Monotonic solutions to the experts aggregation problem

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The problem

- An amount of money needs to be divided among a group of tasks.
- Experts (judges) recommend independent divisions of the money.
- An aggregator takes into account these recommendations and provides an exact division.

Related literature

- Traditional social choice impossibility results starting with Arrow[1951, 1963].
- Large literature on probability aggregation: Bates[1969], Genes[1980],McConway[1981], Bordley and Wolf[1981], Dickson[1972], Morris[1977, 1974]...
- Recent literature on pooling of expert opinions: List and Pettit[2002], Dokow and Holzman[2005], Nehring and Puppe[2005]...Mostly discrete.
- **Precursors of abstract aggregation:** Wilson[1975], Fishburn and Rubistein[1985].
- Dividing a dollar impartially: De Clippel, Moulin and Tideman[2007]
- Mathematically equivalent to non-manipulable division rules in claim problems: Ju, Miyagawa and Sakai[2007]

Two traditional requirements:

- **Unanimity:** If all experts agree on the value of a task, that task should be allocated that value.
- **Monotonicity:** Aggregator is monotonic on expert's reports for that task (independent on the other tasks!).
  In spirit similar to Arrow's IIA: The value on the task should only depend on the suggested allocations for that task.

Model 1: Judges are experts on all tasks

Let $K = \{1, \ldots, k\}$ the set of tasks that need money.
Let $N = \{1, \ldots, n\}$ the set of experts.
Let M the amount of money to divide.
Let $\Delta_M = \{x \in \mathbb{R}^K_+ | \sum_{i \in K} x_i = M\}$

Definition

An aggregator is a function

$$\varphi : (\Delta_M)^N \to \Delta_M$$

- **Unanimity:** For any reports $x^1, \ldots, x^n \in \Delta_M$, if $x_k^i = x_k^j = \bar{x}_k$ for all $i, j \in N$, then

$$\varphi_k(x^1, \ldots, x^n) = \bar{x}_k$$

- **Monotonicity:** For two different reports of agent i, x^i and $\tilde{x}^i$, and reports of the other agents $x^{-i} = (x^1, \ldots, x^{i-1}, x^{i+1}, \ldots, x^n)$. If $x_k^i \geq \tilde{x}_k^i$ for some task k, then:

$$\varphi_k(x^i, x^{-i}) \geq \varphi_k(\tilde{x}^i, x^{-i}).$$

Theorem 1. Assume $|K| \geq 3$. An aggregator meets monotonicity if and only if there exist: costants $A \in \Delta_M$, $\lambda \in [0,1]$, and weights $\pi^1, \ldots, \pi^n \geq 0$ such that $\sum_i \pi^i = 1$ and:

$$\varphi(x^1, \ldots, x^n) = \lambda(\sum_i \pi^i x^i) + (1-\lambda)A.$$

Corollary 1. Assume $|K| \geq 3$. An aggregator meets unanimity and monotonicity if and only if there exist weights $\pi^1, \ldots, \pi^n \geq 0$ and $\sum_i \pi^i = 1$ such that:

$$\varphi(x^1, \ldots, x^n) = \sum_i \pi^i x^i.$$

Sketch of the proof

Consider $x^1, \ldots, x^n \in \Delta_M$.
By monotonicity:

$$\varphi(x^1, \ldots, x^n) = (f^1(x_1^1, \ldots, x_1^n), \ldots, f^k(x_k^1, \ldots, x_k^n)).$$

Therefore:

$$f^1(x_1^1, \ldots, x_1^n) + \cdots + f^{k-1}(x_{k-1}^1, \ldots, x_{k-1}^n)$$

is constant and additive.
By unanimity $f^1(0, \ldots, 0) = 0$. Thus $f^1 = f^2 = \cdots = f^{k-1}$.
Finally, any bounded and additive function has to be linear.

Aggregators for $K = 2$ meeting Un and Mon

Let $f : [0,1]^N \to [0,1]$ increasing such that $f(x, \dots, x) = x$.

$$\varphi(x^1, \dots, x^n) = (f(x_1^1, \dots, x_1^n), 1 - f(x_1^1, \dots, x_1^n)).$$

Independence of the conditions.

- **Aggregator without unanimity:**
  - $\tilde{\varphi}(x^1, \ldots, x^n) = a$ for some fixed a.
- **Aggregators without monotonicity.**
  Allocate unanonymous reports. Then divide the rest of the money proportionally (e.g. to maximal report).

Model 2: Aggregators with specialized experts

Expert i only informed about $S^i \subset K$.
$\cup_{i \in N} S^i = K$.
Let $\bar{\Delta}_M^{S^i} = \{x \in \mathbb{R}_+^{S^i} | \sum_{j \in S^i} x_j \leq M\}$ the space of reports of expert i.

Definition

An aggregator is a function

$$\varphi : \Pi_{i \in N} \bar{\Delta}_M^{S^i} \rightarrow \Delta_M$$

Catch: I should allocate the full amount of money M even if it is not needed!

- **Strong Monotonicity:** Fix reports x^{-i} and consider two reports of agent i, x^i and $\tilde{x}^i$, such that $x^i_S \geq \tilde{x}^i_S$ for some subset $S \subset S^i$, then:

$$\varphi_S(x^i, x^{-i}) \geq \varphi_S(\tilde{x}^i, x^{-i}).$$

  - Rules out priority solutions.
- When the experts do not have a common intersection, there is no compelling definition of unanimity!

Quasi-linear aggregators

For any agent i, an extension is a function

$$f^i : \bar{\Delta}_M^{S^i} \rightarrow \bar{\Delta}_M^{K \setminus S^i}$$

such that $\sum_{j \in S^i} x_j + \sum_{l \in K \setminus S^i} f_l^i(x) = M$.

Definition

Given an arbitrary set of weights $\pi^1, \ldots, \pi^n \geq 0$ such that $\sum_i \pi^i = 1$, constants $A \in \Delta_M$, $\lambda \in [0,1]$ and arbitrary extensions as above $f^1, \ldots, f^n$, a quasi-linear aggregator is such that:

$$\varphi(x^1, \ldots, x^n) = \lambda(\sum_{i \in N} \pi^i(x^i, f^i(x^i))) + (1-\lambda)A.$$

Examples

- Same distortion for uninformed tasks:

$$f^i(x) = (\frac{M - \sum_i x_i}{k - s^i})_{k \in N \setminus S^i}$$

- Priority to remaining tasks:

$$f^i_j(x) = M - \sum_i x_i$$

where j has highest priority on $N \setminus S^i$ with some arbitrarily order σ^i.

Examples (cont'd)

Quasi-linear aggregators are very inefficient:
Let $N = \{1, 2, 3\}$ and consider $S^1 = 1$ and $S^2 = N$.
Let $\pi^1 = 1$, $\pi^2 = 0$, $f_1(0) = (M, 0)$.
Let $u_1 = (0)$, $u_2 = (0, 0, M)$,
then $\varphi(u_1, u_2) = (0, M, 0)$.

Main result

Theorem 2. Assume $|K| \geq 3$ and all experts are *connected*. An aggregator meets strong-monotonicity if and only if it is a quasi-linear aggregator.

Efficiency:
If $\sum_{k \in K} \max(x_k^i)_{\{i|k \in S^i\}} < M$ then
$\varphi_k(x^1, \dots, x^n) \geq \max(x_k^i)_{\{i|k \in S^i\}}$ for each k.

Corollary 2.
If $|S^i| \leq K - 2$ for all i. Then there is no aggregator that meets strong-monotonicity and efficiency.

Sketch of the Proof:

By theorem 2, for every i there is a weight π^i and extension f^i such that

$$\varphi(x^1, \ldots, x^n) = \sum_{i \in N} \pi^i(x^i, f^i(x^i)).$$

Consider i such that $\pi^i > 0$. Let $x^i = \vec{0}$ and set $y^i = f^i(x^i)$. Since $|S^i| \leq N - 2$ then $y^i_k < M$ for some $k \notin S^i$.
Let j such that $k \in S^j$. Consider $x^j = (M - \epsilon, \vec{0}_{-k})$.
Let $x^l = \vec{0}$ for $l \neq k$.
Then

$$\varphi_k(x^i, x^j, x^{-i,j}) = \pi^i y^i_k + \sum_{\{l | k \notin S^l, l \neq i,j\}} \pi^l f^l_k(\vec{0}) + \pi^j(M - \epsilon) < M - \epsilon$$

for ϵ close to zero.

Aggregator without independence

Follow priority of goods and agents.
Serve S^1, $(S^1 \cup S^2) \setminus (S^1)$, $(S^1 \cup S^2 \cup S^3) \setminus (S^1 \cup S^2)$, etc...

Aggregators when experts are not connected

$|S^i| = 1$ for all $i \in N$, then too many SM aggregators!
Partition N in the indifference classes of K.
Let P_k the agents in N that reports on task k.
Let $\varphi : [0, M]^N \to \delta_M^K$ such that $\varphi_k(x)$ is increasing in coordinates P^k.

Open problems:

- Appealing definition of Unanimity.
- How innefficient are quasi-linear aggregators? Can we find the optimal mechanism?