b.

Proof: We will start by relaxing the integer optimization to include real-valued a, b.

Next, suppose, that $a_m + b_m = m < n$ are the maximizers under the constraint that $a + b = m$. We will show that $a_n = na_m/m$, $b_n = nb_m/m$ lead to a larger value than a_m, b_m. Combining this with the observation that $a_n + b_n = n$ then completes the proof.

$$\frac{a_n}{a_n+b_n} + \frac{1-(1-p)^{b_n}}{1-(1-p)^{a_n+b_n}} = \frac{a_m}{a_m+b_m} + \frac{1-(1-p)^{n/m}(1-p)^{b_m}}{1-(1-p)^{n/m}(1-p)^{a_m+b_m}}$$

Now, it is enough to show that

$$\frac{1-(1-p)^{n/m}(1-p)^{b_m}}{1-(1-p)^{n/m}(1-p)^{a_m+b_m}} \geq \frac{1-(1-p)^{b_m}}{1-(1-p)^{a_m+b_m}}$$

A bit of algebra shows that this holds as long as $(1-p)^{n/m} \leq 1$, which is always true in our setting since $p \in [0,1]$.
□

Now, we know that $b = n - a$. So, we need only calculate a.

LEMMA 3.

$$a^* = \arg\max_{0 \leq a \leq n-1} \left\{ \frac{a}{n} + \frac{1-(1-p)^{n-a}}{1-(1-p)^n} \right\} = \begin{cases} n-1, & p = 1; \\ n - \frac{\log\left(\frac{n\log(1/(1-p))}{1-(1-p)^n}\right)}{\log(1/(1-p))}, & p < 1. \end{cases}$$

Proof: For the case of $p = 1$, the result is immediate. In the case when $p \neq 1$, we will determine the maximizer by simply differentiating. Differentiating with respect to a gives:

$$\frac{1}{n} - \frac{(1-p)^{n-a}\log(1/(1-p))}{1-(1-p)^n}$$

Setting the derivative equal to zero, then gives

$$(1-p)^{n-a} = \frac{n\log(1/(1-p))}{1-(1-p)^n}$$

Solving for a, we obtain

$$a = n - \frac{\log\left(\frac{n\log(1/(1-p))}{1-(1-p)^n}\right)}{\log(1/(1-p))}$$

which completes the proof.
□