Dynamics and Equilibrium

Sergiu Hart

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DYNAMICS AND EQUILIBRIUM

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Papers

Papers

- Hart and Mas-Colell, *Econometrica* 2000
- Hart and Mas-Colell, *J Econ Theory* 2001
- Hart and Mas-Colell, *Amer Econ Rev* 2003
- Hart, *Econometrica* 2005
- Hart and Mas-Colell, *Games Econ Behav* 2006
- Hart and Mansour, *Games Econ Behav* 2009 ?
- Hart, *Center for Rationality DP* 2008

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- Hart and Mas-Colell, *Econometrica* 2000
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Nash Equilibrium

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John Nash, Ph.D. Dissertation, Princeton 1950

Nash Equilibrium

EQUILIBRIUM POINT:

John Nash, Ph.D. Dissertation, Princeton 1950

Nash Equilibrium

EQUILIBRIUM POINT:

"Each player's strategy is optimal against those of the others."

John Nash, Ph.D. Dissertation, Princeton 1950

Dynamics

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FACT

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There are no general, natural dynamics leading to Nash equilibrium

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- ***"general"***

Dynamics

FACT

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- ***"general"*** : **in all games**

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- ***"general"*** : **in all games**
  rather than: in specific classes of games

Dynamics

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- *"general"* : **in all games**
  rather than: in specific classes of games:
  - two-person zero-sum games
  - two-person potential games
  - supermodular games
  - . . .

Dynamics

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- ***"leading to Nash equilibrium"***

Dynamics

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- ***"leading to Nash equilibrium"*** **:**
  at a Nash equilibrium (or close to it) from some time on

Dynamics

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- ***"natural"***

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- *"natural"* :
  - **adaptive** (reacting, improving, ...)

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  - **simple and efficient**

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    - **computation** (performed at each step)

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  bounded rationality

Dynamics

Dynamics that are **NOT** ***"natural"*** :

Dynamics

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- **exhaustive search**
  (deterministic or stochastic)

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- **broadcasting** the private information and then performing **joint** computation

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Dynamics that are **NOT** ***"natural"*** :

- **exhaustive search**
  (deterministic or stochastic)
- using a **mediator**
- **broadcasting** the private information and then performing **joint** computation
- **fully rational learning**
  (prior beliefs on the strategies of the opponents, Bayesian updating, optimization)

Dynamics

<u>**FACT**</u>

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  - adaptive
  - simple and efficient:
    - **computation** (performed at each step)
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Hart and Mas-Colell, AER 2003

Natural Dynamics: Information

UNCOUPLED DYNAMICS :

Each player knows *only* his own payoff (utility) function

(does *not* know the payoff functions of the other players)

(privacy-preserving, decentralized, distributed ...)

Hart and Mas-Colell, AER 2003

Games

N-person game in strategic (normal) form:

- **Players**

$$i = 1, 2, \ldots, N$$

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$$a^i \text{ in } A^i$$

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N**-person game in strategic (normal) form**:

- **Players**

$$i = 1, 2, ..., N$$

- For each player i: **Actions**

$$a^i \text{ in } A^i$$

- For each player i: **Payoffs (utilities)**

$$u^i(a) \equiv u^i(a^1, a^2, ..., a^N)$$

Dynamics

- **Time**

$$t = 1, 2, \ldots$$

Dynamics

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$$t = 1, 2, \ldots$$

- At period t each player i chooses an **action**

$$a_t^i \text{ in } A^i$$

Dynamics

- **Time**

$$t = 1, 2, \ldots$$

- At period t each player i chooses an **action**

$$a_t^i \text{ in } A^i$$

according to a probability distribution

$$\sigma_t^i \text{ in } \Delta(A^i)$$

Dynamics

Fix the set of players $1, 2, ..., N$ and their action spaces $A^1, A^2, ..., A^N$

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Uncoupled Dynamics

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Uncoupled Dynamics

Fix the set of players $\mathbf{1}, \mathbf{2}, \ldots, \boldsymbol{N}$ and their action spaces $\boldsymbol{A}^{\mathbf{1}}, \boldsymbol{A}^{\mathbf{2}}, \ldots, \boldsymbol{A}^{\boldsymbol{N}}$

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$$\boldsymbol{\sigma}_t^i \equiv \boldsymbol{\sigma}_t^i(\text{HISTORY}\,;\ \text{GAME})$$

$$\equiv \boldsymbol{\sigma}_t^i(\text{HISTORY}\,;\ \boldsymbol{u}^{\mathbf{1}}, \ldots, \boldsymbol{u}^{i}, \ldots, \boldsymbol{u}^{\boldsymbol{N}})$$

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$$\sigma_t^i \equiv f^i(a_{t-1}; u^i)$$

where $a_{t-1} = (a_{t-1}^1, a_{t-1}^2, ..., a_{t-1}^N) \in A$
are the actions of all the players
in the previous period

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  are the actions of all the players
  in the previous period
  - Only last period matters (“*1-recall*”)

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where $\boldsymbol{a}_{t-1} = (\boldsymbol{a}_{t-1}^1, \boldsymbol{a}_{t-1}^2, ..., \boldsymbol{a}_{t-1}^N) \in \boldsymbol{A}$
are the actions of all the players
in the previous period

  - Only last period matters ("*1-recall*")
  - Time t does not matter ("*stationary*")

Impossibility

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Theorem. *There are* **NO** *uncoupled dynamics with 1-recall*

$$\boldsymbol{\sigma}_t^i \equiv \boldsymbol{f}^i(\boldsymbol{a}_{t-1}; \boldsymbol{u}^i)$$

that yield almost sure convergence of play to pure Nash equilibria of the stage game in all games where such equilibria exist.

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that yield almost sure convergence of play to pure Nash equilibria of the stage game in all games where such equilibria exist.

Hart and Mas-Colell, GEB 2006

Proof

Consider the following two-person game, which has a unique pure Nash equilibrium

	C1	C2	C3
R1	1,0	0,1	1,0
R2	0,1	1,0	1,0
R3	0,1	0,1	1,1

Proof

Consider the following two-person game, which has a unique pure Nash equilibrium **(R3,C3)**

	C1	C2	**C3**
R1	1,0	0,1	1,0
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Assume ***by way of contradiction*** that we are given an uncoupled, 1-recall, stationary dynamic that yields almost sure convergence to pure Nash equilibria when these exist

Proof

- Suppose the play at time $t-1$ is **(R1,C1)**

	C1	C2	C3
R1	1,0	0,1	1,0
R2	0,1	1,0	1,0
R3	0,1	0,1	1,1

Proof

- Suppose the play at time $t-1$ is **(R1,C1)**
- **ROWENA is best replying at (R1,C1)**

	C1	C2	C3
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Proof

- Suppose the play at time $t-1$ is **(R1,C1)**
- **ROWENA is best replying at (R1,C1)**
- $\Rightarrow$ **ROWENA will play R1 also at** t

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Proof

- Suppose the play at time $t-1$ is **(R1,C1)**
- **ROWENA is best replying at (R1,C1)**
- **$\Rightarrow$ ROWENA will play R1 also at t**

  Proof:

  - Change the payoff function of COLIN so that **(R1,C1)** is the unique pure Nash eq.

	C1	C2	C3
R1	1,0	0,1	1,0
R2	0,1	1,0	1,0
R3	0,1	0,1	1,1

Proof

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Proof:

- Change the payoff function of COLIN so that **(R1,C1)** is the unique pure Nash eq.
- In the new game, ROWENA ***must*** play **R1** after **(R1,C1)** (by *1-recall*, *stationarity*, and *a.s. convergence* to the pure Nash eq.)

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  - In the new game, ROWENA ***must*** play **R1** after **(R1,C1)** (by *1-recall*, *stationarity*, and *a.s. convergence* to the pure Nash eq.)
  - By *uncoupledness*, the same holds in the original game

Proof

- Suppose the play at time $t-1$ is **(R1,C1)**
- **ROWENA is best replying at (R1,C1)**
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Proof

- ROWENA is best replying at $t-1$
- $\Longrightarrow$ ROWENA will play the same action at t

Proof

Similarly for COLIN:

A player who is best replying cannot switch

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	C1	C2	C3
R1	1,0 $\longleftrightarrow$	0,1	1,0
R2	0,1	1,0	1,0
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Proof

Similarly for COLIN:

A player who is best replying cannot switch

	C1	C2	C3
R1	1,0 ⟷	0,1 ↕	1,0 ⟷
R2	0,1 ↕	1,0 ⟷	1,0 ⟷
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Proof

Similarly for COLIN:

A player who is best replying cannot switch

	C1	C2	C3
R1	1,0 ↔	0,1 ↕	1,0 ↔
R2	0,1 ↕	1,0 ↔	1,0 ↔
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$\Rightarrow$ (R3,C3) **cannot be reached**

Proof

Similarly for COLIN:

A player who is best replying cannot switch

	C1	C2	C3
R1	1,0 ↔	0,1 ↕	1,0 ↔
R2	0,1 ↕	1,0 ↔	1,0 ↔
R3	0,1 ↕	0,1 ↕	1,1

$\Rightarrow$ (R3,C3) **cannot be reached**
(unless we start there)

Possibility

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Theorem. THERE EXIST *uncoupled dynamics with* 2-RECALL

$$\sigma_t^i \equiv f^i(a_{t-2}, a_{t-1}; u^i)$$

that yield almost sure convergence of play to pure Nash equilibria of the stage game in every game where such equilibria exist.

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Define the strategy of each player i as follows:

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IF:

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THEN: At t player i *plays a^i again*: $a^i_t = a^i$

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ELSE: At t player i *randomizes uniformly over A^i*

Possibility

"Good":

Possibility

"Good":

- simple

Possibility

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"Bad":

Possibility

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- exhaustive search

Possibility

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- all players must use it

Possibility

"Good":

- simple

"Bad":

- exhaustive search
- all players must use it
- takes a long time

Dynamics

FACT

There are no general, natural dynamics leading to Nash equilibrium

- *"natural"*:
  - adaptive
  - simple and efficient:
    - computation
    - time
    - information

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    - **computation**: *finite recall* ✓
    - **time** to reach equilibrium ?
    - **information**: *uncoupledness* ✓

Natural Dynamics: Time

HOW LONG TO EQUILIBRIUM?

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Estimate the number of time periods it takes until a Nash equilibrium is reached

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- How?
- An uncoupled dynamic

  $\approx$

  A distributed computational procedure

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  A distributed computational procedure
- $\Rightarrow$ **COMMUNICATION COMPLEXITY**

Communication Complexity

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- ***Distributed computational procedures***

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- **COMMUNICATION COMPLEXITY** $=$ the minimal number of rounds needed

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Yao 1979, Kushilevitz and Nisan 1997

How Long to Equilibrium

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Conitzer and Sandholm 2004

How Long to Equilibrium

- An uncoupled dynamic leading to Nash equilibria is **TIME-EFFICIENT** if

How Long to Equilibrium

- An uncoupled dynamic leading to Nash equilibria is **TIME-EFFICIENT** if its **COMMUNICATION COMPLEXITY** is **POLYNOMIAL** in the number of players (rather than: exponential)

How Long to Equilibrium

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Theorem. *There are* **NO TIME-EFFICIENT** *uncoupled dynamics that reach a pure Nash equilibrium in all games where such equilibria exist.*

How Long to Equilibrium

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Hart and Mansour, GEB 2009 (?)

How Long to Equilibrium

- **Intuition:**

How Long to Equilibrium

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  - different games have different equilibria

How Long to Equilibrium

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  - different games have different equilibria
  - the dynamic procedure must distinguish between them

How Long to Equilibrium

- **Intuition:**
  - different games have different equilibria
  - the dynamic procedure must distinguish between them
  - no single player can do so by himself

Dynamics and Nash Equilibrium

Dynamics and Nash Equilibrium

FACT

There are* NO *general, natural dynamics leading to Nash equilibrium

Dynamics and Nash Equilibrium

FACT

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RESULT

There* CANNOT BE *general, natural dynamics leading to Nash equilibrium

Dynamics and Nash Equilibrium

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Dynamics and Nash Equilibrium

RESULT

There CANNOT BE general, natural dynamics leading to Nash equilibrium

- **Perhaps we are asking too much?**

Dynamics and Nash Equilibrium

RESULT

There CANNOT BE general, natural dynamics leading to Nash equilibrium

- **Perhaps we are asking too much?**
- For instance, the size of the data (the payoff functions) is *exponential* rather than polynomial in the number of players

Correlated Equilibrium

Correlated Equilibrium

CORRELATED EQUILIBRIUM

Aumann, JME 1974

Correlated Equilibrium

CORRELATED EQUILIBRIUM :

Nash equilibrium when players receive payoff-irrelevant information before playing the game

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Aumann, JME 1974

Correlated Equilibrium

A **Correlated Equilibrium** is a Nash equilibrium when players receive payoff-irrelevant signals before playing the game

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- **Examples**:

Correlated Equilibrium

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- **Examples**:
  - Independent signals

Correlated Equilibrium

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- **Examples**:
  - Independent signals $\Leftrightarrow$ Nash equilibrium

Correlated Equilibrium

A **Correlated Equilibrium** is a Nash equilibrium when players receive payoff-irrelevant signals before playing the game

- **Examples**:
  - Independent signals $\Leftrightarrow$ Nash equilibrium
  - Public signals ("sunspots")

Correlated Equilibrium

A **Correlated Equilibrium** is a Nash equilibrium when players receive payoff-irrelevant signals before playing the game

- **Examples**:
  - Independent signals $\Leftrightarrow$ Nash equilibrium
  - Public signals ("sunspots") $\Leftrightarrow$ convex combinations of Nash equilibria

Correlated Equilibrium

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Correlated Equilibria

"Chicken" game

	LEAVE	STAY
LEAVE	$5, 5$	$3, 6$
STAY	$6, 3$	$0, 0$

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Correlated Equilibria

"Chicken" game

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STAY	$6, 3$	$0, 0$

0	$1/2$
$1/2$	0

a **(publicly) correlated equilibrium**

Correlated Equilibria

"Chicken" game

	LEAVE	STAY
LEAVE	$5, 5$	$3, 6$
STAY	$6, 3$	$0, 0$

	L	S
L	$1/3$	$1/3$
S	$1/3$	0

another **correlated equilibrium**

- after signal L play LEAVE
- after signal S play STAY

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  - Boston Celtics' front line

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A joint distribution z is a **correlated equilibrium**

$$\Leftrightarrow$$

$$\sum_{s^{-i}} u(j, s^{-i}) z(j, s^{-i}) \geq \sum_{s^{-i}} u(k, s^{-i}) z(j, s^{-i})$$

for all $i \in N$ and all $j, k \in S^i$

Dynamics & Correlated Equilibria

Dynamics & Correlated Equilibria

RESULT

THERE EXIST ***general, natural dynamics***
leading to CORRELATED EQUILIBRIA

Dynamics & Correlated Equilibria

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THERE EXIST ***general, natural dynamics***
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- ***Regret Matching***

Hart and Mas-Colell, Ec'ca 2000

Dynamics & Correlated Equilibria

<u>**RESULT**</u>

THERE EXIST ***general, natural dynamics***

leading to **CORRELATED EQUILIBRIA**

- ***Regret Matching***
- General regret-based dynamics

Hart and Mas-Colell, Ec'ca 2000, JET 2001

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- **"REGRET"**: the increase in past payoff, if any, if a different action would have been used

- **"MATCHING"**: switching to a different action with a probability that is proportional to the regret for that action

Dynamics & Correlated Equilibria

THERE EXIST ***general, natural dynamics***
leading to CORRELATED EQUILIBRIA

Dynamics & Correlated Equilibria

THERE EXIST *general*, *natural dynamics*
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- ***"general"***: in all games

Dynamics & Correlated Equilibria

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Dynamics & Correlated Equilibria

THERE EXIST ***general,*** ***natural*** ***dynamics***
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- ***"general"***: in all games
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Dynamics & Correlated Equilibria

THERE EXIST ***general***, ***natural*** ***dynamics***
leading to CORRELATED EQUILIBRIA

- ***"general"***: in all games
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  - **simple and efficient**:
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Dynamics & Correlated Equilibria

THERE EXIST ***general, natural dynamics***
leading to CORRELATED EQUILIBRIA

- ***"general"***: in all games
- ***"natural"***:
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  - **simple and efficient**:
    computation, time, information
- ***"leading to correlated equilibria"***:
  statistics of play become close to
  CORRELATED EQUILIBRIA

Dynamics and Equilibrium

Dynamics and Equilibrium

- **NASH EQUILIBRIUM**:
  a ***fixed-point*** of a non-linear map

Dynamics and Equilibrium

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- CORRELATED EQUILIBRIUM:
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Dynamics and Equilibrium

- **NASH EQUILIBRIUM**:
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- **CORRELATED EQUILIBRIUM**:
  a solution of finitely many ***linear inequalities***

- ***set-valued*** fixed-point (curb sets)?

Dynamics and Equilibrium

Dynamics and Equilibrium

"<u>LAW OF CONSERVATION OF COORDINATION</u>":

Dynamics and Equilibrium

"LAW OF CONSERVATION OF COORDINATION":

There must be some COORDINATION **—**

Dynamics and Equilibrium

"LAW OF CONSERVATION OF COORDINATION":

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either in the **EQUILIBRIUM** ***notion,***

Dynamics and Equilibrium

"<u>LAW OF CONSERVATION OF COORDINATION</u>":

There must be some* COORDINATION *—

either in the* EQUILIBRIUM *notion,

***or in the* DYNAMIC**

The "Program"

The "Program"

A. **Demarcate** **the** **BORDER** **between**

The "Program"

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- classes of dynamics where convergence to equilibria CAN be obtained, and
- classes of dynamics where convergence to equilibria CANNOT be obtained

The "Program"

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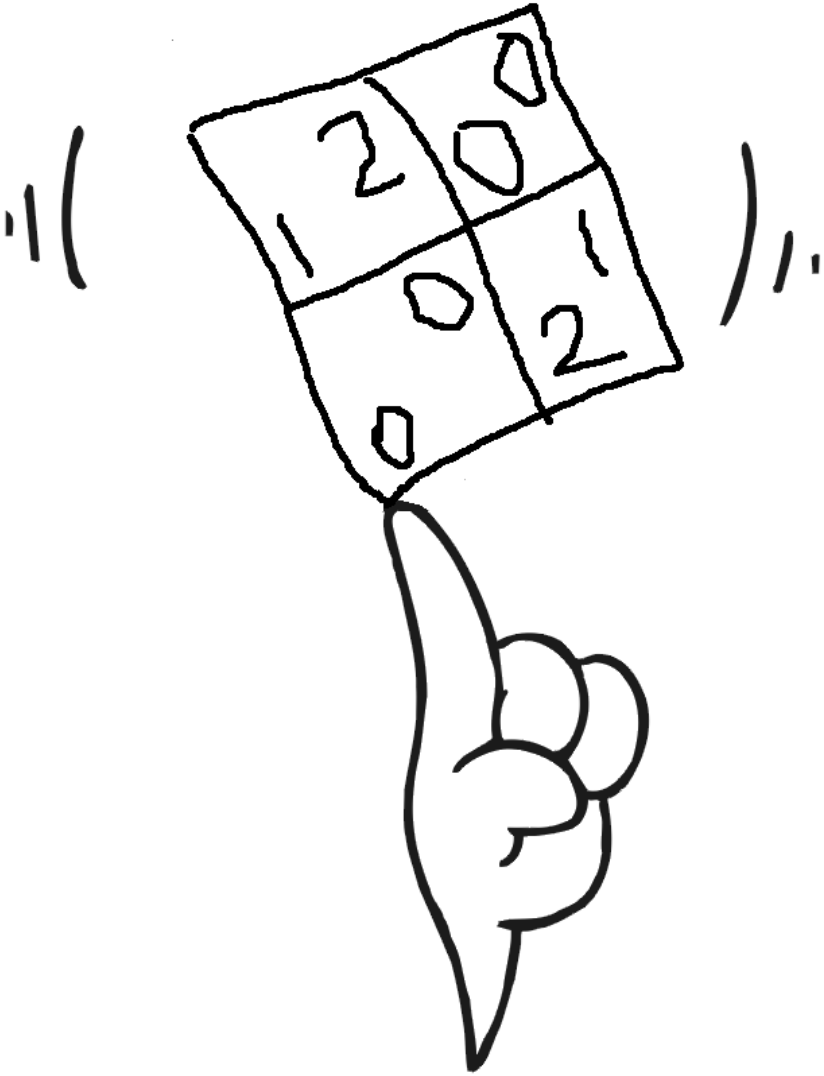
- **classes of dynamics where convergence to equilibria CAN be obtained, and**
- **classes of dynamics where convergence to equilibria CANNOT be obtained**

B. **Find NATURAL dynamics for the various equilibrium concepts**

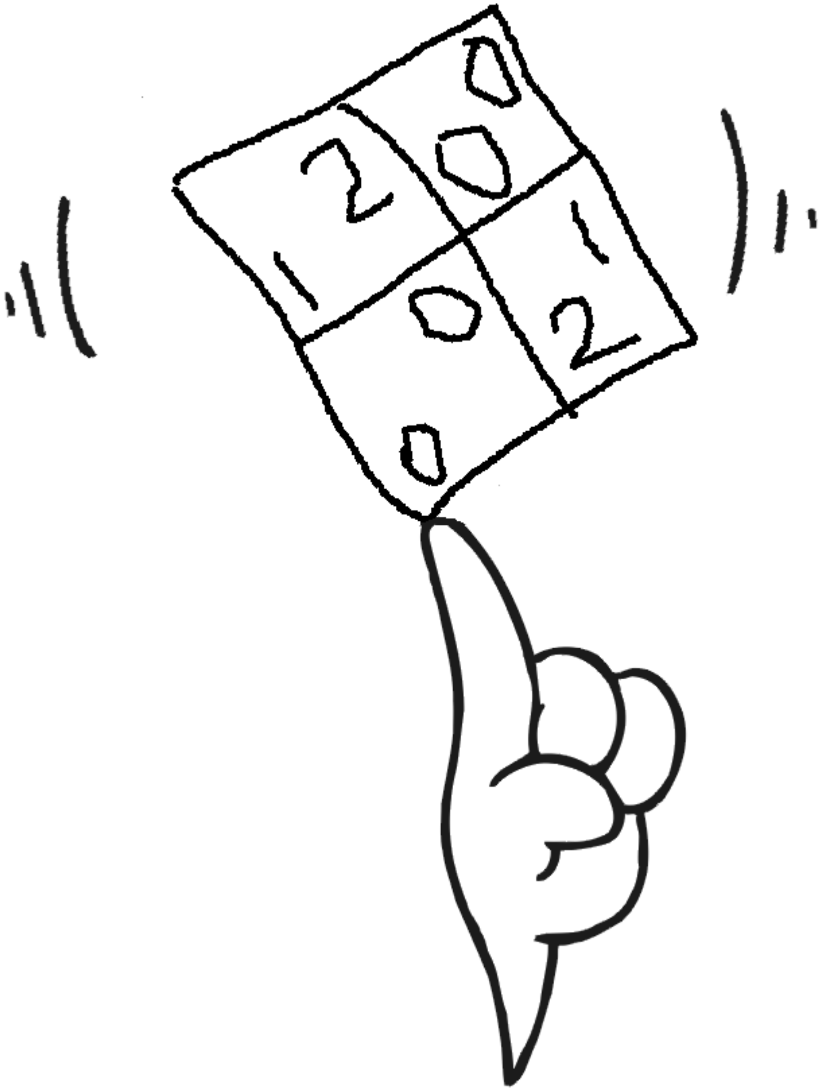
Dynamics and Equilibrium

Dynamics and Equilibrium

Dynamics and Equilibrium



Dynamics and Equilibrium



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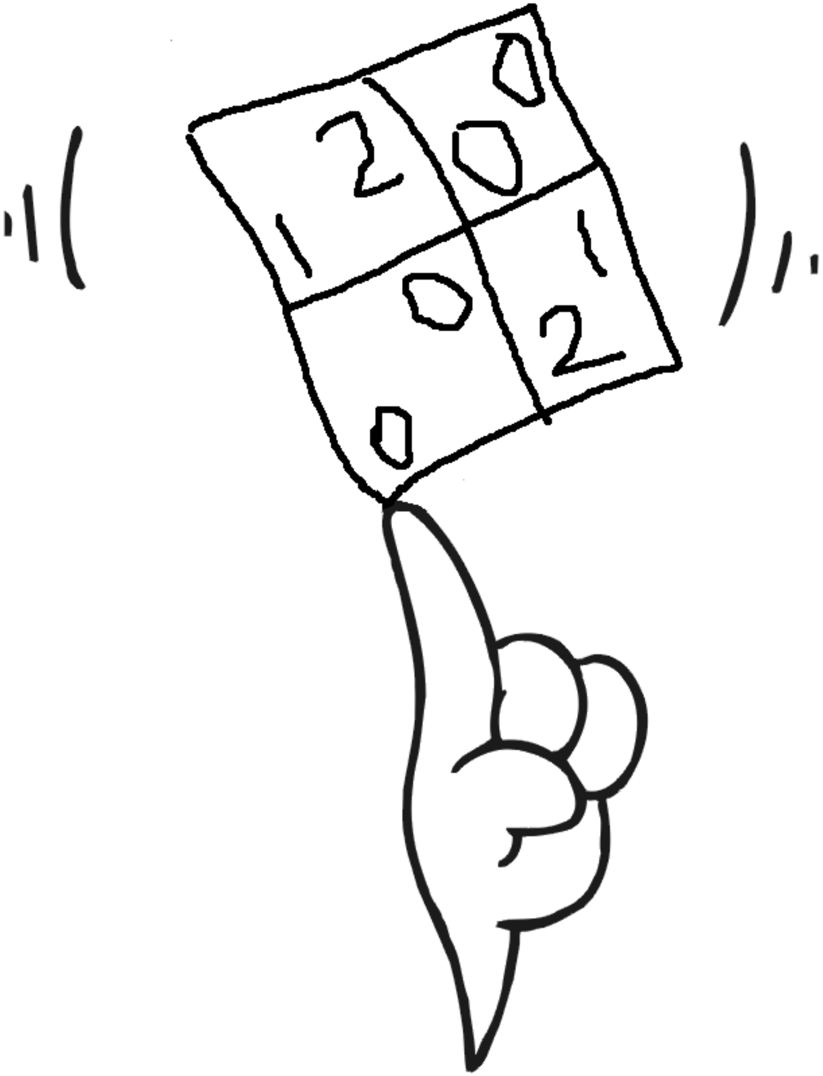

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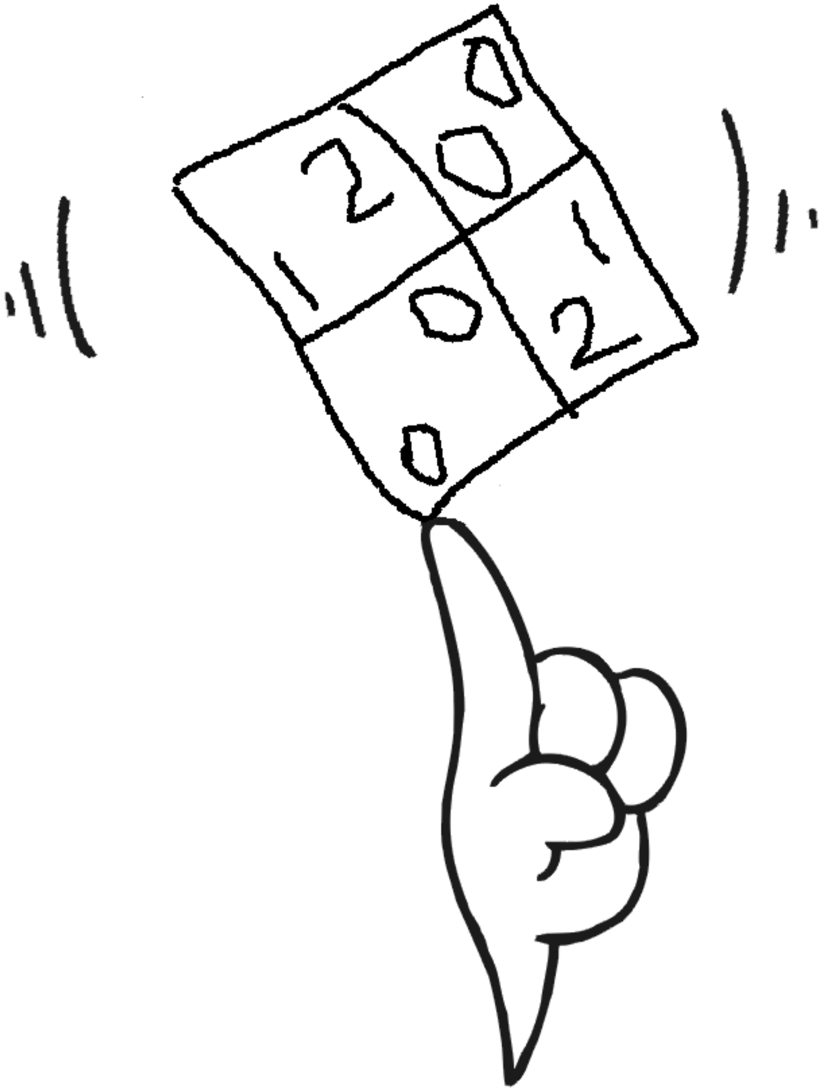

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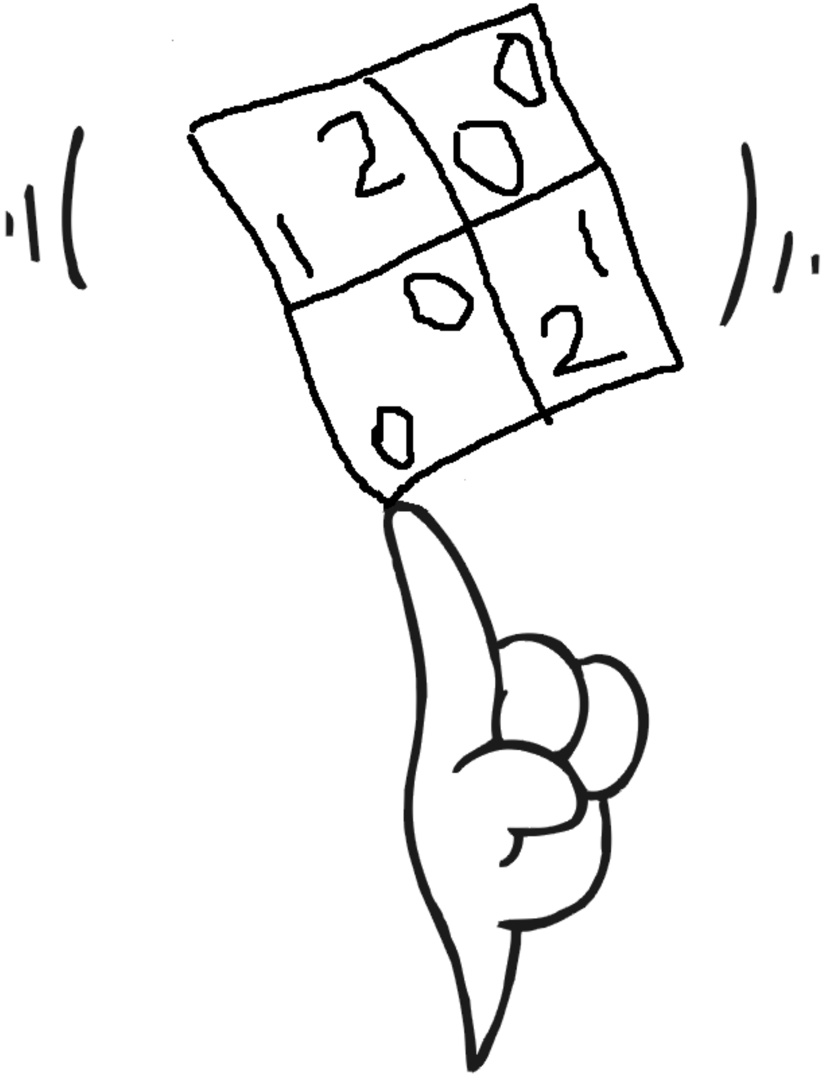

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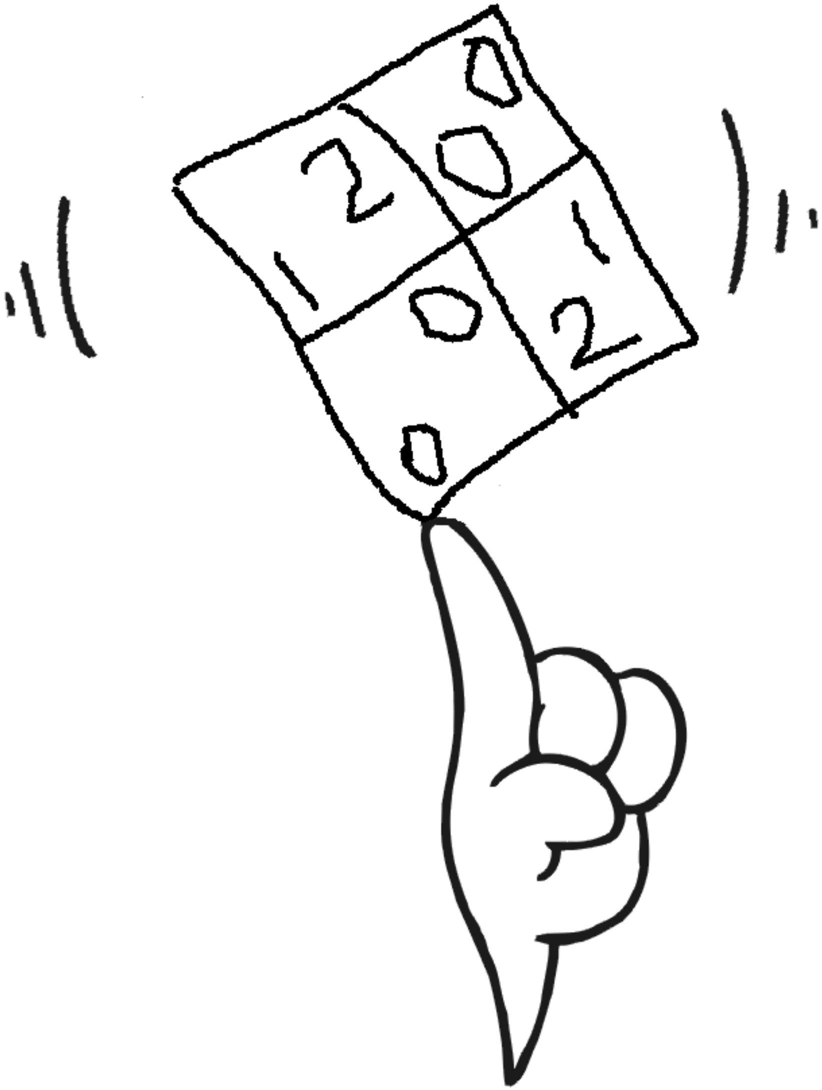

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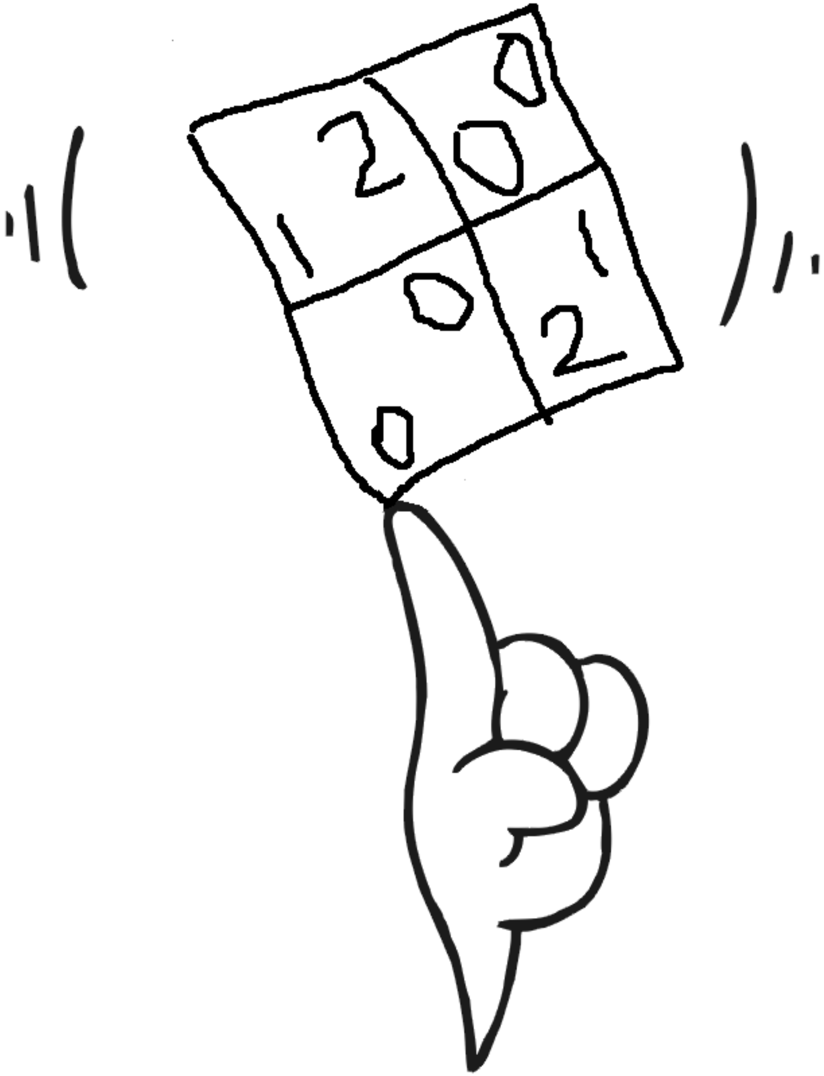

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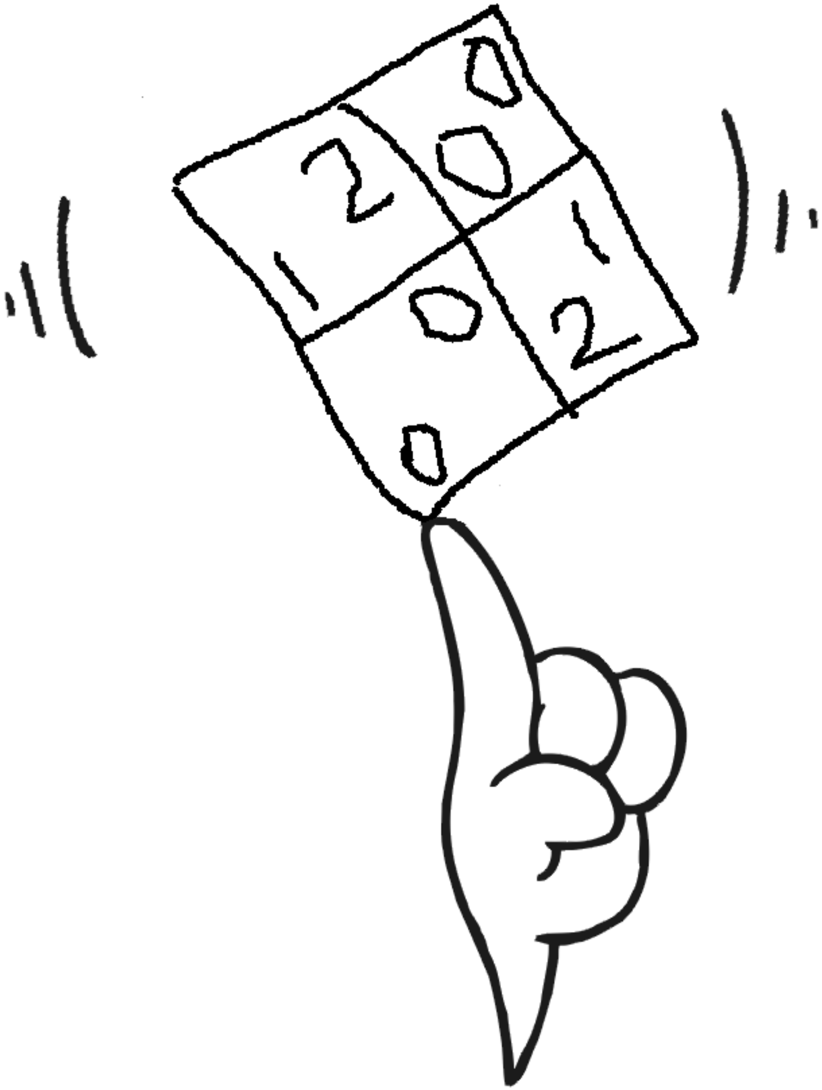

My Game Theory

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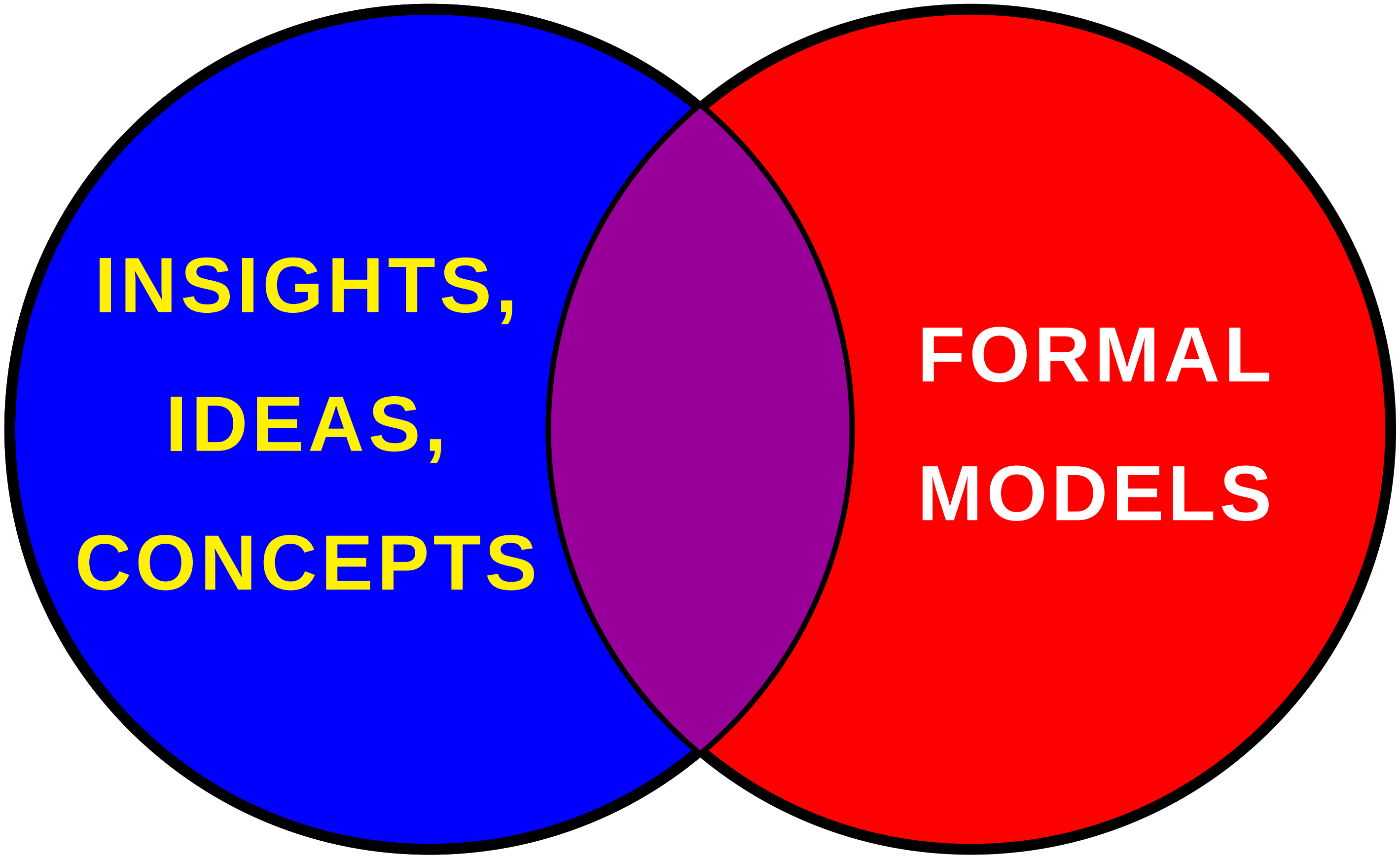

My Game Theory

INSIGHTS,
IDEAS,
CONCEPTS

FORMAL
MODELS

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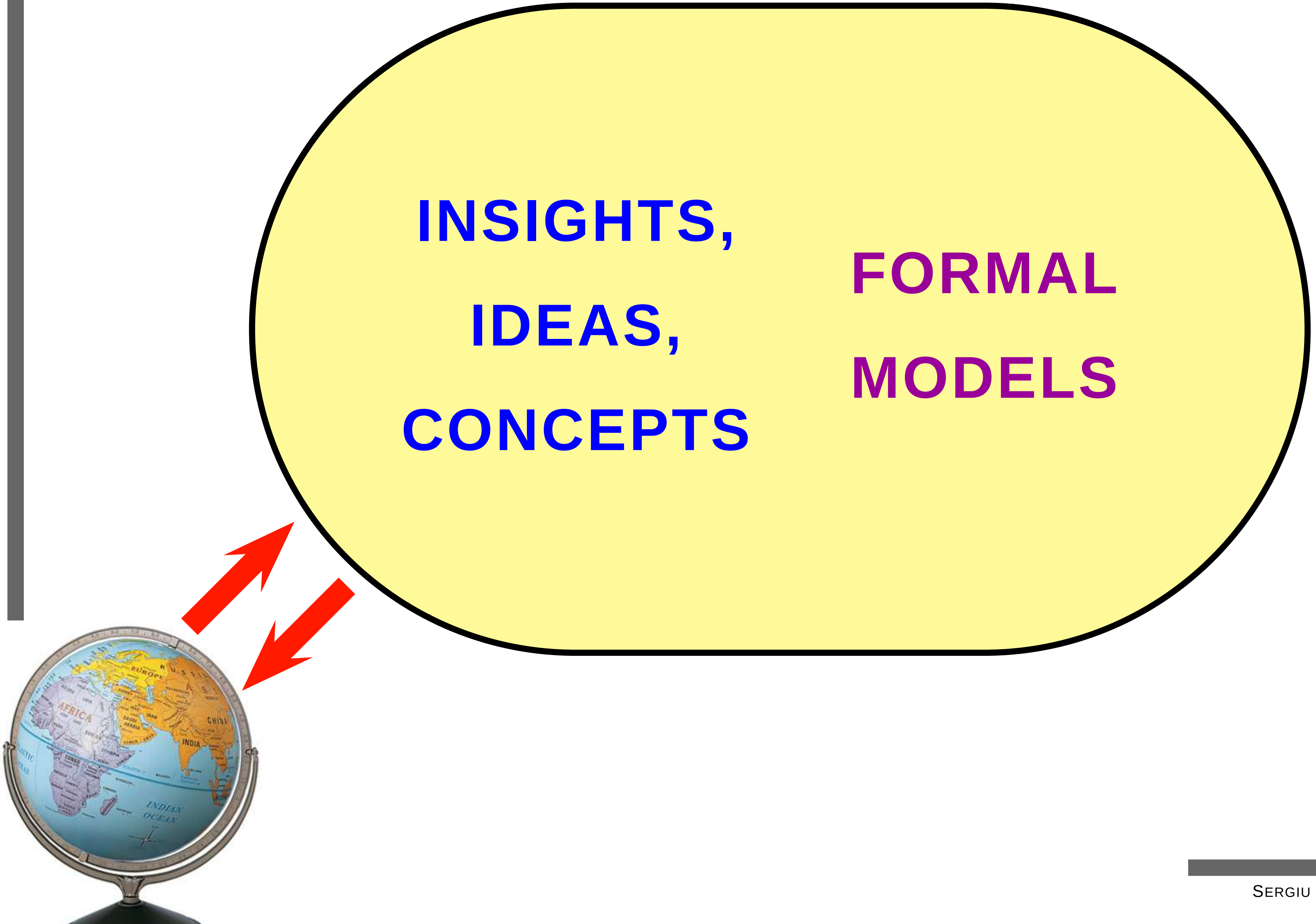

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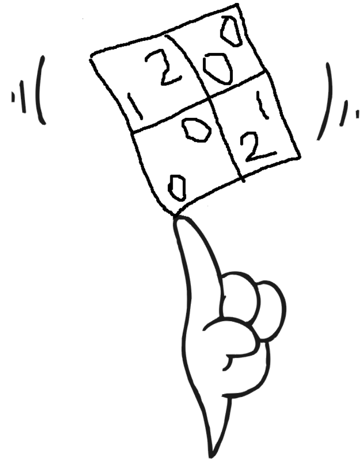

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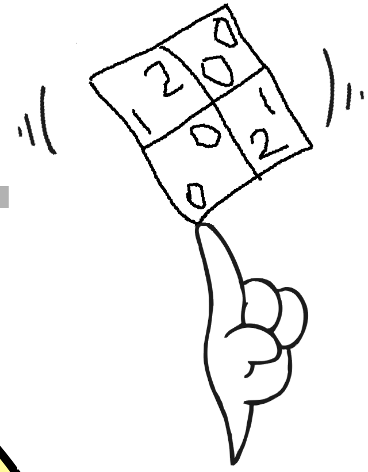

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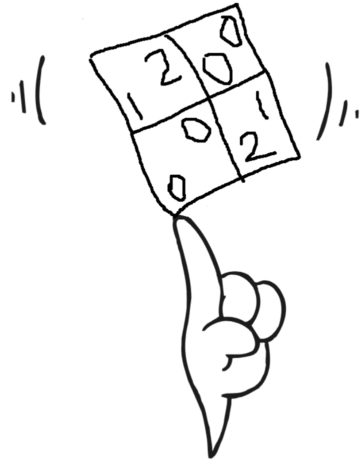

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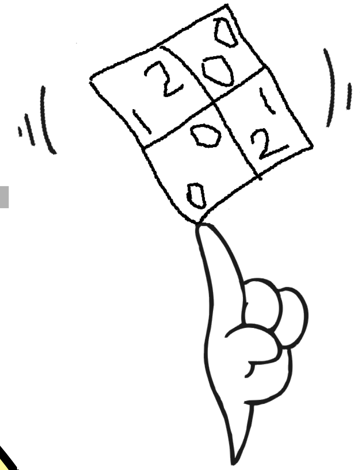

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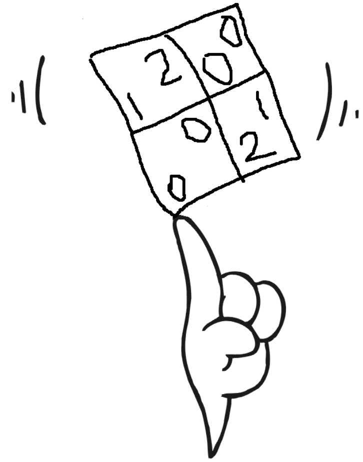

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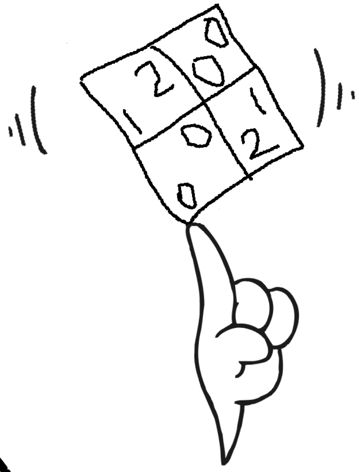

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