Strongly Rational Sets for Normal-Form Games*

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January 13, 2009

Abstract

Curb sets [Basu and Weibull, Econ. Letters 36 (1991), 141-146] are product sets of pure strategies containing all individual best-responses against beliefs restricted to the recommendations to the remaining players. Prep sets [Voorneveld, Games Econ. Behav. 48 (2004), 403-414] only require that the product sets contain at least one best-response to such beliefs. While the concepts of curb and prep sets are set-theoretic coarsenings of the notion of Nash equilibrium, we introduce the concepts of strong curb sets and strong prep sets which are set-theoretic coarsenings of the notion of strong Nash equilibrium. We require the set to be immune not only against individual deviations, but also against group deviations. We show that every game has at least one minimal strong curb (prep) set. Minimal strong curb (prep) sets are compared with strong Nash equilibria, coalition-proof Nash equilibria and the set of coalitionally rationalizable strategies. Finally, we provide a dynamic learning process leading the players to playing strategies from a minimal strong curb set.

JEL Classification: C72.

Keywords: Set-valued solution concept, Coalitional deviations, Strong curb set, Strong prep set, Learning.

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1 Introduction

The main solution concept in noncooperative game theory, Nash equilibrium, requires stability only with respect to individual deviations by players. It does not take into account the possibility that groups of players might coordinate their moves, in order to achieve an outcome that is better for all of them. Aumann (1959) was first to incorporate this consideration into the theory of noncooperative games by proposing the notion of strong Nash equilibrium. A strategy profile is a strong Nash equilibrium if it is immune not only to individual deviations, but also to coalitional deviations. More recently, Bernheim, Peleg and Whinston (1987) have proposed the notion of coalition-proof Nash equilibrium. A strategy profile is a coalition-proof Nash equilibrium if it is immune to coalitional deviations which are themselves immune to further deviations by subcoalitions. The main weakness of strong Nash equilibrium and coalition-proof Nash equilibrium is that existence is not guaranteed in a natural class of games, as opposed to the Nash equilibrium concept.

Basu and Weibull (1991) have extended the point-valued strict Nash equilibrium concept to a set-valued concept: minimal curb (closed under rational behavior) sets. This set-valued solution concept combines a standard rationality condition, stating that the set of recommended strategies of each player must contain all best responses to whatever belief he may have that is consistent with the recommendations to the other players, with players' aim at simplicity, which encourages them to maintain a set of strategies as small as possible. Recently, Voorneveld (2004) has proposed the notion of minimal prep ('prep' is short for 'preparation') sets which are product sets of pure strategies containing not all, but at least one best response against beliefs restricted to the recommendations to the remaining players.[1] The concepts of minimal curb sets and minimal prep sets are set-theoretic coarsenings of the notion of Nash equilibrium.

In this paper we introduce the concepts of strong curb sets and strong prep sets which are set-theoretic coarsenings of the notion of strong Nash equilibrium. We require the sets to be immune not only against individual deviations, but also against group deviations. We show that every game has at least one minimal strong curb (prep) set. Every strong curb set is a strong prep set, so if a strong curb set is contained in a minimal strong prep set, the two sets are necessarily equal. Minimal strong curb (prep) sets are compared with strong Nash equilibria, coalition-proof Nash equilibria and the set of coalitionally

[1] Voorneveld, Kets and Norde (2005) have provided axiomatizations of minimal prep sets and minimal curb sets. Kalai and Samet (1984) have proposed the notion of persistent retracts which also require the recommendations to each player to contain at least one best response to beliefs in a small neighborhood of the beliefs restricted to the recommendations to the remaining players. Voorneveld (2005) has shown that, in generic games, persistent retracts, minimal prep sets and minimal curb sets coincide.

rationalizable strategies.

Hurkens (1995) has proposed a dynamic learning process where players have bounded memory and they play best responses against beliefs, formed on the basis of strategies used in the recent past. This learning process leads the players to playing strategies from a minimal curb set.[2] Recently, Kets and Voorneveld (2008) have provided an alternative dynamic learning process in which players display a bias towards recent choices and choose best responses to beliefs supported by observed play in the recent past. The limit behavior of this learning process is shown to eventually settle down in minimal prep sets. We propose a learning process similar to the one proposed by Hurkens (1995) except that now groups of players may coordinate their actions, and we show that this learning process leads the players to playing strategies from a minimal strong curb set.

The paper is organized as follows. Section 2 contains preliminaries. In section 3 strong curb sets and strong prep sets are formally defined. Minimal strong curb sets and minimal strong prep sets are shown to exist in general. In section 4, minimal strong curb sets and minimal strong prep sets are compared with strong Nash equilibria, coalition-proof Nash equilibria and coalitional rationalizability. In Section 5 a learning process leading the players to playing strategies from a minimal strong curb set is provided. Section 6 concludes.

2 Preliminaries

Strict set inclusion is denoted by $\subsetneq$ and weak set inclusion is denoted by $\subseteq$. A **normal-form game** is a tuple $G = \langle N, \{A_i\}_{i\in N}, \{u_i\}_{i\in N}\rangle$, where $N = \{1, 2, \ldots, n\}$ is a finite set of players, each player $i \in N$ has a nonempty, finite set of pure strategies (or actions) A_i and a von Neumann-Morgenstern utility function $u_i : A \to \mathbb{R}$, where $A = \times_{j\in N} A_j$. The set of all games is denoted by Γ. **Coalitions** are nonempty subsets of players (J such that $J \subseteq N$ and $J \neq \varnothing$). Let $\mathbb{J}$ be the finite set of coalition structures. A coalition structure $\mathbf{J} = (J_1, J_2, \ldots, J_M)$ is a partition of the player set $N = \{1, 2, \ldots, n\}$: $J_k \cap J_l = \varnothing$ for $k \neq l$ and $\cup_{k=1}^{M} J_k = N$. For every $X \subseteq A$, let $X_{-i} = \times_{j\in N\setminus\{i\}} X_j$, $\forall i \in N$ and $X_{-J} = \times_{j\in N\setminus J} X_j$, $\forall J \subseteq N$. The **subgame** obtained from G by restricting the action set of each player $i \in N$ to a subset $X_i \subseteq A_i$ is denoted – with a minor abuse of notation from restricting the domain of the utility functions u_i to $\times_{j\in N} X_j$ – by $G_X = \langle N, \{X_i\}_{i\in N}, \{u_i\}_{i\in N}\rangle$. The set of mixed strategies of player $i \in N$ with support in $X_i \subseteq A_i$ is denoted by $\Delta(X_i)$. Payoffs are extended to mixed strategies in the usual way. As usual, (a_i, α_{-i}) is the profile of strategies where player $i \in N$ plays $a_i \in A_i$ and his opponents play according to the mixed strategy

[2] See also Young (1998).

profile $\alpha_{-i} = (\alpha_j)_{j \in N \setminus \{i\}} \in \times_{j \in N \setminus \{i\}} \Delta(A_j)$. A mixed strategy profile for coalition $J \subseteq N$ is denoted $\alpha_J = (\alpha_i)_{i \in J} \in \times_{i \in J} \Delta(A_i)$, and (α_J, α_{-J}) is the profile of strategies where players belonging to coalition J play according to the mixed strategy profile $\alpha_J \in \times_{i \in J} \Delta(A_i)$ and their opponents play according to the mixed strategy profile $\alpha_{-J} \in \times_{j \in N \setminus J} \Delta(A_j)$. For every $J \subseteq N$, $i \in J$, $X \subseteq A$, $\alpha_{-i} = (\alpha_j)_{j \in N \setminus \{i\}} \in \times_{j \in N \setminus \{i\}} \Delta(X_j)$, we denote by α_{-i}^{-J} the marginal distribution of α_{-i} over X_{-J}.

Let $G = \langle N, \{A_i\}_{i \in N}, \{u_i\}_{i \in N} \rangle$ be a game, let $\varnothing \neq J \subseteq N$, and let $\sigma \in \times_{i \in N} \Delta(A_i)$. The **reduced game** of G with respect to J and σ is the game $G^{J,\sigma} = \langle J, \{A_i\}_{i \in J}, \{u_i^\sigma\}_{i \in J} \rangle$ where $u_i^\sigma(\alpha_J) = u_i(\alpha_J, \sigma_{-J})$ for all $\alpha_J \in \times_{i \in J} \Delta(A_i)$ and $i \in J$. This definition of reduced games is due to Peleg and Tijs (1996) and has a straightforward interpretation. Let $\varnothing \neq J \subseteq N$ and $\sigma \in \times_{i \in N} \Delta(A_i)$. If it is commonly known among the members of J that the members of $N \setminus J$ have chosen the mixed strategies σ_i, $i \in N \setminus J$, then the members of J are faced with the game $G^{J,\sigma}$.

For $i \in N$ and $\alpha_{-i} \in \times_{j \in N \setminus \{i\}} \Delta(A_j)$,

$$BR^i(\alpha_{-i}) = \{a_i \in A_i \mid u_i(a_i, \alpha_{-i}) \geq u_i(a_i', \alpha_{-i}) \text{ for each } a_i' \in A_i\}$$

is the set of pure best responses of player i against α_{-i}.

The notion of **strong Nash equilibrium** is due to Aumann (1959). A strong Nash equilibrium is a Nash equilibrium such that there is no nonempty set of players who could all gain by deviating together to some other combination of strategies that is jointly feasible for them, when the other players who are not in this set are expected to stay with their equilibrium strategies. Formally, the notion of strong Nash equilibrium is defined as follows. The strategy profile $\alpha^* \in \times_{i \in N} \Delta(A_i)$ is a strong Nash equilibrium if and only if, $\forall J \subseteq N$, $\forall \alpha_J \in \times_{j \in J} \Delta(A_j)$, $\exists i \in J$ such that $u_i(\alpha^*) \geq u_i(\alpha_J, \alpha^*_{-J})$.

Basu and Weibull (1991) have introduced the concept of strategy subset closed under rational behavior (curb), which is a set-theoretic coarsening of the notion of strict Nash equilibrium. Formally, **curb sets** are defined as follows.

Definition 1 A curb set is a product set $X = \times_{i \in N} X_i$ where

(*a*) for each $i \in N$, $X_i \subseteq A_i$ is a nonempty set of pure strategies;

(*b*) for each $i \in N$ and each belief α_{-i} of player i with support in X_{-i}, the set X_i contains all best responses of player i against his belief:

$$\forall i \in N, \forall \alpha_{-i} \in \times_{j \in N \setminus \{i\}} \Delta(X_j), BR^i(\alpha_{-i}) \subseteq X.$$

Since the full strategy space is always curb, particular attention is devoted to minimal curb sets. A curb set X is minimal if no curb set is a proper subset of X. Basu and

Weibull (1991) have shown that every game G possesses at least one minimal curb set. The set-valued solution concept that assigns to each game its collection of minimal curb sets is denoted by min-curb. Hence,

$$\text{min-curb}(G) = \{X \subseteq A \mid X \text{ is a minimal curb set of } G\}.$$

Similarly,

$$\text{curb}(G) = \{X \subseteq A \mid X \text{ is a curb set of } G\}.$$

Voorneveld (2004) has proposed another set-valued solution concept, **prep sets**, which are formally defined as follows.

Definition 2 A prep set is a product set $X = \times_{i \in N} X_i$ where

(a) for each $i \in N$, $X_i \subseteq A_i$ is a nonempty set of pure strategies;

(b) for each $i \in N$ and each belief α_{-i} of player i with support in X_{-i}, the set X_i contains at least one best response of player i against his belief:

$$\forall i \in N, \forall \alpha_{-i} \in \times_{j \in N \setminus \{i\}} \Delta(X_j), BR^i(\alpha_{-i}) \cap X_i \neq \varnothing.$$

Prep sets are product sets of pure strategies such that each player's set of recommended strategies must contain at least one best-reply to whatever belief he may have that is consistent with the recommendations to the other players, while curb sets are product sets of pure strategies containing not just some, but all best responses against beliefs restricted to the opponents' recommendations. A prep set X is **minimal** if no prep set is a proper subset of X. Voorneveld (2004) has shown that every game G possesses at least one minimal prep set. The set-valued solution concept that assigns to each game its collection of minimal prep sets is denoted by min-prep. Hence,

$$\text{min-prep}(G) = \{X \subseteq A \mid X \text{ is a minimal prep set of } G\}.$$

Similarly,

$$\text{prep}(G) = \{X \subseteq A \mid X \text{ is a prep set of } G\}.$$

3 Strong curb sets and strong prep sets

While the concepts of curb sets and prep sets are set-theoretic coarsenings of the notion of Nash equilibrium, we introduce the concepts of strong curb sets and strong prep sets which are set-theoretic coarsenings of the notion of strong Nash equilibrium. That is, we require the set to be immune not only against individual deviations (as for curb and prep sets), but also against group deviations. For $\boldsymbol{\alpha} = (\alpha_{-i})_{i \in N}$ with $\alpha_{-i} \in \times_{j \in N \setminus \{i\}} \Delta(A_j)$, we denote by $CBR^J(\boldsymbol{\alpha})$ the set of **coalitional best-responses** of coalition $J \subseteq N$ which is:

$$CBR^J(\boldsymbol{\alpha}) = \{\sigma_J \in \times_{i \in J} \Delta(A_i) \mid (i) \forall i \in J,\ u_i(a_i, \alpha_{-i}) \leq u_i(\sigma_J, \alpha_{-i}^{-J}),\ \forall a_i \in A_i \text{ and}$$
$$(ii) \nexists\, \sigma'_J \in \times_{i \in J} \Delta(A_i) \text{ such that } \forall i \in J, u_i(\sigma_J, \alpha_{-i}^{-J}) < u_i(\sigma'_J, \alpha_{-i}^{-J})\}.$$

It is the set of strategy profiles of coalition J that are preferred by each member i of J to any individual play, given its beliefs α_{-i}. Intuitively, a set X is a strong curb or a strong prep set if the belief that only strategies in X are played imply that players and coalitions have no incentives to use other strategies than those belonging to X.

Formally, **strong curb sets** are defined as follows.

Definition 3 A strong curb set is a product set $X = \times_{i \in N} X_i$ where

(a) for each $i \in N$, $X_i \subseteq A_i$ is a nonempty set of pure strategies;

(b) for each $J \subseteq N$ and each vector of beliefs $\boldsymbol{\alpha} = (\alpha_{-1}, ..., \alpha_{-N})$ of the players with each belief α_{-i} having support in X_{-i}, the product set $X_J = \times_{j \in J} X_j$ contains the support of all coalitional best responses of coalition J against the beliefs of its members:

$$\forall J \subseteq N, \forall \boldsymbol{\alpha} = (\alpha_{-1}, ..., \alpha_{-n}) \text{ with } \alpha_{-i} \in \times_{l \in N \setminus \{i\}} \Delta(X_l),\ i \in N,$$
$$\text{either } CBR^J(\boldsymbol{\alpha}) = \varnothing \text{ or } CBR^J(\boldsymbol{\alpha}) \subseteq \times_{j \in J} \Delta(X_j).$$

Formally, **strong prep sets** are defined as follows.

Definition 4 A strong prep set is a product set $X = \times_{i \in N} X_i$ where

(a) for each $i \in N$, $X_i \subseteq A_i$ is a nonempty set of pure strategies;

(b) for each $J \subseteq N$ and each vector of beliefs $\boldsymbol{\alpha} = (\alpha_{-1}, ..., \alpha_{-N})$ of the players with each belief α_{-i} having support in X_{-i}, the product set $X_J = \times_{j \in J} X_j$ contains the support of at least one coalitional best response of coalition J against the beliefs of its members:

$$\forall J \subseteq N, \forall \boldsymbol{\alpha} = (\alpha_{-1}, ..., \alpha_{-n}) \text{ with } \alpha_{-i} \in \times_{l \in N \setminus \{i\}} \Delta(X_l),\ i \in N,$$
$$\text{either } CBR^J(\boldsymbol{\alpha}) = \varnothing \text{ or } CBR^J(\boldsymbol{\alpha}) \cap \times_{j \in J} \Delta(X_j) \neq \varnothing.$$

Strong prep sets are product sets of pure strategies such that each player's set of recommended strategies must contain at least one coalitional best-response of each coalition, if some exist, to whatever belief each coalition member may have that is consistent with the recommendations to the other players, while strong curb sets are product sets of pure strategies containing not just some, but all coalitional best-responses of each coalition against beliefs of each coalition member that are consistent with the recommendations to the other players.

A set $X \subseteq A$ is not a strong prep set if there exists a coalition having a deviation outside the set of recommended strategies such that each coalitional member is better off by deviating for at least one possible belief concerning the play of others in the set of recommended strategies. Notice that each coalitional member is allowed to have a different belief concerning the play of others in the set of recommended strategies to assess the profitability of the deviation. Thus, the coalitional members may disagree on where the deviation leads to. Under the strong curb concept, a deviation is blocked if we can find one player who is strictly better off by blocking the deviation. Thus a deviation occurs only if all coalitional members are at least as well off by deviating. Under the strong prep concept, a deviation is blocked if at least one player is weakly better off by blocking the deviation. Thus a deviation occurs if all coalitional members are strictly better off by deviating.

A strong curb set X is **minimal** if no strong curb set is a proper subset of X. The set-valued solution concept that assigns to each game its collection of minimal strong curb sets is denoted by min-strong-curb. Hence,

$$\text{min-strong-curb}(G) = \{X \subseteq A \mid X \text{ is a minimal strong curb set of } G\}.$$

Similarly,

$$\text{strong-curb}(G) = \{X \subseteq A \mid X \text{ is a strong curb set of } G\}.$$

A strong prep set X is **minimal** if no strong prep set is a proper subset of X. The set-valued solution concept that assigns to each game its collection of minimal strong prep sets is denoted by min-strong-prep. Hence,

$$\text{min-strong-prep}(G) = \{X \subseteq A \mid X \text{ is a minimal strong prep set of } G\}.$$

Similarly,

$$\text{strong-prep}(G) = \{X \subseteq A \mid X \text{ is a strong prep set of } G\}.$$

Theorem 1 *Every normal-form game G has a minimal strong curb set and a minimal strong prep set.*

Establishing existence of minimal strong curb sets (or minimal strong prep sets) in finite games is simple: the entire pure-strategy space A is a strong curb set (or a strong prep set). Hence the collection of strong curb sets (strong prep sets) is nonempty, finite (since A is finite) and partially ordered by set inclusion. Consequently, a minimal strong curb set (minimal strong prep set) exists. In the appendix we show that the existence result holds in general. Every strong curb set is a strong prep set, so if a strong curb set is contained in a minimal strong prep set, the two sets are necessarily equal.

If X is a minimal strong prep set of $G = \langle N, \{A_i\}_{i\in N}, \{u_i\}_{i\in N}\rangle$, then it is a minimal strong prep set of the subgame $G_X = \langle N, \{X_i\}_{i\in N}, \{u_i\}_{i\in N}\rangle$.

Theorem 2 *If $X \in$ min-strong-prep(G) then $X \in$ min-strong-prep(G_X).*

Proof. Let $X \in$ min-strong-prep(G). X is a trivial strong prep set of the subgame G_X: $X \in$ strong-prep(G_X). We will show that there is no $Y \subsetneq X$ such that $Y \in$ strong-prep(G_X). Suppose, on the contrary, that there exists $Y \subsetneq X$ such that $Y \in$ strong-prep(G_X). Since Y is not a minimal strong prep set of G, there exists a vector of beliefs concentrated on Y and a coalition $J \subseteq N$ such that each member of the coalition prefers to play a strategy profile outside the set Y rather than playing a best-response in Y to his belief. Formally, since $Y \notin$ strong-prep(G) there exists $J \subseteq N$, $\sigma_J \in \times_{j\in J}\Delta(A_j\backslash Y_j)$ and $\boldsymbol{\alpha} = (\alpha_{-1}, ..., \alpha_{-N})$ with $\alpha_{-i} \in \times_{j\in N\backslash\{i\}}\Delta(Y_j)$, $i \in N$, such that $u_j(\sigma_J, \alpha_{-j}^{-J}) > u_j(a_j, \alpha_{-j})$ for all $j \in J$ for all $a_j \in Y_j$. Since $Y \in$ strong-prep(G_X), the aforementioned deviation of coalition J does not belong to $\times_{j\in J}\Delta(X_j\backslash Y_j)$, we have $\sigma_J \in \times_{j\in J}\Delta(A_j\backslash X_j)$. Since $X \in$ strong-prep(G) and $\alpha_{-i} \in \times_{j\in N\backslash\{i\}}\Delta(X_j)$ (since $\times_{j\in N\backslash\{i\}}\Delta(Y_j) \subsetneq \times_{j\in N\backslash\{i\}}\Delta(X_j)$), at least one member $j^* \in J$ prefers to play a best-response in X against the belief α_{-j^*}than playing according to σ_J. Thus, we have $u_{j^*}(b_{j^*}, \alpha_{-j^*}) \geq u_{j^*}(\sigma_J, \alpha_{-j^*}^{-J})$ for some $b_{j^*} \in X_j$. Since $u_{j^*}(\sigma_J, \alpha_{-j^*}^{-J}) > u_{j^*}(a_{j^*}, \alpha_{-j^*})$ for all $a_{j^*} \in Y_{j^*}(Y \notin$ strong-prep$(G))$, we have $u_{j^*}(b_{j^*}, \alpha_{-j^*}) > u_{j^*}(a_{j^*}, \alpha_{-j^*})$ for some $b_{j^*} \in X_{j^*}$, for all $a_{j^*} \in Y_{j^*}$. This contradicts the fact that $Y \in$ strong-prep(G_X) since we have identified a belief $\boldsymbol{\alpha}$ which is such that $BR^{j^*}(\alpha_{-j^*}) \cap Y = \varnothing$. ■

If X is a minimal strong curb set of $G = \langle N, \{A_i\}_{i\in N}, \{u_i\}_{i\in N}\rangle$, then it is a minimal strong curb set of the subgame $G_X = \langle N, \{X_i\}_{i\in N}, \{u_i\}_{i\in N}\rangle$.

Theorem 3 *If $X \in$ min-strong-curb(G) then $X \in$ min-strong-curb(G_X).*

Proof. Let $X \in$ min-strong-curb(G). X is a trivial strong curb set of the subgame G_X: $X \in$ strong-curb(G_X). We will show that there is no $Y \subsetneq X$ such that $Y \in$ strong-curb(G_X). Suppose, on the contrary, that there exists $Y \subsetneq X$ such that $Y \in$ strong-curb(G_X). Since Y is not a minimal strong curb set of G, there exists a vector of beliefs concentrated on Y and a coalition $J \subseteq N$ such that each member of the coalition prefers to play a strategy profile outside the set Y rather than playing a best-response in Y to his belief. Formally, since $Y \notin$ min-strong-curb(G), there exists $J \subseteq N$, $\sigma_J \in \times_{j\in J}\Delta(A_j\backslash Y_j)$ and

$\boldsymbol{\alpha} = (\alpha_{-1}, ..., \alpha_{-N})$ with $\alpha_{-i} \in \times_{j \in N\setminus\{i\}} \Delta(Y_j)$, $i \in N$, such that $u_j(\sigma_J, \alpha_{-j}^{-J}) \geq u_j(a_j, \alpha_{-j})$ for all $j \in J$, for all $a_j \in Y_j$. Since $Y \in$ strong-curb(G_X), the aforementioned deviation of coalition J does not belong to $\times_{j \in J} \Delta(X_j \setminus Y_j)$, we have $\sigma_J \in \times_{j \in J} \Delta(A_j \setminus X_j)$. Since $X \in$ strong-curb(G) and $\alpha_{-i} \in \times_{j \in N\setminus\{i\}} \Delta(X_j)$ (since $\times_{j \in N\setminus\{i\}} \Delta(Y_j) \subsetneq \times_{j \in N\setminus\{i\}} \Delta(X_j)$), at least one member $j^* \in J$ prefers to play a best-response in X against the belief α_{-j^*} than playing according to σ_J. Thus, we have $u_{j^*}(b_{j^*}, \alpha_{-j^*}) > u_{j^*}(\sigma_J, \alpha_{-j^*}^{-J})$ for some $b_{j^*} \in X_j$. Since $u_{j^*}(\sigma_J, \alpha_{-j^*}^{-J}) \geq u_{j^*}(a_{j^*}, \alpha_{-j^*})$ for all $a_{j^*} \in Y_{j^*}(Y \notin$ strong-curb$(G))$, we have $u_{j^*}(b_{j^*}, \alpha_{-j^*}) \geq u_{j^*}(a_{j^*}, \alpha_{-j^*})$ for some $b_{j^*} \in X_{j^*}$, for all $a_{j^*} \in Y_{j^*}$. This contradicts the fact that $Y \in$ strong-curb(G_X) since we have identified a belief $\boldsymbol{\alpha}$ which is such that $BR^{j^*}(\alpha_{-j^*}) \nsubseteq Y$. ■

4 Relationships with other solution concepts

In this section we relate the concepts of (minimal) strong curb sets and (minimal) strong prep sets to the concepts of (strict) strong Nash equilibrium, coalition-proof Nash equilibrium and coalitional rationalizability. We have that (i) the product set of actions chosen in every strict strong Nash equilibrium is a minimal strong curb set; (ii) the product set of actions chosen in every strong Nash equilibrium in pure strategies is a minimal strong prep set. Conversely, (i) for every minimal strong curb set composed of one action per player, the strategy profile in which each player selects this action is a strict strong Nash equilibrium; (ii) for every minimal strong prep set composed of one action per player, the strategy profile in which each player selects this action is a strong Nash equilibrium. Since every strict strong Nash equilibrium is in pure strategies, the support of every strict strong Nash equilibrium belongs to a minimal strong curb set. One may wonder whether every strategy in the support of a strong Nash equilibrium in mixed strategies belongs to a minimal strong prep set. We show in Example 1 that it is not necessarily the case.

Example 1 Consider the following normal-form game G_1.

	L	R
U	$4,1$	$0,0$
M	$0,0$	$4,1$
D	$3,2$	$3,2$

We have that min-strong-curb$(G_1) = \{\{U\} \times \{L\}; \{M\} \times \{R\}\} =$ min-strong-prep(G_1). The mixed strategy profile $(\alpha_1(D) = 1,\ \alpha_2(L) = x)$ is a strong Nash equilibrium of G_1 for $1/4 \leq x \leq 3/4$. Thus, the action D is used in some strong Nash equilibria but does not belong to any minimal strong curb set or any minimal strong prep set.□

The main weakness of the strong Nash equilibrium concept is that it fails to exist in a natural class of games. We have shown that the existence of minimal strong curb sets and minimal strong prep sets is guaranteed in general. One question addressed here is whether minimal strong curb sets or minimal strong prep sets allow us to improve predictions in games in which a (strict) strong Nash equilibrium does not exist. In Example 2, we provide a game in which a (strict) strong Nash equilibrium does not exist but the minimal strong prep (curb) set is a proper subset of the full strategy space.

Example 2 Consider the following normal-form game G_2.

	L	C	R
U	$4,4$	$0,5$	$0,0$
M	$0,3$	$2,2$	$0,0$
D	$0,0$	$0,0$	$a,1$

For $a < 4$ the game G_2 has no strong Nash equilibrium while the minimal strong prep (curb) set is unique: min-strong-prep$(G_2) = \{\{U,M\} \times \{L,C\}\} =$ min-strong-curb(G_2). Indeed, when each player believes that the other player plays in the set, each player's individual best-responses lie in the set. In addition, all coalitional moves outside the set are blocked by player 2. □

Notice that the set of minimal strong prep sets may be composed of more elements than the product set of actions chosen in every strong Nash equilibria even when strong Nash equilibria exist. Consider again the game G_2 for $a \geq 4$. The strategy profile (D,R) is the unique strong Nash equilibrium of the game. The set composed of those actions is thus also a minimal strong prep set. However, $\{U,M\} \times \{L,C\}$ is another minimal strong prep set. As a consequence, the unique strong Nash equilibrium may not be the only reasonable prediction in this game.[3]

The following proposition establishes that if $X \subseteq A$ is a strong prep set and $\alpha \in \times_{i \in N} \Delta(X_i)$ is a strong (or coalition-proof) Nash equilibrium of the subgame restricted to X, then α is a strong (or coalition-proof) Nash equilibrium of the original game.[4]

Proposition 1 *For every game $G = \langle N, (A_i)_{i \in N}, (u_i)_{i \in N} \rangle$, if $X \subseteq A$ is a strong prep set of G and $\alpha \in \times_{i \in N} \Delta(X_i)$ is a strong (coalition-proof) Nash equilibrium of the subgame $G_X = \langle N, (X_i)_{i \in N}, (u_i)_{i \in N} \rangle$, then α is a strong (coalition-proof) Nash equilibrium of the original game G.*

[3]For $a > 4$, the strategy profile (D,R) is the unique strict strong Nash equilibrium of G_2, and the set composed of those actions is thus also a minimal strong curb set. But, $\{U,M\} \times \{L,C\}$ is another minimal strong curb set.

[4]Similarly, it holds that if $X \subseteq A$ is a strong curb set and $\alpha \in \times_{i \in N} \Delta(X_i)$ is a strict strong Nash equilibrium of the subgame restricted to X, then α is a strict strong Nash equilibrium of the original game.

Proof. Consider a game $G = \langle N, (A_i)_{i \in N}, (u_i)_{i \in N} \rangle$. By contradiction, suppose $X \subseteq A$ is a strong prep set of G, $\alpha \in \times_{i \in N} \Delta(X_i)$ is a strong (coalition-proof) Nash equilibrium of the subgame $G_X = \langle N, (X_i)_{i \in N}, (u_i)_{i \in N} \rangle$ but α is not a strong (coalition-proof) Nash equilibrium of the original game G. Since α is not a strong (coalition-proof) Nash equilibrium of the original game G, there exists a coalition $J \subseteq N$ and a strategy profile $\sigma_J \in \times_{j \in J} \Delta(A_j)$ which satisfies $u_i(\sigma_J, \alpha_{-J}) > u_i(\alpha)$ $\forall i \in J$. Since X is a strong prep set of the original game, strategies in the support of the coalitional deviation σ_J belong to the set X_J ($\sigma_J \notin \times_{j \in J} \Delta(A_j \backslash X_j)$). This contradicts the fact that α is a strong (coalition-proof) Nash equilibrium of the subgame $G_X = \langle N, (X_i)_{i \in N}, (u_i)_{i \in N} \rangle$. ■

Corollary 1 *In any two-player game G, every strong prep set contains the support of a coalition-proof Nash equilibrium.*

Proof. Any subgame $G_X = \langle N, (X_i)_{i \in N}, (u_i)_{i \in N} \rangle$ obtained from the original two-player game G by restricting the action set of each player $i \in N$ to a subset $X_i \subseteq A_i$ satisfies the conditions of the Glicksberg's (1952) theorem for existence of a Nash equilibrium in mixed strategy. In two-player games, the set of coalition-proof Nash equilibria is equivalent to the set of Nash equilibria that are not Pareto dominated by another Nash equilibrium. Any subgame G_X of the original game thus contains at least one coalition-proof Nash equilibrium of that subgame. If $X \subseteq A$ is a strong prep set, it follows from Proposition 1 that the coalition-proof Nash equilibria of the subgame G_X are coalition-proof Nash equilibria of the original game. ■

Since the existence of a strict coalition-proof Nash equilibrium in any subgame is not guaranteed, it does not hold that every strong curb set contains the support of a strict coalition-proof Nash equilibrium. But, if X is a strong curb set, it is also a strong prep set, and thus X contains the support of a coalition-proof Nash equilibrium but not necessarily a strict one. Corollary 1 does not generalize to games with more than two players since the existence of coalition-proof Nash equilibria is not guaranteed. However, even when a (strict) coalition-proof Nash equilibrium exists, its support is not necessarily contained in a minimal strong prep (curb) set. This is illustrated through the following example.

Example 3 (Ambrus, QJE 2006) Consider the following normal-form game G_3.

l	L	C	R
U	$2,1,0$	$0,0,0$	$-9,-9,-9$
M	$2,0,1$	$1,0,2$	$-9,-9,-9$
D	$-9,-9,-9$	$-9,-9,-9$	$-9,-9,-9$

c	L	C	R
U	$1,2,0$	$0,2,1$	$-9,-9,-9$
M	$0,0,0$	$0,1,2$	$-9,-9,-9$
D	$-9,-9,-9$	$-9,-9,-9$	$-9,-9,-9$

r	L	C	R
U	$-9,-9,-9$	$-9,-9,-9$	$-9,-9,-9$
M	$-9,-9,-9$	$-9,-9,-9$	$-9,-9,-9$
D	$-9,-9,-9$	$-9,-9,-9$	$-8,-8,-8$

The unique (strict) coalition-proof Nash equilibrium of G_3 is (D,R,r), while the unique minimal strong prep set and the unique minimal strong curb set is: min-strong-curb(G_3) = min-strong-prep$(G_3) = \{\{U,M\} \times \{L,C\} \times \{l,c\}\}$. The predictions obtained under the minimal strong prep (curb) set seem more reasonable than the one given by the coalition-proof Nash equilibrium.□

Bernheim (1984) and Pearce (1984) have proposed an iterative procedure in which at each round strategies that are never best-response are deleted. Strategies that survive this iterative procedure are rationalizable. Contrary to curb sets and prep sets,[5] strong curb sets or strong prep sets may include strategies that are strictly dominated or even not rationalizable.

Example 4 Consider the prisoners dilemma G_4.

	L	R
U	$2,2$	$0,3$
D	$3,0$	$1,1$

We have that the action U (L) is strictly dominated for player 1 (2) but belongs to the unique minimal strong prep (curb) set of G_4: min-strong-curb$(G_4) = \{\{U,D\} \times \{L,R\}\}$ = min-strong-prep(G_4). □

Ambrus (2006) has extended the concept of rationalizability by considering also the deletion of strategies by groups of players. Strategies that survive the iterative deletion

[5] Basu and Weibull (1991) and Voorneveld (2004) have shown that every strategy contained in a minimal curb set or in a minimal prep set is rationalizable. The set of rationalizable strategies coincide with the maximal tight curb set where tight curb sets are curb sets which are identical with their own best replies.

of never-best response strategies of coalitions are coalitionally rationalizable. We first provide an example where minimal strong curb sets and minimal strong prep sets have more cutting power than the concept of coalitional rationalizability.

Example 5 (Ambrus, QJE 2006) Consider the following normal-form game G_5.

	L	C	R
U	$-2, 1$	$-1, 0$	$1, -2$
M	$0, -1$	$0, 0$	$0, -1$
D	$1, -2$	$-1, 0$	$-2, 1$

In G$_5$ the strategy profile (M, C) is a strict strong Nash equilibrium and min-strong-curb(G_5) = min-strong-prep(G_5) = $\{\{M\} \times \{C\}\}$. However, any action profile is coalitionally rationalizable.□

We now provide an example showing that the converse may also be true: coalitional rationalizability may have more cutting power than minimal strong curb sets and minimal strong prep sets.

Example 6 (Ambrus, QJE 2006) Consider the following normal-form game G_6.

l	L	R
U	$2, 2, 2$	$0, 0, 0$
D	$0, 0, 0$	$3, 3, 0$

r	L	R
U	$0, 0, 0$	$0, 0, 0$
D	$0, 0, 0$	$1, 1, 1$

The game G_6 has a unique coalitionally rationalizable strategy profile which is (D, R, r). Player 1 and player 2 both recognize that they have a dominant strategy profile (D, R). Anticipating this choice, player 3 selects r. On the other hand, $\{D\} \times \{R\} \times \{r\}$ is nor a strong prep set nor a strong curb set since the deviation of the three players from it to (U, L, l) is Pareto improving. The only strong prep set and strong curb set of G_6 is A, the full strategy space.□

5 Learning by forgetful players

In this section we propose a dynamic learning process similar to the one proposed by Hurkens (1995) by adding the possibility that groups of players may coordinate their actions. Players observe actions played recently and form their beliefs upon these observations. Groups of players may select a joint action if by doing so, the expected payoff of each member of the group is increased with respect to the payoff he would have obtained by playing an individual best-response to his belief. We will show that this learning process leads the players to playing strategies from a minimal strong curb set.

A finite Markov chain is characterized by a pair (X, P), where X is a finite state space and $P : X \times X \to [0, 1]$ is a transition matrix. The interpretation is that $P(x, x')$ is the probability that the process will move from x to x' in one period. We will denote $x \rightsquigarrow x'$ if there exists $k \in \mathbb{N} \cup \{0\}$, $x_0, ..., x_k \in X$ with $x_0 = x$, $x_k = x'$, and $P(x_i, x_{i+1}) > 0$ $(i = 0, ..., k-1)$. Now $\rightsquigarrow$ defines a weak order on X. We can define an equivalence relation on X: $x \sim y \Leftrightarrow x \rightsquigarrow y$ and $y \rightsquigarrow x$. Let $[x]$ denote the equivalence class that contains x and let $Q = \{[x] \mid x \in X\}$ denote the set of equivalence classes. A partial order $\preceq$ on Q is given by: $[x] \preceq [y] \Leftrightarrow y \rightsquigarrow x$. The minimal elements with respect to the order $\preceq$ are called ergodic sets. The other elements are called transient sets. If the process leaves a transient set it can never return to that set. And if the process is in an ergodic set it can never leave this set. The elements of these sets are called ergodic and transient states, respectively. A very useful result is that in any finite Markov chain, no matter where the process starts, the probability after k steps that the process is in an ergodic state tends to 1 as k tends to infinity.

Fix a positive integer K. Suppose we have a finite population of individuals that is partitioned into non-empty classes $V_1, ..., V_n$. The members of V_i are candidates to play role i in the game, and they all have the same payoff function u_i. Let $t = 0, 1, 2, ...$ denote successive time periods. The game G is played once every period. In period t, one individual is drawn from each class V_i, and players are partitioned into coalitions to form a coalition structure. Every possible coalition structure has a positive probability to occur at each period. We refer to the individual that is drawn from V_i to play the game in the current period as player i, although the identity of this player may change from time to time. Player i receives some, but not necessarily all, information about play in the recent K periods. Then he chooses a strategy according to some rule. Then the players are put back in their class. This ends period t and we move up to period $t + 1$.

Since we assume that all the rules are time-independent, this learning process can be described by a stationary Markov chain on the state space $H = A^K$. Call $\widehat{h} \in H$ a successor of $h \in H$ if $\widehat{h}$ is obtained from h by deleting the leftmost element and by adding some element $a \in A$ to the right. Let $r(\widehat{h})$ denote the rightmost element of $\widehat{h} \in H$. For $h = (a^{-K}, ..., a^{-1}) \in H$, let $\pi_i(h) = \{a_i^{-K}, ..., a_i^{-1}\}$ denote the set of strategies played by player i in the recent past.

The learning process is described by a transition matrix $P \in \mathcal{P}$, where $\mathcal{P}$ is the set of transition matrices P that satisfy for all histories h, $\widehat{h} \in H$, $P(h, \widehat{h}) > 0$ if and only if (i) $\widehat{h}$ is a successor of h, (ii) for some $\mathbf{J} \in \mathbb{J}$ and $\boldsymbol{\alpha} = (\alpha_{-1}, ..., \alpha_{-n})$ with $\alpha_{-i} \in \times_{l \in N \setminus \{i\}} \Delta(\pi_l(h))$,

$i \in N$, we have $r(\widehat{h}) = \cup_{J \in \mathbf{J}} a_J$ such that $a_J \in supp(\beta_J)$ for $\beta_J \in CBR^J(\boldsymbol{\alpha})$ if $CBR^J(\boldsymbol{\alpha}) \neq \varnothing$, while $a_J \in \cup_{i \in J} BR^i(\alpha_{-i})$ if $CBR^J(\boldsymbol{\alpha}) = \varnothing$.[6] In words, from period to period each player chooses a strategy. This strategy can be chosen individually or in group, and is chosen by observing recent past play. When groups of players coordinate their move, they choose a strategy profile which is such that all members of the group benefit from playing jointly. In state h, if coalition $J \subseteq N$ has a best-response in mixed strategy $(\beta_J \in CBR^J(\boldsymbol{\alpha}))$ given a profile of beliefs with support in the set of strategies played in the recent past ($\boldsymbol{\alpha} = (\alpha_{-1}, ..., \alpha_{-n})$ with $\alpha_{-i} \in \times_{l \in N \backslash \{i\}} \Delta(\pi_l(h))$, $i \in N$), then there is a positive probability to move from state h to state h' in which each member of coalition J plays an action $a_i \in A_i$ in the support of the best-response β_J of coalition J ($a_J \in supp(\beta_J)$).

We assume that a sufficient level of diversity exists in the population. Let $C \in$ min-strong-curb(G) be a minimal strong curb set of G. We say that $h \in H$ is a C-history if $h \in C^K$. We call h a strong curb history if it is a C-history for some minimal strong curb set C.

Theorem 4 *There exists $\underline{K} \in \mathbb{N}$ such that for all $K \geq \underline{K}$ and every Markov chain with transition matrix $P \in \mathcal{P}$:*

(i) If $Z \subseteq H$ is an ergodic set then $Z \subseteq C^K$ for some minimal strong curb set C.

(ii) For every minimal strong curb set C there exists exactly one subset $Z \subseteq C^K$ that is ergodic.

Proof. Take $\underline{K} \geq 2M = 2(\Sigma_{i=1}^n |A_i| - (n-1))$ and let $K \geq \underline{K}$. Let $P \in \mathcal{P}$. Let $h^t = (x^{K-t}, ..., x^1, a^1, ..., a^t)$ be a particular history and assume that $span(\{a^1, .., a^t\})$ is not a strong curb set. Then, there exists a partition $\mathbf{J} \in \mathbb{J}$, a profile of beliefs $\boldsymbol{\alpha} = (\alpha_{-1}, ..., \alpha_{-n})$ with $\alpha_{-i} \in \times_{l \in N \backslash \{i\}} \Delta(span\{a_l^1, ..., a_l^t\})$, $i \in N$, and a profile of action $a^{t+1} \in A \backslash span(\{a^1, .., a^t\})$ such that $a^{t+1} = \cup_{J \in \mathbf{J}} a_J^{t+1}$ with $a_J^{t+1} \in supp(\beta_J)$ for $\beta_J \in CBR^J(\boldsymbol{\alpha})$ if $CBR^J(\boldsymbol{\alpha}) \neq \varnothing$, while $a_J \in \cup_{i \in J} BR^i(\alpha_{-i})$ if $CBR^J(\boldsymbol{\alpha}) = \varnothing$. Let $h^{t+1} = (x^{K-t+1}, ..., x^1, a^1, ..., a^{t+1})$. Then $P(h^t, h^{t+1}) > 0$. Starting from an arbitrary history h^1 we can apply this argument repeatedly.

Let $a^1, ..., a^T \in A$ be such that $a^{t+1} \notin span(\{a^1, ..., a^t\})$ for all $t = 1, ..., T-1$. Then $T \leq M$. There exists a $T \leq M$ such that $h^1 \rightsquigarrow h^T = (x^{K-T}, ..., x^1, a^1, ..., a^T)$ and $span(\{a^1, ..., a^T\})$ is a strong curb set. Let $C \subseteq span(\{a^1, ..., a^T\})$ be a minimal strong curb set. Since every strategy in a minimal strong curb set is an element in the support of a coalitional best reply to some belief concentrated on the set and since $K \geq 2M$, there exists

[6] We implictly assume that if a coalition has to move, each member of that coalition is myopic in the sense that it does not try to gain information from the decision made by other players of the coalition.

a set $\{b^1, ..., b^M\}$ that spans C and such that $h^T \rightsquigarrow h^{T+M} = (..., a^1, ..., a^T, b^1, ..., b^M) \rightsquigarrow h^{T+K} = (b^1, ..., b^M, b^M, ..., b^M) \rightsquigarrow h^{T+K+1} = (b^2, ..., b^M, b^M, ..., b^M)$.[7] In words, from h^T, there is a positive probability that each player $i \in N$ draws specific beliefs from $\times_{j \in N \setminus \{i\}} \Delta(span(\{a_j^1, ..., a_j^T\}))$ and are assigned in specific coalitions during M periods in a row such that each player (possibly in coalition) chooses a (an element in the support of a coalitional) best-response to its belief in each period and the process reach $h^{T+M} = (..., a^1, ..., a^T, b^1, ...b^M)$ after M periods. Following the same argument, once in h^{T+M}, with positive probability players draw the right belief from $\times_{j \in N \setminus \{i\}} \Delta(span(\{b_j^1, ..., b_j^M\})$ and are assigned in specific coalitions during $K - M$ periods in a row and the process moves from h^{t+M} to $h^{t+K} = (b^1, ..., b^M, b^M, ..., b^M)$.

We have shown that from any history h, the process can reach in at most $T+K$ periods another history $h^{T+K} = (b^1, ..., b^M, b^M, ..., b^M)$ such that $span(\{b^1, ..., b^M, b^M, ..., b^M\}) = C$. Once in h^{T+K}, players will select actions from C only by definition of $\mathcal{P}$. This implies that the set of C-histories contains an ergodic set, for any minimal strong curb set C. To show that the ergodic set $Z \subseteq C^K$ is unique, notice that we have shown that there exists an history $h \in C^K$ such that for all $h' \in C^K$, we have $h' \rightsquigarrow h$. This implies that h belongs to every ergodic set included in C. Then, the ergodic set included in C has to be unique since the intersection between two ergodic sets is empty. ■

6 Conclusion

Basu and Weibull (1991) have introduced the notion of curb sets which are product sets of pure strategies containing all individual best-responses against beliefs restricted to the recommendations to the remaining players. Voorneveld (2004) has introduced the notion of prep sets which only require that the product sets contain at least one best-response to such beliefs. In this paper we have introduced the concepts of strong curb sets and strong prep sets which are set-theoretic coarsenings of the notion of strong Nash equilibrium. Strong curb sets and strong prep sets require sets to be immune not only against individual deviations, but also against group deviations. We have shown that every game has at least one minimal strong curb (prep) set. We have compared minimal strong curb (prep) sets with strong Nash equilibria, coalition-proof Nash equilibria and the set of coalitionally rationalizable strategies. Finally, we have provided a dynamic learning process leading the players to playing strategies from a minimal strong curb set.

Acknowledgments

[7] We impose $h^{T+K} \rightsquigarrow h^{T+K+1}$ to ensure that b^M can be justified for beliefs concentrated on C.

Ana Mauleon and Vincent Vannetelbosch are Research Associates of the National Fund for Scientific Research (FNRS), Belgium. Vincent Vannetelbosch is Associate Fellow of CEREC, Facultés Universitaires Saint-Louis. Financial support from Spanish Ministerio de Educación y Ciencia under the project SEJ2006-06309/ECON, support from the Belgian French Community's program Action de Recherches Concertée 05/10-331 (UCL), and support of a SSTC grant from the Belgian State - Belgian Science Policy under the IAP contract P6/09 are gratefully acknowledged.

Appendix

A Existence of strong curb sets

We will show that the existence of minimal strong curb sets holds in general. A normal-form game is a tuple $G = \langle N, \{A_i\}_{i\in N}, \{u_i\}_{i\in N}\rangle$, where $N = \{1, 2, \ldots, n\}$ is a finite set of players, each player $i \in N$ has a nonempty set of pure strategies (or actions) A_i and a von Neumann-Morgenstern utility function $u_i : A \to \mathbb{R}$, where $A = \times_{j\in N} A_j$ and $A_{-i} = \times_{j\in N\backslash\{i\}} A_j$. Payoffs are extended to mixed strategies in the usual way. Assuming each A_i to be a topological space, $\Delta(A_i)$ denotes the set of Borel probability measures over A_i. Using a common, minor abuse of notation, α_{-i} denotes both an element of $\times_{j\in N\backslash\{i\}}\Delta(A_j)$ specifying a profile of mixed strategies of the opponents of player $i \in N$, and the probability measure it induces over the set A_{-i} of pure strategy profiles of his opponents. Beliefs of player i take the form of such a mixed strategy profile. Similarly, if $B_i \subseteq A_i$ is a Borel set, then $\Delta(B_i)$ denotes the set of Borel probability measures with support in B_i: $\Delta(B_i) = \{\alpha_i \in \Delta(A_i) \mid \alpha_i(B) = 1\}$. As usual, (a_i, α_{-i}) is the profile of strategies where player $i \in N$ plays $a_i \in A_i$ and his opponents play according to the mixed strategy profile $\alpha_{-i} = (\alpha_j)_{j\in N\backslash\{i\}} \in \times_{j\in N\backslash\{i\}}\Delta(A_j)$. A mixed strategy profile for coalition $J \subseteq N$ is denoted $\alpha_J = (\alpha_i)_{i\in J} \in \times_{i\in J}\Delta(A_i)$, and (α_J, α_{-J}) is the profile of strategies where players belonging to coalition J play according to the mixed strategy profile $\alpha_J \in \times_{i\in J}\Delta(A_i)$ and their opponents play according to the mixed strategy profile $\alpha_{-J} \in \times_{j\in N\backslash J}\Delta(A_j)$. For every $J \subseteq N$, $i \in J$, $B \subseteq A$, $\alpha_{-i} = (\alpha_j)_{j\in N\backslash\{i\}} \in \times_{j\in N\backslash\{i\}}\Delta(B_j)$, we denote by α_{-i}^{-J} the marginal distribution of α_{-i} over B_{-J}.

Let $\mathcal{G}$ be the class of normal-form games $G = \langle N, \{A_i\}_{i\in N}, \{u_i\}_{i\in N}\rangle$ where for each player $i \in N$: (i) A_i is a compact Hausdorff topoligical space; (ii) u_i is sufficiently measurable; (iii) u_i is upper semicontinuous on A_i. Remember that for $i \in N$ and $\alpha_{-i} \in \times_{j\in N\backslash\{i\}}\Delta(A_j)$, $BR^i(\alpha_{-i}) = \{a_i \in A_i \mid u_i(a_i, \alpha_{-i}) \geq u_i(a_i', \alpha_{-i}) \text{ for each } a_i' \in A_i\}$ is the set of pure best responses of player i against α_{-i}. Since every upper semicontinuous function on a compact set achieves a maximum, it follows that each player in a game

$G \in \mathcal{G}$ always has a nonempty set of best responses against an arbitrary belief. However, the set of coalitional best-responses may be empty. Remember that for $\boldsymbol{\alpha} = (\alpha_{-i})_{i \in N}$ with $\alpha_{-i} \in \times_{j \in N \setminus \{i\}} \Delta(A_j)$, $CBR^J(\boldsymbol{\alpha}) = \{\sigma_J \in \times_{i \in J} \Delta(A_i) \mid \forall i \in J,\ u_i(b_i, \alpha_{-i}) \leq u_i(\sigma_J, \alpha_{-i}^{-J})$, with $b_i \in BR_i(\alpha_{-i})\}$ is the set of coalitional best-responses of coalition $J \subseteq N$. We provide an example where the set of coalitional best-reponses is empty:

	L	R
U	$2, 0$	$0, 0$
D	$0, 0$	$0, 2$

Consider the beliefs $\boldsymbol{\alpha} = (\alpha_{-1}, \alpha_{-2})$ with $\alpha_{-1}(L) = 1$ and $\alpha_{-2}(D) = 1$. Then, $BR_1(\alpha_{-1}) = \{U\}$ and $BR_2(\alpha_{-2}) = \{R\}$ and the expected payoffs are $u_1(U, \alpha_{-1}) = 2$ and $u_2(R, \alpha_{-2}) = 2$. Thus, we have that $CBR_{\{1,2\}}(\boldsymbol{\alpha}) = \varnothing$.

A strong curb set is a product set $X = \times_{i \in N} X_i$ where (a) for each $i \in N$, $X_i \subseteq A_i$ is a nonempty set of pure strategies; (b) for each $J \subseteq N$ and each vector of beliefs $\alpha = (\alpha_{-1}, ..., \alpha_{-N})$ of the players with each belief α_{-i} having support in X_{-i}, the product set $X_J = \times_{j \in J} X_j$ contains the support of all coalitional best responses of coalition J against the beliefs of its members: $\forall J \subseteq N, \forall \boldsymbol{\alpha} = (\alpha_{-1}, ..., \alpha_{-N})$ with $\alpha_{-i} \in \times_{l \in N \setminus \{i\}} \Delta(X_l)$, $i \in N$, either $CBR^J(\alpha) = \varnothing$ or $CBR^J(\alpha) \subseteq \times_{j \in J} \Delta(X_j)$.

Theorem Every game $G \in \mathcal{G}$ has a minimal strong curb set.

Proof. Let Q = strong-curb(G) denote the collection of all strong curb sets of G. A is a strong curb set of G since for every $J \subseteq N$ and $\boldsymbol{\alpha} = (\alpha_{-1}, ..., \alpha_{-N})$ with $\alpha_{-i} \in \times_{l \in N \setminus \{i\}} \Delta(A_l)$, $i \in N$, we have either $CBR^J(\alpha) = \varnothing$ or $CBR^J(\alpha) \subseteq \times_{j \in J} \Delta(A_j)$. So Q is nonempty and partially ordered via set inclusion. According to the Hausdorff Maximality Principle, Q contains a maximal nested subset R. For each $i \in N$, let $X_i = \cap_{Y \in R} Y_i$ be the intersection of player i's strategies in the nested set R. The set X_i is nonempty since the conditions of the Cantor intersection principle[8] are satisfied, i.e. (i) the collection $\{Y_i \mid Y \in R\}$ is nested and thus satisfies the finite intersection property and (ii) each Y_i is nonempty and compact for each strong curb set. It remains to prove that $X = \times_{i \in N} X_i$ is a minimal strong curb set. Take $\boldsymbol{\alpha} = (\alpha_{-1}, ..., \alpha_{-N})$ with $\alpha_{-i} \in \times_{l \in N \setminus \{i\}} \Delta(X_l)$, $i \in N$. We have that $CBR^J(\boldsymbol{\alpha}) \cap \times_{j \in J} \Delta(A_j \backslash X_j) = \emptyset$ for $J \subseteq N$ since $CBR^J(\boldsymbol{\alpha}) \cap \times_{j \in J} \Delta(A_j \backslash X_j) = CBR^J(\boldsymbol{\alpha}) \cap (\cup_{Y \in R} \times_{j \in J} \Delta(A_j \backslash Y_j)) = \cup_{Y \in R}(CBR^J(\boldsymbol{\alpha}) \cap \times_{j \in J} \Delta(A_j \backslash Y_j))$ and $CBR^J(\boldsymbol{\alpha}) \cap \times_{j \in J} \Delta(A_j \backslash Y_j) = \emptyset$ for all $Y \in R$ (Y is a strong curb set). This establishes that X is a strong curb set. The fact that it is

[8] In words, the Cantor intersection principle tells us that to show that the intersection of an infinite number of elements of a set Z is nonempty and compact, we just need to show that the intersection is nonempty and compact for every subset of Z composed of finite elements.

minimal follows directly from the fact that R is a maximal nested subset of Q. ■

B Other proofs not in the main text

Proposition For every game $G = \langle N, (A_i)_{i\in N}, (u_i)_{i\in N}\rangle$, if $X \subseteq A$ is a strong curb set of G and $\alpha \in \times_{i\in N}\Delta(X_i)$ is a strict strong Nash equilibrium of the subgame $G_X = \langle N, (X_i)_{i\in N}, (u_i)_{i\in N}\rangle$, then α is a strict strong Nash equilibrium of the original game G.

Proof. Consider a game $G = \langle N, (A_i)_{i\in N}, (u_i)_{i\in N}\rangle$. By contradiction, suppose $X \subseteq A$ is a strong curb set of G, $\alpha \in \times_{i\in N}\Delta(X_i)$ is a strict strong Nash equilibrium of the subgame $G_X = \langle N, (X_i)_{i\in N}, (u_i)_{i\in N}\rangle$ but α is not a strict strong Nash equilibrium of the original game G. Since α is not a strict strong Nash equilibrium of the original game G, there exists a coalition $J \subseteq N$ and a strategy profile $\sigma_J \in \times_{j\in J}\Delta(A_j)$ which satisfies $u_i(\sigma_J, \alpha_{-J}) \geq u_i(\alpha)\ \forall i \in J$. Since X is a strong curb set of the original game, strategies in the support of the coalitional deviation σ_J belongs to the set X_J ($\sigma_J \notin \times_{j\in J}\Delta(A_j \backslash X_j)$). This contradicts the fact that α is a strict strong Nash equilibrium of the subgame $G_X = \langle N, (X_i)_{i\in N}, (u_i)_{i\in N}\rangle$. ■

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