Revision Games

KAMADA, Yuichiro[*] and KANDORI, Michihiro[†]
[*]*Department of Economics, Harvard University*
[†]*Faculty of Economics, University of Tokyo*

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Abstract

We analyze a situation where players in advance prepare their actions in a game. After the initial preparation, they have some opportunities to revise their actions, which arrive stochastically. Prepared actions are assumed to be mutually observable. We show that players can achieve a certain level of cooperation in such a class of games.

1 Introduction

It is often the case that agents must prepare their strategies in advance before they interact. For example, consider researchers who are competing to win research grants. Their strategies correspond to their proposals, which are to be submitted by a prespecified deadline. In such a situation, researchers typically prepare their proposal in advance, and proposals are usually subject to some revisions before submission. Since they have other obligations, such as teaching, committee work and so on, they do not always have an opportunity to revise their proposals. Rather, opportunities for revision may be arrive stochastically over time. Researchers may also have some information about what kind of proposals are being prepared by their rivals. Based on such information, researchers revise their proposals when they have time (i.e., when a revision opportunity arrives), and they submit what they have when the deadline comes.

In the present paper, we analyze such a situation, where a "component game" is played only once, and players must in advance prepare their actions. They have some opportunities to revise their prepared actions, and the opportunities for revision arrive stochastically. Prepared actions are assumed to be mutually observable. We show that, under some conditions, players can achieve a certain level of cooperation in such a class of games, which we refer to as Revision Games.

The basic logic to sustain cooperation in our model is closely related to the theory of finitely repeated games. The simple class of revision games, where the component game is played (once and for all) with a constant positive probability in each period, is actually identical to infinitely repeated games. We make this point by a simple example in the next section.

In the main part of our paper (Section 3 -), we consider revision games with a fixed deadline. A "component game" is played at time 0, but players must prepare some actions at time $-T$. Between $-T$ and the deadline 0, opportunities to revise their actions arrive via a Poisson process. Prepared actions at each moment of time is observable. We show that, when the component game has continuous actions and the payoff function "smooth", a certain level of cooperation is sustained. In particular, we show that the best symmetric "trigger strategy" equilibrium can be characterized by a simple differential equation. Furthermore, the expected payoff associated with such an equilibrium is independent of the Poisson arrival rate, if the time horizon (T) is sufficiently long.

Our framework can be applied to many standard games in the economics literature: We provide analyses on prisoner's dilemma, Cournot oligopoly. It will be shown that revision games sometimes manage to achieve quite high degree of cooperation or collusion. For example, in the Cournot duopoly game with linear inverse demand function and a constant marginal cost, 99.9% of the collusive payoff can be achieved in a subgame perfect equilibrium that we construct.

There is a body of literature which is related to our model. Pitchford and Snyder (2004) considered a hold-up game, where the seller makes an investment k, and the buyer pays p. The payoffs are $-k$ for the seller and $u(k) - p$ for the buyer. In the one-shot game where k and p are made simultaneously, the Nash equilibrium is inefficient with $k = 0$ and $p = 0$. They considered the following dynamic process of preparing actions in the hold-up game (i.e., making investments and payments). At each moment of time $t = 0, 1, 2, ...$ the seller invests Δk_t and then the buyer pays Δp_t. At the end of each period, there is a fixed probability θ to terminate the process. If the process is terminated at time T, the component game (the hold-up game) is played with the cumulative investment $\sum_{t=0}^{T} \Delta k_t$ and payment $\sum_{t=0}^{T} \Delta p_t$. They show that, as $\theta \to 0$, the first best level of investment is achieved by a subgame perfect equilibrium.[1] This is closely related to our revision games I. An important difference is that prepared action is irreversible (cumulative investment and payment can only increase) in their model, but actions can be changes in any directions in our model. If their model had increasing probability of termination $\theta_t \to 1$ as $t \to \infty$ (and if the actions were reversible), it would be similar in spirit to our revision games II. Our revision games II are also similar to Bernheim and Dasgupta (1995), who considered infinitely repeated games with decreasing discount factor ($\delta_t \to 0$ as $t \to \infty$). They show that if (i) δ_t tends to 0 sufficiently slowly and (ii) the action space is continuous and payoff function is smooth, then the efficient outcome can be sustained by a subgame perfect equilibrium. Our paper is also closely related to Ambrus and Lu (2008), who examined multilateral bargaining played in continuous time $t \in [-T, 0]$. Each player $i = 1, ..., N$ receives opportunities to make an offer, which arrive by a Poisson process (independent across players). If an agreement is reached at any time, the game ends then. If no offer is accepted until the deadline 0, players receive zero payoff. They showed that there is a unique subgame perfect equilibrium, and the share of the surplus a player can expect is proportional to her arrival rate. Our model is different in that players always have a possibility to revise their action in the entire period $[-T, 0]$

[1] Admatti and Perry (1991) cosidered a related dynamic model with the joint project game, in which the completion of a joint project requies a fixed sum of investment.

and, the actions are implemented at the deadline 0. Another difference lies in the component games. Unlike their model, we obtain multiple equilibria, just as in repeated games.

The plan of the paper is as follows. The next section presents a simple class of revision games to help the reader to build up some intuition. The main model is presented in Section 3. The results on general setting with one-dimensitonal continuous strategies is given in Section 4. Section 5 provides a number of applications.

2 Revision Games I - with Stationarity

The purpose of our paper is to analyze a class of games where (i) a component game is played only once, (ii) players must prepare for their actions in advance, (iii) prepared actions are observable, and (iv) opportunity of revising their actions arrive stochastically. We refer those games as *Revision Games*. In this section, we start with a simple case, where the problem is stationary in the sense that in each period $t = 0, 1, 2, ...$ there is a fixed, positive probability p that a game is played (Revision Games I). This class will turn out to be isomorphic to a familiar class of games, and it helps to build some intuition on how revision games in general work. The point we make is a simple one, so we just present an example of Revision Game I.

Suppose a rural village faces an attack of bandits. In each period $t = 0, 1, 2, ...$ the bandits attack the village with probability $p \in (0, 1)$ around midnight. They only attack once. There are two villagers, $i = 1, 2$, and they must prepare to defend the village (to show up at the village gate around midnight) or not (to hide away). Hence in each period they observe each other's prepared actions. The acts of preparation themselves (showing up and hiding away) have negligible effects on the villagers' payoffs. When the bandits attack, however, their prepared actions have huge impacts to their payoffs;

	Defend	Hide
Defend	2, 2	-1, 3
Hide	3, -1	0, 0

This is a Prisoner's Dilemma game. Now consider player i's expected payoff. We denote player i's payoff by $\pi_i(t)$, when the bandit attack occurs at time t. We also assume that players have a common discount factor $\delta \in (0, 1)$.

Player i's expected payoff is

$$\begin{aligned} & p\pi_i(0) + \delta(1-p)p\pi_i(1) + \delta^2(1-p)^2 p\pi_i(2) + \cdots \\ = \; & p\sum_{t=0}^{\infty} \overline{\delta}^t \pi_i(t), \end{aligned}$$

where $\overline{\delta} \equiv \delta(1-p)$. Hence, *revision games I are isomorphic to infinitely repeated games*, and cooperation can be sustained as a subgame perfect equilibrium. Even though the component game is played only once, when (i) players prepare their action in advance, (ii) prepared actions are observable, and (iii) opportunity of revising their actions arrive stochastically, then players manage to cooperate. The mechanism to sustain cooperation works, for example, as follows. As long as the villegers have been showing up at the gate, they continue to do so (to prepare to defend the village). If anyone hides away, however, they stop to prepare to defend. The next section deals with our main model, where there is a fixed deadline to prepare action in the component game. We will show that some cooperation can be sustained in such games (Revision Games II), and the basic mechanism is essentially the same as in this bandits story.

3 Revision Games II - with a Fixed Deadline

Consider a normal form game with players $i = 1, ..., N$. Player i's strategy and payoff are denoted by $a_i \in A_i$ and $\pi_i(a_1, ..., a_N)$. This game is played at time 0, but players have to prepare their strategies in this game in advance, and they also have some (stochastic) opportunities to revise their prepared strategies. Hence, technically the game under consideration is a dynamic game with preparation and revisions of strategies, where the normal form game π is played at the end of the dynamic game (time 0). Specifically, the timing of the game is as follows: At period $-T < 0$, each player i chooses an action from A_i. In time interval $[-T, 0]$, revision opportunities arrive at each player with Poisson arrival rate λ. We consider the following two cases: (i) the synchronous case, where revision opportunities are common to all players (i.e., players revise their actions simultaneously) and (ii) the asynchronous case, where the arrival of revision opportunities is independent across players. There is no cost of revision. At period 0, the payoffs $\pi(a') = (\pi_1(a'_1), \ldots, \pi_N(a'_N))$ materialize, where a'_i is i's latest revised action. To distinguish the entire dynamic game and its component π, the former is referred to as a *revision game* and π is referred to as the *component game*.

We consider subgame perfect equilibria of the revision game. When the component game has a pure strategy Nash equilibrium, one obvious subgame perfect equilibrium is the game in which players choose a static Nash action at time $-T$, and they don't revise their actions until time 0. In what follows, we show that, under some conditions, revision games have other subgame perfect equilibria, where players are better off than in the static Nash equilibrium.

4 One-dimensional Continuous Strategies

In this section, we consider a case with two players with one-dimensional continuous strategies. This case subsumes, for example, public goods provision games, Cournot, and so forth. These applications are discussed in Section 5. We assume two players, but this is just to simplify the exposition: The results below easily extend to the case of N-player.

We analyze a general two-person symmetric component game with action $x_i \in X_i$ and payoff function π_i. Two players are denoted $i = 1, 2$, and a player's action space is an interval in $\mathbf{R}$: $X_i = [\underline{x}_i, \overline{x}_i]$, $[\underline{x}_i, \infty)$, $(-\infty, \overline{x}_i]$, or $(-\infty, \infty)$. Symmetry means $X_1 = X_2$ and $\pi_1(x, x') = \pi_2(x', x)$ for all x, x'. We assume that the component game has a symmetric pure strategy Nash equilibrium x^N, whose payoff is π^N. We first consider the case where revision opportunities arrive simultaneously to all players (the synchronous case). Here we confine our attention to symmetric equilibrium $x_1(t) = x_2(t) = x(t)$ that uses "the trigger strategy". The action path $x(t)$ means that, when a revision opportunity arrives at time $-t$, players are supposed to choose action $x(t)$. If any player deviates and does not choose the prescribed action $x(t)$, then in the future players revert to the Nash equilibrium of the component game, wherever a revision opportunity arrives. This is what we mean by the trigger strategy in revision games. Below we identify the best symmetric equilibrium in the class of "trigger strategy equilibria". Define

$$\begin{aligned} \pi(x) &\equiv \pi_1(x, x) \text{ and} \\ \widehat{\pi}(x) &\equiv \pi_1(BR(x), x), \end{aligned}$$

where $BR(x)$ is the best reply to the opponent's action x. Let $x^* = \arg\max_x \pi_1(x, x)$ denote the best symmetric action in the component game and let $\pi^* = \pi_1(x^*, x^*)$ (the best symmetric payoff). We assume that the following properties are satisfied.

(A0) Pure strategy symmetric Nash equilibrium (x^N, x^N) exists, and it is different from the best symmetric action profile (x^*, x^*).

(A1) The symmetric payoff $\pi_1(x_1, x_2)$ is twice continuously differentiable.

(A2) There is a unique best reply $BR(x)$ for all x.

(A3) At the best reply, the first and second order conditions are satisfied:

$$\frac{\partial \pi_1(BR(x), x)}{\partial x_1} = 0, \quad \frac{\partial^2 \pi_1(BR(x), x)}{\partial^2 x_1} < 0$$

(A4) $BR'(x^N) \neq 1$, or equivalently (by the implicit function theorem[2]),

$$-\frac{\partial^2 \pi_1(x^N, x^N)}{\partial^2 x_1} \neq \frac{\partial^2 \pi_1(x^N, x^N)}{\partial x_1 \partial x_2.}$$

(A5) $\pi(x)$ is single peaked: $\pi'(x) > 0$ if $x < x^*$ and $\pi'(x) < 0$ if $x^* < x$.

(A6) The gain from deviation

$$d(x) \equiv \pi_1(BR(x), x) - \pi_1(x, x) = \widehat{\pi}(x) - \pi(x)$$

has the single bottom at the Nash action: $d'(x) < 0$ if $x < x^N$ and $d'(x) > 0$ if $x^N < x$.

Those condition hold, for example, Cournot duopoly with linear demand and constant marginal cost. The next theorem gives a characterization of the best symmetric trigger strategy equilibrium.

Theorem 1 *Under (A0)-(A6), there is a solution $x^0(t)$ to differential equation*

$$\lambda(\widehat{\pi}(x) - \pi^N) = d'(x)\dot{x} \tag{1}$$

with initial condition $x(0) = x^N$, such that $x^0(t^0) = x^*$ for some finite time t^0. Furthermore, $x^0(\cdot)$ is monotone on $[0, t^0]$. When $T < t^0$, the best symmetric trigger strategy equilibrium is given by $x^0(t)$ for $t \in [0, T]$. When $t^0 \leq T$, it is given by

$$x(t) = \begin{cases} x^0(t) & \text{for } t \in [0, t^0] \\ x^* & \text{for } t \in [t^0, T] \end{cases} .$$

[2] The implicit function theorem, applied to the first order condition $\frac{\partial \pi_1(BR(x),x)}{\partial x_1} = 0$, shows $BR' = -\frac{\partial^2 \pi_1}{\partial x_1 \partial x_2} / \frac{\partial^2 \pi_1}{\partial^2 x_1}$.

The theorem shows that the best symmetric trigger strategy equilibrium is basically given by a solution to the differential equation (1), which starts with the Nash equilibrium action x^N and monotonically approaches the best symmetric action x^*. If there is enough time to reach x^*, after reaching x^*, the action path stays there. Note that we measure time in the inverse order so that $x(t)$ is the prescribed action at time $-t$. The above strategy means (given the horizon is long enough: $t^0 \leq T$) the following pattern of actions in the revision game. Players start with the best action x^*, and even if a revision opportunity arrives, they do not revise their actions until time $-t^0$ is reached. After that, if a revision opportunity arrives, they choose an action $x^0(t)$, which is closer to the Nash action. The closer is the revision opportunity to the end of the game, the closer is the revised action $x^0(t)$ to the Nash equilibrium. At the end of the game, the actions chosen at the last revision opportunity are implemented. Hence the best symmetric trigger strategy equilibrium induces a probability distribution of actions over (i) the best action, (ii) the Nash action, and (iii) any actions in between. The nature of this equilibrium distribution will be examined in the following propositions (Propositions 1 and 2).

Proof. The equilibrium payoff at time $-t$ is

$$V_C(t) \equiv \pi(x(t))e^{-\lambda t} + \int_0^t \pi(x(s))\lambda e^{-\lambda s} ds.$$

The first term represents the payoff when there is no opportunity of revision in the future. This happens with probability $e^{-\lambda t}$, and the current action $x(t)$ will be implemented at the end of the game. Similarly, with probability density λ a revision opportunity arrives at time s, and with probability $e^{-\lambda s}$ this is the last revision opportunity. If that happens, action $x(s)$ will be implemented and the realized payoff is equal to $\pi(x(s))$. In contrast, the optimal deviation payoff is

$$V_D(t) \equiv \widehat{\pi}(x(t))e^{-\lambda t} + \pi^N(1 - e^{-\lambda t}),$$

where the second term shows that whenever at least one revision opportunity arrives in the future (which happens with probability $(1 - e^{-\lambda t})$), players revert to the Nash equilibrium and the Nash payoff π^N realizes at the end of the game. The incentive constraint is $V_C(t) \geq V_D(t)$ for all t. Given the future action path $x(s)$, $s \in (t, 0]$, the optimal current action $x(t)$ is given by

$$\max_{x(t)} V_C(t)$$
$$\text{s.t. } V_C(t) \geq V_D(t), \tag{2}$$

or equivalently

$$\max_{x(t)} \pi(x(t))$$
$$\text{s.t. } \int_0^t \pi(x(s))\lambda e^{-\lambda s} ds - \pi^N(1 - e^{-\lambda t}) \geq d(x(t))e^{-\lambda t}.$$

Note that the left hand side of the constraint (a constant in this optimization problem) is the loss of future profit when a player deviates, while the right hand side represents the gain from deviation. Consider the case $x^N < x^*$ (symmetric arguments apply to the other case). Since the Nash action satisfies the incentive constraint, the maximized value is no less than π^N, and it cannot exceed π^* (the maximal possible symmetric payoff). By (A5), for any $\pi \in (\pi^N, \pi^*)$, there are at most two x' and x" such that $\pi = \pi(x') = \pi(x")$, where $x^N < x' < x^* < x"$. By (A6), x' has a smaller gain from deviation ($d(x') < d(x")$). Hence we can always find a solution to the above optimization in $[x^N, x^*]$. Also note that, by (A5) and (A6), both $\pi(x)$ and $d(x)$ are increasing in this region. Hence, the solution to the above optimization problem is either (i) $x(t) = x^N$, (ii) $x(t) = x^*$, or (iii) $x(t) < x^*$ and the constraint is binding. This means that, the best symmetric trigger strategy is characterized by the binding incentive constraint $V_C(t) = V_D(t)$ (until optimal action x^* is reached). Trivially, only the Nash action can be sustained at $t = 0$, and we have the boundary condition $V_C(0) = V_D(0)(= \pi^N)$. Then, condition $V_C'(t) = V_D'(t)$ ensures that the incentive constraint is always binding, i.e., $V_C(t) = V_D(t)$ for all t. We have

$$V_C' = -\lambda \pi e^{-\lambda t} + \pi' \dot{x} e^{-\lambda t} + \lambda \pi e^{-\lambda t} = \pi' \dot{x} e^{-\lambda t}, \text{ and}$$

$$V_D' = -\lambda \widehat{\pi} e^{-\lambda t} + \widehat{\pi}' \dot{x} e^{-\lambda t} + \lambda \pi^N e^{-\lambda t},$$

and $V_C'(t) = V_D'(t)$ reduces to

$$\lambda(\widehat{\pi} - \pi^N) = d' \dot{x}, \tag{3}$$

where we used $\widehat{\pi}' - \pi' = d'$. The initial condition of this differential equation is $x(0) = x^N$. This differential equation always has a trivial solution $x(t) \equiv$

x^N. Assumptions (A1)-(A6) ensure that it has another solution x^0, where the fully cooperative action x^* is achieved for some $t^0 < \infty$.

Let us continue to confine our attention to the case $x^N < x^*$. By (A6), $d'(x) \neq 0$ for $x \neq x^N$, so that we can rewrite the differential equation (3) for $x \neq x^N$ as

$$\dot{x} = \lambda \frac{\widehat{\pi} - \pi^N}{d'}. \tag{4}$$

When $x^N < x^*$, the right hand side is strictly positive for all $x \in (x^N, x^*]$. This follows from, for for all $x \in (x^N, x^*]$, $\widehat{\pi}(x) - \pi^N \geq \pi(x) - \pi^N > 0$ (the weak inequality holds by definition and the strict inequality follows from (A5)) and $d'(x) > 0$ ((A6)).

Hence, if we show that $\dot{x} > 0$ is also satisfied at the initial point $x(0) = x^N$, then we find strictly increasing path from $x(0) = x^N$ to reach the optimal action x^*, and the proof is basically completed. We elaborate on this point in what follows.

At the initial point $x(0) = x^N$, the right hand side of differential equation (4) is not well-defined, because $d'(x^N) = 0$. The reason is as follows. First, note that

$$\begin{aligned} d'(x) &= (\pi_1(BR(x), x) - \pi_1(x, x))' \\ &= \frac{\partial \pi_1(BR(x), x)}{\partial x_2} - \frac{\partial \pi_1(x, x)}{\partial x_1} - \frac{\partial \pi_1(x, x)}{\partial x_2}, \end{aligned}$$

where we used (A3) to eliminate $\frac{\partial \pi_1(BR(x),x)}{\partial x_1} BR'(x) = 0$ (the envelope theorem). Firstly, at the Nash action x^N, again by (A3) the first order condition is satisfied and $\frac{\partial \pi_1(x^N,x^N)}{\partial x_1} = 0$. Secondly, we have $\frac{\partial \pi_1(BR(x^N),x^N)}{\partial x_2} = \frac{\partial \pi_1(x^N,x^N)}{\partial x_2}$ and we obtained $d'(x^N) = 0$. In summary, basically because the first order condition is satisfied at the Nash action ((A3)) , we have

$$d'(x^N) = 0$$

and the right hand side of the second differential equation (4) is not well-defined at the initial point $x(0) = x^N$. To show the non-trivial solution to our original differential equation (3) exists, we set

$$\dot{x}(0) = \lim_{x \to x^N} \lambda \frac{\widehat{\pi}(x) - \pi^N}{d'(x)}$$

and show that the right hand side of this equation exists and strictly positive.

Since both the numerator and denominator of $\lambda\frac{\widehat{\pi}(x)-\pi^N(x)}{d'(x)}$ tend to zero as $x \to x^N$, we use de l'Hospital theorem

$$\lim_{x\to x^N}\lambda\frac{\widehat{\pi}(x)-\pi^N}{d'(x)} = \lim_{x\to x^N}\lambda\frac{(\widehat{\pi}(x)-\pi^N)'}{d''(x)}$$

$$= \frac{\partial\pi_1(x^N,x^N)}{\partial x_2}\Bigg/\frac{\left(-\frac{\partial^2\pi_1(x^N,x^N)}{\partial^2 x_1}-\frac{\partial^2\pi_1(x^N,x^N)}{\partial x_1\partial x_2}\right)^2}{-\frac{\partial^2\pi_1(x^N,x^N)}{\partial^2 x_1}}. \tag{5}$$

The last equality follows from direct computation and some rearrangements, and the details are found in Appendix A.1. We now show that the value of this expression (5) is strictly positive. First, by (A5), we have $\pi'(x^N) > 0$ (recall that we are in the case $x^N < x^*$). Since the first order condition is satisfied at the Nash action x^N ((A3)), we have $\pi'(x^N) = \frac{\partial\pi_1(x^N,x^N)}{\partial x_2}$. Therefore, the numerator of (5) is strictly positive. On the other hand, the second order condition in (A3) shows that $-\frac{\partial^2\pi_1(x^N,x^N)}{\partial^2 x_1} > 0$. Furthermore, (A4) shows that $\left(-\frac{\partial^2\pi_1(x^N,x^N)}{\partial^2 x_1}-\frac{\partial^2\pi_1(x^N,x^N)}{\partial x_1\partial x_2}\right)^2 > 0$. Hence the denominator of (5)) is also strictly positive.

In summary, we have obtained a differential equation

$$\dot{x} = f(x) > 0 \text{ for } x \in [x^N, x^*],$$

where $f(x)$ is a continuous function $\lambda\frac{\widehat{\pi}(x)-\pi^N}{d'(x)}$ for $x \in (x^N, x^*]$ and $f(x^N) = \lim_{x\downarrow x^N} f(x)$. Since f is strictly positive and continuous on $[x^N, x^*]$, it has the minimum value $\underline{f} > 0$. The fact $\dot{x} \geq \underline{f} > 0$ shows that there is a strictly increasing solution $x^0(t)$ with initial condition $x(0) = x^N$, and in a finite time it reaches the optimal symmetric action x^*. This solution satisfies the original differential equation (3) and therefore the binding incentive constraint $V_C(t) = V_D(t)$. Hence we conclude that the strategy to "follow $x^0(t)$ until the best symmetric action x^* is reached" satisfies the optimality condition (2) for the best symmetric trigger strategy equilibrium. This completes the proof. ■

One might expect that the outcome of the component game, and hence the payoffs, depend on the arrival rate λ. The next theorem shows that this is not the case:

Proposition 1 ***(Arrival Rate Invariance)*** *Under the best symmetric trigger strategy equilibrium, the probability distribution of action profile in* *period 0 is independent of the Poisson arrival rate λ, provided that the time horizon T is long enough. Specifically, Let $t^0(\lambda)$ be the (first) time to reach optimal symmetric action, stated in Theorem 1. Then, as long as $t^0(\lambda)$ $\leq T$, the probability distribution of action profile at period 0 is independent of the Poisson arrival rate λ.*

Proof. Let us denote revision game with Poisson arrival rate λ and time horizon T by $R(\lambda, T)$. Now choose a new unit of measurement of time and represent this game with respect to new time $s = \lambda t$, where t is the original variable to represent time. Since the probability that k revision opportunities arrive within time Δt is given by Poisson distribution

$$\frac{1}{k!}(\lambda \Delta t)^k e^{-\lambda \Delta t} = \frac{1}{k!}(\Delta s)^k e^{-\Delta s},$$

under the new unit of measurement of time the Poisson arrival rate is 1. Hence, the new representation of $R(\lambda, T)$, with the new unit of measurement of time $s = \lambda t$, is identical to $R(1, \lambda T)$. Let us denote the best symmetric trigger strategy equilibrium in $R(\lambda, T)$ by $x(t)$. Since $R(1, \lambda T)$ is identical to $R(\lambda, T)$ represented by a new time variable $s = \lambda t$, $x(t)$ must be mapped, by $s = \lambda t$, to the best symmetric trigger strategy equilibrium $y(\cdot)$ in $R(1, \lambda T)$. That is, $y(t) = x(\frac{t}{\lambda})$. Since x and y represent the same strategy (with different units of measurement of time), they must induce the same probability distribution over the action profiles in the component game.

Now suppose $x(t)$ reaches the optimal symmetric action x^* at time $t^0(\lambda)$ $\leq T$. Then, the first time $y(t)$ $(= x(\frac{t}{\lambda}))$ reaches $x^*$is $t^0(1) = \lambda t^0(\lambda) < \lambda T$. Since players never revise their actions (they stick to x^*) in $[t^0(1), \lambda T]$ under $y(t)$ in revision game $R(1, \lambda T)$, the probability distribution of action profile induced by y (and hence x) is identical to the one that is induced by the best symmetric trigger strategy equilibrium in $R(1, t^0(1))$, which is independent of the Poisson arrival rate λ. ■

Proposition 1 shows the following feature of revision games: The framework of revision games gives us a unique prediction that does not depend on the detailed specification of the dynamics, λ. In particular, even if λ is sufficiently high (so that there are many chances to revise actions right before the component game), the expected outcome in the component game is the same as in the case of low λ.

The proof also shows that, when there is enough time for the best symmetric trigger strategy equilibrium to reach the best symmetric action x^*, the probability distribution of action profile in the component game (which is independent of arrival rate λ) can be calculated as follows. It is equal to the distribution induced by the best symmetric trigger strategy equilibrium in $R(1, t^0(1))$ (revision game with arrival rate $\lambda = 1$, where there is just enough time to reach the best symmetric action x^*). First, we find the action path $x(t)$, which is a non-trivial solution ($x(t)$ not identical to x^N) to the differential equation

$$\widehat{\pi}(x) - \pi^N = d'(x)\dot{x}.$$

We explain the procedure to compute the action profile distribution when $x^* < x^N$ (the other case is similar). In this case $x(t)$ is decreasing, starting with $x(0) = x^N$ and reaches x^* at a finite time $t^0(1)$ ($x(t^0(1)) = x^*$). The cumulative distribution function of symmetric action x in the component game, denoted by $F(x)$, can be computed as follows.

First, note that the support of the distribution is $[x^*, x^N]$. Action x^* realizes if and only if the initial action x^* at time $t^0(1)$ is never revised (recall that we are measuring time in the inverse order, so that $t^0(1)$ is the beginning of the game and $t = 0$ is the end). This happens with probability $e^{-t^0(1)}$. Hence

$$F(x) = \begin{cases} 0 \text{ if } x < x^* \\ e^{-t^0(1)} \text{ if } x = x^* \end{cases}.$$

For $x \in (x^*, x^N]$, $F(x) - F(x^*) = \int_{\{t|x(t)\leq x\}} e^{-t}dt = \int_{t_x}^{t^0(1)} e^{-t}dt = e^{-t_x} - e^{-t^0(1)}$, where $x(t_x) = x$. The first equality follows from the fact that the density of action $x(t) \leq x$ is the product of

- 1 (the density of revision at time t) and
- e^{-t} (the probability that the revised action at time t, $x(t)$, will never be revised again).

Let us summarize our arguments. To simplify notation, we denote $t^0(1)$, the first time the action path reaches x^*, by t_{x^*}.

Proposition 2 *Suppose that (i) the best symmetric trigger strategy equilibrium is played and (ii) time horizon is long enough so that the efficient action x^* is chosen at the beginning of the revision game. When $x^* < x^N$, the cumulative distribution function of the symmetric action in the component game is given by*

$$F(x) = \begin{cases} 0 & \text{if } x < x^* \\ e^{-t_x} & \text{if } x^* \leq x \leq x^N \\ 1 & \text{if } x^N < x \end{cases},$$

where t_x is the time for the best symmetric trigger strategy action path to reach $x \in [x^*, x^N]$, when the arrival rate is $\lambda = 1$. When $x^N < x^*$, $1 - F(x)$ is equal to the right hand side of the above equality.

5 Applications

In this section, we use the general framework given in the previous section to analyze a widely studied class of games. We study prisoner's dilemma with continuous actions, Cournot oligopolies, and Bertrand competition with product differentiation. Other possible applications are discussed in the last subsection.

5.1 Continuous Prisoner's Dilemma

Let the payoff function be $\pi_1(x_1, x_2) = ax_2 - x_1^b$ and $\pi_2(x_1, x_2) = ax_1 - x_2^b$, where $a > 0$ and $b > 1$. The action space is $x_i \in [0, \infty)$. This game represents the following situation. Two players $i = 1, 2$ exchange goods they produce. That is, player 1 produces one unit of good and give it to player 2 (and *vice versa*). The quality of the good player i produces is equal to x_i, and the cost to provide a good with quality x_i is a convex function x_i^b. Alterenatively, one can interpret x_i as the quantity of goods i provides and assume that x_i^b is the cost to produce x_i units of goods. Note that $x_i = 0$ is the dominant strategy, while the best symmetric action $x^* = \left(\frac{a}{b}\right)^{\frac{1}{b-1}}$ is strictly positive. Hence this can be regarded as a version of the prisoner's dilemma game with continuous actions. It is easy to check that this game satisfies our assumptions (A0)-(A6).

By using the formula obtained in Theorem 1, we can easily obtain the following:

Proposition 3 *Under the best symmetric trigger strategy equilibrium, action path is given by $x(t) = \left(\frac{a}{b}\right)^{\frac{1}{b-1}}$ for $t \geq \frac{1}{(b-1)\lambda}$ and $x(t) = \left(\frac{a(b-1)\lambda t}{b}\right)^{\frac{1}{b-1}}$ for $t < \frac{1}{(b-1)\lambda}$. Furthermore, this is the best symmetric equilibrium.*

Proof. The differential equation in Theorem 1, $\lambda(\widehat{\pi}(x) - \pi^N) = d'(x)\dot{x}$,

reduces to

$$\lambda ax = bx^{b-1}\dot{x},$$

which has a non-trivial solution $x(t) = \left(\frac{a(b-1)\lambda t}{b}\right)^{\frac{1}{b-1}}$, and the efficient action is $x^* = \left(\frac{a}{b}\right)^{\frac{1}{b-1}}$. The first part of the proposition directly follows from Theorem 1. The optimality of this equilibrium follows from the fact that the punishment in the trigger strategy, reversion to Nash action $(0, 0)$, is the strongest possible punishment, where players receive their minimax payoff. ■

To understand the proposition, we state the two corollaries below:

Corollary 1 *If $a = 1$ and $b = 2$, the symmetric efficient equilibrium is $c(t) = \frac{1}{2}$ for $t \geq \frac{1}{\lambda}$ and $c(t) = \frac{\lambda}{2}t$ for $t < \frac{1}{\lambda}$.*

Thus, with quadratic cost, the efficient equilibrium path is linear. Now, recall that π^* is the perfectly collusive payoff.

Corollary 2 *The expected payoff associated with the best symmetric trigger strategy tends to π^* as $b \to \infty$.*

Thus, as the convexity of the cost rises, it becomes easier to achieve collusive outcomes. The reasoning here is that as b goes to infinity, the time when the full cooperation starts to collapse, $\frac{1}{(b-1)\lambda}$, becomes closer and closer to the deadline. Denote $x(t) = kt^{\frac{1}{b-1}}$, with $k = \left(\frac{a(b-1)\lambda}{b}\right)^{\frac{1}{b-1}}$. Notice that when b is very large, k approaches 1, while the derivative of $t^{\frac{1}{b-1}}$ with respect to t near $t = 0$ becomes larger and larger. This is why large b implies that the players stick to the best symmetric action until a time shortly before the deadline. The proof of Corollary 2 is relegated to Appendix A.2.

5.2 Asynchronous Revision in the Continuous Prisoner's Dilemma

In the previous section, we considered a situation where opportunity of revision arrives at the same time for both players. Here we consider the case where revision opportunities are independent across players. The specification of the model is the same as before, except that we let λ_i, $i = 1, 2$, be each player's arrival rate. We assume that player i observes when revision opportunities arrived to player j, so that i can see if j has actually followed the equilibrium action path $x_j(t)$.

One can show the following:

Proposition 4 *The symmetric efficient equilibrium in Proposition 3 also constitute the symmetric efficient equilibrium in the case of independent revisions with equal arrival rates $\lambda_i = \lambda_j = \lambda$.*

Proof. Consider the following simple trigger strategy defined by action plan $x_i(t)$, $t \in [0, T]$, $i = 1, 2$. Players stick to this action plan as long as they have done so, and otherwise they switch to $x_i = 0$.

We will show that we can find i's equilibrium revision plan $x(t)$, which depends only on time, because of the separability of the payoff function.

When a revision opportunity arrives at time $-t$ to player i, the expected payoff to player i is given by the following formula, if both players follow the trigger strategies ($x_j(s)$ denotes the opponent's fixed action chosen before time $-t$ $(s > t)$).

$$W_C^i(t) = \int_0^t ax_j(\tau)\lambda_j \mathbf{e}^{-\lambda_j \tau} d\tau - \int_0^t x_i(\tau)^b \lambda_i \mathbf{e}^{-\lambda_i \tau} d\tau + [ax_j(s)\mathbf{e}^{-\lambda_j} t - x_i(t)^b \mathbf{e}^{-\lambda_i \tau}]$$

When i deviates at time $-t$, the optimal deviation is to take the dominant action in the stage game ($x_i = 0$) forever. Then the opponent j retaliates by switching to 0. Hence, the optimal deviation payoff at time $-t$ is given by

$$W_D^i(t) = ac_j(s)\mathbf{e}^{-\lambda_j t}$$

Those payoffs are similar to the ones when revision opportunities are perfectly synchronized with arrival rate λ. In the synchronized case, the equilibrium continuation payoff is

$$V_i^C(t) = \int_0^t (ax_j(\tau) - x_i(\tau)^b)\lambda \mathbf{e}^{-\lambda \tau} d\tau + [ax_j(s) - x_i(t)^b]\mathbf{e}^{-\lambda \tau}$$

while the optimal defection payoff is

$$V_i^D(t) = ax_j(s)e^{-\lambda t}.$$

The trigger strategies in the case of asynchronous revision constitute a subgame perfect equilibrium if the following incentive constraints are satisfied:

$$W_i^C(t) - W_i^D(t) \geq 0 \quad \text{for all} \quad t \in [T, 0], i = 1, 2.$$

One can immediately see, when $\lambda_i = \lambda_j = \lambda$, we have $W_i^C(t) - W_i^D(t) = V_i^C(t) - V_i^D(t)$, and therefore the two models have identical incentive constraints. ■

This shows that the same incentive constraints hold in both games, hence the same symmetric efficient equilibrium arises. We note that this result crucially depends on the assumption that the *payoff function is additively separable with respect to each player's action*. In general, i's revision plan depends not only on time to revise but also on the fixed action of the opponent at the time of revision (hence a revision plan is represented by a function $x_i(t, x_j)$, where x_j is the fixed action of the opponent at revision time $-t$). As our payoff is separable across players' actions (no cross effects between x_i and x_j, as we have $\pi_i(x_i, x_j) = ax_j - x_i^b$), we can effectively ignore the dependence of action path with respect to the opponent's action. One can easily extend this result to the case of any revision games with additively separable payoff functions.

5.3 Cournot Duopoly

In this subsection, we consider Cournot duopoly with synchronized revisions: The payoff for agent i is $\pi_i = (a - b(q_i + q_j) - c)\, q_i$, where q_i denotes agent i's final announced quantity. Implicit in this payoff function is the Cournot game in which $a - bQ$ is the inverse demand function and c is the (constant) marginal cost, where Q denotes the total quantity. We suppose $a, b > 0$.

Proposition 5 *Under the best symmetric trigger strategy equilibrium, the action path is* $q_i = q_j = \frac{a-c}{4b}$ *if* $t \geq \hat{t}$, *and* $q_i = q_j = \frac{a-c}{3b} \cdot r(t)$ *if* $t < \hat{t}$, *where we let* $\hat{t} = \frac{9}{2\lambda} \ln\left(\frac{12167}{12096}\right)$ *and* $r(t)$ *solves* $\frac{(r(t)+5)^3}{r(t)+1} = 108\mathbf{e}^{\frac{2}{9}\lambda t}$.

The proof is relegated to Appendix A.3.

Note that $r(t)$ is the ratio of the equilibrium quantity to the static equilibrium quantity $\frac{a-c}{3b}$.

We note that this model is more complicated than the continuous prisoner's dilemma. This is because now the payoff function is not additively separable. Hence, player 1's best response in a component game does depend on player 2's action. This was not the case in the continuous prisoner's dilemma: $x_1 = 0$ is always the best response in the componet game.

We also note that the Cournot revision game with asynchronous revisions would be much more difficult to analyze, as it is not necessarily an optimal deviation to play the best response against the opponent's current quantity.

Next, we consider the welfare implication of the revision game in Cournot oligopolies. It turns out that surprisingly high degree of collusion can be achieved in the Cournot revision game. Note that our Cournot game has linear demand $P = a - bQ$ and constant marginal cost c, but the following statements are independent of parameters a, b, and c.

Proposition 6 *Under the best symmetric trigger strategy equilibrium, the expected profit* $\overline{\pi}$ *in the Cournot revision game is more than 99.9% of the fully collusive payoff* π^*.

The proof is relegated to Appendix A.4. Another way of evaluating the collusive power of the Cournot revision game is to examine the improvement of payoff relative to the static Nash equilibrium. Relative to static Nash equilibrium, $\pi^* - \pi^N$ is the maximum possible payoff improvement, and the next corollary shows that a substantial fraction of the potential improvement $\pi^* - \pi^N$ is achieved in the Cournot revision game.

Corollary 3 *Let $\overline{\pi}$ be the expected payoff associated with the best symmetric trigger strategy equilibrium in the Cournot revision game.* $\overline{\pi} - \pi^N \geq 0.9925(\pi^* - \pi^N)$.

The proof is relegated to Appendix A.5.

5.4 Other applications

Beyond the applications that we provided thus far, there are other places to which our framework can be applied: we can apply our setting to the analysis of bargaining game, chicken game, coordination game, and so forth.[3]

References

[1] Admatti, A. and M. Perry (1991): "Joint Project without Commitment," Review of Economic Studies, 58, pp. 259-276.

[2] Ambrus, A. and S. E. Lu (2008): "A Continuous Model of Multilateral Bargaining," *mimeo.*

[3] Bernheim, B. D. and A. Dasgupta (1995): "Repeated Games with Asymptotically Finite Horizons," *Journal of Economic Theory*, 67(1): pp.129-152

[4] Pitchford, R. and C. Snyder (2004): "A Solution to the Hold-up Problem Involving Gradual Investment," Journal Economic Theory, 114(2), pp. 88-103.

[3] The detailed analyses can be obtained by the authors upon request.

A Appendix

A.1 Calculation in the proof of Theorem 1

By the first order condition ((A3)), $(\widehat{\pi}(x) - \pi^N)' = \frac{\partial \pi_1(BR(x),x)}{\partial x_2}$, and at $x = x^N$, this is equal to $\frac{\partial \pi_1(x^N,x^N)}{\partial x_2}$. On the other hand, the denominator is

$$\begin{aligned} d''(x) &= \left(\frac{\partial \pi_1(BR(x),x)}{\partial x_2} - \frac{\partial \pi_1(x,x)}{\partial x_1} - \frac{\partial \pi_1(x,x)}{\partial x_2}\right)' \\ &= \frac{\partial^2 \pi_1(BR(x),x)}{\partial x_1 \partial x_2} BR'(x) + \frac{\partial^2 \pi_1(BR(x),x)}{\partial^2 x_2} \\ &\quad - \frac{\partial^2 \pi_1(x,x)}{\partial^2 x_1} - 2\frac{\partial^2 \pi_1(x,x)}{\partial x_1 \partial x_2} - \frac{\partial^2 \pi_1(x,x)}{\partial^2 x_2}. \end{aligned}$$

We evaluate this expression at x^N, by noting (i) $\frac{\partial^2 \pi_1(BR(x^N),x^N)}{\partial^2 x_2} = \frac{\partial^2 \pi_1(x^N,x^N)}{\partial^2 x_2}$ and (ii) the implicit function theorem $BR' = -\frac{\partial^2 \pi_1}{\partial x_1 \partial x_2} / \frac{\partial^2 \pi_1}{\partial^2 x_1}$;

$$\begin{aligned} d''(x^N) &= -\frac{\partial^2 \pi_1(x^N,x^N)}{\partial^2 x_1} + \frac{\partial^2 \pi_1(x^N,x^N)}{\partial x_1 \partial x_2}(BR'(x^N) - 2) \\ &= \frac{\left(-\frac{\partial^2 \pi_1(x^N,x^N)}{\partial^2 x_1} - \frac{\partial^2 \pi_1(x^N,x^N)}{\partial x_1 \partial x_2}\right)^2}{-\frac{\partial^2 \pi_1(x^N,x^N)}{\partial^2 x_1}}. \end{aligned}$$

A.2 Proof of Corollary 2

Proof. The expected payoff can be written as:

$$\int_0^{\frac{1}{(b-1)\lambda}} \pi(x(t))\lambda e^{-\lambda t} dt + \pi^* e^{-\lambda \cdot \frac{1}{(b-1)\lambda}}.$$

The ratio of the second term to the fully collusive payoff π^* is:

$$e^{-\lambda \cdot \frac{1}{(b-1)\lambda}} = \left(\frac{1}{e}\right)^{\frac{1}{b-1}}.$$

Note that this approximates zero as $b \to 1$ while it approximates 1 as $b \to \infty$. Hence, when b is sufficiently large, the collusive payoff can be approximated.
■

A.3 Proof of Proposition 5

Proof. First, we calculate the static Nash equilibrium. This can be obtained by taking derivative of π_i with respect to q_i, setting it equal to zero, and then substituting $q_j = q_i$, by symmetry:

$$\frac{\partial\left[(a - b(q_i + q_j) - c)\, q_i\right]}{\partial q_i} = a - b(2q_i + q_j) - c = 0.$$

Substituting $q_j = q_i$, we have

$$q_i = q_j = \frac{a-c}{3b}.$$

Next, consider the collusive optimal strategy. The total profit, Q, is maximized at Q^* such that:

$$\frac{\partial\left[(a - bQ^* - c)\, Q^*\right]}{\partial Q} = a - 2bQ^* - c = 0.$$

Substituting $q_i = q_j = Q^*/2$, we have:

$$q_i = q_j = \frac{a-c}{4b}$$

Note that this amount is less than that of the equilibrium quantity: The ratio is 3/4 (we will make use of this ratio later).

Now, consider the symmetric grim-trigger strategy $q(t)$ that depends only on time. In particular, it doesn't depend on the history. We suppose that once a player observes a deviation by the opponent or by himself, then he sticks to the static Nash equilibrium. Supposing that the opponent follows this strategy, the expected payoff from cooperation at period $-t$ is:

$$C_i(t) = (a - 2bq(t) - c)\, q(t)\mathbf{e}^{-\lambda t} + \int_0^t (a - 2bq(\tau) - c)\, q(\tau)\lambda \mathbf{e}^{-\lambda\tau} d\tau.$$

On the other hand, the expected payoff from deviation is maximized by deviating to the best response to the current quantity of the opponent, which is $\frac{a-c}{2b} - \frac{q(t)}{2}$ by the first order condition. The expected payoff from deviation is then:

$$\begin{aligned} D_i(t) &= \left(a - b\left(\frac{a-c}{2b} + \frac{q(t)}{2}\right) - c\right)\left(\frac{a-c}{2b} - \frac{q(t)}{2}\right)\mathbf{e}^{-\lambda t} + \left(a - 2b\frac{a-c}{3b} - c\right)\frac{a-c}{3b}(1 - \mathbf{e}^{-\lambda t}) \\ &= b\left(\frac{a-c}{2b} - \frac{q(t)}{2}\right)^2 \mathbf{e}^{-\lambda t} + b\left(\frac{a-c}{3b}\right)^2 (1 - \mathbf{e}^{-\lambda t}). \end{aligned}$$

Note that $C_i(0) = D_i(0)$ if $q(0) = \frac{a-c}{3b}$, which is the equilibrium quantity in the static Cournot game. Now, the sufficient condition for the cooperation to be sustained is $C_i'(t) \geq D_i'(t)$ for all t. Taking derivatives, we have:

$$C_i'(t) = -\lambda\left(a - 2bq(t) - c\right)q(t)\mathbf{e}^{-\lambda t} + \left((a-c)q'(t) - 4bq(t)q'(t)\right)\mathbf{e}^{-\lambda t} + \left(a - 2bq(t) - c\right)q(t)\lambda\mathbf{e}^{-\lambda t},$$

and

$$D_i'(t) = -\lambda b\left(\frac{a-c}{2b} - \frac{q(t)}{2}\right)^2\mathbf{e}^{-\lambda t} - b\left(\frac{a-c}{2b} - \frac{q(t)}{2}\right)q'(t)\mathbf{e}^{-\lambda t} + \lambda b\left(\frac{a-c}{3b}\right)^2\mathbf{e}^{-\lambda t}.$$

Hence,

$$\begin{aligned}
& C_i'(t) \geq D_i'(t) \\
\iff & -\lambda\left(a - 2bq(t) - c\right)q(t)\mathbf{e}^{-\lambda t} + \left((a-c)q'(t) - 4bq(t)q'(t)\right)\mathbf{e}^{-\lambda t} + \left(a - 2bq(t) - c\right)q(t)\lambda\mathbf{e}^{-\lambda t} \\
& \quad \geq -\lambda b\left(\frac{a-c}{2b} - \frac{q(t)}{2}\right)^2\mathbf{e}^{-\lambda t} - b\left(\frac{a-c}{2b} - \frac{q(t)}{2}\right)q'(t)\mathbf{e}^{-\lambda t} + \lambda b\left(\frac{a-c}{3b}\right)^2\mathbf{e}^{-\lambda t} \\
\iff & (a-c)q'(t) - 4bq(t)q'(t) \\
& \quad \geq -\lambda b\left(\frac{a-c}{2b} - \frac{q(t)}{2}\right)^2 - b\left(\frac{a-c}{2b} - \frac{q(t)}{2}\right)q'(t) + \lambda b\left(\frac{a-c}{3b}\right)^2 \\
\iff & \left(\frac{3(a-c)}{2} - \frac{9}{2}bq(t)\right)q'(t) \geq -\lambda b\left(\frac{a-c}{2b} + \frac{q(t)}{2}\right)^2 + \lambda b\left(\frac{a-c}{3b}\right)^2 \\
\iff & \left(\frac{3(a-c)}{2} - \frac{9}{2}bq(t)\right)q'(t) \geq \lambda b\left(\left(\frac{a-c}{3b}\right)^2 - \left(\frac{a-c}{2b} + \frac{q(t)}{2}\right)^2\right) \\
\iff & \left(q(t) - \frac{(a-c)}{3b}\right)q'(t) \leq \frac{\lambda}{9}\left(q(t) + 5\frac{a-c}{3b}\right)\left(q(t) + \frac{a-c}{3b}\right) \\
\iff & \frac{\partial r(t)}{\partial t} \leq \frac{\lambda}{9}\frac{\left(r(t)+5\right)\left(r(t)+1\right)}{r(t)-1},
\end{aligned}$$

where we let $r(t) = \frac{3b}{a-c}q(t)$. Note that $r(t)$ is the ratio of the quantity to the static equilibrium quantity.

With the initial condition $r(0) = \frac{3b}{a-c}q(0) = \frac{3b}{a-c} \cdot \frac{a-c}{3b} = 1$, we have:

$$\frac{1}{2}\left(3\ln(r(t)+5) - \ln(r(t)+1)\right) = \frac{\lambda}{9}t + \ln(2) + \frac{3}{2}\ln(3),$$

which implies:

$$\frac{\left(r(t)+5\right)^3}{r(t)+1} = \mathbf{e}^{\frac{2}{9}\lambda t + 2\ln(2) + 3\ln(3)} = 108\mathbf{e}^{\frac{2}{9}\lambda t}. \tag{6}$$

The right hand side of the equation (6) is increasing in t. The derivative of the left hand side with respect to r is $\frac{2(r+5)^2(r-1)}{(r+1)^2}$, which is nonpositive if and only if r is smaller than 1. Hence it must be the case either that r is monotonically decreasing, or that r is monotonically increasing. We are interested in the case where r is decreasing with respect to t. Now, to solve for the optimal path, substitute r with the ratio of collusive quantity to the equilibrium quantity:

$$\frac{\left(\frac{3}{4}+5\right)^3}{\frac{3}{4}+1} = 108\mathbf{e}^{\frac{2}{9}\lambda t},$$

which is equivalent to:

$$t = \frac{9}{2\lambda}\ln\left(\frac{12167}{12096}\right) := \hat{t}.$$

■

A.4 Proof of Proposition 6

Proof. We will solve for the value of:

$$\frac{\int_0^{\hat{t}}(a-c-2bq(t))q(t)\lambda\mathbf{e}^{-\lambda t}dt + \mathbf{e}^{-\lambda\hat{t}}\pi^*}{\pi^*} := R. \tag{7}$$

Recall that: $a-Q(t)$ is the inverse demand function; $Q(t)$ is the sum of outputs; $q(t)$ is the individual output (hence $Q(t) = 2q(t)$); c is the (constant) marginal cost; λ is the arrival rate of the Poisson process; $-\hat{t}$ is the date when players start departing from the fully collusive actions; and π^* is the fully collusive profit.

From the proof of Proposition 5, it is immediate that

$$\mathbf{e}^{-\lambda\hat{t}} = \left(\frac{12167}{12096}\right)^{-\frac{9}{2}}. \tag{8}$$

Equation (7), (8), and $\pi^* = (a-c-2bq(\hat{t}))q(\hat{t})$ implies:

$$R = \int_0^{\hat{t}} \frac{(a-c-2bq(t))}{(a-c-2bq(\hat{t}))}\frac{q(t)}{q(\hat{t})}\lambda\mathbf{e}^{-\lambda t}dt + \left(\frac{12167}{12096}\right)^{-\frac{9}{2}}. \tag{9}$$

Recalling that $r(t) = \frac{q(t)}{q(\hat{t})}$, we have:

$$R = \frac{a-c}{a-c-2bq(\hat{t})}\int_0^{\hat{t}} r(t)\lambda\mathbf{e}^{-\lambda t}dt - \frac{2bq(\hat{t})}{a-c-2bq(\hat{t})}\int_0^{\hat{t}} r(t)^2\lambda\mathbf{e}^{-\lambda t}dt + \left(\frac{12167}{12096}\right)^{-\frac{9}{2}}. \tag{10}$$

Note that $q(\hat{t}) = \frac{a-c}{4b}$. This implies:

$$R = 2\int_0^{\hat{t}} r\lambda \mathbf{e}^{-\lambda t} dt - \int_0^{\hat{t}} r^2 \lambda \mathbf{e}^{-\lambda t} dt + \left(\frac{12167}{12096}\right)^{-\frac{9}{2}}. \tag{11}$$

(For notational simplicity suppress the dependence of r on t.) Now, in the proof of Proposition 5, we have shown that

$$\frac{(r+5)^3}{r+1} = 108\mathbf{e}^{\frac{2}{9}\lambda t}. \tag{12}$$

Equation (12) implies:

$$\mathbf{e}^{-\lambda t} = \left(\frac{(r+5)^3}{108(r+1)}\right)^{-\frac{9}{2}}. \tag{13}$$

Equation (12) also implies:

$$t = \frac{9}{2\lambda}\left(3\ln(r+5) - \ln(r+1) - \ln(108)\right),$$

which implies:

$$\frac{dt}{dr} = \frac{9}{2\lambda}\left(\frac{3}{r+5} - \frac{1}{r+1}\right). \tag{14}$$

Note that

$$r(0) = 1 \quad \text{and} \quad r(\hat{t}) = \frac{3}{4}. \tag{15}$$

Substituting Equations (13), (14), and (15) into Equation (11), we obtain:

$$\begin{aligned}
R = {} & 2\int_1^{\frac{3}{4}} r\lambda \left(\frac{(r+5)^3}{108(r+1)}\right)^{-\frac{9}{2}} \frac{9}{2\lambda}\left(\frac{3}{r+5} - \frac{1}{r+1}\right) dr \\
& - \int_1^{\frac{3}{4}} r^2\lambda \left(\frac{(r+5)^3}{108(r+1)}\right)^{-\frac{9}{2}} \frac{9}{2\lambda}\left(\frac{3}{r+5} - \frac{1}{r+1}\right) dr + \left(\frac{12167}{12096}\right)^{-\frac{9}{2}} \\
= {} & 2\int_1^{\frac{3}{4}} r \left(\frac{(r+5)^3}{108(r+1)}\right)^{-\frac{9}{2}} \frac{9}{2}\left(\frac{3}{r+5} - \frac{1}{r+1}\right) dr \\
& - \int_1^{\frac{3}{4}} r^2 \left(\frac{(r+5)^3}{108(r+1)}\right)^{-\frac{9}{2}} \frac{9}{2}\left(\frac{3}{r+5} - \frac{1}{r+1}\right) dr + \left(\frac{12167}{12096}\right)^{-\frac{9}{2}}.
\end{aligned}$$

Note that each of the followings holds:

$$\int \left(\frac{(r+5)^3}{r+1}\right)^{-\frac{9}{2}} \frac{r}{r+5} dr =$$

$$\frac{(1+r)^6\sqrt{\frac{(5+r)^3}{1+r}}(-161936615+453422522r+110940424r^2+19300442r^3+2401420r^4+209930r^5+12292r^6+434r^7+7r^8)}{11473347600(5+r)^{15}},$$

$$\int\left(\frac{(r+5)^3}{r+1}\right)^{-\frac{9}{2}}\frac{r}{r+1}dr =$$

$$\frac{(1+r)^3(-16767869+33535738r+9417508r^2+1837666r^3+252520r^4+24106r^5+1528r^6+58r^7+x^8)}{764889840(5+r)^8 r(\frac{(5+r)^3}{1+x})^{3/2}},$$

$$\int\left(\frac{(r+5)^3}{r+1}\right)^{-\frac{9}{2}}\frac{r^2}{r+5}dr =$$

$$\frac{(1+r)^6\sqrt{\frac{(5+r)^3}{1+r}}(9905395-27735106r+63394528r^2+11028824r^3+1372240r^4+119960r^5+7024r^6+248r^7+4r^8)}{2007835830(5+r)^{15}},$$

$$\int\left(\frac{(r+5)^3}{r+1}\right)^{-\frac{9}{2}}\frac{r^2}{r+1}dr =$$

$$\frac{((1+r)^3(2354377-4708754r+9417508r^2+1837666r^3+252520r^4+24106r^5+1528r^6+58r^7+r^8)}{267711444(5+r)^8(\frac{(5+r)^3}{1+r})^{3/2}}.$$

The above equations imply that we can calculate the value of R explicitly. Combining, the value of R can be shown to be equal to:

$$3673320192\sqrt{3}\left(\frac{151027921669891\sqrt{161}}{96383073058941656687022180}-\frac{5374729\sqrt{3}}{409745772031115520}\right)$$

$$-7346640384\sqrt{3}\left(-\frac{84420743476307\sqrt{161}}{321276910196472188956740600}-\frac{10499}{97558517150265600\sqrt{3}}\right)$$

$$+\frac{513781372230303744\sqrt{21}}{21914624432020321\,23^{\frac{3}{2}}}.$$

Numerically, this is approximately equal to 0.99917208940755. ■

A.5 Proof of Corollary 3

Proof. We consider the following value:

$$\frac{\pi-\pi^N}{\pi^*-\pi^N} := R',$$

where π denotes the equilibrium expected profit and π^N is a static Nash equilibrium profit. Note that R' is equal to:

$$\frac{R - \frac{\pi^N}{\pi^*}}{1 - \frac{\pi^N}{\pi^*}}$$

Note that

$$\pi^* = (a - c - 2b\frac{a-c}{4b})\frac{a-c}{4b} \quad \text{and} \quad \pi^N = (a - c - 2b\frac{a-c}{3b})\frac{a-c}{3b},$$

so

$$\pi^* = \frac{(a-c)^2}{8b} \quad \text{and} \quad \pi^N = \frac{(a-c)^2}{9b}.$$

Thus

$$\frac{\pi^N}{\pi^*} = \frac{8}{9},$$

so that

$$R' = \frac{R - \frac{8}{9}}{1 - \frac{8}{9}} = 9R - 8.$$

This is equal to:

$$\frac{648231996306044005409433 6\sqrt{3}\sqrt{23}\sqrt{161} - 870886995513457331589631973\sqrt{23} + 859657206625475724155289600\sqrt{21}}{937057654739710551123826 75\sqrt{23}}.$$

Numerically, this is approximately 0.99254880466798. ■