Efficiency in a Class of Multi-Unit Auctions

Gagan Ghosh*
University of Iowa

May 2009

Abstract

We analyze discriminatory auctions with symmetric bidders having demand for two units and single-dimensional signals of private valuations, that is, the valuation for the second unit is a known function of the valuation for the first unit. We show that if the distribution of signals and the valuation function are differentiable then there exists a unique symmetric equilibrium which is differentiable. This equilibrium leads to an inefficient allocation with positive probability.

JEL Classification Numbers: D44, C72

Keywords: multi unit, discriminatory, auctions, efficiency, differentiability

1 Introduction

The study and design of multi-unit auctions has gathered considerable momentum over the past decade. Understanding the efficiency of various auction formats in environments where multiple units of a good are up for sale and the buyers have multi-unit demands is important. In the individual independent private values (IPV) paradigm it has been shown that the Vickrey Auction and the Ausubel Auction are efficient [1].

Despite the allocative efficiency of the Vickrey and the Ausubel auctions, there have been few real-world instances where these formats have been put to use. For

*I am grateful to Professor Srihari Govindan for suggesting the topic for this paper and his inputs. I also thank Professor Ayca Kaya for her continuous encouragement. Discussions with Professor Dan Kovenock greatly aided in improving the paper. Any remaining errors are mine alone.

multi-unit auctions, the use of discriminatory and uniform-price auctions has been more widespread, both being used, for example in the US Treasury Auctions and electricity markets in England and Wales. In this paper we investigate the efficiency of discriminatory auctions.

A first price auction with one unit for sale and symmetric bidders in an IPV paradigm is efficient. However the multi-unit analogue of these auctions is not efficient [4][6], even with symmetric bidders. Bid-pooling i.e. bidders submitting the same bid on multiple units even if they have different valuations for them, it is argued in the literature, results in inefficiency [6]. The question then is whether this inefficiency disappears in the absence of bid-pooling almost everywhere.

To answer this, we develop a model in which there is almost no bid-pooling. By developing such a model we can also investigate the main driving force behind the inefficiency of this auction. We achieve this by comparing this auction with its single unit analogue, the first-price auction.

The two differences between the first price auction and discriminatory auction in an IPV paradigm are the multi-unit nature of demand of bidders and the multi-dimensionality of valuations. To identify the main driving force of inefficiency we remove multi-dimensionality, and analyze the auction for efficiency. We achieve this by assuming that each bidder has a downward sloping demand for different quantities of the object for sale and that valuation for the additional unit is a known function of the valuation for the first unit. We find that efficiency is still not achieved even with single-dimensional private values.

We adopt the model presented in [7]. In this paper the authors provide a necessary condition for inefficiency of these auctions, though it was not the main emphasis of the paper. We are able to strengthen their result by providing a sufficient condition as well. Explicit derivation of equilibrium strategies is not possible, even under the assumption of a known relation between the valuations. However, we are able to prove that the set of distributions and valuation functions that lead to an efficient allocation is empty.

The rest of the paper is organized as follows. Section 2 introduces the model. Section 3 establishes the existence of equilibrium and provides the first order conditions for the bidders maximization problem. Section 4 provides necessary and sufficient conditions for efficiency of such auctions. Section 5 concludes.

2 The Model

Two units of a good are sold in a single-round discriminatory auction with N bidders. Bidder i's valuation is given by the vector $V^i = (v_1^i, v_2^i)$ where v_k^i is bidder i's valuation for the k-th unit for $k = 1, 2$. Each v_1^i is distributed independently and identically over $[0, 1]$ according to a c.d.f., $F(\cdot)$ with density function $f(\cdot)$ for all i. We assume there exists a function $g(\cdot)$ such that $v_2^i = g(v_1^i)$ for all i. The assumptions that will be maintained throughout the paper are

A1 The marginal valuations for the first and second units are given by v and $g(v)$, respectively, where $g(0) = 0$, $g(v) < v$ for $v \in (0, 1)$ and $g(1) \leq 1$

A2 F and g are once differentiable and strictly increasing. Also, $f(v) > 0$ for all v.

The auction uses a pay-your-bids format. Each bidder submits two sealed bids for the two units. After receiving $2N$ bids, two from each bidder, the seller announces the winning bids, which are the two highest bids, and these are the payments that must be made. A bidder can win zero, one or two units. The tie-breaking rule is an equi-probability rule as per the existence theorem [10].

3 Equilibrium

3.1 Existence

The equilibrium concept we use is Bayes-Nash Equilibrium. Reny [10] proves the existence of a pure-strategy equilibrium in which the bidders employ monotonic bidding functions in multi-unit, pay-your-bid auctions. This is done in a general setting with possibly asymmetric bidders and $K \geq 2$ units. Our model is a special case of the set up there, hence we can apply its existence result to the existence of β, which is a symmetric equilibrium strategy, where, $\beta : [0, 1] \to \mathbb{R}_+^2$, and $\beta_k(\cdot)$ is the bid on the k-th unit

We provide proofs of continuity and differentiability of the equilibrium bidding functions in the N bidder case in Proposition A.5 in Appendix.

Proposition 1 *β is differentiable.*

Note that this proves the uniqueness of equilibrium in our auction game, since any symmetric equilibrium is differentiable and the first order conditions (f.o.c.)

stated below give a unique solution with the initial condition $\beta(0) = (0, 0)$.[1]

3.2 First-Order Conditions

Let b_k^i represent bidder i's bid on the k-th unit. Suppose the other players are using an equilibrium bidding strategy $\beta = (\beta_1, \beta_2)$. Note that $\beta_1(v) \geq \beta_2(v)$. The first (highest) bid of bidder i competes with the second order statistic of the $2(N-1)$ bids, and the second bid competes with the highest of the $2(N-1)$ bids. Given these details, it is easy to write down the expected payoff function, Π_i for bidder i. For notational parsimony we will write v as bidder i's valuation for the first unit. Hence we have,

$$\Pi = (\text{Pr of winning first unit})(v - b_1) + (\text{Pr of winning second unit})(g(v) - b_2) \quad (1)$$

Bidder i views the rival bids as random variables that depend on the value distributions of the other bidders. Let $H_j(b)$ denote the c.d.f. of j-th order statistic of bids of the opponents. Therefore the probability of winning one unit is $H_2(b_1)$ and the probability of winning a second unit is $H_1(b_2)$ where

$$\begin{aligned} H_2(b_1) &= F(\beta_1^{-1}(b_1))^{N-1} + (N-1)F(\beta_1^{-1}(b_1))^{N-2}(F(\beta_2^{-1}(b_1) - F(\beta_1^{-1}(b_1)) \\ H_1(b_2) &= F(\beta_1^{-1}(b_2))^{N-1} \end{aligned}$$

The first term in $H_2(b_1)$ is the probability that bidder i's first bid defeats $N-1$ first bids. The second term is the probability the bidder i's first bid beats $N-2$ bidders' first bids and one bidder's second bid but not her first. The second states that the only way bidder i can win the second unit as well is, if her second bid defeats everyone's first bid.

We have the payoff function of the bidder in terms of her bids. By taking first order conditions with respect to b_1 and b_2, subject to $b_1 \geq b_2$ we get two equations which we shall use for deriving many of the results in the paper. These equations

[1]Given, that the equilibrium is strictly increasing and the distribution of valuations has positive density everywhere, the system of ordinary differential equations which charecterize the equations satisfy Lipshitz's condition. Therefore by Picard's existence theorem we know the solution is unique.

are as follows[2]

$$\frac{(N-2)(F(\beta_2^{-1}(b_1)) - F(\beta_1^{-1}(b_1)))f(\beta_1^{-1}(b_1))}{F(\beta_1^{-1}(b_1))\beta_1'(\beta_1^{-1}(b_1))} + \frac{f(\beta_2^{-1}(b_1))}{\beta_2'(\beta_2^{-1}(b_1))}$$
$$= \frac{(N-1)F(\beta_2^{-1}(b_1)) - (N-2)F(\beta_1^{-1}(b_1))}{(N-1)(v-b_1)} \qquad (2)$$

$$\frac{(N-1)F(\beta_1^{-1}(b_2))^{N-2}f(\beta_1^{-1}(b_2))}{\beta_1'(\beta_1^{-1}(b_2))} = \frac{F(\beta_1^{-1}(b_2))^{N-1}}{(g(v)-b_2)} \qquad (3)$$

where equation (2) is the differential equation with respect to b_1 and the equation (3) is with respect to b_2. Using these first order conditions and the fact that the bid functions are differentiable we can prove some useful facts about the bid functions.

Lemma 2 *The equilibrium satisfies the following properties*

(i) $\beta_1(v) > \beta_2(v)$ *for all* $v \in (0,1)$ *and*

(ii) $\beta_1(v) = \beta_2(v)$ *for* $v = 0, 1$.

Proof: (i) Let $v' = \inf\{v \mid \beta_1(v) = \beta_2(v),\ 0 < v < 1\}$. Suppose $v' \in (0,1)$. Then $\beta_1(v') = \beta_2(v') = b'$ and, for all $v \in (0, v')$, $\beta_1(v) > \beta_2(v)$. Since β_1 and β_2 are continuous the f.o.c. (2) and (3) will hold with equality at v'. Substituting for symmetric equilibrium and $\beta_1(v') = \beta_2(v')$ in equations (2) and (3) we get

$$\beta_1'(v') = \frac{(N-1)F(v')^{N-2}f(v)(g(v)-b')}{F(v')^{N-1}} \qquad (4)$$

$$\beta_2'(v') = \frac{(N-1)F(v')^{N-2}f(v)(v-b')}{F(v')^{N-1}} \qquad (5)$$

From here it can be easily seen that $\beta_2'(v') > \beta_1'(v')$. Since β is differentiable (Proposition A.5), this implies that $\beta_2(v+\epsilon) > \beta_1(v+\epsilon)$ for all small $\epsilon > 0$ which is impossible since $\beta_1 \geq \beta_2$ by definition. Hence, $v' = 0$.

Suppose there exists $\gamma \in (0,1)$ such that $\beta_1(\gamma) = \beta_2(\gamma)$. Then it must be the case that for all $v \in [0,\gamma]$, $b_1 = b_2$. If not, then there exists $\hat{v} \in (0,\gamma)$ such that $\beta_1(\hat{v}) > \beta_2(\hat{v})$. So, the f.o.c. hold with equality at $\hat{v}$. Then, by a similar argument as in the previous paragraph, there can be no point $v \in (\hat{v}, 1)$ such that $\beta_1(v) = \beta_2(v)$,

[2] We are omitting writing the f.o.c.s if $b_1 = b_2$. As will be proved presently the condition $b_1 \geq b_2$ never binds in the interior of the support.

as this would imply $\beta_2'(v) > \beta_1'(v)$ for the infimum of such points. Hence the only way bid-pooling can occur in this model is in an interval around zero.

Suppose such a $\gamma > 0$ exists. Therefore, there exists a $\hat{b} > 0$, such that $H_1(b) = H_2(b) = H(b)$ for all $b \leq \hat{b} = \beta_1(\gamma) = \beta_2(\gamma)$. We can let γ be arbitrarily small.

Consider a bidder with valuation $v \in (0, \gamma)$. We will prove that this bidder will bid differently on the two units if his opponents pool bids on $[0, \gamma]$. This bidder's payoff from bidding the same on the two units is given by

$$\Pi = H(b)(v + g(v) - 2b)$$

Now, suppose this bidder decides to bid ϵ more on the first unit and ϵ less on the second unit such that $b + \epsilon \leq \hat{b}$, $b - \epsilon \geq 0$. The new payoff is now

$$\bar{\Pi} = H(b + \epsilon)(v - b - \epsilon) + H(b - \epsilon)(g(v) - b + \epsilon)$$

The relative change in payoff is given by

$$\frac{\bar{\Pi} - \Pi}{\epsilon} = \frac{H(b + \epsilon) - H(b)}{\epsilon}(v - b) - \frac{H(b) - H(b - \epsilon)}{\epsilon}(g(v) - b) + H(b - \epsilon) - H(b + \epsilon)$$

Evaluating the above expression in the limit

$$lim_{\epsilon \to 0} \frac{\bar{\Pi} - \Pi}{\epsilon} = h(b)(v - g(v)) > 0$$

where h is the density of the bid distribution which exists since the bid distributions are differentiable and the inequality follows from the fact that $v > g(v)$. Hence there can be no such interval. ■

(ii) Suppose not. Then $\beta_1(1) > \beta_2(1)$. Let $v = \beta_1^{-1}(\beta_2(1))$ and $\beta_1(v) = b$. Recall, the probability of winning at least one unit is

$$H_2(b) = F(\beta_1^{-1}(b))^{N-1} + (N - 1)F(\beta_1^{-1}(b))^{N-2}(F(\beta_2^{-1}(b)) - F(\beta_1^{-1}(b)))$$

For $b' \geq b$, the term $F(\beta_2^{-1}(b'))$ is equal to one. We know that H_2 must be differentiable for all b from Corollary A.6 in the Appendix. If we take the left hand

and right density of H_2 at b, and equate them we get

$$\frac{f(1)}{\beta_2'(1)} = 0$$

We know that $f(1) > 0$. Hence the above equation can be satisfied if $\beta_2'(1) = \infty$, which is a contradiction to Proposition A.5. ■

We now know that in our model there is no bid pooling occurring except at the boundaries. Hence, the bids are separated almost everywhere.

4 Efficiency

Efficiency in our model requires that the two units be allocated to the two highest of the $2N$ values. In this section we provide the efficiency condition along with the main result of the paper.

Theorem 3 *The discriminatory auction leads to an inefficient allocation with positive probability.*

This theorem will be proved through a series of lemmas. We first provide a necessary and sufficient condition for efficiency in terms of the equilibrium bid functions. This condition, along with the f.o.c.s, implies a further condition on F and g which then is shown to be violated. We now formally state the lemmas and provide a proof of Theorem 3.

For our model, the necessary and sufficient condition for efficiency in terms of the bid functions is summarized in the following proposition.

Lemma 4 *The equilibrium allocation is efficient if and only if $\beta_2 = \beta_1 \circ g$.*

Proof : We first prove the necessity of the condition. Suppose $\beta_2(v) > \beta_1(g(v))$ for some v. Let $\epsilon = \beta_2(v) - \beta_1(g(v)) > 0$. Now consider two individuals with valuations for the first unit being v and v' where $v' \in (g(v), \beta_1^{-1}(\beta_2(v))$. This means $g(v) < v' < v$, which implies bidder both bidders should get one unit. However, note that $\beta_2(v) > \beta_1(v')$. Hence, the bidder with valuation v gets both units. Therefore the allocation is inefficient. We can similarly prove a contradiction if $\beta_2(v) < \beta_1(g(v))$.

The sufficiency condition is easily checked. Since $\beta_1(g(v)) = \beta_2(v)$ for all v and for bidders and the bid function is increasing and continuous, the auction allocation will always be efficient given the assumption that $v > g(v)$. ■

This condition implies that the allocation of a discriminatory auction is efficient if and only if the bidders use a single increasing function, in this case $\beta_1(\cdot)$, for bidding on both of the units. This corresponds with Proposition 13.3 in Krishna [[3], p. 184] Using this efficiency condition and the first order conditions we can state the following lemma.

Lemma 5 *A discriminatory auction is efficient only if the c.d.f.s of the valuations satisfy $F(v) = cF(g(v))$ for all v and some $c > 1$.*

Proof: To prove necessity of the lemma we apply conditions of symmetry of equilibrium and $\beta_1(g(\cdot)) = \beta_2(\cdot)$ to the f.o.c.s. From equation (2) we get

$$\beta_1(g(v)) = \frac{1}{M(v)} \int_0^v g(x)dM(x) \tag{6}$$

and from equation (3) we get

$$\beta_1(g(v)) = \frac{1}{F(g(v))^{N-1}} \int_0^v g(x)dF(g(x))^{N-1} \tag{7}$$

where $M(v) = (N-1)F(v)F(g(v))^{N-2} - (N-2)F(g(v))^{N-1}$. Hence

$$\frac{1}{M(v)} \int_0^v g(x)dM(x) = \frac{1}{F(g(v))^{N-1}} \int_0^v g(x)dF(g(x))^{N-1} \, \forall v \tag{8}$$

Since this is true for all v, the slope of the LHS function must be equal to slope of the RHS function at all v. Taking derivatives on both sides and simplifying further we get

$$\frac{f(v)}{F(v)} = \frac{f(g(v))g'(v)}{F(g(v))} \tag{9}$$

i.e.

$$\log F(v) = \log cF(g(v)) \tag{10}$$

where c is some constant. Also, since $g(v) < v$ we know that $c > 1$. Hence we get our desired results $F(v) = cF(g(v))$. ■

Using Lemma 5 we can prove Theorem 3. The idea is that condition $F(v) = cF(g(v))$ cannot be satisfied if $g(1) = 1$. Hence $g(1) < 1$ is required for efficiency. However this fact provides a contradiction.

Proof of Theorem 3: Lemma 6 implies the auction can be efficient only if $g(1) < 1$ since $F(1) = cF(g(1))$ for efficiency and $c > 1$.

We know that $\beta_1(1) = \beta_2(1)$. Efficiency requires $\beta_2(v) = \beta_1(g(v))$, for all v. Specifically, $\beta_2(1) = \beta_1(g(1))$. Since β_1 is strictly increasing and $g(1) < 1$, $\beta_1(g(1)) < \beta_1(v)$ for all $v \in (g(1), 1)$. But $\beta_1(v) < \beta_1(1) = \beta_2(1)$ which gives us the desired contradiction. Hence no function pair (F, g) can lead to efficiency. ■

5 Conclusion

Working within the independent private values paradigm we examined a single-round discriminatory auction of two units to N bidders who have demand for both the units. These auctions have been proven to be typically inefficient. We looked for the source of this inefficiency by converting the multi-dimensional valuations to a single dimension. We showed that the source of inefficiency is the multi-unit nature of the demand and not the multi-dimensionality of signals.

Appendix

First, we will show that the equilibrium bid functions are strictly increasing. This will make proving the continuity of the equilibrium straightforward.

Lemma A.1 *$\beta_1(v)$ and $\beta_2(v)$ are strictly increasing for all $v \in [0, 1]$*

Proof. The idea is that there will be no 'flat' portions in the bid functions and hence they will be invertible. From Reny[10] we know that there exists a monotonically increasing equilibrium bidding strategy for a discriminatory auction. We want to show that this equilibrium is in fact strictly monotone. First, let us prove this for β_1.

From equation (1) it can be seen that the expected payoff from winning the first unit can be written as

$$\Pi_1 = H_2(b)(v - b)$$

where v is the marginal valuation of the first unit and b is first bid.

Now suppose there exist $v' > v$ such that $\beta_1(v) = \beta_1(v') = b$. Since β_1 is monotonic, $\beta_1(v'') = b$ for all $v'' \in [v, v']$. For all bidders with marginal valuations on the first unit in this interval there is positive probability of tieing with some other bidder or bidders[3]. Let this probability be $H_{2t}(b) > 0$. We can show that any bidder with valuation in this interval would prefer to bid strictly more than b

[3] Bids on first units compete with bids on second units as well, however for this proof we do not need β_2 to be strictly monotonic. We can prove a contradiction without involving the second bids.

Consider a bidder with valuation $\bar{v} \in [v, v']$. If he decides to bid $\epsilon > 0$ more on his first unit, there is a discrete jump in his probability of winning the first unit of at least $H_{2t}(b)$. So his gain from increasing his bid is greater than or equal to $H_{2t}(b)(\bar{v}' - b - \epsilon)$ (since he wins with a higher probability) and his loss from having to pay more is $\epsilon H_2(b+\epsilon)$. It is easily verified that there exists an $\epsilon > 0$ such that $\epsilon H_2(b+\epsilon) < H_{2t}(b)(\bar{v}' - b - \epsilon)$. Hence β_1 is strictly increasing. Carrying out an analogous argument it can be proven that β_2 is strictly increasing. ■

Now, to prove continuity we will use the fact that this equilibrium is a Nash equilibrium, that it is monotonically increasing and that $\beta_1(v)$ is greater than or equal to $\beta_2(v)$ for all v .

Lemma A.2 *$\beta_1(v)$ is a continuous function for all $v \in [0, 1]$.*

Proof. We will prove the left continuity of β_1. The right continuity can be proved similarly.

Suppose β_1 is not left continuous at some $v \in (0, 1]$, then $\lim_{v_n \to v_1} \beta_1(v_n) \neq \beta_1(v)$ for some sequence $\{v_n\}_{n=0}^{\infty}$. Since β_1 is monotonic $\lim_{v_n \to v_1} \beta_1(v_n) = b_1$ exists and is smaller than $\beta_1(v)$. Consider the interval $\Delta b = (b_1, \beta_1(v))$ We claim that, $\beta_2(v') \cap \Delta b = \phi$ for all v'. Indeed if there exists $\bar{v}$, such that $\beta_2(\bar{v}) \in \Delta b$, then the individual whose valuation is $\bar{v}$, can do strictly better by reducing his bid by $\epsilon > 0$, as his second bid will still defeat the same mass of first bids but he will pay less for the second unit.

There is no second bid in Δb. Now consider the bidder whose valuation is v. This individual can do strictly better by bidding $\beta_1(v) - \epsilon$, where $\epsilon > 0$. The first bid of any bidder competes with $N-2$ first bids and 1 second bid. However in the region Δb there are no first or second bids. So the bidder can bid less, still defeat the same mass of first and second bids and pay less if he wins. In the limit, $\beta_1(v) = \lim_{v_n \to v_1} \beta_1(v_n)$. This proves the left-continuity of β_1. Similarly we can prove that β_1 will be right continuous. Hence β_1 is continuous. ■

Lemma A.3 *$\beta_2(v)$ is a continuous function for all $v \in [0, 1]$*

Proof. We will prove the left continuity of β_2. The right continuity can be proven analogously.

Suppose there exists v' such that $b = \lim_{v_n \to v} \beta_2(v_n) < \beta_2(v') = b'$. Consider a bidder with a valuation for the first unit equal to $v = \beta_1^{-1}(b)$. The probability that this bidder wins one unit is given by

$$H_2(b) = F(\beta_1^{-1}(b))^{N-1} + (N-1)F(\beta_1^{-1}(b))^{N-2}(F(\beta_2^{-1}(b)) - F(\beta_1^{-1}(b)))$$

Notice that since β_2 is discontinuous at v', H_2 will not be differentiable at b. Specifically, $h_{2-}(b) > h_{2+}(b)$[4], where $h_{2-}(b)$ is the left hand slope of $H_2(b)$ and $h_{2+}(b)$ is the right hand

[4]Since the distribution F has positive density everywhere it is easy to show that if β_2 is discontinuous at v' then $h_{2-}(b) \neq h_{2+}(b)$

slope.

It is this difference in bid densities that causes bidder with valuation v to bid differently. Let us consider the expected payoff of the bidder from winning one unit

$$\Pi_1 = H_2(b)(v - b)$$

If this bidder bids $\epsilon > 0$ more or less on first unit his payoffs are approximately

$$\begin{aligned}\Pi_{1+} &= (H_2(b) + \epsilon h_{2+})(v - b - \epsilon) \\ \Pi_{1-} &= (H_2(b) - \epsilon h_{2-})(v - b + \epsilon)\end{aligned}$$

respectively. It is easily verified that if $\Pi_{1+} \leq \Pi_1$ then it must be the case that $\Pi_{1-} > \Pi_1$, a contradiction to the fact that the bidder is maximizing his payoff by bidding b on the first unit. ■

In order to prove differentiability of the bidding functions we need to show that slope of the functions if well defined at a point will be positive and finite. Since the bid functions will be increasing therefore the slopes will be positive. We now provided the proof for finiteness of the slopes.

Lemma A.4 *β_1 has positive and finite slope, i.e.*

(i) $\lim_{v_k \to v} \frac{\beta_1(v) - \beta_1(v_k)}{v - v_k} < +\infty$, *for all* $v \in [0, 1]$.

(ii) $\lim_{v_k \to v} \frac{\beta_1(v) - \beta_1(v_k)}{v - v_k} > 0$, *for all* $v \in (0, 1]$.

Proof. (i) Any bidder's bid on the first unit will always be less than the valuation for the first unit. Hence β_1 will always be below the diagonal. Also, $\beta_1(0) = 0$. Therefore the slope of β_1 at zero will be less than one.

Suppose there exists a $v \in (0, 1]$, such that $\lim_{v_k \to v} \frac{\beta_1(v) - \beta_1(v_k)}{v - v_k} = +\infty$ for some sequence $\{v_k\}_{k=1}^{\infty}$. Now consider a bidder whose valuation is given by,[5] $\beta_2^{-1}(\beta_1(v)) = \hat{v}$. Consider this bidder at $\hat{v}$ reducing his bid on the second unit to $\beta_1(v_k)$, where $v_k \uparrow v$. The benefit of this decrease for any k is given by

$$(\beta_1(v) - \beta_1(v_k))F(v_k)^{N-1} \tag{11}$$

The cost of this bid reduction is given by

$$(g(v) - \beta_1(v))(F(v)^{N-1} - F(v_k)^{N-1}) \tag{12}$$

[5]At this point we allow for $\beta_1(1) \neq \beta_2(1)$. We will subsequently prove that this can not be the case. Hence if $\beta_1(v) > \beta_2(1)$ then we can look at the bidder whose valuation for the first unit is v and the same proof with adjustments to the payoff function will work.

For large k the second term in the above expression can be approximated by the density $(N-1)F(v)^{N-2}f(v)(v-v_k)$, The ratio of the benefits to the cost is

$$\frac{(\beta_1(v)-\beta_1(v_k))}{(v-v_k)}\frac{F(v_k)^{N-1}}{(g(v)-\beta_1(v))(N-1)F(v)^{N-2}f(v)}. \tag{13}$$

The second term in the above expression is positive and finite. By our supposition the first term is tending towards infinity as k is increasing. Hence for large K, this ratio is greater than 1. This means that the net benefit of the bid reduction was positive. Hence the bidder wants to reduce his bid on the second unit. A contradiction.

(ii) The proof of this part works in a similar way as that of the previous part. Hence we mention the intuition. Suppose there exists a $v \in (0,1]$, such that $\lim_{v_k \to v} \frac{\beta_1(v)-\beta_1(v_k)}{v-v_k} = 0$ for some sequence $\{v_k\}_{k=1}^{\infty}$. That is, the slope of the bidding function becomes very flat as we get closer to v. Now consider a bidder whose valuation is given by $\beta_2^{-1}(\beta_1(v)) = \hat{v}$. If this bidder[5] increases his bid on the second unit by a little bit then, the mass of bidders (first bids) he defeats is large since the slope at v is zero, and he pays only a little more on winning, hence he would prefer bidding more. This makes β_2 discontinuous, a contradiction. ■

Proposition A.5 *$\beta_1(v)$ and $\beta_2(v)$ are both differentiable functions over* $[0,1]$.

Proof. Differentiability of β_1 at the boundary points is known by the fact that the bid function has finite slopes at these points. We will prove differentiability in the interior of the support.

Suppose $\beta_1(\cdot)$ is not differentiable at some v i.e., the left hand derivative is not equal to the right hand derivative. We can take these derivative since the derivative (slope) at any point (if it exists) is finite (Lemma A.4). We can write the following inequality

$$\beta_1'(v)_- > \beta_1'(v)_+$$

where the right hand side is the right hand derivative which is given by

$$\beta_1'(v)_+ = \lim_{\epsilon \to 0} \frac{\beta_1(v+\epsilon)-\beta_1(v)}{\epsilon}$$

which can be approximated by

$$\beta_1'(v)_+ \approx \frac{\beta_1(v+\epsilon)-\beta_1(v)}{\epsilon}, \text{ for } \epsilon > 0, \ \epsilon \text{ small} \tag{14}$$

and the left hand side is similarly given (approximately) by

$$\beta_1'(v)_- \approx \frac{\beta_1(v)-\beta_1(v-\epsilon)}{\epsilon}, \text{ for } \epsilon > 0, \ \epsilon \text{ small} \tag{15}$$

The CDF of the first order statistic of the first valuations, other than the bidder's own valuation is given by $F(v)^{N-1}$. Let us call $J(v) = F(v)^{N-1}$. Then $j(v) = (N-1)F(v)^{N-2}f(v)$, is a well defined density function.
Consider a bidder whose valuation for the first unit is given by

$$v' = \beta_2^{-1}(\beta_1(v)) \tag{16}$$

then his bid on the second unit is $\beta_2(v') = \beta_1(v)$. We will show a contradiction here[6]
The payoff on the second unit from bidding is

$$\Pi_2 = F(\beta_1^{-1}(\beta_2(v')))^{N-1}(g(v') - \beta_2(v')) \tag{17}$$

which can be rewritten as

$$\Pi_2 = J(v)(g(v') - \beta_1(v))$$

Suppose bidder at v' decides to bid $\beta_2(v') + \epsilon\beta_1'(v)_+$. His payoff on the second unit is

$$\Pi_{2+} = F(\beta_1^{-1}(\beta_2(v') + \epsilon\beta_1'(v)_+)^{N-1}(g(v') - \beta_2(v') - \epsilon\beta_1'(v)_+) \tag{18}$$

This expression can be simplified after approximating the bids

$$\Pi_{2+} \approx \Pi_2 - J(v)\epsilon\beta_1'(v)_+ + \epsilon j(v)(g(v') - \beta_1(v+\epsilon)) \tag{19}$$

Similarly if the bidder with valuation v' bids $\beta_2(v') - \epsilon\beta_1'(v)_-$ on his second unit then the payoff is given by

$$\Pi_{2-} \approx \Pi_2 + J(v)\epsilon\beta_1'(v)_- - \epsilon j(v)(g(v') - \beta_1(v-\epsilon)) \tag{20}$$

If the bidder with valuation v' is bidding optimally on his second unit then it must be the case that

$$-J(v)\epsilon\beta_1'(v)_+ + \epsilon j(v)(g(v') - \beta_1(v+\epsilon)) \leq 0 \tag{21}$$

taking limits as ϵ goes to zero

$$J(v)\beta_1'(v)_+ = j(v)(g(v') - \beta_1(v)) \tag{22}$$

Since $\beta_1'(v)_- > \beta_1'(v)_+$

$$J(v)\beta_1'(v)_- > j(v)(g(v') - \beta_1(v)) \tag{23}$$

and therefore

$$\epsilon J(v)\beta_1'(v)_- - \epsilon j(v)(g(v') - \beta_1(v-\epsilon)) > 0 \tag{24}$$

[6] If there is no such bidder then we can consider the bidder whose valuation for the first unit is v and prove that β_1 would be discontinuous.

where ϵ is some small positive number. Therefore $\Pi_{2-} > \Pi_2$.A contradiction to the fact that $\beta_2(v')$ is optimal.

Hence β_1, must be differentiable.

Using a similar argument as for β_1 and proving that the slope of β_2 is positive and finite, if β_2 is not differentiable at some point v, then β_1 will not be differentiable at $\beta_1^{-1}(\beta_2(v))$, a contradiction. Hence β_2 is differentiable. ∎

Corollary A.6 *H_1 and H_2 are differentiable.*

Proof. The differentiability of H_1 follows from the differentiability of β_1.

Suppose $H_2(v)$is not differentiable at some b. Without loss of generality assume that at b the left hand density of $H_2(v)$ given by $h_{2-}(b)$ is less than the right hand density given by $h_{2+}(b)$. The payoff from bidding b on the first unit is given by

$$\Pi_1 = H_2(b)(v - b)$$

Now, we will consider the change in the payoff from bidding ϵ more and less on the first unit. The idea is the same as the previous proposition and Lemma A.3. It can be shown the bidder prefers to change his bid from b on the first unit to something lower. This would be a contradiction to the fact that the bidder is maximizing his payoff by bidding b on the first unit. ∎

References

[1] Ausubel, L.M. (2004)"An Efficient Ascending-Bid Auction for Multiple Objects."American Economic Review, Vol. 94, No. 5, 1452-1475

[2] Ausubel, L.M & Cramton P.(1998)" Demand Reduction and Inefficiency in Multi-Unit Auctions"

[3] Chakraborty I.(2006) "Characterization of equilibrium in pay-as-bid auctions for multiple units" Economic Theory. 29: 197–211

[4] Engelbrecht-Wiggans, R., Kahn, C.M.: Multi-unit pay-as-bid auctions with variable awards. Games & Economic Behavior 23, 25–42 (1998a)

[5] Hickman, B (2008), "Electronic Auctions", TYPESCRIPT, University of Iowa.

[6] Krishna, V.(2002). "Auction Theory".Academic Press

[7] Lebrun, B (1999) "First Price Auction in the Asymmetric N Bidder case", International Economic Review, Vol. 40, No. 1, 125-142

[8] Lebrun, B. & Tremblay, M.C.(2003) ”Multiunit Pay-Your-Bid Auctions with One Dimensional Multiunit Demands”, International Economic Review, Vol 44, No. 3, 1135-1172.

[9] Maskin, E. & Riley, J. (2000), ”Asymmetric Auctions”, Review of Economic Studies Vol 67 Issue 3, 413-438.

[10] Reny, P. (1999), “On the Existence of Pure and Mixed Strategy Nash Equilibria in Discontinuous Games”, Econometrica, Volume 67, 1029–1056.

[11] Swinkels, J.M. (1999), ”Asymptotic Efficiency for Discriminatory Private Value Auctions”, Review of Economic Studies, Volume 66 Issue 3 Page 509-528

[12] Vickrey, W. (1961): “Counterspeculation, Auctions, and Competitive Sealed-Tenders,” Journal of Finance, 16, 8–37.