A Recursive Algorithm to Compute the Stochastically Stable Distribution of a Perturbed Markov Matrix

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joint work with John R. Wicks

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Game 1: Typewriter

	d	q
D	5,5	0,0
Q	0,0	4,4

This game has two pure coordination equilibria:

- Play D and d, respectively;
- Play Q and q.

Equilibrium Selection Problem

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- Play D and d, respectively;
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Question

Is there a natural model in which players learn to coordinate?

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Adaptive Learning

- Markov model of learning in a repeated game.
- A variant of Fictitious Play (Brown, 1951).
- Finite memory m and sample size s (with $s \leq m$).
- Play a best-response to the empirical distribution of the sample.

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- Play a best-response to the empirical distribution of the sample.

The transition matrix M_0 for the Typewriter Game with $m = s = 1$:

M_0	Dd	Qd	Dq	Qq
Dd	1	0	0	0
Qd	0	0	1	0
Dq	0	1	0	0
Qq	0	0	0	1

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Dq	0	1	0	0
Qq	0	0	0	1

M_0 has three pure "equilibria" (i.e., stationary distributions):

$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \text{ and } \begin{pmatrix} 0 \\ \frac{1}{2} \\ \frac{1}{2} \\ 0 \end{pmatrix}$$

Every stationary distribution of M_0 is a convex combination of these.

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Perturbed Adaptive Learning

- Play a best-response with probability $1 - \epsilon$.
- Play arbitrarily with probability ϵ.

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Perturbed Adaptive Learning

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- Play arbitrarily with probability ϵ.

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M_ϵ	Dd	Qd	Dq	Qq
Dd	$(1-\epsilon)(1-\epsilon)$	$(1-\epsilon)\epsilon$	$\epsilon(1-\epsilon)$	ϵ^2
Qd	$\epsilon(1-\epsilon)$	ϵ^2	$(1-\epsilon)(1-\epsilon)$	$(1-\epsilon)\epsilon$
Dq	$(1-\epsilon)\epsilon$	$(1-\epsilon)(1-\epsilon)$	ϵ^2	$\epsilon(1-\epsilon)$
Qq	ϵ^2	$\epsilon(1-\epsilon)$	$(1-\epsilon)\epsilon$	$(1-\epsilon)(1-\epsilon)$

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Qd	$\epsilon(1-\epsilon)$	ϵ^2	$(1-\epsilon)(1-\epsilon)$	$(1-\epsilon)\epsilon$
Dq	$(1-\epsilon)\epsilon$	$(1-\epsilon)(1-\epsilon)$	ϵ^2	$\epsilon(1-\epsilon)$
Qq	ϵ^2	$\epsilon(1-\epsilon)$	$(1-\epsilon)\epsilon$	$(1-\epsilon)(1-\epsilon)$

- M_ϵ is an example of a perturbed Markov matrix (PMM).
- For $\epsilon > 0$, M_ϵ has a unique stationary distribution.
- Letting $\epsilon \to 0$, we can select a stationary distribution of M_0, the so-called stochastically stable distribution (SSD) of the PMM.
- We give an efficient algorithm to compute the SSD of a PMM.

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Preliminary Experimental Results

Game 2: Typewriter'

	d	q
D	5,5	0,3
Q	3,0	4,4

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Theorem (Young, 1998) For $2s \leq m$ sufficiently large, the stochastically stable states of the repeated version of a 2×2 coordination game correspond to risk-dominant conventions.

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Experimental Results

- For $2 \leq s \leq m \leq 4$, the stochastically stable distribution is concentrated at $Q \cdots Qq \cdots q$.
- These results suggest that it may be possible to strengthen Young's theorem.

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Preliminary Experimental Results (cont'd)

Game 3: Generic Coordination

	l	c	r
T	3,3	0,0	0,0
M	0,0	2,2	0,0
B	0,0	0,0	1,1

Preliminary Experimental Results (cont'd)

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Experimental Results

- Perhaps surprisingly, when $s = m = 3$, the SSD is $\Pr[TTTlll] = \frac{6}{7}, \Pr[MMMccc] = \frac{1}{7}$.
- Are there values of s and m for which the SSD is concentrated at $T \cdots Tl \cdots l$?
- Either way, reasonable dynamics may put sufficiently high probability on $T \cdots Tl \cdots l$.

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Related Experimental Results

- Edgeworth (1881) proposed a model for contracting within a simple housing economy.
- Serrano and Volij (2003) have augmented Edgeworth's model, allowing agents to make mistakes.
- Kaihatsu and Milionis in the Brown Department of Economics have applied our algorithm to Serrano and Volij's model to show that the various core allocations of the economy will emerge in the long-run with different relative frequencies.

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Markov Matrices and Stationary Distributions

A matrix $M \in \mathbb{R}^{n \times n}$ is Markov iff

- its columns sum to 1, and
- its entries are non-negative

$$M = \begin{pmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \\ 0 & \frac{1}{4} & 0 \\ \frac{3}{4} & \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

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A vector $v \in \mathbb{R}^n$ is a distribution iff

- its entries sum to 1, and
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$$v = \begin{pmatrix} \frac{2}{5} \\ 0 \\ \frac{3}{5} \end{pmatrix}$$

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Theorem Every Markov matrix M has a stationary (or invariant) distribution v: i.e., $Mv = v$.

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Theorem Every Markov matrix M has a stationary (or invariant) distribution v: i.e., $Mv = v$.

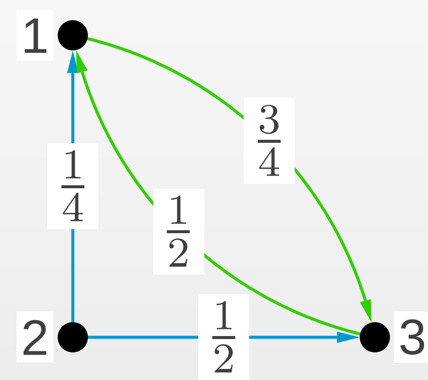
Theorem Every unichain Markov matrix M has a unique stationary distribution v.

The Geometry of Markov Matrices

$$M$$

$$\begin{pmatrix} \cdot & \frac{1}{4} & \frac{1}{2} \\ 0 & \cdot & 0 \\ \frac{3}{4} & \frac{1}{2} & \cdot \end{pmatrix}$$

$$G\,(M)$$



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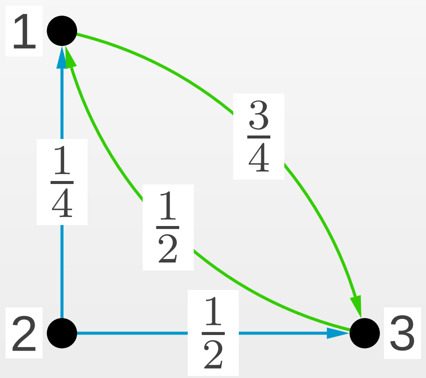
The Geometry of Markov Matrices

M

$$\begin{pmatrix} \cdot & \frac{1}{4} & \frac{1}{2} \\ 0 & \cdot & 0 \\ \frac{3}{4} & \frac{1}{2} & \cdot \end{pmatrix}$$

$G(M)$



- A communicating class is a maximal set of nodes such that every node in the set is accessible from every other node: $\{1, 3\}$, $\{2\}$
- A closed class is a communicating class from which no nodes outside the class are accessible: $\{1, 3\}$
- The transients are not members of any closed class: $\{2\}$

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The Geometry of Markov Matrices

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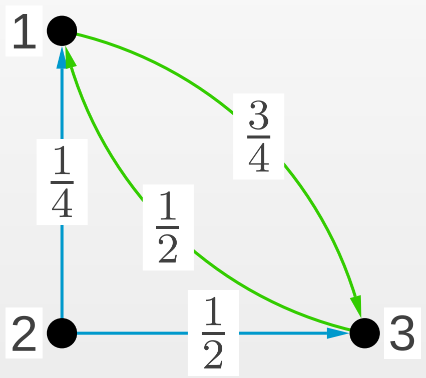

- A communicating class is a maximal set of nodes such that every node in the set is accessible from every other node: $\{1, 3\}$, $\{2\}$
- A closed class is a communicating class from which no nodes outside the class are accessible: $\{1, 3\}$
- The transients are not members of any closed class: $\{2\}$

A Markov matrix M is unichain if it has exactly one closed class.

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Exponentially Convergent Functions

Definition

f is asymptotically equal to g (i.e., $f \sim g$) iff $\lim_{\epsilon \to 0^+} \frac{f(\epsilon)}{g(\epsilon)} = 1$.

Examples

$f(\epsilon)$	$g(\epsilon)$
$\frac{1}{2}$	$\frac{1}{2}$
$\epsilon - \epsilon^2$	ϵ
$2\epsilon^2 - 3\epsilon^4$	$2\epsilon^2$

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Examples

$f(\epsilon)$	$g(\epsilon)$
$\frac{1}{2}$	$\frac{1}{2}$
$\epsilon - \epsilon^2$	ϵ
$2\epsilon^2 - 3\epsilon^4$	$2\epsilon^2$

Definition

Let $\mathbb{C}^+$ be the set of functions $f(\epsilon)$ that are asymptotically equal to a positive exponential: i.e., $f(\epsilon) \sim c\epsilon^r$ for some $c, r \geq 0$.

Examples

$f(\epsilon)$	c	r
$\frac{1}{2}$	$\frac{1}{2}$	0
$\epsilon - \epsilon^2$	1	1
$2\epsilon^2 - 3\epsilon^4$	2	2

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Perturbed Markov Matrices and Stochastically Stable Distributions

A matrix M_ϵ with entries in $\mathbb{C}^+$ is called perturbed Markov iff

- it is Markov
- it is unichain

for sufficiently small $\epsilon > 0$.

$$\begin{pmatrix} \cdot & \frac{1}{2} & 0 & \epsilon \\ \frac{1}{2} & \cdot & 0 & 0 \\ 0 & 2\epsilon & \cdot & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & \cdot \end{pmatrix}$$

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Theorem Every perturbed Markov matrix M_ϵ has a unique stationary distribution v_ϵ with entries in $\mathbb{C}^+$.

$$\begin{pmatrix} \frac{1+4\epsilon}{6+8\epsilon} \\ \frac{1}{6+8\epsilon} \\ \frac{1+2\epsilon}{3+4\epsilon} \\ \frac{1}{3+4\epsilon} \end{pmatrix}$$

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Theorem Every perturbed Markov matrix M_ϵ has a (unique) **stochastically stable** distribution $v_0 = \lim_{\epsilon \to 0} v_\epsilon$.

$$\begin{pmatrix} \frac{1}{6} \\ \frac{1}{6} \\ \frac{1}{3} \\ \frac{1}{3} \end{pmatrix}$$

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$$\begin{pmatrix} \frac{1}{6} \\ \frac{1}{6} \\ \frac{1}{3} \\ \frac{1}{3} \end{pmatrix}$$

We give an efficient algorithm for computing the SSD of a PMM.

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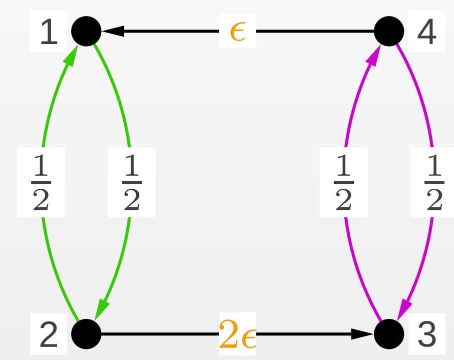
Numerical Method

M_ϵ

$$\begin{pmatrix} \cdot & \frac{1}{2} & 0 & \epsilon \\ \frac{1}{2} & \cdot & 0 & 0 \\ 0 & 2\epsilon & \cdot & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & \cdot \end{pmatrix}$$

$G(M_\epsilon)$



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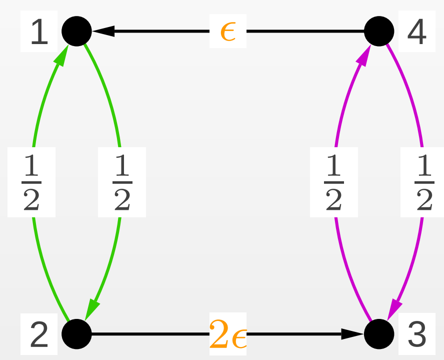


Idea Compute v_ϵ for very small ϵ and extrapolate to infer v_0

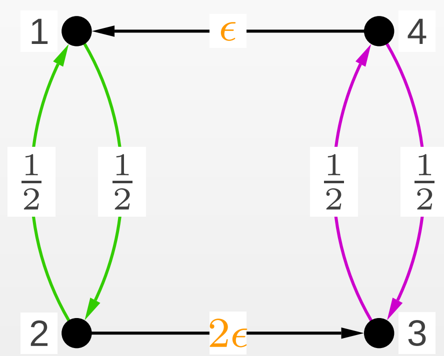
ϵ	10^{-1}	10^{-2}	10^{-3}	$\cdots$	10^{-9}
v_ϵ	$\begin{pmatrix} 0.206 \\ 0.147 \\ 0.353 \\ 0.294 \end{pmatrix}$	$\begin{pmatrix} 0.171 \\ 0.164 \\ 0.336 \\ 0.329 \end{pmatrix}$	$\begin{pmatrix} 0.167 \\ 0.166 \\ 0.334 \\ 0.333 \end{pmatrix}$	$\cdots$	$\begin{pmatrix} 0.167 \\ 0.167 \\ 0.333 \\ 0.333 \end{pmatrix}$

Numerical Method

$$M_\epsilon$$

$$\begin{pmatrix} \cdot & \frac{1}{2} & 0 & \epsilon \\ \frac{1}{2} & \cdot & 0 & 0 \\ 0 & 2\epsilon & \cdot & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & \cdot \end{pmatrix}$$

$$G(M_\epsilon)$$



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v_ϵ	$\begin{pmatrix} 0.206 \\ 0.147 \\ 0.353 \\ 0.294 \end{pmatrix}$	$\begin{pmatrix} 0.171 \\ 0.164 \\ 0.336 \\ 0.329 \end{pmatrix}$	$\begin{pmatrix} 0.167 \\ 0.166 \\ 0.334 \\ 0.333 \end{pmatrix}$	$\cdots$	$\begin{pmatrix} 0.167 \\ 0.167 \\ 0.333 \\ 0.333 \end{pmatrix}$

$$v_0 = \begin{pmatrix} \frac{1}{6} \\ \frac{1}{6} \\ \frac{1}{3} \\ \frac{1}{3} \end{pmatrix}$$

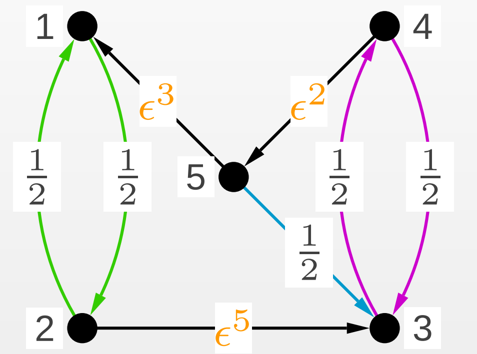
Success!

Numerical Method (cont'd)

$$M_\epsilon$$

$$\begin{pmatrix} \cdot & \frac{1}{2} & 0 & 0 & \epsilon^3 \\ \frac{1}{2} & \cdot & 0 & 0 & 0 \\ 0 & \epsilon^5 & \cdot & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & \cdot & 0 \\ 0 & 0 & 0 & \epsilon^2 & \cdot \end{pmatrix}$$

$$G\left(M_\epsilon\right)$$



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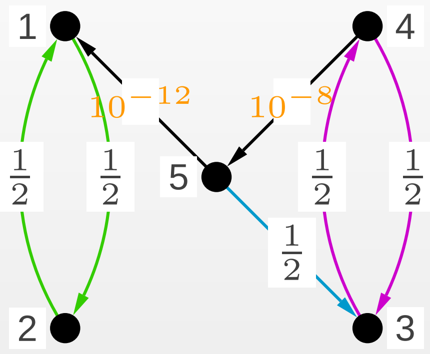
Numerical Method (cont'd)

$$M_{10^{-4}}$$

$$\begin{pmatrix} \cdot & \frac{1}{2} & 0 & 0 & 10^{-12} \\ \frac{1}{2} & \cdot & 0 & 0 & 0 \\ 0 & 0 & \cdot & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & \cdot & 0 \\ 0 & 0 & 0 & 10^{-8} & \cdot \end{pmatrix}$$

$$G\left(M_{10^{-4}}\right)$$



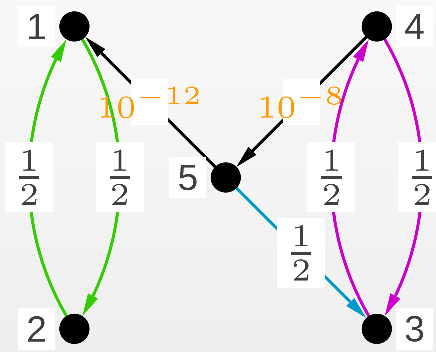
Problem At double-precision, when $\epsilon = 10^{-4}$, the probability of transitioning from 2 to 3 falls below machine precision.

Numerical Method (cont'd)

$$M_{10^{-4}}$$

$$\begin{pmatrix} \cdot & \frac{1}{2} & 0 & 0 & 10^{-12} \\ \frac{1}{2} & \cdot & 0 & 0 & 0 \\ 0 & 0 & \cdot & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & \cdot & 0 \\ 0 & 0 & 0 & 10^{-8} & \cdot \end{pmatrix}$$

$$G\left(M_{10^{-4}}\right)$$



Problem At double-precision, when $\epsilon = 10^{-4}$, the probability of transitioning from 2 to 3 falls below machine precision.

The unique stationary distribution of $M_{10^{-4}}$ is:

$$\cdots, \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 0 \\ 0 \\ 0 \end{pmatrix}, \cdots \not\to \begin{pmatrix} \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{6} \\ \frac{1}{6} \\ 0 \end{pmatrix} = v_0$$

Failure!

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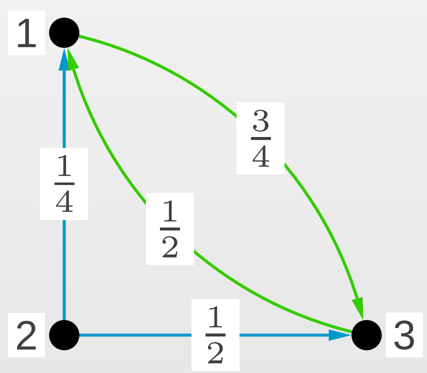
Markov Chain Tree Theorem The stationary distribution of a unichain Markov matrix M is proportional to the vector w_M of sums of aggregated weights of its directed spanning subtrees.

M

$$\begin{pmatrix} \cdot & \frac{1}{4} & \frac{1}{2} \\ 0 & \cdot & 0 \\ \frac{3}{4} & \frac{1}{2} & \cdot \end{pmatrix}$$

$G(M)$



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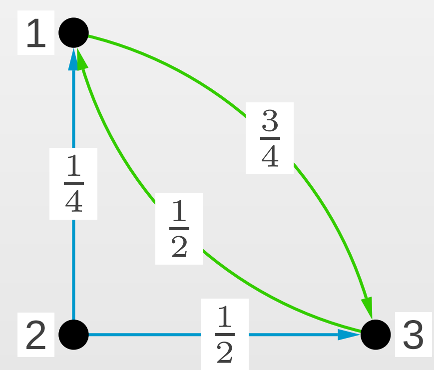
$$\begin{pmatrix} \cdot & \frac{1}{4} & \frac{1}{2} \\ 0 & \cdot & 0 \\ \frac{3}{4} & \frac{1}{2} & \cdot \end{pmatrix}$$

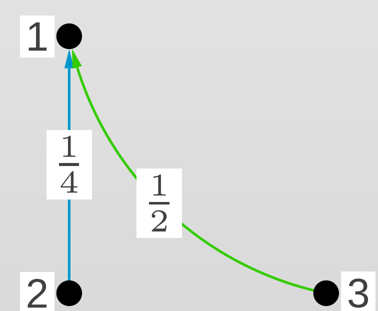
$G(M)$



w_M

$$\begin{pmatrix} \frac{1}{4} \cdot \frac{1}{2} \\ \\ \\ \end{pmatrix}$$

$T \subset G(M)$



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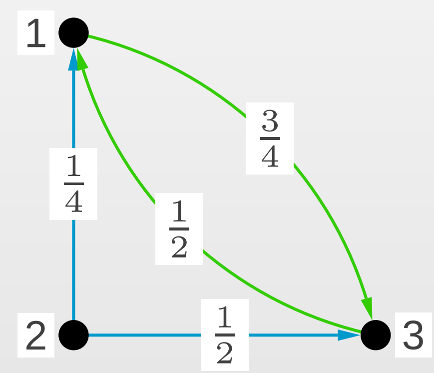
Markov Chain Tree Theorem The stationary distribution of a unichain Markov matrix M is proportional to the vector w_M of sums of aggregated weights of its directed spanning subtrees.

M

$$\begin{pmatrix} \cdot & \frac{1}{4} & \frac{1}{2} \\ 0 & \cdot & 0 \\ \frac{3}{4} & \frac{1}{2} & \cdot \end{pmatrix}$$

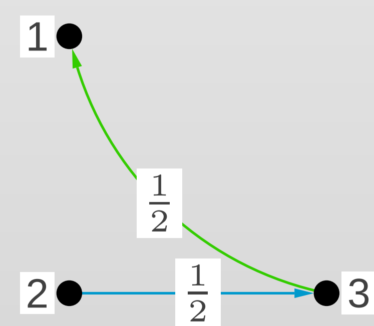
$G(M)$



w_M

$$\begin{pmatrix} \frac{1}{4} \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{2} \\ \\ \\ \end{pmatrix}$$

$T \subset G(M)$



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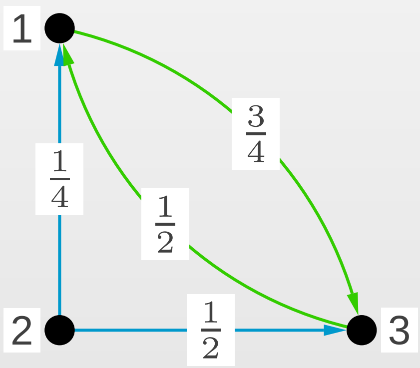
Markov Chain Tree Theorem The stationary distribution of a unichain Markov matrix M is proportional to the vector w_M of sums of aggregated weights of its directed spanning subtrees.

$$M$$

$$\begin{pmatrix} \cdot & \frac{1}{4} & \frac{1}{2} \\ 0 & \cdot & 0 \\ \frac{3}{4} & \frac{1}{2} & \cdot \end{pmatrix}$$

$$G(M)$$



$$w_M$$

$$\begin{pmatrix} \frac{1}{4} \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{2} \\ 0 \\ \\ \end{pmatrix}$$

$$T \subset G(M)$$

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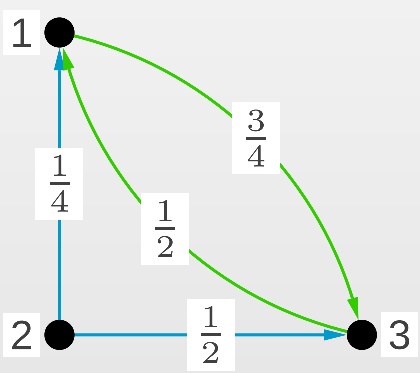
$$M$$

$$\begin{pmatrix} \cdot & \frac{1}{4} & \frac{1}{2} \\ 0 & \cdot & 0 \\ \frac{3}{4} & \frac{1}{2} & \cdot \end{pmatrix}$$

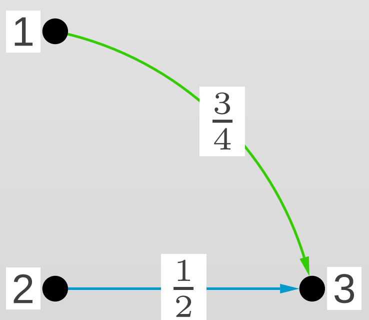
$$G(M)$$



$$w_M$$

$$\begin{pmatrix} \frac{1}{4} \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{2} \\ 0 \\ \frac{1}{2} \cdot \frac{3}{4} \end{pmatrix}$$

$$T \subset G(M)$$



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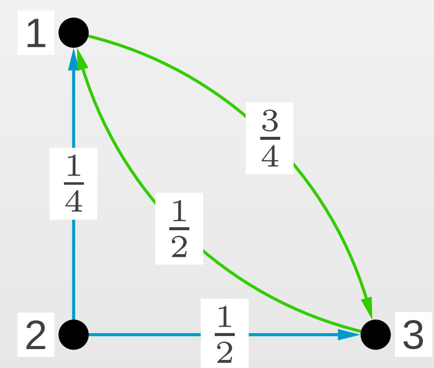
M

$$\begin{pmatrix} \cdot & \frac{1}{4} & \frac{1}{2} \\ 0 & \cdot & 0 \\ \frac{3}{4} & \frac{1}{2} & \cdot \end{pmatrix}$$

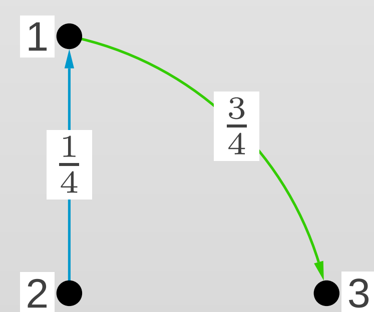
$G(M)$



w_M

$$\begin{pmatrix} \frac{1}{4} \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{2} \\ 0 \\ \frac{1}{2} \cdot \frac{3}{4} + \frac{1}{4} \cdot \frac{3}{4} \end{pmatrix}$$

$T \subset G(M)$



Geometric Approach for MMs

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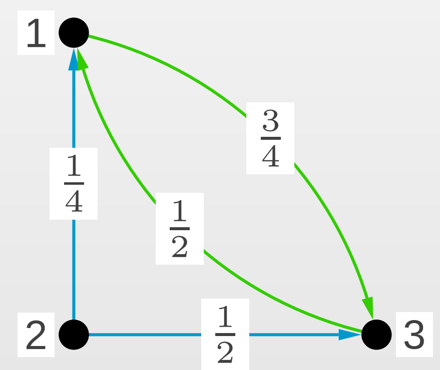
M

$$\begin{pmatrix} \cdot & \frac{1}{4} & \frac{1}{2} \\ 0 & \cdot & 0 \\ \frac{3}{4} & \frac{1}{2} & \cdot \end{pmatrix}$$

$G(M)$



w_M

$$\begin{pmatrix} \frac{6}{16} \\ 0 \\ \frac{9}{16} \end{pmatrix}$$

$\mathrm{inv}(M)$

$$\begin{pmatrix} \frac{2}{5} \\ 0 \\ \frac{3}{5} \end{pmatrix}$$

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Markov Chain Tree Theorem In particular, the stationary distribution of a PMM M_ϵ is proportional to the vector w_{M_ϵ} of sums of aggregated weights of its directed spanning subtrees.

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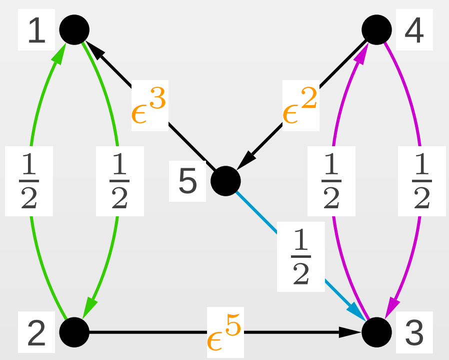
Markov Chain Tree Theorem In particular, the stationary distribution of a PMM M_ϵ is proportional to the vector w_{M_ϵ} of sums of aggregated weights of its directed spanning subtrees.

M_ϵ

$$\begin{pmatrix} \cdot & \frac{1}{2} & 0 & 0 & \epsilon^3 \\ \frac{1}{2} & \cdot & 0 & 0 & 0 \\ 0 & \epsilon^5 & \cdot & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & \cdot & 0 \\ 0 & 0 & 0 & \epsilon^2 & \cdot \end{pmatrix}$$

$G(M_\epsilon)$



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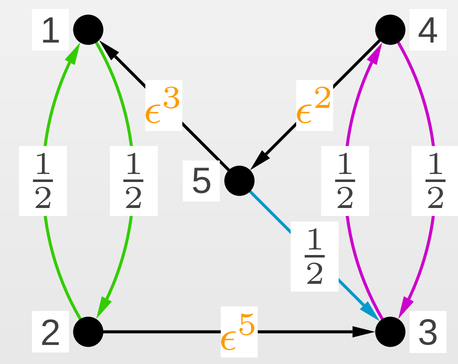
Markov Chain Tree Theorem In particular, the stationary distribution of a PMM M_ϵ is proportional to the vector w_{M_ϵ} of sums of aggregated weights of its directed spanning subtrees.

M_ϵ

$$\begin{pmatrix} \cdot & \frac{1}{2} & 0 & 0 & \epsilon^3 \\ \frac{1}{2} & \cdot & 0 & 0 & 0 \\ 0 & \epsilon^5 & \cdot & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & \cdot & 0 \\ 0 & 0 & 0 & \epsilon^2 & \cdot \end{pmatrix}$$

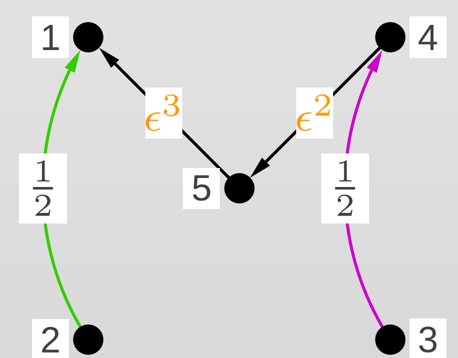
$G(M_\epsilon)$



w_{M_ϵ}

$$\begin{pmatrix} \frac{1}{2} \cdot \epsilon^3 \cdot \epsilon^2 \cdot \frac{1}{2} \\ \\ \\ \\ \\ \end{pmatrix}$$

$T \subset G(M_\epsilon)$



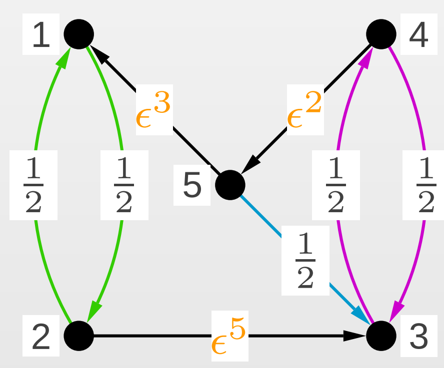
Geometric Approach for PMMs

Markov Chain Tree Theorem In particular, the stationary distribution of a PMM M_ϵ is proportional to the vector w_{M_ϵ} of sums of aggregated weights of its directed spanning subtrees.

M_ϵ

$$\begin{pmatrix} \cdot & \frac{1}{2} & 0 & 0 & \epsilon^3 \\ \frac{1}{2} & \cdot & 0 & 0 & 0 \\ 0 & \epsilon^5 & \cdot & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & \cdot & 0 \\ 0 & 0 & 0 & \epsilon^2 & \cdot \end{pmatrix}$$

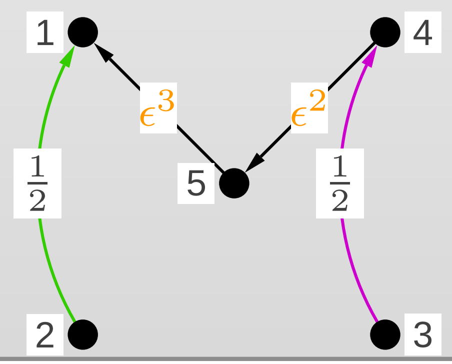
$G(M_\epsilon)$



w_{M_ϵ}

$$\begin{pmatrix} \frac{1}{4}\epsilon^5 \\ \\ \\ \\ \\ \end{pmatrix}$$

$T \subset G(M_\epsilon)$



Geometric Approach for PMMs

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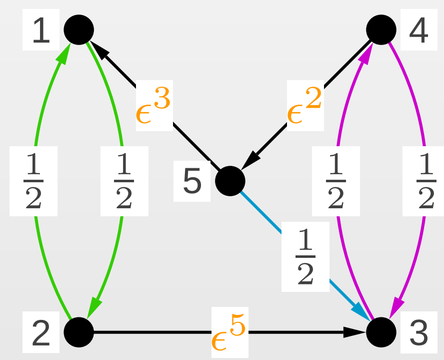
M_ϵ

$$\begin{pmatrix} \cdot & \frac{1}{2} & 0 & 0 & \epsilon^3 \\ \frac{1}{2} & \cdot & 0 & 0 & 0 \\ 0 & \epsilon^5 & \cdot & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & \cdot & 0 \\ 0 & 0 & 0 & \epsilon^2 & \cdot \end{pmatrix}$$

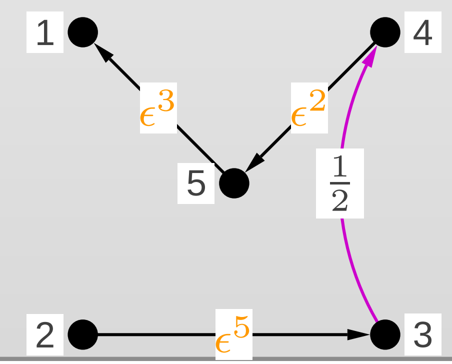
$G\left(M_\epsilon\right)$



w_{M_ϵ}

$$\begin{pmatrix} \frac{1}{4}\epsilon^5 + \epsilon^3 \cdot \epsilon^2 \cdot \frac{1}{2} \cdot \epsilon^5 \\ \\ \\ \\ \\ \end{pmatrix}$$

$T \subset G\left(M_\epsilon\right)$



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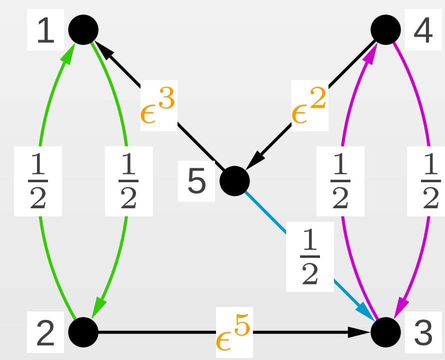
Markov Chain Tree Theorem In particular, the stationary distribution of a PMM M_ϵ is proportional to the vector w_{M_ϵ} of sums of aggregated weights of its directed spanning subtrees.

M_ϵ

$$\begin{pmatrix} \cdot & \frac{1}{2} & 0 & 0 & \epsilon^3 \\ \frac{1}{2} & \cdot & 0 & 0 & 0 \\ 0 & \epsilon^5 & \cdot & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & \cdot & 0 \\ 0 & 0 & 0 & \epsilon^2 & \cdot \end{pmatrix}$$

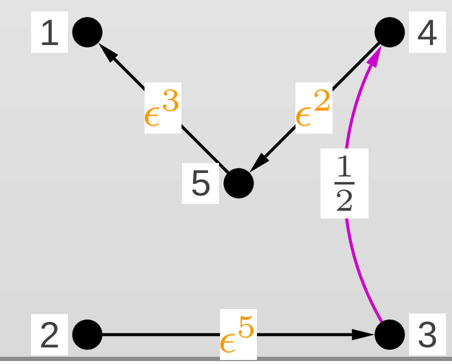
$G(M_\epsilon)$



w_{M_ϵ}

$$\begin{pmatrix} \frac{1}{4}\epsilon^5 + \frac{1}{2}\epsilon^{10} \\ \\ \\ \\ \end{pmatrix}$$

$T \subset G(M_\epsilon)$



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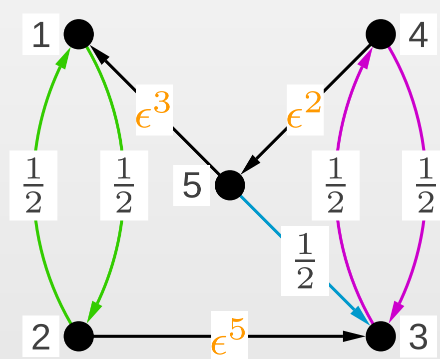
M_ϵ

$$\begin{pmatrix} \cdot & \frac{1}{2} & 0 & 0 & \epsilon^3 \\ \frac{1}{2} & \cdot & 0 & 0 & 0 \\ 0 & \epsilon^5 & \cdot & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & \cdot & 0 \\ 0 & 0 & 0 & \epsilon^2 & \cdot \end{pmatrix}$$

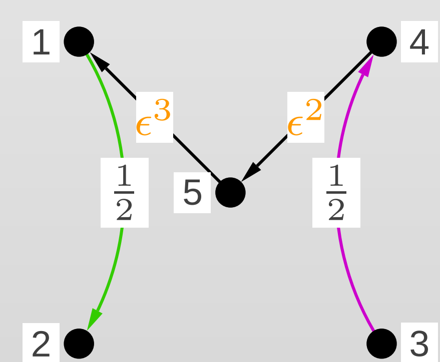
$G\left(M_\epsilon\right)$



w_{M_ϵ}

$$\begin{pmatrix} \frac{1}{4}\epsilon^5 + \frac{1}{2}\epsilon^{10} \\ \frac{1}{2} \cdot \epsilon^3 \cdot \epsilon^2 \cdot \frac{1}{2} \\ \\ \\ \\ \end{pmatrix}$$

$T \subset G\left(M_\epsilon\right)$



Geometric Approach for PMMs

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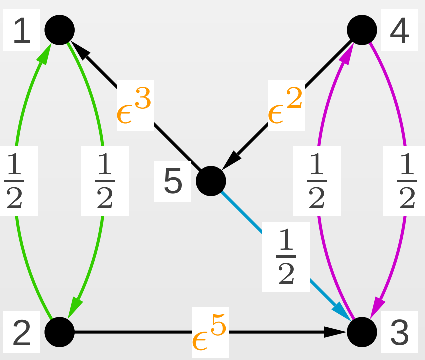
M_ϵ

$$\begin{pmatrix} \cdot & \frac{1}{2} & 0 & 0 & \epsilon^3 \\ \frac{1}{2} & \cdot & 0 & 0 & 0 \\ 0 & \epsilon^5 & \cdot & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & \cdot & 0 \\ 0 & 0 & 0 & \epsilon^2 & \cdot \end{pmatrix}$$

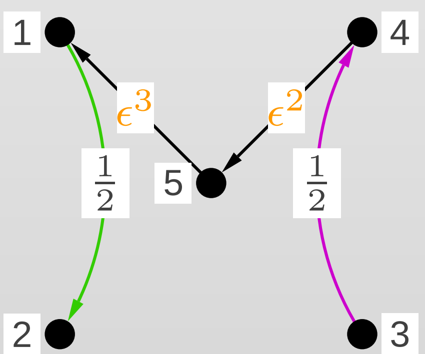
$G(M_\epsilon)$



w_{M_ϵ}

$$\begin{pmatrix} \frac{1}{4}\epsilon^5 + \frac{1}{2}\epsilon^{10} \\ \frac{1}{4}\epsilon^5 \\ \\ \\ \\ \end{pmatrix}$$

$T \subset G(M_\epsilon)$



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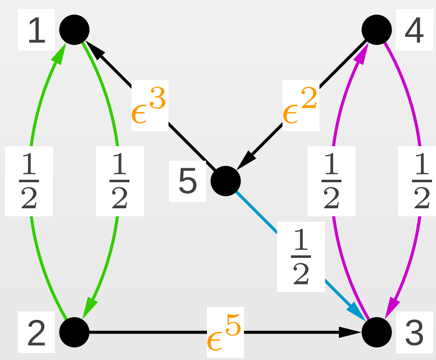
M_ϵ

$$\begin{pmatrix} \cdot & \frac{1}{2} & 0 & 0 & \epsilon^3 \\ \frac{1}{2} & \cdot & 0 & 0 & 0 \\ 0 & \epsilon^5 & \cdot & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & \cdot & 0 \\ 0 & 0 & 0 & \epsilon^2 & \cdot \end{pmatrix}$$

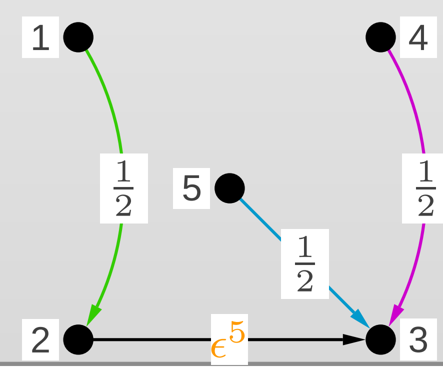
$G(M_\epsilon)$



w_{M_ϵ}

$$\begin{pmatrix} \frac{1}{4}\epsilon^5 + \frac{1}{2}\epsilon^{10} \\ \frac{1}{4}\epsilon^5 \\ \frac{1}{2} \cdot \epsilon^5 \cdot \frac{1}{2} \cdot \frac{1}{2} \\ \\ \\ \end{pmatrix}$$

$T \subset G(M_\epsilon)$



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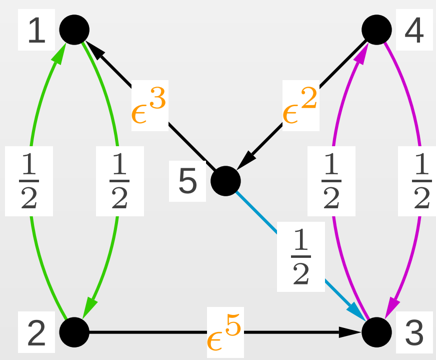
$$M_\epsilon$$

$$\begin{pmatrix} \cdot & \frac{1}{2} & 0 & 0 & \epsilon^3 \\ \frac{1}{2} & \cdot & 0 & 0 & 0 \\ 0 & \epsilon^5 & \cdot & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & \cdot & 0 \\ 0 & 0 & 0 & \epsilon^2 & \cdot \end{pmatrix}$$

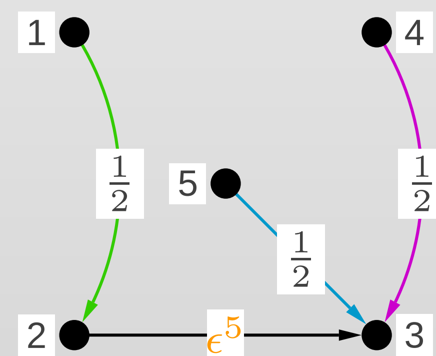
$$G\left(M_\epsilon\right)$$



$$w_{M_\epsilon}$$

$$\begin{pmatrix} \frac{1}{4}\epsilon^5 + \frac{1}{2}\epsilon^{10} \\ \frac{1}{4}\epsilon^5 \\ \frac{1}{8}\epsilon^5 \\ \\ \\ \end{pmatrix}$$

$$T \subset G\left(M_\epsilon\right)$$



Geometric Approach for PMMs

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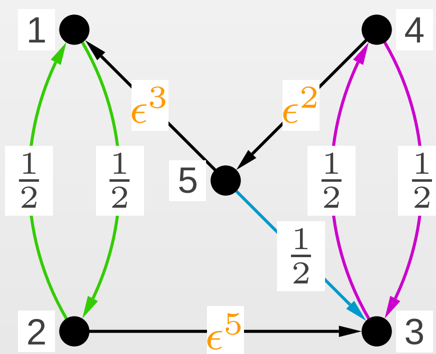
$$M_\epsilon$$

$$\begin{pmatrix} \cdot & \frac{1}{2} & 0 & 0 & \epsilon^3 \\ \frac{1}{2} & \cdot & 0 & 0 & 0 \\ 0 & \epsilon^5 & \cdot & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & \cdot & 0 \\ 0 & 0 & 0 & \epsilon^2 & \cdot \end{pmatrix}$$

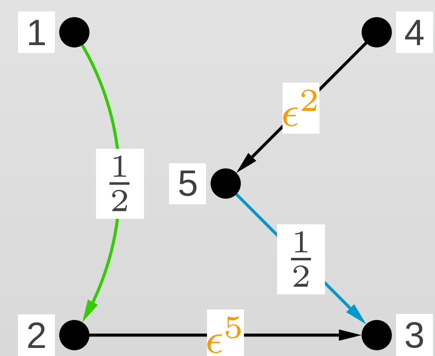
$$G(M_\epsilon)$$



$$w_{M_\epsilon}$$

$$\begin{pmatrix} \frac{1}{4}\epsilon^5 + \frac{1}{2}\epsilon^{10} \\ \frac{1}{4}\epsilon^5 \\ \frac{1}{8}\epsilon^5 + \frac{1}{2}\cdot\epsilon^5\cdot\frac{1}{2}\cdot\epsilon^2 \\ \\ \\ \end{pmatrix}$$

$$T \subset G(M_\epsilon)$$



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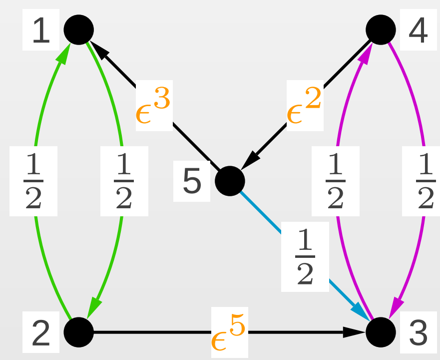
Markov Chain Tree Theorem In particular, the stationary distribution of a PMM M_ϵ is proportional to the vector w_{M_ϵ} of sums of aggregated weights of its directed spanning subtrees.

M_ϵ

$$\begin{pmatrix} \cdot & \frac{1}{2} & 0 & 0 & \epsilon^3 \\ \frac{1}{2} & \cdot & 0 & 0 & 0 \\ 0 & \epsilon^5 & \cdot & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & \cdot & 0 \\ 0 & 0 & 0 & \epsilon^2 & \cdot \end{pmatrix}$$

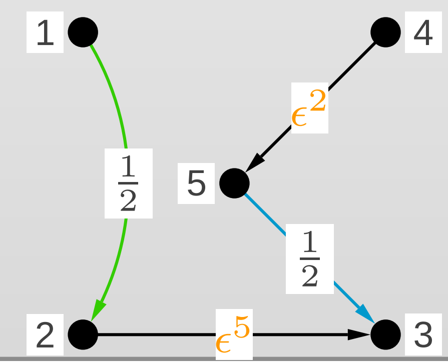
$G(M_\epsilon)$



w_{M_ϵ}

$$\begin{pmatrix} \frac{1}{4}\epsilon^5 + \frac{1}{2}\epsilon^{10} \\ \frac{1}{4}\epsilon^5 \\ \frac{1}{8}\epsilon^5 + \frac{1}{4}\epsilon^7 \\ \\ \\ \end{pmatrix}$$

$T \subset G(M_\epsilon)$



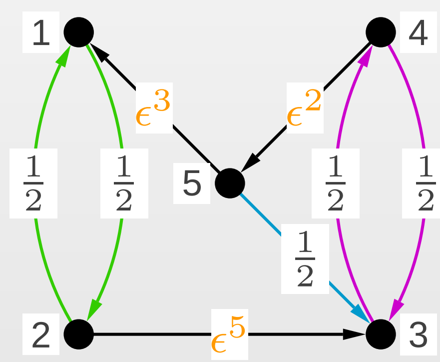
Geometric Approach for PMMs

Markov Chain Tree Theorem In particular, the stationary distribution of a PMM M_ϵ is proportional to the vector w_{M_ϵ} of sums of aggregated weights of its directed spanning subtrees.

$$M_\epsilon$$

$$\begin{pmatrix} \cdot & \frac{1}{2} & 0 & 0 & \epsilon^3 \\ \frac{1}{2} & \cdot & 0 & 0 & 0 \\ 0 & \epsilon^5 & \cdot & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & \cdot & 0 \\ 0 & 0 & 0 & \epsilon^2 & \cdot \end{pmatrix}$$

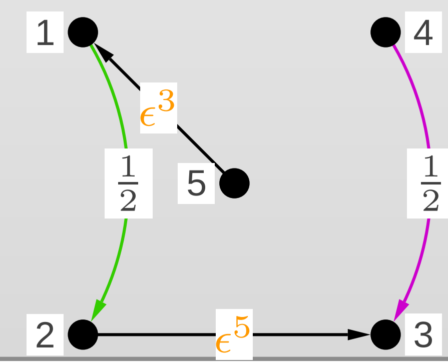
$$G\left(M_\epsilon\right)$$



$$w_{M_\epsilon}$$

$$\begin{pmatrix} \frac{1}{4}\epsilon^5 + \frac{1}{2}\epsilon^{10} \\ \frac{1}{4}\epsilon^5 \\ \frac{1}{8}\epsilon^5 + \frac{1}{4}\epsilon^7 + \frac{1}{4}\epsilon^8 \\ \\ \end{pmatrix}$$

$$T \subset G\left(M_\epsilon\right)$$



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Markov Chain Tree Theorem In particular, the stationary distribution of a PMM M_ϵ is proportional to the vector w_{M_ϵ} of sums of aggregated weights of its directed spanning subtrees.

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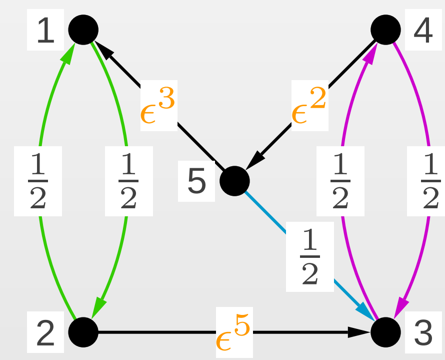
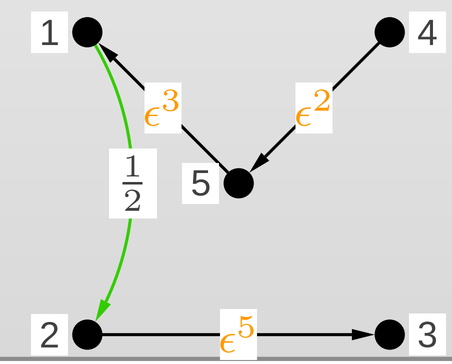
$$G\left(M_\epsilon\right)$$



$$w_{M_\epsilon}$$

$$\begin{pmatrix} \frac{1}{4}\epsilon^5 + \frac{1}{2}\epsilon^{10} \\ \frac{1}{4}\epsilon^5 \\ \frac{1}{8}\epsilon^5 + \frac{1}{4}\epsilon^7 + \frac{1}{4}\epsilon^8 + \frac{1}{2}\epsilon^{10} \\ \\ \end{pmatrix}$$

$$T \subset G\left(M_\epsilon\right)$$



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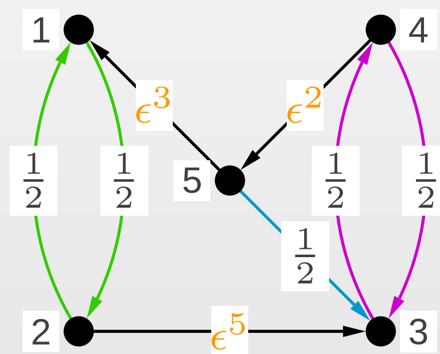
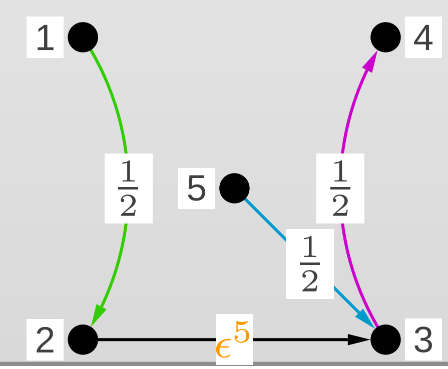
$G(M_\epsilon)$



w_{M_ϵ}

$$\begin{pmatrix} \frac{1}{4}\epsilon^5 + \frac{1}{2}\epsilon^{10} \\ \frac{1}{4}\epsilon^5 \\ \frac{1}{8}\epsilon^5 + \frac{1}{4}\epsilon^7 + \frac{1}{4}\epsilon^8 + \frac{1}{2}\epsilon^{10} \\ \frac{1}{8}\epsilon^5 \\ \end{pmatrix}$$

$T \subset G(M_\epsilon)$



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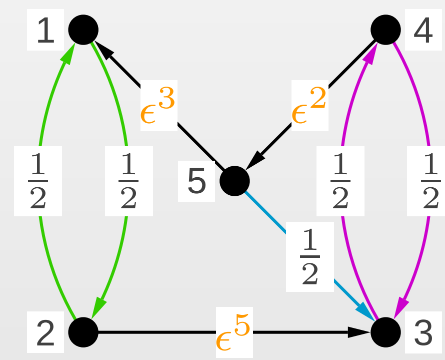
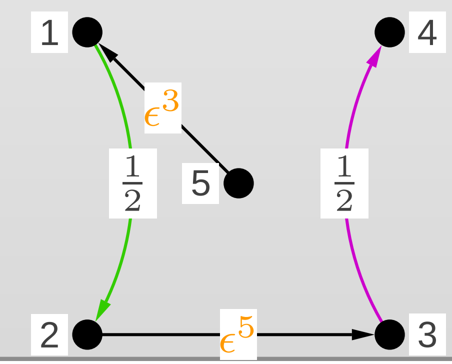
$G(M_\epsilon)$



w_{M_ϵ}

$$\begin{pmatrix} \frac{1}{4}\epsilon^5 + \frac{1}{2}\epsilon^{10} \\ \frac{1}{4}\epsilon^5 \\ \frac{1}{8}\epsilon^5 + \frac{1}{4}\epsilon^7 + \frac{1}{4}\epsilon^8 + \frac{1}{2}\epsilon^{10} \\ \frac{1}{8}\epsilon^5 + \frac{1}{4}\epsilon^8 \end{pmatrix}$$

$T \subset G(M_\epsilon)$



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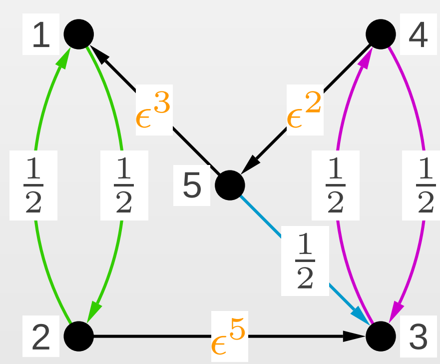
$$M_\epsilon$$

$$\begin{pmatrix} \cdot & \frac{1}{2} & 0 & 0 & \epsilon^3 \\ \frac{1}{2} & \cdot & 0 & 0 & 0 \\ 0 & \epsilon^5 & \cdot & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & \cdot & 0 \\ 0 & 0 & 0 & \epsilon^2 & \cdot \end{pmatrix}$$

$$G\left(M_\epsilon\right)$$



$$w_{M_\epsilon}$$

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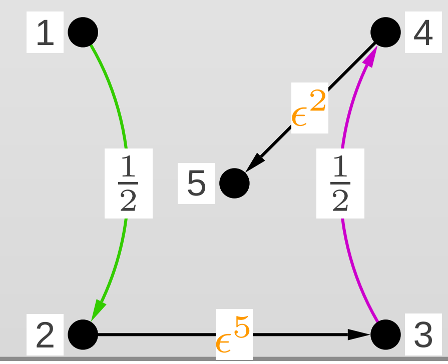

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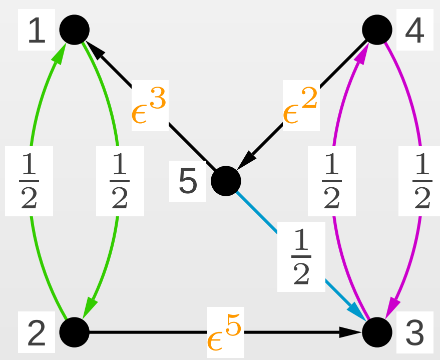
$$M_\epsilon$$

$$\begin{pmatrix} \cdot & \frac{1}{2} & 0 & 0 & \epsilon^3 \\ \frac{1}{2} & \cdot & 0 & 0 & 0 \\ 0 & \epsilon^5 & \cdot & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & \cdot & 0 \\ 0 & 0 & 0 & \epsilon^2 & \cdot \end{pmatrix}$$

$$G(M_\epsilon)$$



$$\epsilon^{-5} w_{M_\epsilon}$$

$$\begin{pmatrix} \frac{1}{4} + \frac{1}{2}\epsilon^5 \\ \frac{1}{4} \\ \frac{1}{8} + \frac{1}{4}\epsilon^2 + \frac{1}{4}\epsilon^3 + \frac{1}{2}\epsilon^5 \\ \frac{1}{8} + \frac{1}{4}\epsilon^3 \\ \frac{1}{4}\epsilon^2 \end{pmatrix}$$

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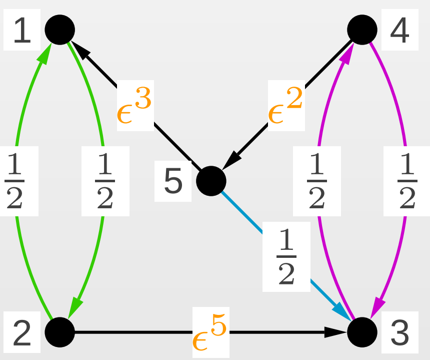
M_ϵ

$$\begin{pmatrix} \cdot & \frac{1}{2} & 0 & 0 & \epsilon^3 \\ \frac{1}{2} & \cdot & 0 & 0 & 0 \\ 0 & \epsilon^5 & \cdot & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & \cdot & 0 \\ 0 & 0 & 0 & \epsilon^2 & \cdot \end{pmatrix}$$

$G(M_\epsilon)$



$$\lim_{\epsilon \to 0} \epsilon^{-5} w_{M_\epsilon}$$

$$\begin{pmatrix} \frac{1}{4} \\ \frac{1}{4} \\ \frac{1}{8} \\ \frac{1}{8} \\ 0 \end{pmatrix}$$

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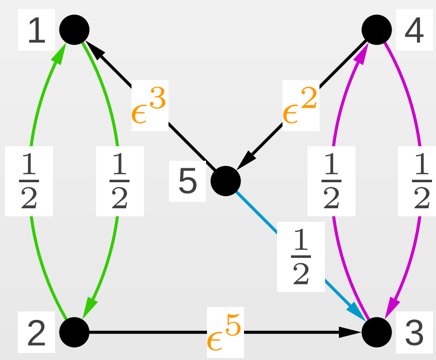
$$G(M_\epsilon)$$



$$\lim_{\epsilon \to 0} \epsilon^{-5} w_{M_\epsilon}$$

$$\begin{pmatrix} \frac{1}{4} \\ \frac{1}{4} \\ \frac{1}{8} \\ \frac{1}{8} \\ 0 \end{pmatrix}$$

$$\text{ssd}(M_\epsilon)$$

$$\begin{pmatrix} \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{6} \\ \frac{1}{6} \\ 0 \end{pmatrix}$$

Algebraic Approach for MMs

Gaussian Elimination

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ -\frac{3}{4} & \frac{1}{4} & \frac{1}{2} & 0 \\ 0 & -\frac{3}{4} & 0 & 0 \end{array}\right) \xrightarrow[\textcircled{3} + \frac{3}{4}\textcircled{2'}]{\textcircled{2} + \frac{3}{4}\textcircled{1}} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & \frac{5}{4} & \frac{3}{4} \\ 0 & 0 & \frac{15}{16} & \frac{9}{16} \end{array}\right)$$

$$\downarrow \tfrac{15}{16}\textcircled{3}$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \frac{9}{15} \end{array}\right) \xleftarrow{\textcircled{2} - \frac{5}{4}\textcircled{3}} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & \frac{5}{4} & \frac{3}{4} \\ 0 & 0 & 1 & \frac{9}{15} \end{array}\right)$$

$$\downarrow \textcircled{1} - \textcircled{2}$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \frac{9}{15} \end{array}\right) \xrightarrow{\textcircled{1} - \textcircled{3}} \left(\begin{array}{ccc|c} 1 & 0 & 0 & \frac{6}{15} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \frac{9}{15} \end{array}\right)$$

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Gaussian Elimination

Example $M_\epsilon = \begin{pmatrix} \cdot & \frac{\epsilon}{1+2\epsilon} \\ 2\epsilon & \cdot \end{pmatrix}$

$$\left(\begin{array}{cc|c} 1 & 1 & 1 \\ -2\epsilon & \frac{\epsilon}{1+2\epsilon} & 0 \end{array}\right) \xrightarrow{2\epsilon \, (1) + (2)} \left(\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & \frac{\epsilon(3+4\epsilon)}{1+2\epsilon} & 2\epsilon \end{array}\right)$$

$$\downarrow \frac{1+2\epsilon}{\epsilon(3+4\epsilon)} \, (2)$$

$$\left(\begin{array}{cc|c} 1 & 0 & \frac{1}{3+4\epsilon} \\ 0 & 1 & \frac{2+4\epsilon}{3+4\epsilon} \end{array}\right) \xleftarrow{(1) - (2)} \left(\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 1 & \frac{2+4\epsilon}{3+4\epsilon} \end{array}\right)$$

$$\operatorname{ssd}(M_\epsilon) = \lim_{\epsilon \to 0} \begin{pmatrix} \frac{1}{3+4\epsilon} \\ \frac{2+4\epsilon}{3+4\epsilon} \end{pmatrix} = \begin{pmatrix} \frac{1}{3} \\ \frac{2}{3} \end{pmatrix}$$

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Asymptotic Equivalence

Definition Two PMMs, M_ϵ and M'_ϵ, are asymptotically equal (a.e.) iff their entries are: i.e., $(M_\epsilon)_{i,j} \sim (M'_\epsilon)_{i,j}$, for all i, j.

Definition Two PMMs are stochastically equal (s.e.) iff their stationary distributions are asymptotically equal.

Theorem If two PMMs are a.e., then they are also s.e.. In particular, they have the same SSD.

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Asymptotic Equivalence

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Definition Two PMMs are stochastically equal (s.e.) iff their stationary distributions are asymptotically equal.

Theorem If two PMMs are a.e., then they are also s.e.. In particular, they have the same SSD.

Example

$$M_\epsilon = \begin{pmatrix} \cdot & \frac{\epsilon}{1+2\epsilon} \\ 2\epsilon & \cdot \end{pmatrix} \quad \text{and} \quad M'_\epsilon = \begin{pmatrix} \cdot & \epsilon \\ 2\epsilon & \cdot \end{pmatrix}$$

$$\operatorname{inv}(M_\epsilon) = \begin{pmatrix} \frac{1}{3+4\epsilon} \\ \frac{2+4\epsilon}{3+4\epsilon} \end{pmatrix} \quad \text{and} \quad \operatorname{inv}(M'_\epsilon) = \begin{pmatrix} \frac{1}{3} \\ \frac{2}{3} \end{pmatrix}$$

$$\operatorname{ssd}(M_\epsilon) = \begin{pmatrix} \frac{1}{3} \\ \frac{2}{3} \end{pmatrix} = \operatorname{ssd}(M'_\epsilon)$$

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Efficient Representation

Key Idea Represent the PMM M_ϵ by a pair of real-valued matrices, C and R, such that $M_\epsilon \sim C\epsilon^R$.

Example

$$M_\epsilon = \begin{pmatrix} \cdot & \frac{\epsilon}{1+2\epsilon} \\ 2\epsilon & \cdot \end{pmatrix} \sim \begin{pmatrix} \cdot & \epsilon \\ 2\epsilon & \cdot \end{pmatrix} = M'_\epsilon$$

$$M_\epsilon = \begin{pmatrix} \cdot & \frac{\epsilon}{1+2\epsilon} \\ 2\epsilon & \cdot \end{pmatrix} \sim C\epsilon^R \quad \text{where} \quad C = \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix} \quad \text{and} \quad R = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

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Key Idea Represent the PMM M_ϵ by a pair of real-valued matrices, C and R, such that $M_\epsilon \sim C\epsilon^R$.

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$$M_\epsilon = \begin{pmatrix} \cdot & \frac{\epsilon}{1+2\epsilon} \\ 2\epsilon & \cdot \end{pmatrix} \sim \begin{pmatrix} \cdot & \epsilon \\ 2\epsilon & \cdot \end{pmatrix} = M'_\epsilon$$

$$M_\epsilon = \begin{pmatrix} \cdot & \frac{\epsilon}{1+2\epsilon} \\ 2\epsilon & \cdot \end{pmatrix} \sim C\epsilon^R \quad \text{where} \quad C = \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix} \quad \text{and} \quad R = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Theorem Addition and multiplication are well-defined on asymptotic equivalence classes, while subtraction and division are only defined in restricted cases: $2\epsilon - 3\epsilon, \frac{\epsilon^2}{\epsilon^3} \notin \mathbb{C}^+$.

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Geometric Approach: MCTT

- Using our representation, one can compute the SSD using MCTT because $\mathbb{C}^+$ is closed under addition and multiplication.
- But this approach is combinatorial ($O(n^n)$).

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Geometric Approach: MCTT

- Using our representation, one can compute the SSD using MCTT because $\mathbb{C}^+$ is closed under addition and multiplication.
- But this approach is combinatorial ($O(n^n)$).

Algebraic Approach: GE

- Using our representation, one cannot compute the SSD using GE because $\mathbb{C}^+$ is not closed under subtraction and division.

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Geometric Approach: MCTT

- Using our representation, one can compute the SSD using MCTT because $\mathbb{C}^+$ is closed under addition and multiplication.
- But this approach is combinatorial ($O(n^n)$).

Algebraic Approach: GE

- Using our representation, one cannot compute the SSD using GE because $\mathbb{C}^+$ is not closed under subtraction and division.
- Our algorithm can be viewed as a careful implementation of GE.
  - We subtract and divide only by elements of $\mathbb{C}^+$ for which the entries of the PMM remain exponentially convergent.
  - E.g., We invert only unperturbed Markov matrices ($O(n^3)$).

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- Characterization of Stochastically Stable Distributions
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- Recovering Dynamics
- Algorithm: Step 1
- Scale
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- Algorithm: Step 2
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Recursive Algorithm

A recursive algorithm calls itself on smaller problem instances.

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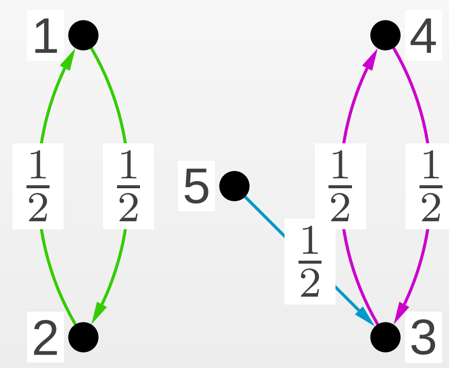
Characterization of Stationary Distributions

$$M$$

$$\begin{pmatrix} \cdot & \frac{1}{2} & 0 & 0 & 0 \\ \frac{1}{2} & \cdot & 0 & 0 & 0 \\ 0 & 0 & \cdot & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & \cdot & 0 \\ 0 & 0 & 0 & 0 & \cdot \end{pmatrix}$$

$G(M)$



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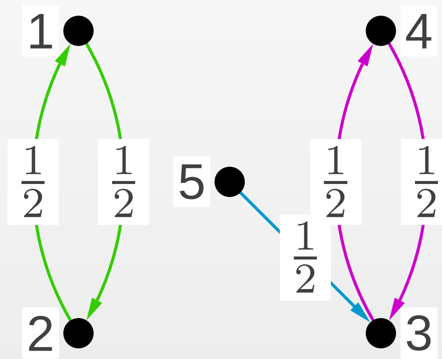
Characterization of Stationary Distributions

$$M$$

$$\begin{pmatrix} \cdot & \frac{1}{2} & 0 & 0 & 0 \\ \frac{1}{2} & \cdot & 0 & 0 & 0 \\ 0 & 0 & \cdot & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & \cdot & 0 \\ 0 & 0 & 0 & 0 & \cdot \end{pmatrix}$$

$G(M)$



Theorem For any Markov matrix M:

- M restricted to each of its closed classes has a unique stationary distribution.
- Every stationary distribution of M is a convex combination of the stationary distributions of its closed classes.
- Every stationary distribution of M puts weight 0 on transients.

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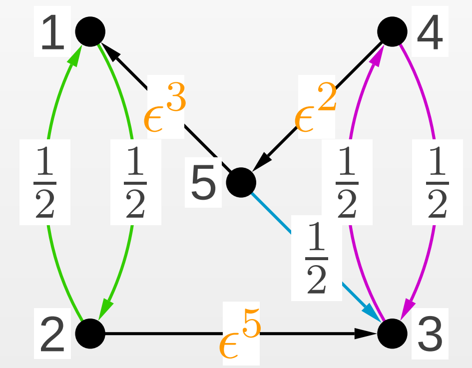
Characterization of Stochastically Stable Distributions

M_ϵ

$$\begin{pmatrix} \cdot & \frac{1}{2} & 0 & 0 & \epsilon^3 \\ \frac{1}{2} & \cdot & 0 & 0 & 0 \\ 0 & 0 & \cdot & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & \cdot & 0 \\ 0 & \epsilon^5 & 0 & \epsilon^2 & \cdot \end{pmatrix}$$

$G(M_\epsilon)$

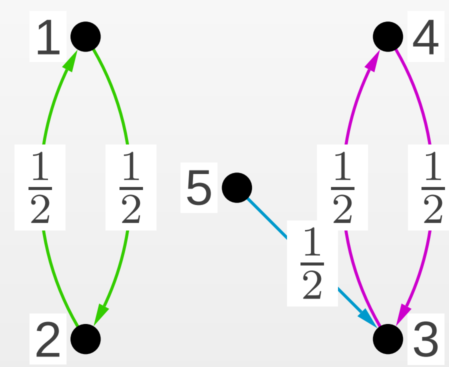


Characterization of Stochastically Stable Distributions

M_0

$$\begin{pmatrix} \cdot & \frac{1}{2} & 0 & 0 & 0 \\ \frac{1}{2} & \cdot & 0 & 0 & 0 \\ 0 & 0 & \cdot & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & \cdot & 0 \\ 0 & 0 & 0 & 0 & \cdot \end{pmatrix}$$

$G(M_0)$



Theorem For any PMM M_ϵ:

- Its SSD is one of the stationary distributions of the unperturbed matrix M_0: $M_\epsilon v_\epsilon = v_\epsilon \rightarrow M_0 v_0 = v_0$.
- Hence, the is a convex combination of the stationary distributions of the closed classes of M_0.

$$v_0 = \begin{pmatrix} \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{6} \\ \frac{1}{6} \\ 0 \end{pmatrix} = \frac{2}{3} \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 0 \\ 0 \\ 0 \end{pmatrix} + \frac{1}{3} \begin{pmatrix} 0 \\ 0 \\ \frac{1}{2} \\ \frac{1}{2} \\ 0 \end{pmatrix}$$

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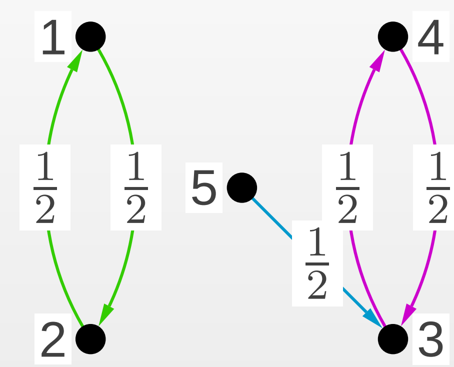
Characterization of Stochastically Stable Distributions

M_0

$$\begin{pmatrix} \cdot & \frac{1}{2} & 0 & 0 & 0 \\ \frac{1}{2} & \cdot & 0 & 0 & 0 \\ 0 & 0 & \cdot & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & \cdot & 0 \\ 0 & 0 & 0 & 0 & \cdot \end{pmatrix}$$

$G(M_0)$



Theorem For any PMM M_ϵ:

- Its SSD is one of the stationary distributions of the unperturbed matrix M_0: $M_\epsilon v_\epsilon = v_\epsilon \;\rightarrow\; M_0 v_0 = v_0$.
- Hence, the is a convex combination of the stationary distributions of the closed classes of M_0.

Corollary

If M_0 is unichain, its unique stationary distribution is the SSD of M_ϵ.

Algorithm: Base Case

A recursive algorithm calls itself on smaller problem instances.

Base Case

- If M_0 is unichain, then the SSD of M_ϵ is the unique stationary distribution of M_0.

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Key Construction #1: Reduce

Reduce

- Reduce M_ϵ to $\widehat{M}_\epsilon$ by eliminating indices of M_ϵ, so that $\widehat{M}_\epsilon$ contains only one representative of each SCC of M_0.
- Along the way, record (multiples of) the stationary distributions of the closed classes of M_0 in the columns of a matrix i_0.
- Key to the Recursion: The entries of the SSD of $\widehat{M}_\epsilon$ are the coefficients that combine the columns of i_0 into the SSD of M_ϵ.

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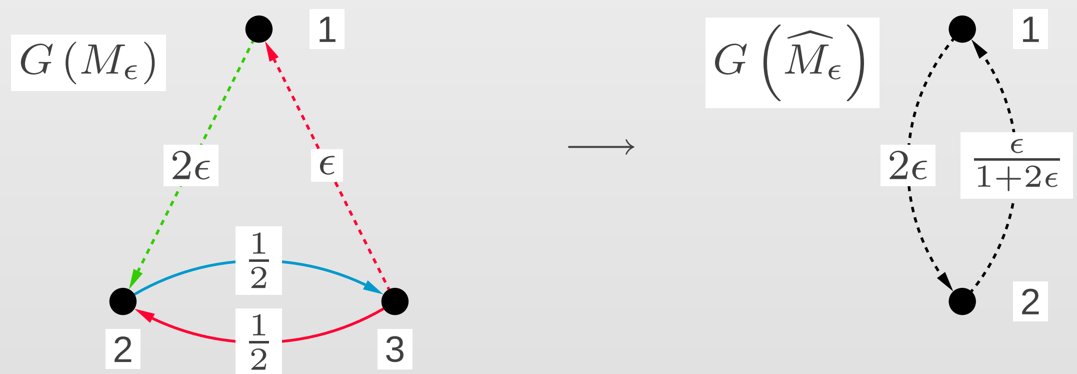
Reduce Graphically

- Reduce M_ϵ to $\widehat{M}_\epsilon$ by eliminating indices of M_ϵ so that $\widehat{M}_\epsilon$ contains only one representative of each SCC of M_0.

$$M_\epsilon = \begin{pmatrix} \cdot & 0 & \epsilon \\ 2\epsilon & \cdot & \frac{1}{2} \\ 0 & \frac{1}{2} & \cdot \end{pmatrix} \longrightarrow \widehat{M}_\epsilon = \begin{pmatrix} \cdot & \frac{\epsilon}{1+2\epsilon} \\ 2\epsilon & \cdot \end{pmatrix}$$



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Summary

Reduce Algebraically

$$\widehat{M}_\epsilon = \text{reduce}_\epsilon\,(M_\epsilon)$$

1. $\Lambda_\epsilon = M_\epsilon - I$
2. $\widehat{\Lambda}_\epsilon = p_\epsilon \Lambda_\epsilon i_\epsilon$, where

$$p_\epsilon = \begin{pmatrix} I & -\overline{N_\epsilon}\,\overline{\Lambda_\epsilon}^{-1} \end{pmatrix} P \quad \text{and} \quad i_\epsilon = P^t \begin{pmatrix} I \\ -\overline{\Lambda_\epsilon}^{-1}\widetilde{N_\epsilon} \end{pmatrix}$$

3. $\widehat{M}_\epsilon = \widehat{\Lambda}_\epsilon + I$

Here, P is a permutation matrix taking a subset of indices $s \subset \{1, \ldots, n\}$ of Λ_ϵ to the last $|s|$ indices of Λ_ϵ: $P\Lambda_\epsilon P^t \;=\; \begin{pmatrix} \widetilde{\Lambda_\epsilon} & \overline{N_\epsilon} \\ \widetilde{N_\epsilon} & \overline{\Lambda_\epsilon} \end{pmatrix}$.

Reduce Algebraically (cont'd)

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Example $s = \{3\}$

$$\Lambda_\epsilon = \begin{pmatrix} \cdot & 0 & \epsilon \\ 2\epsilon & \cdot & \frac{1}{2} \\ 0 & \frac{1}{2} & \cdot \end{pmatrix} \longrightarrow \widehat{\Lambda_\epsilon} = \begin{pmatrix} \cdot & \frac{\epsilon}{1+2\epsilon} \\ 2\epsilon & \cdot \end{pmatrix}$$

$$p_\epsilon = \left(I \quad -\overline{N_\epsilon}\,\overline{\Lambda_\epsilon}^{-1} \right) P = \begin{pmatrix} 1 & 0 & \frac{2\epsilon}{1+2\epsilon} \\ 0 & 1 & \frac{1}{1+2\epsilon} \end{pmatrix}$$

$$i_\epsilon = P^t \begin{pmatrix} I \\ -\overline{\Lambda_\epsilon}^{-1} \widetilde{N_\epsilon} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & \frac{1}{1+2\epsilon} \end{pmatrix}$$

$$\widehat{\Lambda_\epsilon} = p_\epsilon \Lambda_\epsilon i_\epsilon$$

$$= \begin{pmatrix} 1 & 0 & \frac{2\epsilon}{1+2\epsilon} \\ 0 & 1 & \frac{1}{1+2\epsilon} \end{pmatrix} \begin{pmatrix} \cdot & 0 & \epsilon \\ 2\epsilon & \cdot & \frac{1}{2} \\ 0 & \frac{1}{2} & \cdot \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & \frac{1}{1+2\epsilon} \end{pmatrix}$$

$$= \begin{pmatrix} -2\epsilon & \frac{\epsilon}{1+2\epsilon} & 0 \\ 2\epsilon & -\frac{\epsilon}{1+2\epsilon} & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & \frac{1}{1+2\epsilon} \end{pmatrix} = \begin{pmatrix} \cdot & \frac{\epsilon}{1+2\epsilon} \\ 2\epsilon & \cdot \end{pmatrix}$$

Recovering Dynamics

Theorem When reducing one of M_0's closed classes, i_0 records (a multiple of) the stationary distributions of that class.

Example

$$\lim_{\epsilon \longrightarrow 0} i_\epsilon = \lim_{\epsilon \longrightarrow 0} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & \frac{1}{1+2\epsilon} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{pmatrix} = i_0$$

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Recovering Dynamics

Theorem When reducing one of M_0's closed classes, i_0 records (a multiple of) the stationary distributions of that class.

Example

$$\lim_{\epsilon \longrightarrow 0} i_\epsilon = \lim_{\epsilon \longrightarrow 0} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & \frac{1}{1+2\epsilon} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{pmatrix} = i_0$$

Theorem Up to normalization, i_0 maps $\mathrm{ssd}\left(\widehat{M}_\epsilon\right)$ to $\mathrm{ssd}\left(M_\epsilon\right)$.

Example

$$\mathrm{ssd}\left(\widehat{M}_\epsilon\right) \propto \begin{pmatrix} 1 \\ 2 \end{pmatrix} \xrightarrow{\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{pmatrix}} \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} \propto \mathrm{ssd}\left(M_\epsilon\right)$$

Recovering Dynamics (cont'd)

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An Example

Summary

Observation By definition, $\widehat{\Lambda}_\epsilon = \widetilde{\Lambda}_\epsilon - \overline{N}_\epsilon \overline{\Lambda}_\epsilon^{-1} \widetilde{N}_\epsilon$, but $\widehat{\Lambda}'_\epsilon \equiv \widetilde{\Lambda}_\epsilon - \overline{N}_\epsilon \overline{\Lambda}_0^{-1} \widetilde{N}_\epsilon$ yields essentially the same result.

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Recovering Dynamics (cont'd)

Observation By definition, $\widehat{\Lambda}_\epsilon = \widetilde{\Lambda}_\epsilon - \overline{N}_\epsilon \overline{\Lambda}_\epsilon^{-1} \widetilde{N}_\epsilon$, but $\widehat{\Lambda}'_\epsilon \equiv \widetilde{\Lambda}_\epsilon - \overline{N}_\epsilon \overline{\Lambda}_0^{-1} \widetilde{N}_\epsilon$ yields essentially the same result.

Theorem $\widehat{M}_\epsilon$ and $\widehat{M}'_\epsilon$ are stochastically equal.

Check

$$\widehat{M}_\epsilon = \begin{pmatrix} \cdot & \frac{\epsilon}{1+2\epsilon} \\ 2\epsilon & \cdot \end{pmatrix} \quad \text{and} \quad \widehat{M}'_\epsilon = \begin{pmatrix} \cdot & \epsilon \\ 2\epsilon & \cdot \end{pmatrix}$$

$$\operatorname{inv}\left(\widehat{M}_\epsilon\right) = \begin{pmatrix} \frac{1}{3+4\epsilon} \\ \frac{2+4\epsilon}{3+4\epsilon} \end{pmatrix} \quad \text{and} \quad \operatorname{inv}\left(\widehat{M}'_\epsilon\right) = \begin{pmatrix} \frac{1}{3} \\ \frac{2}{3} \end{pmatrix}$$

$$\operatorname{ssd}(M_\epsilon) = \begin{pmatrix} \frac{1}{3} \\ \frac{2}{3} \end{pmatrix} = \operatorname{ssd}(M'_\epsilon)$$

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Recovering Dynamics (cont'd)

Observation By definition, $\widehat{\Lambda}_\epsilon = \widetilde{\Lambda}_\epsilon - \overline{N}_\epsilon \, \overline{\Lambda}_\epsilon^{-1} \widetilde{N}_\epsilon$, but $\widehat{\Lambda}'_\epsilon \equiv \widetilde{\Lambda}_\epsilon - \overline{N}_\epsilon \, \overline{\Lambda}_0^{-1} \widetilde{N}_\epsilon$ yields essentially the same result.

Theorem $\widehat{M}_\epsilon$ and $\widehat{M}'_\epsilon$ are stochastically equal.

Check

$$\widehat{M}_\epsilon = \begin{pmatrix} \cdot & \frac{\epsilon}{1+2\epsilon} \\ 2\epsilon & \cdot \end{pmatrix} \quad \text{and} \quad \widehat{M}'_\epsilon = \begin{pmatrix} \cdot & \epsilon \\ 2\epsilon & \cdot \end{pmatrix}$$

$$\operatorname{inv}\left(\widehat{M}_\epsilon\right) = \begin{pmatrix} \frac{1}{3+4\epsilon} \\ \frac{2+4\epsilon}{3+4\epsilon} \end{pmatrix} \quad \text{and} \quad \operatorname{inv}\left(\widehat{M}'_\epsilon\right) = \begin{pmatrix} \frac{1}{3} \\ \frac{2}{3} \end{pmatrix}$$

$$\operatorname{ssd}\left(M_\epsilon\right) = \begin{pmatrix} \frac{1}{3} \\ \frac{2}{3} \end{pmatrix} = \operatorname{ssd}\left(M'_\epsilon\right)$$

Corollary Up to normalization, i_0 maps $\operatorname{ssd}\left(\widehat{M}'_\epsilon\right)$ to $\operatorname{ssd}\left(M_\epsilon\right)$.

Algorithm: Step 1

A recursive algorithm calls itself on smaller problem instances.

Base Case

- If M_0 is unichain, then the SSD of M_ϵ is the unique stationary distribution of M_0.

Step

- Reduce each of M_0's SCCs to one representative, recording the corresponding transformations i_0 to recover the SSD.

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Problem

What is the SSD of $\widehat{M}'_\epsilon = \begin{pmatrix} \cdot & \epsilon \\ 2\epsilon & \cdot \end{pmatrix}$? (NB: $\widehat{M}'_0$ is not unichain.)

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Key Construction #2: Scale

Problem

What is the SSD of $\widehat{M}'_\epsilon = \begin{pmatrix} \cdot & \epsilon \\ 2\epsilon & \cdot \end{pmatrix}$? (NB: $\widehat{M}'_0$ is not unichain.)

Scale

- Introduce at least one edge into $G(M_0)$ exiting a closed class. The result is one fewer closed class or one additional SCC.
- Record the corresponding transformation i_0 to recover the SSD.
- Two types of scaling: uniform and non-uniform.

Uniform Scale

Example “Divide” Λ_ϵ by 2ϵ.

$$M_\epsilon = \begin{pmatrix} \cdot & \epsilon \\ 2\epsilon & \cdot \end{pmatrix} \longrightarrow M'_\epsilon = \begin{pmatrix} \cdot & \frac{1}{2} \\ 1 & \cdot \end{pmatrix}$$

Uniform Scale

Example "Divide" Λ_ϵ by 2ϵ.

$$M_\epsilon = \begin{pmatrix} \cdot & \epsilon \\ 2\epsilon & \cdot \end{pmatrix} \longrightarrow M'_\epsilon = \begin{pmatrix} \cdot & \frac{1}{2} \\ 1 & \cdot \end{pmatrix}$$

Observation

$G(M_0)$ contains two singleton SCCs.

$G(M'_0)$ contains a non-singleton SCC.

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Uniform Scale

Example "Divide" Λ_ϵ by 2ϵ.

$$M_\epsilon = \begin{pmatrix} \cdot & \epsilon \\ 2\epsilon & \cdot \end{pmatrix} \longrightarrow M'_\epsilon = \begin{pmatrix} \cdot & \frac{1}{2} \\ 1 & \cdot \end{pmatrix}$$

Observation

$G(M_0)$ contains two singleton SCCs.

$G(M'_0)$ contains a non-singleton SCC.

Theorem $\mathrm{inv}(M_\epsilon) = \mathrm{inv}(M'_\epsilon)$.

Corollary $\mathrm{ssd}(M_\epsilon) = \mathrm{ssd}(M'_\epsilon)$.

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Scale Algebraically

$$M' = \text{scale}\,(M_\epsilon, D_\epsilon)$$

1. $\Lambda_\epsilon = M_\epsilon - I$
2. $\Lambda'_\epsilon = \Lambda_\epsilon D_\epsilon$
3. $M'_\epsilon = \Lambda'_\epsilon + I$

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Scale Algebraically

$M' = \text{scale}\,(M_\epsilon, D_\epsilon)$

1. $\Lambda_\epsilon = M_\epsilon - I$
2. $\Lambda'_\epsilon = \Lambda_\epsilon D_\epsilon$
3. $M'_\epsilon = \Lambda'_\epsilon + I$

Example If $M_\epsilon = \begin{pmatrix} \cdot & \epsilon \\ 2\epsilon & \cdot \end{pmatrix}$ and $D_\epsilon = \begin{pmatrix} \frac{1}{2\epsilon} & 0 \\ 0 & \frac{1}{2\epsilon} \end{pmatrix}$,

then $M'_\epsilon = \text{scale}\,(M_\epsilon, D_\epsilon) = \begin{pmatrix} \cdot & \frac{1}{2} \\ 1 & \cdot \end{pmatrix}$.

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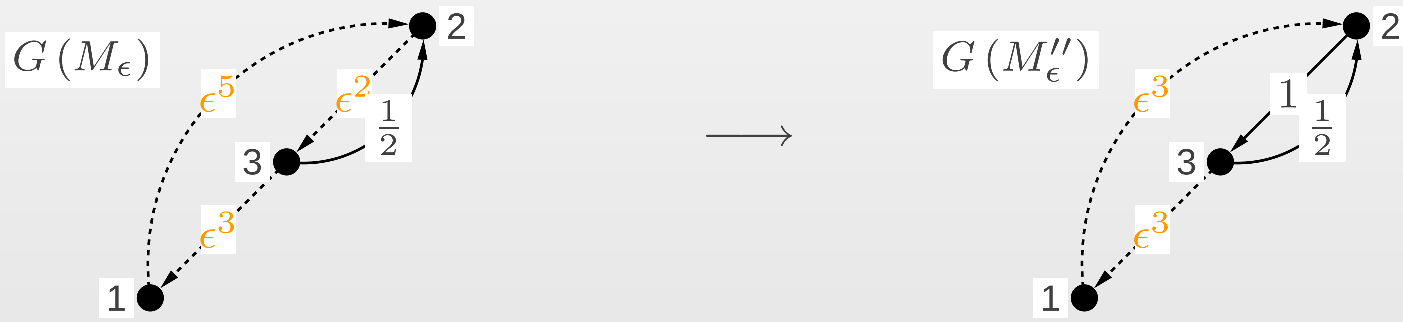
Non-uniform Scale

Example “Divide” only the first two columns of Λ_ϵ by ϵ^2.

$$M_\epsilon = \begin{pmatrix} \cdot & 0 & \epsilon^3 \\ \epsilon^5 & \cdot & \frac{1}{2} \\ 0 & \epsilon^2 & \cdot \end{pmatrix} \longrightarrow M''_\epsilon = \begin{pmatrix} \cdot & 0 & \epsilon^3 \\ \epsilon^3 & \cdot & \frac{1}{2} \\ 0 & 1 & \cdot \end{pmatrix}$$



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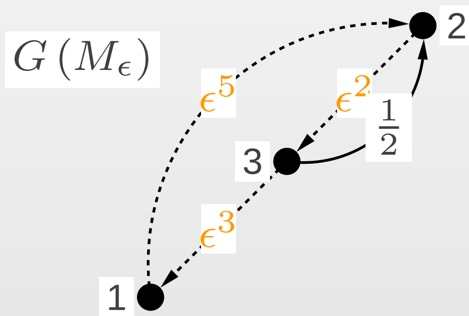
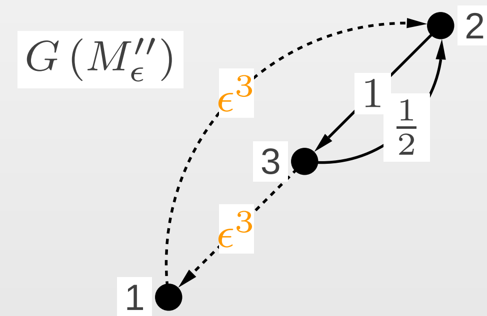
Non-uniform Scale

Example “Divide” only the first two columns of Λ_ϵ by ϵ^2.

$$M_\epsilon = \begin{pmatrix} \cdot & 0 & \epsilon^3 \\ \epsilon^5 & \cdot & \frac{1}{2} \\ 0 & \epsilon^2 & \cdot \end{pmatrix} \longrightarrow M''_\epsilon = \begin{pmatrix} \cdot & 0 & \epsilon^3 \\ \epsilon^3 & \cdot & \frac{1}{2} \\ 0 & 1 & \cdot \end{pmatrix}$$



$\longrightarrow$



$$D_\epsilon = \begin{pmatrix} \frac{1}{\epsilon^2} & 0 & 0 \\ 0 & \frac{1}{\epsilon^2} & 0 \\ 0 & 0 & \frac{1}{\epsilon^2} \end{pmatrix} \quad \text{and} \quad i_\epsilon = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \epsilon^2 \end{pmatrix}$$

$$M'_\epsilon = \text{scale}\left(M_\epsilon, D_\epsilon\right)$$

$$M''_\epsilon = \text{scale}\left(M'_\epsilon, i_\epsilon\right)$$

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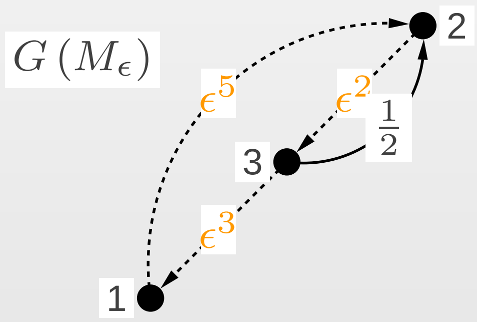
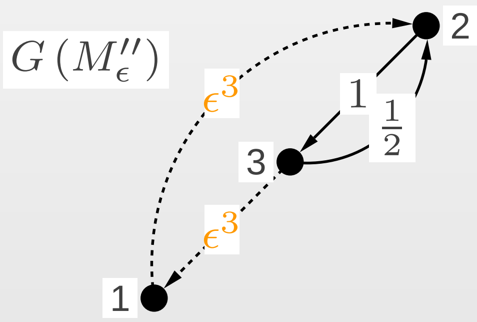
Non-uniform Scale

Example "Divide" only the first two columns of Λ_ϵ by ϵ^2.

$$M_\epsilon = \begin{pmatrix} \cdot & 0 & \epsilon^3 \\ \epsilon^5 & \cdot & \frac{1}{2} \\ 0 & \epsilon^2 & \cdot \end{pmatrix} \longrightarrow M''_\epsilon = \begin{pmatrix} \cdot & 0 & \epsilon^3 \\ \epsilon^3 & \cdot & \frac{1}{2} \\ 0 & 1 & \cdot \end{pmatrix}$$



$\longrightarrow$



Observation

$G(M_0)$ contains three singleton SCCs.

$G(M''_0)$ contains a non-singleton SCC.

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● Characterization of Stochastically Stable Distributions
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● Reduce Algebraically
● Recovering Dynamics
● Algorithm: Step 1
● Scale
● Uniform Scale
● Scale Algebraically
● Non-uniform Scale
● Algorithm: Step 2
● Pseudocode

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Non-uniform Scale

Example "Divide" only the first two columns of Λ_ϵ by ϵ^2.

$$M_\epsilon = \begin{pmatrix} \cdot & 0 & \epsilon^3 \\ \epsilon^5 & \cdot & \frac{1}{2} \\ 0 & \epsilon^2 & \cdot \end{pmatrix} \longrightarrow M''_\epsilon = \begin{pmatrix} \cdot & 0 & \epsilon^3 \\ \epsilon^3 & \cdot & \frac{1}{2} \\ 0 & 1 & \cdot \end{pmatrix}$$

Observation

$G(M_0)$ contains three singleton SCCs.

$G(M''_0)$ contains a non-singleton SCC.

Theorem Up to normalization, i_ϵ maps $\mathrm{inv}(M''_\epsilon)$ to $\mathrm{inv}(M_\epsilon)$.

Corollary Up to normalization, i_0 maps $\mathrm{ssd}(M''_\epsilon)$ to $\mathrm{ssd}(M_\epsilon)$.

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Algorithm: Step 2

A recursive algorithm calls itself on smaller problem instances.

Base Case

- If M_0 is unichain, then the SSD of M_ϵ is the unique stable distribution of M_0.

Step

- Reduce each of M_0's SCCs to one representative, recording the corresponding transformations i_0 to recover the SSD.
- Once all M_0's SCCs are singletons, scale to introduce at least one edge into $G\left(M_0\right)$ exiting a closed class, recording the corresponding transformation i_0 to recover the SSD.

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Pseudocode

function SSD (M_ϵ)

1. Calculate the communicating classes C of M_0,
   marking each as closed, transient, and/or singletons
2. If M_0 has only 1 closed class
3. **return**$(\text{inv}(M_0))$
4. If M_0 has a non-singleton SCC
5. $\left(\widehat{\Lambda}_\epsilon, i_0\right) = \text{collapse}_0(\Lambda_\epsilon, C)$
6. **return**$\left(i_0^*\left(\text{SSD}\left(\widehat{M}_\epsilon\right)\right)\right)$
7. Else (if all M_0's SCCs are singletons)
8. $(\Lambda'_\epsilon, i_0) = \text{nonUniformScale}(\Lambda_\epsilon, C)$
9. $\Lambda''_\epsilon = \text{uniformScale}(\Lambda'_\epsilon, C)$
10. **return**$(i_0^*(\text{SSD}(M''_\epsilon)))$

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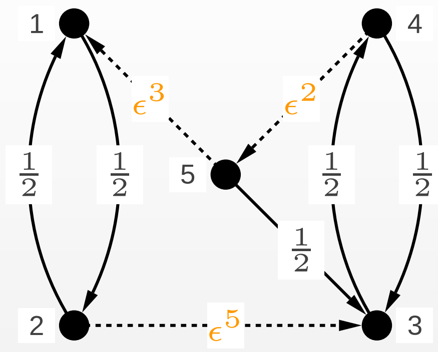
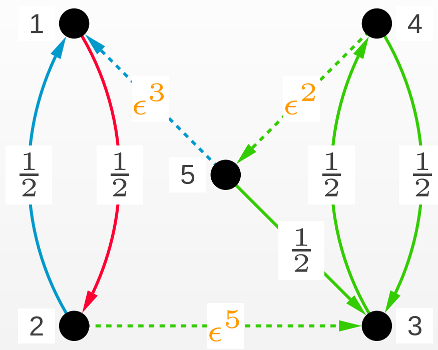


$$\begin{pmatrix} \cdot & \frac{1}{2} & 0 & 0 & \epsilon^3 \\ \frac{1}{2} & \cdot & 0 & 0 & 0 \\ 0 & \epsilon^5 & \cdot & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & \cdot & 0 \\ 0 & 0 & 0 & \epsilon^2 & \cdot \end{pmatrix}$$

An Example



$$\begin{pmatrix} -\frac{1}{2} & \frac{1}{2} & 0 & 0 & \epsilon^3 \\ \frac{1}{2} & \cdot & 0 & 0 & 0 \\ 0 & \epsilon^5 & \cdot & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & \cdot & 0 \\ 0 & 0 & 0 & \epsilon^2 & \cdot \end{pmatrix}$$

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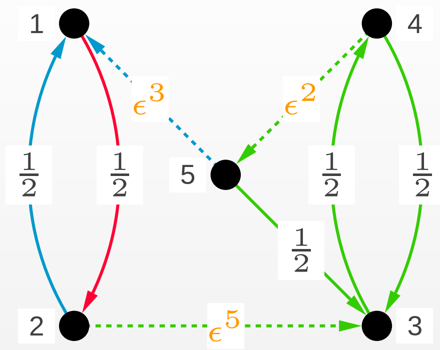
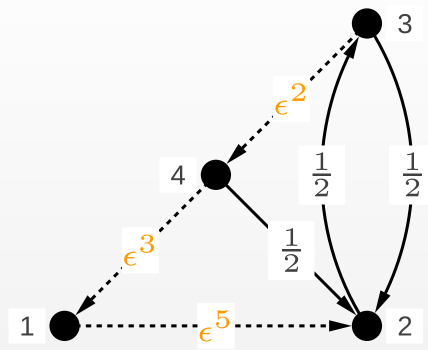


$$\begin{pmatrix} -\frac{1}{2} & \frac{1}{2} & 0 & 0 & \epsilon^3 \\ \frac{1}{2} & \cdot & 0 & 0 & 0 \\ 0 & \epsilon^5 & \cdot & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & \cdot & 0 \\ 0 & 0 & 0 & \epsilon^2 & \cdot \end{pmatrix}$$

reduce to $\begin{pmatrix} \cdot & 0 & 0 & 0 \\ \epsilon^5 & \cdot & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & \cdot & 0 \\ 0 & 0 & \epsilon^2 & \cdot \end{pmatrix} - \begin{pmatrix} \frac{1}{2} \\ 0 \\ 0 \\ 0 \end{pmatrix} \left(-\frac{1}{2} \right)^{-1} \begin{pmatrix} \frac{1}{2} & 0 & 0 & \epsilon^3 \end{pmatrix}$

An Example



$$\begin{pmatrix} -\frac{1}{2} & \frac{1}{2} & 0 & 0 & \epsilon^3 \\ \frac{1}{2} & \cdot & 0 & 0 & 0 \\ 0 & \epsilon^5 & \cdot & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & \cdot & 0 \\ 0 & 0 & 0 & \epsilon^2 & \cdot \end{pmatrix} \underset{i_1}{\overset{\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}}{\longrightarrow}} \begin{pmatrix} \cdot & 0 & 0 & \epsilon^3 \\ \epsilon^5 & \cdot & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & \cdot & 0 \\ 0 & 0 & \epsilon^2 & \cdot \end{pmatrix}$$

reduce to $$\begin{pmatrix} \cdot & 0 & 0 & 0 \\ \epsilon^5 & \cdot & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & \cdot & 0 \\ 0 & 0 & \epsilon^2 & \cdot \end{pmatrix} - \begin{pmatrix} \frac{1}{2} \\ 0 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} -\frac{1}{2} \end{pmatrix}^{-1} \begin{pmatrix} \frac{1}{2} & 0 & 0 & \epsilon^3 \end{pmatrix}$$

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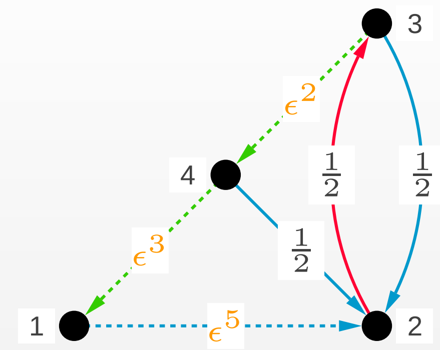
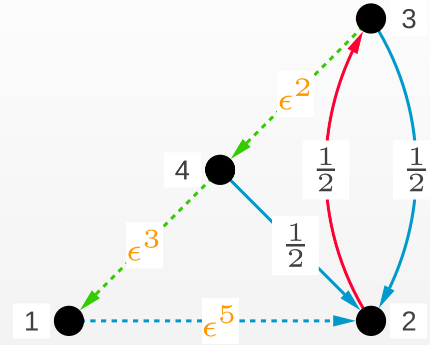


$$\begin{pmatrix} -\frac{1}{2} & \frac{1}{2} & 0 & 0 & \epsilon^3 \\ \frac{1}{2} & \cdot & 0 & 0 & 0 \\ 0 & \epsilon^5 & \cdot & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & \cdot & 0 \\ 0 & 0 & 0 & \epsilon^2 & \cdot \end{pmatrix} \underset{i_1}{\overset{\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}}{\longrightarrow}} \begin{pmatrix} \cdot & 0 & 0 & \epsilon^3 \\ \epsilon^5 & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & \cdot & 0 \\ 0 & 0 & \epsilon^2 & \cdot \end{pmatrix}$$

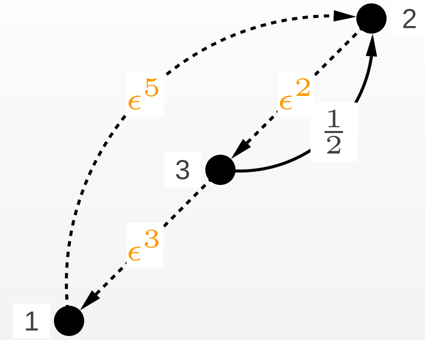
An Example



$$\begin{pmatrix} -\frac{1}{2} & \frac{1}{2} & 0 & 0 & \epsilon^3 \\ \frac{1}{2} & \cdot & 0 & 0 & 0 \\ 0 & \epsilon^5 & \cdot & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & \cdot & 0 \\ 0 & 0 & 0 & \epsilon^2 & \cdot \end{pmatrix} \underset{\overrightarrow{i_1}}{\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}} \begin{pmatrix} \cdot & 0 & 0 & \epsilon^3 \\ \epsilon^5 & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & \cdot & 0 \\ 0 & 0 & \epsilon^2 & \cdot \end{pmatrix}$$

reduce to $\begin{pmatrix} \cdot & 0 & \epsilon^3 \\ 0 & \cdot & 0 \\ 0 & \epsilon^2 & \cdot \end{pmatrix} - \begin{pmatrix} 0 \\ \frac{1}{2} \\ 0 \end{pmatrix} \begin{pmatrix} -\frac{1}{2} \end{pmatrix}^{-1} \begin{pmatrix} \epsilon^5 & \frac{1}{2} & \frac{1}{2} \end{pmatrix}$

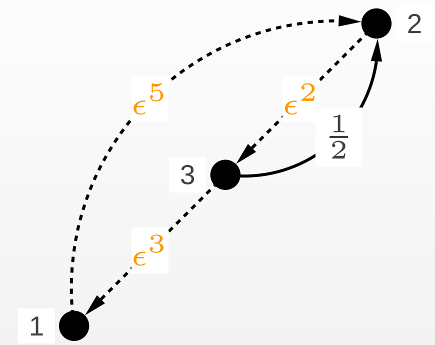
An Example



$$\begin{pmatrix} -\frac{1}{2} & \frac{1}{2} & 0 & 0 & \epsilon^3 \\ \frac{1}{2} & \cdot & 0 & 0 & 0 \\ 0 & \epsilon^5 & \cdot & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & \cdot & 0 \\ 0 & 0 & 0 & \epsilon^2 & \cdot \end{pmatrix} \underset{i_1}{\overset{\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}}{\longrightarrow}} \begin{pmatrix} \cdot & 0 & 0 & \epsilon^3 \\ \epsilon^5 & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & \cdot & 0 \\ 0 & 0 & \epsilon^2 & \cdot \end{pmatrix} \underset{i_2}{\overset{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}{\longrightarrow}} \begin{pmatrix} \cdot & 0 & \epsilon^3 \\ \epsilon^5 & \cdot & \frac{1}{2} \\ 0 & \epsilon^2 & \cdot \end{pmatrix}$$

reduce to $\begin{pmatrix} \cdot & 0 & \epsilon^3 \\ 0 & \cdot & 0 \\ 0 & \epsilon^2 & \cdot \end{pmatrix} - \begin{pmatrix} 0 \\ \frac{1}{2} \\ 0 \end{pmatrix} \begin{pmatrix} -\frac{1}{2} \end{pmatrix}^{-1} \begin{pmatrix} \epsilon^5 & \frac{1}{2} & \frac{1}{2} \end{pmatrix}$

An Example



$$\begin{pmatrix} -\frac{1}{2} & \frac{1}{2} & 0 & 0 & \epsilon^3 \\ \frac{1}{2} & \cdot & 0 & 0 & 0 \\ 0 & \epsilon^5 & \cdot & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & \cdot & 0 \\ 0 & 0 & 0 & \epsilon^2 & \cdot \end{pmatrix} \underset{i_1}{\overset{\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}}{\longrightarrow}} \begin{pmatrix} \cdot & 0 & 0 & \epsilon^3 \\ \epsilon^5 & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & \cdot & 0 \\ 0 & 0 & \epsilon^2 & \cdot \end{pmatrix} \underset{i_2}{\overset{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}{\longrightarrow}} \begin{pmatrix} \cdot & 0 & \epsilon^3 \\ \epsilon^5 & \cdot & \frac{1}{2} \\ 0 & \epsilon^2 & \cdot \end{pmatrix}$$

non-uniform scale by $i_\epsilon = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \epsilon^2 \end{pmatrix}$

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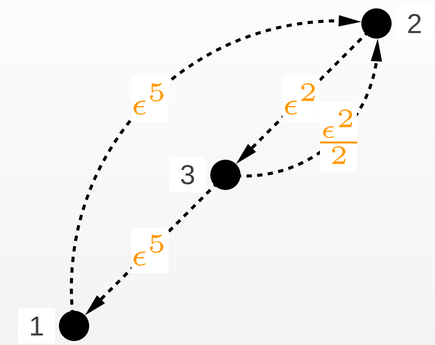

$$\begin{pmatrix} -\frac{1}{2} & \frac{1}{2} & 0 & 0 & \epsilon^3 \\ \frac{1}{2} & \cdot & 0 & 0 & 0 \\ 0 & \epsilon^5 & \cdot & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & \cdot & 0 \\ 0 & 0 & 0 & \epsilon^2 & \cdot \end{pmatrix} \underset{i_1}{\xrightarrow{\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}}} \begin{pmatrix} \cdot & 0 & 0 & \epsilon^3 \\ \epsilon^5 & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & \cdot & 0 \\ 0 & 0 & \epsilon^2 & \cdot \end{pmatrix} \underset{i_2}{\xrightarrow{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}} \begin{pmatrix} \cdot & 0 & \epsilon^3 \\ \epsilon^5 & \cdot & \frac{1}{2} \\ 0 & \epsilon^2 & \cdot \end{pmatrix}$$

$$\underset{i_3}{\xrightarrow{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}}} \begin{pmatrix} \cdot & 0 & \epsilon^5 \\ \epsilon^5 & \cdot & \frac{\epsilon^2}{2} \\ 0 & \epsilon^2 & \cdot \end{pmatrix}$$

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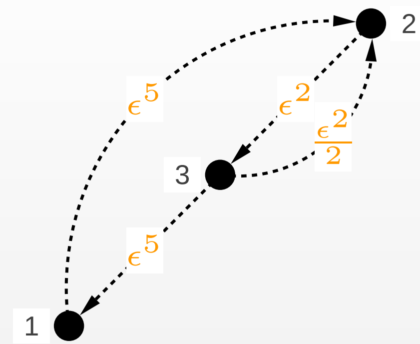

$$\begin{pmatrix} -\frac{1}{2} & \frac{1}{2} & 0 & 0 & \epsilon^3 \\ \frac{1}{2} & \cdot & 0 & 0 & 0 \\ 0 & \epsilon^5 & \cdot & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & \cdot & 0 \\ 0 & 0 & 0 & \epsilon^2 & \cdot \end{pmatrix} \underset{i_1}{\overset{\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}}{\longrightarrow}} \begin{pmatrix} \cdot & 0 & 0 & \epsilon^3 \\ \epsilon^5 & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & \cdot & 0 \\ 0 & 0 & \epsilon^2 & \cdot \end{pmatrix} \underset{i_2}{\overset{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}{\longrightarrow}} \begin{pmatrix} \cdot & 0 & \epsilon^3 \\ \epsilon^5 & \cdot & \frac{1}{2} \\ 0 & \epsilon^2 & \cdot \end{pmatrix}$$

$$\underset{i_3}{\overset{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}}{\longrightarrow}} \begin{pmatrix} \cdot & 0 & \epsilon^5 \\ \epsilon^5 & \cdot & \frac{\epsilon^2}{2} \\ 0 & \epsilon^2 & \cdot \end{pmatrix}$$

uniform scale by $\frac{1}{\epsilon^2} I$

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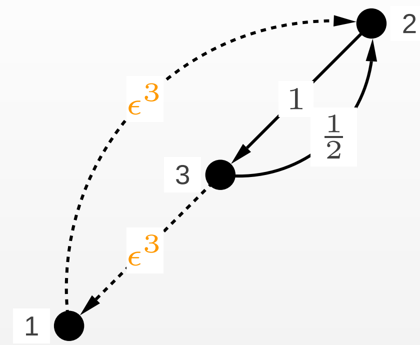

$$\begin{pmatrix} -\frac{1}{2} & \frac{1}{2} & 0 & 0 & \epsilon^3 \\ \frac{1}{2} & \cdot & 0 & 0 & 0 \\ 0 & \epsilon^5 & \cdot & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & \cdot & 0 \\ 0 & 0 & 0 & \epsilon^2 & \cdot \end{pmatrix} \underset{i_1}{\overset{\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}}{\longrightarrow}} \begin{pmatrix} \cdot & 0 & 0 & \epsilon^3 \\ \epsilon^5 & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & \cdot & 0 \\ 0 & 0 & \epsilon^2 & \cdot \end{pmatrix} \underset{i_2}{\overset{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}{\longrightarrow}} \begin{pmatrix} \cdot & 0 & \epsilon^3 \\ \epsilon^5 & \cdot & \frac{1}{2} \\ 0 & \epsilon^2 & \cdot \end{pmatrix}$$

$$\underset{i_3}{\overset{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}}{\longrightarrow}} \begin{pmatrix} \cdot & 0 & \epsilon^5 \\ \epsilon^5 & \cdot & \frac{\epsilon^2}{2} \\ 0 & \epsilon^2 & \cdot \end{pmatrix} \longrightarrow \begin{pmatrix} \cdot & 0 & \epsilon^3 \\ \epsilon^3 & \cdot & \frac{1}{2} \\ 0 & 1 & \cdot \end{pmatrix}$$

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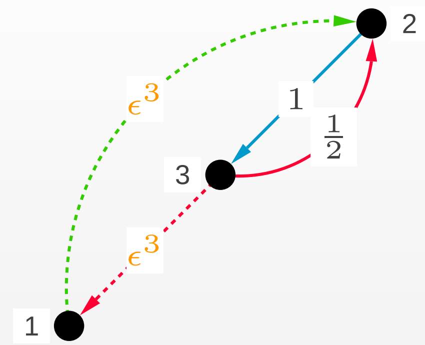

$$\begin{pmatrix} -\frac{1}{2} & \frac{1}{2} & 0 & 0 & \epsilon^3 \\ \frac{1}{2} & \cdot & 0 & 0 & 0 \\ 0 & \epsilon^5 & \cdot & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & \cdot & 0 \\ 0 & 0 & 0 & \epsilon^2 & \cdot \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \underset{i_1}{\longrightarrow} \begin{pmatrix} \cdot & 0 & 0 & \epsilon^3 \\ \epsilon^5 & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & \cdot & 0 \\ 0 & 0 & \epsilon^2 & \cdot \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \underset{i_2}{\longrightarrow} \begin{pmatrix} \cdot & 0 & \epsilon^3 \\ \epsilon^5 & \cdot & \frac{1}{2} \\ 0 & \epsilon^2 & \cdot \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \underset{i_3}{\longrightarrow} \begin{pmatrix} \cdot & 0 & \epsilon^5 \\ \epsilon^5 & \cdot & \frac{\epsilon^2}{2} \\ 0 & \epsilon^2 & \cdot \end{pmatrix} \longrightarrow \begin{pmatrix} \cdot & 0 & \epsilon^3 \\ \epsilon^3 & \cdot & \frac{1}{2} \\ 0 & 1 & -\frac{1}{2}-\epsilon^3 \end{pmatrix}$$

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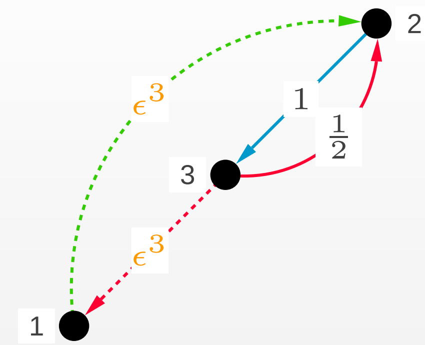

$$\begin{pmatrix} -\frac{1}{2} & \frac{1}{2} & 0 & 0 & \epsilon^3 \\ \frac{1}{2} & \cdot & 0 & 0 & 0 \\ 0 & \epsilon^5 & \cdot & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & \cdot & 0 \\ 0 & 0 & 0 & \epsilon^2 & \cdot \end{pmatrix} \underset{i_1}{\overset{\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}}{\longrightarrow}} \begin{pmatrix} \cdot & 0 & 0 & \epsilon^3 \\ \epsilon^5 & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & \cdot & 0 \\ 0 & 0 & \epsilon^2 & \cdot \end{pmatrix} \underset{i_2}{\overset{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}{\longrightarrow}} \begin{pmatrix} \cdot & 0 & \epsilon^3 \\ \epsilon^5 & \cdot & \frac{1}{2} \\ 0 & \epsilon^2 & \cdot \end{pmatrix}$$

$$\underset{i_3}{\overset{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}}{\longrightarrow}} \begin{pmatrix} \cdot & 0 & \epsilon^5 \\ \epsilon^5 & \cdot & \frac{\epsilon^2}{2} \\ 0 & \epsilon^2 & \cdot \end{pmatrix} \longrightarrow \begin{pmatrix} \cdot & 0 & \epsilon^3 \\ \epsilon^3 & \cdot & \frac{1}{2} \\ 0 & 1 & -\frac{1}{2} - \epsilon^3 \end{pmatrix}$$

reduce to $\begin{pmatrix} \cdot & 0 \\ \epsilon^3 & \cdot \end{pmatrix} - \begin{pmatrix} \epsilon^3 \\ \frac{1}{2} \end{pmatrix} \begin{pmatrix} -\frac{1}{2} \end{pmatrix}^{-1} \begin{pmatrix} 0 & 1 \end{pmatrix}$

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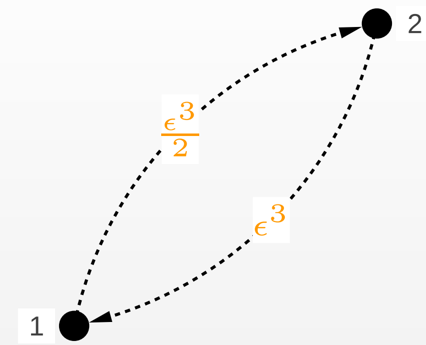
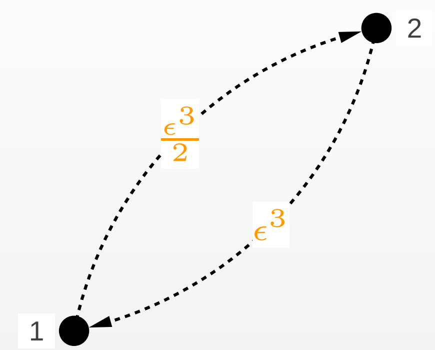


$$\begin{pmatrix} -\frac{1}{2} & \frac{1}{2} & 0 & 0 & \epsilon^3 \\ \frac{1}{2} & \cdot & 0 & 0 & 0 \\ 0 & \epsilon^5 & \cdot & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & \cdot & 0 \\ 0 & 0 & 0 & \epsilon^2 & \cdot \end{pmatrix} \xrightarrow[i_1]{\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}} \begin{pmatrix} \cdot & 0 & 0 & \epsilon^3 \\ \epsilon^5 & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & \cdot & 0 \\ 0 & 0 & \epsilon^2 & \cdot \end{pmatrix} \xrightarrow[i_2]{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}} \begin{pmatrix} \cdot & 0 & \epsilon^3 \\ \epsilon^5 & \cdot & \frac{1}{2} \\ 0 & \epsilon^2 & \cdot \end{pmatrix}$$

$$\xrightarrow[i_3]{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}} \begin{pmatrix} \cdot & 0 & \epsilon^5 \\ \epsilon^5 & \cdot & \frac{\epsilon^2}{2} \\ 0 & \epsilon^2 & \cdot \end{pmatrix} \longrightarrow \begin{pmatrix} \cdot & 0 & \epsilon^3 \\ \epsilon^3 & \cdot & \frac{1}{2} \\ 0 & 1 & -\frac{1}{2} - \epsilon^3 \end{pmatrix} \xrightarrow[i_4]{\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 2 \end{pmatrix}} \begin{pmatrix} \cdot & 2\epsilon^3 \\ \epsilon^3 & \cdot \end{pmatrix}$$

reduce to $\begin{pmatrix} \cdot & 0 \\ \epsilon^3 & \cdot \end{pmatrix} - \begin{pmatrix} \epsilon^3 \\ \frac{1}{2} \end{pmatrix} \begin{pmatrix} -\frac{1}{2} \end{pmatrix}^{-1} \begin{pmatrix} 0 & 1 \end{pmatrix}$

An Example



$$\begin{pmatrix} -\frac{1}{2} & \frac{1}{2} & 0 & 0 & \epsilon^3 \\ \frac{1}{2} & \cdot & 0 & 0 & 0 \\ 0 & \epsilon^5 & \cdot & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & \cdot & 0 \\ 0 & 0 & 0 & \epsilon^2 & \cdot \end{pmatrix} \underset{i_1}{\overset{\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}}{\longrightarrow}} \begin{pmatrix} \cdot & 0 & 0 & \epsilon^3 \\ \epsilon^5 & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & \cdot & 0 \\ 0 & 0 & \epsilon^2 & \cdot \end{pmatrix} \underset{i_2}{\overset{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}{\longrightarrow}} \begin{pmatrix} \cdot & 0 & \epsilon^3 \\ \epsilon^5 & \cdot & \frac{1}{2} \\ 0 & \epsilon^2 & \cdot \end{pmatrix}$$

$$\underset{i_3}{\overset{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}}{\longrightarrow}} \begin{pmatrix} \cdot & 0 & \epsilon^5 \\ \epsilon^5 & \cdot & \frac{\epsilon^2}{2} \\ 0 & \epsilon^2 & \cdot \end{pmatrix} \longrightarrow \begin{pmatrix} \cdot & 0 & \epsilon^3 \\ \epsilon^3 & \cdot & \frac{1}{2} \\ 0 & 1 & -\frac{1}{2}-\epsilon^3 \end{pmatrix} \underset{i_4}{\overset{\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 2 \end{pmatrix}}{\longrightarrow}} \begin{pmatrix} \cdot & 2\epsilon^3 \\ \epsilon^3 & \cdot \end{pmatrix}$$

uniform scale by $\frac{1}{2\epsilon^3} I$

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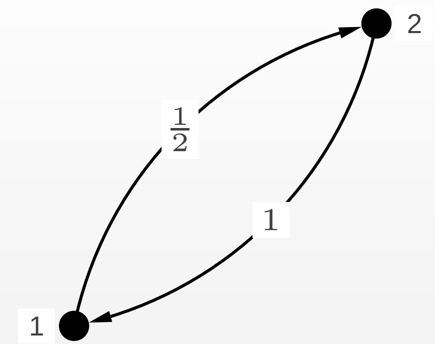
An Example



$$\begin{pmatrix} -\frac{1}{2} & \frac{1}{2} & 0 & 0 & \epsilon^3 \\ \frac{1}{2} & \cdot & 0 & 0 & 0 \\ 0 & \epsilon^5 & \cdot & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & \cdot & 0 \\ 0 & 0 & 0 & \epsilon^2 & \cdot \end{pmatrix} \xrightarrow[i_1]{\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}} \begin{pmatrix} \cdot & 0 & 0 & \epsilon^3 \\ \epsilon^5 & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & \cdot & 0 \\ 0 & 0 & \epsilon^2 & \cdot \end{pmatrix} \xrightarrow[i_2]{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}} \begin{pmatrix} \cdot & 0 & \epsilon^3 \\ \epsilon^5 & \cdot & \frac{1}{2} \\ 0 & \epsilon^2 & \cdot \end{pmatrix}$$

$$\xrightarrow[i_3]{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}} \begin{pmatrix} \cdot & 0 & \epsilon^5 \\ \epsilon^5 & \cdot & \frac{\epsilon^2}{2} \\ 0 & \epsilon^2 & \cdot \end{pmatrix} \longrightarrow \begin{pmatrix} \cdot & 0 & \epsilon^3 \\ \epsilon^3 & \cdot & \frac{1}{2} \\ 0 & 1 & -\frac{1}{2} - \epsilon^3 \end{pmatrix} \xrightarrow[i_4]{\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 2 \end{pmatrix}} \begin{pmatrix} \cdot & 2\epsilon^3 \\ \epsilon^3 & \cdot \end{pmatrix} \longrightarrow \begin{pmatrix} \cdot & 1 \\ \frac{1}{2} & \cdot \end{pmatrix}$$

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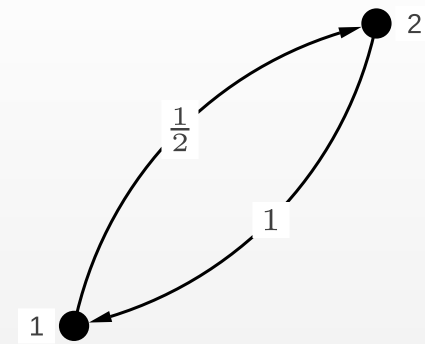

$$\begin{pmatrix} -\frac{1}{2} & \frac{1}{2} & 0 & 0 & \epsilon^3 \\ \frac{1}{2} & \cdot & 0 & 0 & 0 \\ 0 & \epsilon^5 & \cdot & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & \cdot & 0 \\ 0 & 0 & 0 & \epsilon^2 & \cdot \end{pmatrix} \xrightarrow[i_1]{\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}} \begin{pmatrix} \cdot & 0 & 0 & \epsilon^3 \\ \epsilon^5 & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & \cdot & 0 \\ 0 & 0 & \epsilon^2 & \cdot \end{pmatrix} \xrightarrow[i_2]{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}} \begin{pmatrix} \cdot & 0 & \epsilon^3 \\ \epsilon^5 & \cdot & \frac{1}{2} \\ 0 & \epsilon^2 & \cdot \end{pmatrix}$$

$$\xrightarrow[i_3]{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}} \begin{pmatrix} \cdot & 0 & \epsilon^5 \\ \epsilon^5 & \cdot & \frac{\epsilon^2}{2} \\ 0 & \epsilon^2 & \cdot \end{pmatrix} \longrightarrow \begin{pmatrix} \cdot & 0 & \epsilon^3 \\ \epsilon^3 & \cdot & \frac{1}{2} \\ 0 & 1 & -\frac{1}{2}-\epsilon^3 \end{pmatrix} \xrightarrow[i_4]{\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 2 \end{pmatrix}} \begin{pmatrix} \cdot & 2\epsilon^3 \\ \epsilon^3 & \cdot \end{pmatrix} \longrightarrow \begin{pmatrix} \cdot & 1 \\ \frac{1}{2} & \cdot \end{pmatrix}$$

$$v_0 \propto \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

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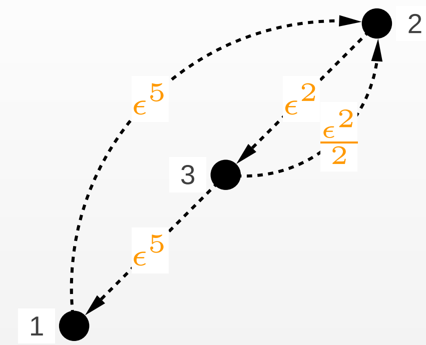

$$\begin{pmatrix} -\frac{1}{2} & \frac{1}{2} & 0 & 0 & \epsilon^3 \\ \frac{1}{2} & \cdot & 0 & 0 & 0 \\ 0 & \epsilon^5 & \cdot & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & \cdot & 0 \\ 0 & 0 & 0 & \epsilon^2 & \cdot \end{pmatrix} \underset{i_1}{\overset{\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}}{\longrightarrow}} \begin{pmatrix} \cdot & 0 & 0 & \epsilon^3 \\ \epsilon^5 & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & \cdot & 0 \\ 0 & 0 & \epsilon^2 & \cdot \end{pmatrix} \underset{i_2}{\overset{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}{\longrightarrow}} \begin{pmatrix} \cdot & 0 & \epsilon^3 \\ \epsilon^5 & \cdot & \frac{1}{2} \\ 0 & \epsilon^2 & \cdot \end{pmatrix}$$

$$\underset{i_3}{\overset{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}}{\longrightarrow}} \begin{pmatrix} \cdot & 0 & \epsilon^5 \\ \epsilon^5 & \cdot & \frac{\epsilon^2}{2} \\ 0 & \epsilon^2 & \cdot \end{pmatrix} \longrightarrow \begin{pmatrix} \cdot & 0 & \epsilon^3 \\ \epsilon^3 & \cdot & \frac{1}{2} \\ 0 & 1 & -\frac{1}{2} - \epsilon^3 \end{pmatrix} \underset{i_4}{\overset{\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 2 \end{pmatrix}}{\longrightarrow}} \begin{pmatrix} \cdot & 2\epsilon^3 \\ \epsilon^3 & \cdot \end{pmatrix} \longrightarrow \begin{pmatrix} \cdot & 1 \\ \frac{1}{2} & \cdot \end{pmatrix}$$

$$v_0 \propto \begin{pmatrix} 2 \\ 1 \end{pmatrix} \underset{i_4}{\longrightarrow} \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$$

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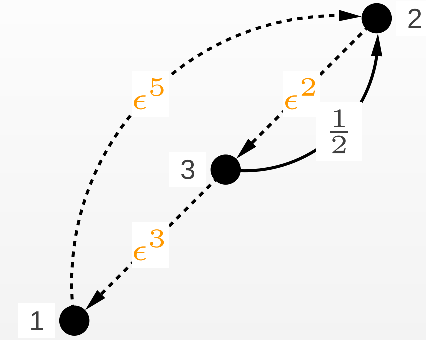
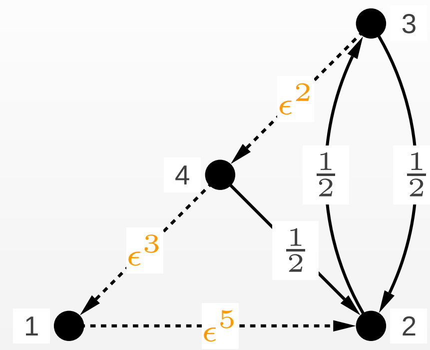


$$\begin{pmatrix} -\frac{1}{2} & \frac{1}{2} & 0 & 0 & \epsilon^3 \\ \frac{1}{2} & \cdot & 0 & 0 & 0 \\ 0 & \epsilon^5 & \cdot & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & \cdot & 0 \\ 0 & 0 & 0 & \epsilon^2 & \cdot \end{pmatrix} \underset{i_1}{\overset{\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}}{\longrightarrow}} \begin{pmatrix} \cdot & 0 & 0 & \epsilon^3 \\ \epsilon^5 & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & \cdot & 0 \\ 0 & 0 & \epsilon^2 & \cdot \end{pmatrix} \underset{i_2}{\overset{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}{\longrightarrow}} \begin{pmatrix} \cdot & 0 & \epsilon^3 \\ \epsilon^5 & \cdot & \frac{1}{2} \\ 0 & \epsilon^2 & \cdot \end{pmatrix}$$

$$\underset{i_3}{\overset{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}}{\longrightarrow}} \begin{pmatrix} \cdot & 0 & \epsilon^5 \\ \epsilon^5 & \cdot & \frac{\epsilon^2}{2} \\ 0 & \epsilon^2 & \cdot \end{pmatrix} \longrightarrow \begin{pmatrix} \cdot & 0 & \epsilon^3 \\ \epsilon^3 & \cdot & \frac{1}{2} \\ 0 & 1 & -\frac{1}{2} - \epsilon^3 \end{pmatrix} \underset{i_4}{\overset{\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 2 \end{pmatrix}}{\longrightarrow}} \begin{pmatrix} \cdot & 2\epsilon^3 \\ \epsilon^3 & \cdot \end{pmatrix} \longrightarrow \begin{pmatrix} \cdot & 1 \\ \frac{1}{2} & \cdot \end{pmatrix}$$

$$v_0 \propto \begin{pmatrix} 2 \\ 1 \end{pmatrix} \underset{i_4}{\longrightarrow} \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} \underset{i_3}{\longrightarrow} \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$$

An Example



$$\begin{pmatrix} -\frac{1}{2} & \frac{1}{2} & 0 & 0 & \epsilon^3 \\ \frac{1}{2} & \cdot & 0 & 0 & 0 \\ 0 & \epsilon^5 & \cdot & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & \cdot & 0 \\ 0 & 0 & 0 & \epsilon^2 & \cdot \end{pmatrix} \underset{\xrightarrow[i_1]{}}{\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}} \begin{pmatrix} \cdot & 0 & 0 & \epsilon^3 \\ \epsilon^5 & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & \cdot & 0 \\ 0 & 0 & \epsilon^2 & \cdot \end{pmatrix} \underset{\xrightarrow[i_2]{}}{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}} \begin{pmatrix} \cdot & 0 & \epsilon^3 \\ \epsilon^5 & \cdot & \frac{1}{2} \\ 0 & \epsilon^2 & \cdot \end{pmatrix}$$

$$\underset{\xrightarrow[i_3]{}}{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}} \begin{pmatrix} \cdot & 0 & \epsilon^5 \\ \epsilon^5 & \cdot & \frac{\epsilon^2}{2} \\ 0 & \epsilon^2 & \cdot \end{pmatrix} \longrightarrow \begin{pmatrix} \cdot & 0 & \epsilon^3 \\ \epsilon^3 & \cdot & \frac{1}{2} \\ 0 & 1 & -\frac{1}{2}-\epsilon^3 \end{pmatrix} \underset{\xrightarrow[i_4]{}}{\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 2 \end{pmatrix}} \begin{pmatrix} \cdot & 2\epsilon^3 \\ \epsilon^3 & \cdot \end{pmatrix} \longrightarrow \begin{pmatrix} \cdot & 1 \\ \frac{1}{2} & \cdot \end{pmatrix}$$

$$v_0 \propto \begin{pmatrix} 2 \\ 1 \end{pmatrix} \xrightarrow[i_4]{} \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} \xrightarrow[i_3]{} \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} \xrightarrow[i_2]{} \begin{pmatrix} 2 \\ 1 \\ 1 \\ 0 \end{pmatrix}$$

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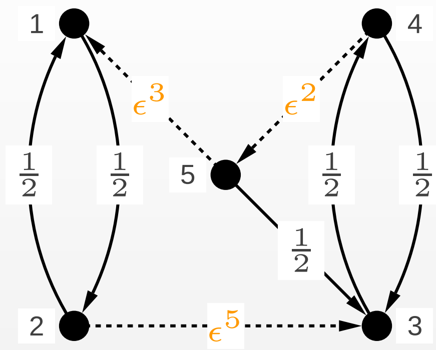

$$\begin{pmatrix} -\frac{1}{2} & \frac{1}{2} & 0 & 0 & \epsilon^3 \\ \frac{1}{2} & \cdot & 0 & 0 & 0 \\ 0 & \epsilon^5 & \cdot & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & \cdot & 0 \\ 0 & 0 & 0 & \epsilon^2 & \cdot \end{pmatrix} \underset{i_1}{\overset{\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}}{\longrightarrow}} \begin{pmatrix} \cdot & 0 & 0 & \epsilon^3 \\ \epsilon^5 & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & \cdot & 0 \\ 0 & 0 & \epsilon^2 & \cdot \end{pmatrix} \underset{i_2}{\overset{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}{\longrightarrow}} \begin{pmatrix} \cdot & 0 & \epsilon^3 \\ \epsilon^5 & \cdot & \frac{1}{2} \\ 0 & \epsilon^2 & \cdot \end{pmatrix}$$

$$\underset{i_3}{\overset{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}}{\longrightarrow}} \begin{pmatrix} \cdot & 0 & \epsilon^5 \\ \epsilon^5 & \cdot & \frac{\epsilon^2}{2} \\ 0 & \epsilon^2 & \cdot \end{pmatrix} \longrightarrow \begin{pmatrix} \cdot & 0 & \epsilon^3 \\ \epsilon^3 & \cdot & \frac{1}{2} \\ 0 & 1 & -\frac{1}{2}-\epsilon^3 \end{pmatrix} \underset{i_4}{\overset{\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 2 \end{pmatrix}}{\longrightarrow}} \begin{pmatrix} \cdot & 2\epsilon^3 \\ \epsilon^3 & \cdot \end{pmatrix} \longrightarrow \begin{pmatrix} \cdot & 1 \\ \frac{1}{2} & \cdot \end{pmatrix}$$

$$v_0 \propto \begin{pmatrix} 2 \\ 1 \end{pmatrix} \underset{i_4}{\longrightarrow} \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} \underset{i_3}{\longrightarrow} \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} \underset{i_2}{\longrightarrow} \begin{pmatrix} 2 \\ 1 \\ 1 \\ 0 \end{pmatrix} \underset{i_1}{\longrightarrow} \begin{pmatrix} 2 \\ 2 \\ 1 \\ 1 \\ 0 \end{pmatrix}$$

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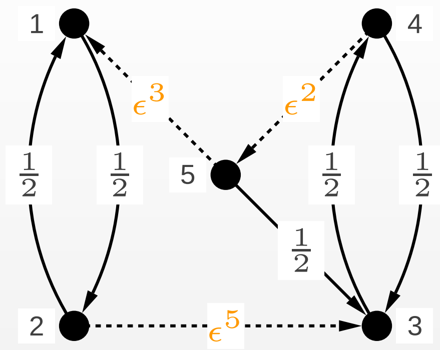

$$\begin{pmatrix} -\frac{1}{2} & \frac{1}{2} & 0 & 0 & \epsilon^3 \\ \frac{1}{2} & \cdot & 0 & 0 & 0 \\ 0 & \epsilon^5 & \cdot & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & \cdot & 0 \\ 0 & 0 & 0 & \epsilon^2 & \cdot \end{pmatrix} \underset{i_1}{\xrightarrow{\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}}} \begin{pmatrix} \cdot & 0 & 0 & \epsilon^3 \\ \epsilon^5 & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & \cdot & 0 \\ 0 & 0 & \epsilon^2 & \cdot \end{pmatrix} \underset{i_2}{\xrightarrow{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}} \begin{pmatrix} \cdot & 0 & \epsilon^3 \\ \epsilon^5 & \cdot & \frac{1}{2} \\ 0 & \epsilon^2 & \cdot \end{pmatrix}$$

$$\underset{i_3}{\xrightarrow{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}}} \begin{pmatrix} \cdot & 0 & \epsilon^5 \\ \epsilon^5 & \cdot & \frac{\epsilon^2}{2} \\ 0 & \epsilon^2 & \cdot \end{pmatrix} \longrightarrow \begin{pmatrix} \cdot & 0 & \epsilon^3 \\ \epsilon^3 & \cdot & \frac{1}{2} \\ 0 & 1 & -\frac{1}{2} - \epsilon^3 \end{pmatrix} \underset{i_4}{\xrightarrow{\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 2 \end{pmatrix}}} \begin{pmatrix} \cdot & 2\epsilon^3 \\ \epsilon^3 & \cdot \end{pmatrix} \longrightarrow \begin{pmatrix} \cdot & 1 \\ \frac{1}{2} & \cdot \end{pmatrix}$$

$$v_0 \propto \begin{pmatrix} 2 \\ 1 \end{pmatrix} \underset{i_4}{\longrightarrow} \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} \underset{i_3}{\longrightarrow} \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} \underset{i_2}{\longrightarrow} \begin{pmatrix} 2 \\ 1 \\ 1 \\ 0 \end{pmatrix} \underset{i_1}{\longrightarrow} \begin{pmatrix} 2 \\ 2 \\ 1 \\ 1 \\ 0 \end{pmatrix} \underset{\text{NORMALIZE}}{\longrightarrow} \begin{pmatrix} \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{6} \\ \frac{1}{6} \\ 0 \end{pmatrix} = v_0$$

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Our Contributions

Geometric Approach: MCTT

- Using our representation, one can compute the SSD using MCTT because $\mathbb{C}^+$ is closed under addition and multiplication.
- But this approach is combinatorial ($O(n^n)$).

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Our Contributions

Geometric Approach: MCTT

- Using our representation, one can compute the SSD using MCTT because $\mathbb{C}^+$ is closed under addition and multiplication.
- But this approach is combinatorial ($O(n^n)$).

Algebraic Approach: GE

- Using our representation, one cannot compute the SSD using GE because $\mathbb{C}^+$ is not closed under subtraction and division.
- We give an efficient algorithm for computing the SSD of a PMM that restricts the use of arithmetic operations.

References

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