Cooperation in Repeated Prisoner's Dilemma with Outside Options*

by

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Abstract: We examine variants of repeated Prisoner's Dilemma from which players can exit by taking an outside option and investigate effects of outside option structures on the sustainability of cooperation. Although mutual cooperation becomes more difficult in the presence of outside options than in ordinary repeated games, whether the options are perturbed or not makes a difference. Stochastic outside options enhance cooperation as compared to deterministic ones, when the possibility of an attractive option tomorrow makes players patient today. This logic applies to both one-sided and two-sided outside option models, but the effects of stochasticity are weaker in the latter.

Key words: outside option, repeated Prisoner's Dilemma, cooperation, perturbation.
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1 Introduction

In many repeated interactions, repetition is not guaranteed but instead must be agreed upon. Workers can quit, customers can walk away, and couples can break-up. If it is possible to strategically exit from a repeated interaction, the ordinary repeated-game framework no longer applies. Ordinary repeated games assume that the same set of players play the same stage game repeatedly for a fixed (possibly infinite) length of time.[1] Therefore no player has a choice to exit from the game. At the other extreme, there is a random matching game model.[2] The random matching games assume that in every period a player is randomly matched with a new partner. Therefore no player has a choice to continue the game with the same partner. However, many economic situations are in an intermediate case where players can play a game repeatedly, but they can also terminate the interaction. There is a growing literature of these "endogenously repeated" games.

In this literature, three issues have been mainly analyzed. First, under complete information, ordinary trigger strategies do not constitute an equilibrium since cooperation from the beginning of a relationship is vulnerable to defection and running away. Instead, gradual cooperation or trust-building strategy becomes an equilibrium. (Datta, 1996, Kranton, 1996a, Fujiwara-Greve, 2002, and Fujiwara-Greve and Okuno-Fujiwara, 2009.) Second, gradual cooperation is also useful in incomplete information models to sort out the types of players. (Ghosh and Ray, 1996, Kranton, 1996a,b, Watson, 2002, and Furusawa and Kawakami, 2008.) Third, a modified folk theorem holds with appropriate lower bounds to the equilibrium payoffs. (Yasuda, 2007.)

We add a new angle to the analysis of the endogenously repeated games by looking at the interaction between in-game behavior and what a player may receive outside of the game. In game theory, often the outside structure of a game is fixed and the analysis is focused on in-game strategic outcomes given the outside structure.[3] By contrast in other research fields such as search theory and operations research, the main interest lies in the effect of outside structural changes on individual behavior/decision-making, but there is no strategic interaction among decision-makers. In this paper we consider strategic interaction of two players under varying outside structures of the game.

[1]The "supergame" framework has been studied from the early stage of game theory, see for example Friedman (1971), which pioneered to show that cooperation can be an equilibrium behavior.

[2]Kandori (1992) showed that even in a random matching game, "contagion of defection" can support cooperation. For related pioneering works, see also Ellison (1994) and Okuno-Fujiwara and Postlewaite (1995).

[3]However, there are some papers that perturb the game structure to see the effect on in-game behavior. See the discussion below.

P1 \ P2	C	D
C	7, 7	0, 9
D	9, 0	1, 1

Table 1: An Example

Specifically, we examine variants of repeated Prisoner's Dilemma from which players can exit by taking an outside option and investigate effects of outside option structures on the sustainability of cooperation. It turns out that the "locked-in" feature of ordinary repeated game is a very strong cooperation enforcement system. The existence of a relevant outside option (greater than the in-game punishment payoff) increases the necessary level of discount factor to sustain cooperation as compared to the one in ordinary repeated games, and in some cases for any discount factor cooperation is not possible. However, within the outside option model, the relative difficulty of repeated cooperation is dependent on the structure of outside options. In particular, if the option values are uncertain, in some cases it is easier to sustain repeated cooperation than when they are certain. Therefore, perturbation of outside options is not always bad for cooperation.

Let us give an example to explain the logic. In each period, as long as the two players are in the game, they play the Prisoner's Dilemma of Table 1. After playing the Prisoner's Dilemma, an outside option is available to Player 1. Player 2 has no such option. The game repeats (Prisoner's Dilemma and then the outside option to exit) as long as Player 1 does not take the outside option. Suppose that in any period the available outside option is the same, and it gives a stationary sequence of payoff $\{6, 6, \ldots\}$ to Player 1 after exit. Player 2's payoff after Player 1 ends the game is normalized to be zero.

Note that if the game is an ordinary repeated game without the outside option, the infinitely repeated cooperation $(C, C), (C, C), \ldots$ is sustainable by the grim trigger strategy if

$$\frac{7}{1-\delta} \geqq 9 + \delta \frac{1}{1-\delta} \iff \delta \geqq \frac{1}{4}.$$

However, if the outside option is available, Player 1 can choose D and take the option $\{6, 6, \ldots\}$. Therefore, Player 1 may not follow the repeated cooperation $(C, C), (C, C), \ldots$ even if δ is not so small. For example, when $\delta = 0.6$,

$$\frac{7}{1-\delta} = 17.5 < 18 = 9 + \delta \frac{6}{1-\delta}.$$

This illustrates that the existence of an outside option greater than the in-game punishment payoff creates a difficulty in achieving cooperation, in the sense that the range of discount factors that sustains repeated cooperation shrinks.

Next, suppose that Player 1 has two possible outside options of the form $\{6+\epsilon, 6+\epsilon, \ldots\}$ and $\{6-\epsilon, 6-\epsilon, \ldots\}$ (where $\epsilon > 0$), and these arrive with equal probability at the end of each period. The average outside option is 6. When ϵ is small (i.e., less than 1), then there is no point of taking any of the outside options if players are to repeat (C, C). When ϵ is large enough, however, the better outside option exceeds the payoff from the repeated (C, C) so that the infinitely repeated cooperation becomes impossible for any δ. However, Player 1 may cooperate until she receives the better option. Let us compute the total expected discounted payoff of cooperation until the better option arrives. Let V be the continuation value at the end of a period, before an option realizes. Then the total expected payoff of repeating (C, C) until $\{6+\epsilon, 6+\epsilon, \ldots\}$ arrives is of the form $7 + \delta V$, where the continuation value V satisfies the following recursive equation.

$$V = \frac{1}{2} \cdot \frac{6+\epsilon}{1-\delta} + \frac{1}{2}(7 + \delta V).$$

For example, when $\epsilon = 1.5$ and $\delta = 0.6$, then $V \approx 18.39$, and the value of the cooperation is increased to $7 + \delta V \approx 18.03 > 17.5 = 7/(1-\delta)$.

The value of a one-shot deviation also needs to be checked more carefully. The optimal exit strategy for Player 1 is either to exit immediately by taking any option or to wait for $\{6+\epsilon, 6+\epsilon, \ldots\}$. If she deviates and then waits for the good option while suffering from the punishment payoff of 1 in the stage game, the total expected payoff is of the form $9 + \delta W$, where the continuation value W satisfies

$$W = \frac{1}{2} \cdot \frac{6+\epsilon}{1-\delta} + \frac{1}{2}(1 + \delta W).$$

Thus $9+\delta W \approx 17.46$ for $\epsilon = 1.5$ and $\delta = 0.6$. If Player 1 defects and then exits immediately by taking any option, the expected payoff is $9 + \delta \frac{6}{1-\delta} = 18$ as before. Therefore, in this example, it is optimal to exit immediately after a deviation. However, $7 + \delta V > 18 = 9 + \delta \frac{6}{1-\delta}$ implies that Player 1 with $\delta = 0.6$ cooperates on the play path, until the better outside option arrives. We call this play path *stochastic cooperation*. It is better than no cooperation, which is the case if the outside option was deterministic.

The above example shows that deterministic or stochastic structure of outside options makes a difference in sustaining cooperation for mid-range discount factors. Moreover, given a discount factor, we can investigate how the spread ϵ of the outside options affects the equilibrium strategy combination. When ϵ is small, there is no change in equilibrium behavior as compared to the deterministic case. As ϵ increases, the value of cooperation while waiting for the good option increases so that stochastic cooperation becomes an equilibrium behavior. As ϵ increases more, e.g., $\epsilon = 2.5$, then $9 + \delta W > 9 + \delta \frac{6}{1-\delta}$ so that after defection, Player 1 wants to wait for the better option. However, $7 + \delta V > 9 + \delta W$ holds so that the stochastic cooperation continues to be an equilibrium behavior. In general, even if no cooperation is possible under the deterministic case, there exists ϵ above which the stochastic cooperation becomes an equilibrium path. Therefore the perturbation of outside options enhances cooperation.

Moreover, we can vary the probability of the binary options and show that the qualitative results are the same: when the better outside option exceeds the cooperation payoff, the stochastic cooperation becomes an equilibrium for mid-range discount factors. Hence, as the probability of the attractive option decreases, the stochastic cooperation lasts longer, and the play path is almost the *eternal cooperation*, in which the players play (C, C) ad infinitum.

There are a few papers which incorporate perturbations into ordinary repeated games. Rotemberg and Saloner (1986) perturb payoffs of the stage game, while Baye and Jansen (1996) and Dal Bó (2007) perturb the discount factor. In these models the optimal (infinitely repeated) cooperation levels are shown to be lower than the one in the absence of perturbation. In this sense, the perturbations are bad for cooperation. Although they did not investigate the lower bound of the discount factors by fixing a level of cooperation, it would be greater than the one under no perturbation. This is clarified in Yasuda and Fujiwara-Greve (2009).

The key is that the players must play the game forever in these models. Therefore, when the perturbation creates a difficulty to cooperate (a high deviation payoff or a low value of the discount factor), the players need to play a non-cooperative action in that period, which *reduces* the on-path payoff, i.e., the incentive to follow the equilibrium strategy. Therefore the players need to be more patient than in the deterministic case.[4]

By contrast, in our model, Player 1 can choose between playing the game forever and stopping. Thus, when the perturbation creates a difficulty to cooperate (a high outside option), it does not mean that Player 1 must endure the low payoff of a non-cooperative action. The difficulty to cooperate means that stopping the game is more beneficial, and hence she can take that option to *increase* the on-path payoff, i.e., the incentive to follow the equilibrium strategy. Therefore lower discount factors are sufficient to sustain the equilibrium as compared to the deterministic case.

In summary, we have shown that there are perturbations that can increase the value of repeated cooperation, and this occurs naturally in the context of outside options in the

[4] A similar argument is noted in Mailath and Samuelson (2006), p.176-177.

endogenously repeated game.

Lastly, we analyze a two-sided outside option model. We find that, although the logic is mostly analogous to the one-sided option model, the effect of perturbation is weakened for the two-sided independent option model: for the case that cooperation is more difficult than the deterministic case, it becomes easier, and vice versa. The logic is as follows. If both players can unilaterally terminate the game, and if the realization of stochastic options is independent across players, one may end up taking a bad option because the partner exits by taking a good option. This reduces the option value in the punishment phase as well as makes players less patient in the cooperation phase. The case where cooperation is more difficult for the one-sided outside option model is when the punishment phase payoff was increased by the perturbation. This effect is weakened by the two-sided options. The case where cooperation is easier for the one-sided outside option model is when the player with the outside options is more patient than in the deterministic case. This effect is weakened since the game may be terminated by the other player, which makes one less patient.

The outline of the paper is as follows. In Section 2, we formulate the basic one-sided outside option model. In Section 3, we show that, in general, the existence of an outside option makes it harder to cooperate than in the ordinary repeated Prisoner's Dilemma. In Section 4, we consider a one-sided stochastic outside option model and show that stochastic outside options may enhance cooperation, as compared to the deterministic options. In Section 5, we analyze a two-sided outside option model. Section 6 gives concluding remarks.

2 A One-Sided Outside Option Model

Consider a two-player dynamic game as follows. Time is discrete and denoted as $t = 1, 2, \ldots$ but the game continues endogenously. At the beginning of period $t = 1, 2, \ldots$ as long as the game continues, two players, called Player 1 and Player 2, simultaneously choose one of the actions from the set $\{C, D\}$ of the Prisoner's Dilemma. The action C is interpreted as a cooperative action and the action D is interpreted as a defective action. We denote the symmetric payoffs associated with each action profile as[5]: $u(C, C) = c$, $u(C, D) = \ell$, $u(D, C) = g$, $u(D, D) = d$ with the ordering $g > c > d > \ell$ and $2c > g + \ell$. See Table 2. The latter inequality implies that (C, C) is efficient among correlated action profiles.

After observing this period's action profile, Player 1 can choose whether to take an outside option and thus terminate the game or not. The game continues to the next period if and

[5]The first coordinate is the player's own action.

P1 \ P2	C	D
C	c, c	ℓ, g
D	g, ℓ	d, d

Table 2: General Prisoner's Dilemma

only if Player 1 does not take an outside option. We assume that all actions are observable to the players. Therefore, in period $t \geqq 2$, players can base their actions on the history of past action profiles. The outline of the dynamic game is depicted in Figure 1.

As the basic setup, let an outside option be a deterministic, stationary stream of payoffs $\{v, v, \ldots\}$, such that $c > v > d$.[6] One can alternatively assume that an outside option is a one-shot payoff of the form $v/(1-\delta)$, where δ is the common discount factor. Other outside option structures (stochastic, two-sided) are discussed in later sections.

Player 2 receives payoff only from the Prisoner's Dilemma as long as the game continues and Player 2 does not have the ability to end the game, as in the ordinary repeated games. Let us also assume that $d \geqq 0$ which implies that Player 2's "outside payoff" 0 is not better than the payoff from (D, D). This simplifies our analysis by making Player 2's deviation not relevant. (To be precise, the qualitative result does not change as long as Player 2's outside payoff is not greater than v.)

There are many economic situations that fit into this model. For example, in buyer-seller relationships, buyers can be Player 1 and sellers can be Player 2 with the interpretation that C is an honest action in transactions and D is a dishonest action. In employment relationships, when firing is not easy as in some countries, Player 1 corresponds to a worker and Player 2 corresponds to a firm.

We assume that both players maximize the discounted sum[7] of the payoff stream with a common discount factor $\delta \in (0, 1)$. For example, if Player 1 takes the outside option at the end of T-th period, her total payoff is

$$\sum_{t=1}^{T} \delta^{t-1} u(a(t)) + \delta^{T} \frac{v}{1-\delta},$$

while Player 2's total payoff is

$$\sum_{t=1}^{T} \delta^{t-1} u(a(t)),$$

[6]If $v > c$, then (C, C) cannot be played at all, and if $v < d$, then the outside option is never taken so that the game essentially reduces to an ordinary repeated Prisoner's Dilemma.

[7]Alternatively one can assume that the players maximize the average payoffs without changing the qualitative results.

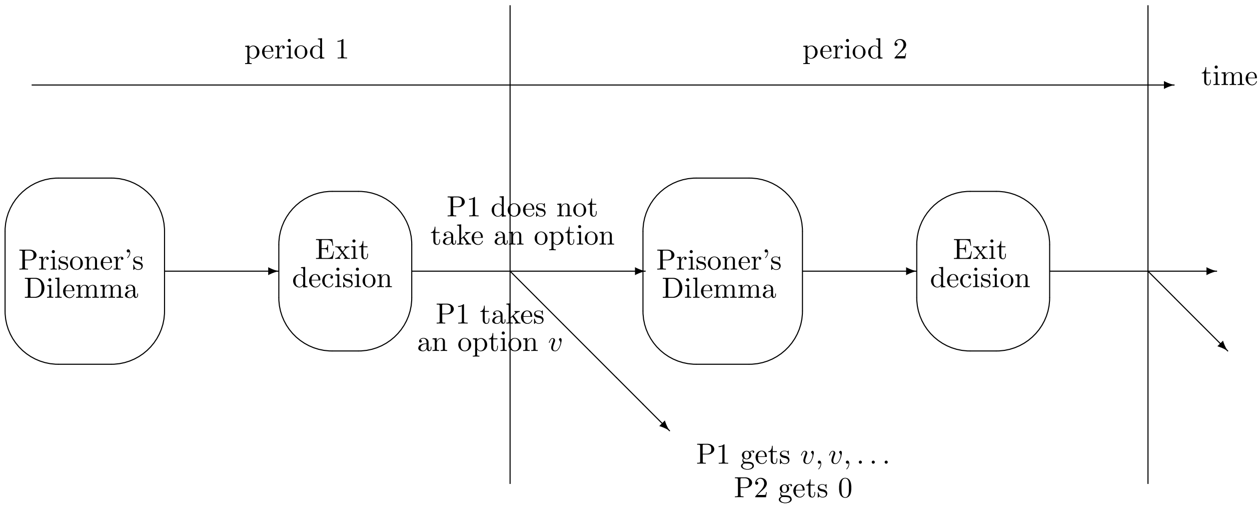


Figure 1: Outline of the Dynamic Game

where $a(t)$ is the action profile in t-th period of the repeated Prisoner's Dilemma.

As the equilibrium concept, we use subgame perfect equilibrium (SPE henceforth). The game is of complete information.

Lemma 1. *The following strategy combination is a SPE for any* $v \in (d, c)$ *and any* $\delta \in (0, 1)$*: In any period of the game, Player 1 and Player 2 play D and Player 1 takes the outside option, regardless of the history.*

Proof: Given the strategy combination, both players get d in every period if they are in the game. Therefore, at any exit decision node, taking the outside option is optimal for Player 1 since $v > d$. Given Player 1's exit strategy, it is optimal for both players to play myopically in every period (if they are still together). □

Notice that Player 1 can guarantee herself the total payoff of $d+\delta\frac{v}{1-\delta}$ against any strategy of Player 2 by choosing D and exiting immediately, while Player 2 can guarantee $d+0$ against any strategy of Player 1. Since the above SPE (called the "myopic SPE" henceforth) achieves exactly these payoffs, it is the maximal equilibrium punishment.

We investigate the range of δ in which repeated mutual cooperation of (C, C) is sustained as long as possible. If the maximal equilibrium punishment does not sustain the on-path action profile, no other punishment would, by the same logic as the optimal penal code in Abreu (1988). Hence it is sufficient to consider the myopic SPE as the punishment. Therefore, in general we consider the following type of strategy combinations, which we call "simple trigger strategy" combinations. Note that Player 1's optimal exit strategy varies depending on the outside option structure.

Cooperation phase: If the history is empty or does not have D, play (C, C) and Player 1 uses an optimal exit strategy given that (C, C) is repeated as long as the game continues.
Punishment phase: If the history contains D, play (D, D) and Player 1 uses an optimal exit strategy given that (D, D) is repeated as long as the game continues.

3 Deterministic Outside Option

When the outside option is deterministic, $c > v$ implies that Player 1's optimal exit strategy in the cooperation phase is not to take the option, and $v > d$ implies that the optimal exit strategy in the punishment phase is to take the option at the first opportunity. Therefore, on the play path of the simple trigger strategy combination, (C, C) is repeated forever, which we call the eternal cooperation. Let us find the lower bound of δ that sustains the eternal cooperation, that is, that makes the simple trigger strategy combination a SPE.

Recall that in the ordinary repeated Prisoner's Dilemma with discounting, the eternal cooperation is sustained by the simple trigger strategy without the exit option if and only if

$$\begin{aligned} \frac{c}{1-\delta} &\geqq g + \frac{\delta d}{1-\delta} \\ \iff \delta &\geqq \frac{g-c}{g-d} =: \underline{\delta}. \end{aligned}$$

In our game, Player 1 does not deviate in the cooperation phase if and only if

$$\frac{c}{1-\delta} \geqq g + \frac{\delta v}{1-\delta} \tag{1}$$

$$\iff \delta \geqq \frac{g-c}{g-v} =: \delta_1^D(v), \tag{2}$$

and Player 2 does not deviate in the cooperation phase if and only if

$$\frac{c}{1-\delta} \geqq g \iff \delta \geqq \frac{g-c}{g} =: \delta_2^D.$$

Let $\delta^D(v) = \max\{\delta_1^D(v), \delta_2^D\}$. Then the simple trigger strategy combination is a SPE if and only if $\delta \geqq \delta^D(v)$. Moreover, $v > d$ implies that $\delta_1^D(v) > \underline{\delta}$, and $d \geqq 0$ implies that $\delta_2^D \leqq \underline{\delta}$. Hence $\delta^D(v) > \underline{\delta}$. This means that, for any $\delta \in [\underline{\delta}, \delta^D(v))$, the existence of outside option makes the eternal cooperation impossible, while it was possible if the game were an ordinary repeated Prisoner's Dilemma. It is also easy to see that $\delta^D(v)$ is increasing in v, implying that better outside option makes it harder to cooperate. Since $\lim_{v\to c} \delta^D(v) = 1$, the range of δ that sustains the eternal cooperation shrinks to the empty set, as the outside option approaches to c.

Proposition 1. *For any $v \in (d, c)$, the eternal cooperation is sustained as the outcome of a SPE if and only if $\delta \geqq \delta^D(v) > \underline{\delta}$. Hence, for any $\delta \in [\underline{\delta}, \delta^D(v))$, the eternal cooperation cannot be sustained in the outside option model, while it is sustainable in the ordinary repeated Prisoner's Dilemma.*

Alternatively, given $\delta \in [\underline{\delta}, 1)$, we can define the highest outside option level $v^*(\delta)$ which makes Player 1 not to deviate in the cooperation phase by

$$\frac{c}{1-\delta} = g + \delta \frac{v^*}{1-\delta}$$
$$\Rightarrow \quad v^*(\delta) := \frac{1}{\delta}\{c - (1-\delta)g\}. \tag{3}$$

Clearly, v^* is increasing in δ, $v^*(\underline{\delta}) = d$, and $\lim_{\delta \to 1} v^*(\delta) = c$. Since Player 2 does not deviate for $\delta \geqq \underline{\delta}$, we have the following corollary.

Corollary 1. *Given $\delta \geqq \underline{\delta}$, the eternal cooperation cannot be sustained if and only if the outside option v exceeds $v^*(\delta)$.*

Two remarks are in order. First, although we focus on the repeated play of (C, C), one might wonder that if players play (D, C) occasionally, it may reduce the sufficient level of the discount factor. Playing (D, C) has two effects. One is that it is possible to lower the sufficient discount factor for Player 1 to follow the strategy. The other is that Player 2 must have incentive to play (D, C). Therefore it is not always the case that playing (D, C) can reduce the sufficient discount factor. In fact, under some parameter condition, (C, C) is the easiest action profile to sustain by a simple trigger strategy combination. For details see Appendix A.

Second, so far we have assumed that the outside option is a single stationary sequence $\{v, v, \ldots\}$. If different sequences become available over time, the cooperation may fall apart, even if most of outside options are unattractive, i.e., below $v^*(\delta)$.

Fix δ and suppose that at the end of each period t, a sequence $\{v(t), v(t), \ldots\}$ is the outside option and there exists the smallest integer $T < \infty$ such that $v(T) > v^*(\delta)$. That is, T is the first time that the outside option exceeds $v^*(\delta)$. Then at the end of period T, Player 1 would exit (given that the players play (C, C) as long as they are in the game), and thus the players would not play (C, C) in T.

Proposition 2. *Given $\delta \geqq \underline{\delta}$, the eternal cooperation cannot be sustained under any deterministic outside option structure such that at some $T < \infty$, an outside option $v(T) > v^*(\delta)$ becomes available.*

That is, deterministic fluctuation of outside options does not make cooperation easier. By contrast, in the next section we consider stochastic outside options such that in each period the actual outside option is random. Even if the players know that eventually an attractive option arrives, the uncertainty of the timing can make the mutual cooperation possible until the realization. This is a striking difference from the above deterministic and fluctuating option case.

4 Stochastic Outside Options

In this section we consider the case that Player 1 receives stochastic outside options at the end of each period from an i.i.d. distribution. The randomness can be interpreted several ways, such as a subjective uncertainty, an external perturbation, or a draw from a distribution of options. To make the comparison with the deterministic case, throughout this section we fix the mean of the distribution equal to $v \in (d, c)$.

The stochasticity of the options leads to two important changes: one is that the values in both the cooperation phase and the punishment phase may increase so that the range of discount factors that sustains cooperation changes, and the other is that even though the mean v is still between d and c, the randomness may create an attractive outside option greater than c occasionally. In this case the eternal cooperation cannot be sustained for any δ, but cooperation until Player 1 exits (the stochastic cooperation) may still be sustained. Note that, in the single deterministic outside option model, under the myopic equilibrium punishment, (C, C) is continuously played for a finite number of periods if and only if the eternal cooperation is sustained. However, in the stochastic outside option model, this equivalence does not hold.

4.1 Symmetric Binary Distributions

In this subsection we focus on a simple binary distribution of outside options: $\{v+\epsilon, v+\epsilon, \ldots\}$ and $\{v - \epsilon, v - \epsilon, \ldots\}$ for some $\epsilon > 0$, which becomes available with probability 0.5 each.[8] With this formulation we can see the effect of the mean v and the spread ϵ of the distribution on the smallest δ that sustains either the eternal cooperation or the stochastic cooperation.

Unlike the deterministic outside option cases analyzed so far, it may not be optimal for

[8] Alternatively we can assume that only the next period option becomes known and future option values are still random. If the one-shot payoff of $v + \frac{\epsilon}{1-\delta}$ and $v - \frac{\epsilon}{1-\delta}$ obtain with equal probability in each period, then taking the option $v + \frac{\epsilon}{1-\delta}$ (resp. $v - \frac{\epsilon}{1-\delta}$) in a period and receiving the same random sequence afterwards gives the same expected payoff of $\frac{v+\epsilon}{1-\delta}$ (resp. $\frac{v-\epsilon}{1-\delta}$).

Player 1 to exit immediately in the punishment phase under the stochastic options, even though the average is still $v > d$. Thus we clarify the optimal exit strategy for Player 1 in the cooperation phase and in the punishment phase respectively.

First, let us derive the optimal exit strategy for Player 1 when repeated (C, C) is expected as long as the game continues. Since $c > v - \epsilon$, either taking only the option of $v + \epsilon$ or not taking any option (in any period) is the optimal strategy. Let V be the continuation payoff, measured at an exit decision node, when Player 1 takes only the option of $v + \epsilon$. While waiting for the good option the players play (C, C) repeatedly. Thus V satisfies the following recursive equation:

$$V = \frac{1}{2}\Big(\frac{v+\epsilon}{1-\delta}\Big) + \frac{1}{2}(c + \delta V).$$

Explicitly, we have

$$V = \frac{\frac{1}{2}(\frac{v+\epsilon}{1-\delta}) + \frac{1}{2}c}{1 - \frac{\delta}{2}} = \frac{\frac{v+\epsilon}{1-\delta} + c}{2-\delta}. \tag{4}$$

If Player 1 does not take any outside option, the continuation payoff is $c/(1-\delta)$. Therefore, not taking any option is optimal in the cooperation phase if and only if

$$\begin{aligned} & \frac{c}{1-\delta} \geqq V \\ \iff \quad & (2-\delta)c \geqq v + \epsilon + (1-\delta)c \qquad \text{from (4)} \\ \iff \quad & c \geqq v + \epsilon. \end{aligned}$$

In summary we have the following characterization of the optimal exit strategy in the cooperation phase.

Lemma 2. *When (C, C) is expected as long as the game continues, not taking any outside option is the optimal exit strategy for Player 1 if $c \geqq v + \epsilon$, and taking only the good option $v + \epsilon$ is optimal otherwise.*

Second, suppose that repeated (D, D) is expected as long as the game continues. Since $d < v + \epsilon$, either taking only the option of $v + \epsilon$ or taking any option is the optimal exit strategy. Let W be the continuation payoff, measured at an exit decision node, when Player 1 takes only the option of $v + \epsilon$. While waiting for the good option the players play (D, D) repeatedly. Thus W satisfies

$$W = \frac{1}{2}\Big(\frac{v+\epsilon}{1-\delta}\Big) + \frac{1}{2}(d + \delta W),$$

and hence

$$W = \frac{\frac{v+\epsilon}{1-\delta} + d}{2-\delta}. \tag{5}$$

		$\epsilon \leqq v - d$		$v - d < \epsilon$
		$\delta \leqq \delta^P(v, \epsilon)$	$\delta^P(v, \epsilon) \leqq \delta$	
$c \geqq v + \epsilon$	cooperation phase	No exit	No exit	
	punishment phase	Take any option	Take only $v + \epsilon$	
$v + \epsilon > c$	cooperation phase	Take only $v + \epsilon$	Take only $v + \epsilon$	
	punishment phase	Take any option	Take only $v + \epsilon$	

Table 3: Player 1's Optimal Exit Strategy

If Player 1 exits immediately by taking any outside option, the continuation payoff is $\frac{1}{2}(\frac{v+\epsilon}{1-\delta}) + \frac{1}{2}(\frac{v-\epsilon}{1-\delta}) = \frac{v}{1-\delta}$. Therefore, waiting for the good option $v+\epsilon$ in the punishment phase is optimal if and only if

$$
\begin{aligned}
& W \geqq \frac{v}{1-\delta} \\
\iff \quad & \frac{\frac{v+\epsilon}{1-\delta} + d}{2-\delta} \geqq \frac{v}{1-\delta} \qquad \text{from (5)} \\
\iff \quad & \delta \geqq \frac{v-d-\epsilon}{v-d}. \qquad (6)
\end{aligned}
$$

Let $\delta^P(v, \epsilon) = \max\{\frac{v-d-\epsilon}{v-d}, 0\}$. The superscript P stands for the punishment phase. In summary, we have the following characterization of the optimal exit strategy in the punishment phase.

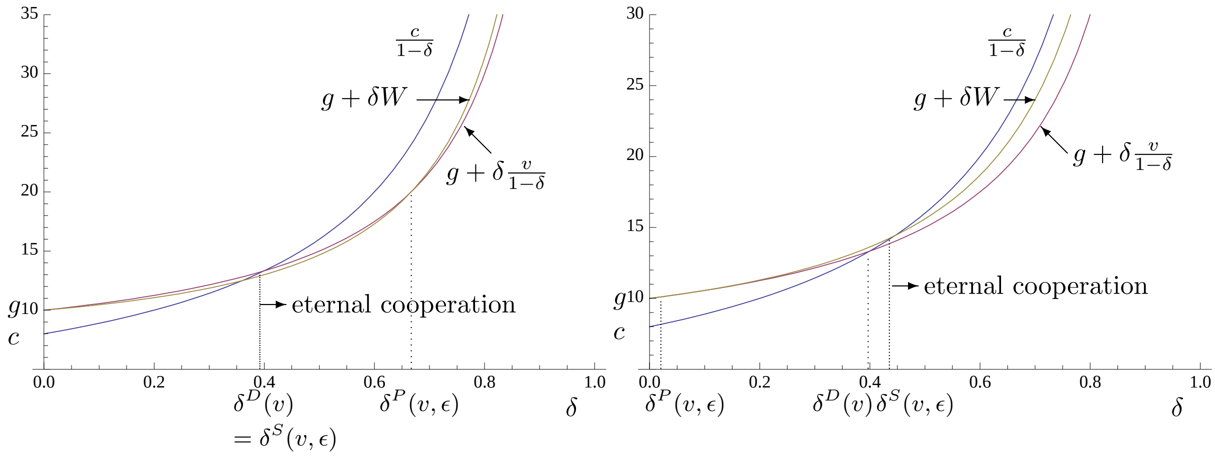
Lemma 3. *When (D, D) is expected as long as the game continues, taking only the good outside option of $v+\epsilon$ is the optimal exit strategy for Player 1 if $\delta \geqq \delta^P(v, \epsilon)$, and taking any outside option is optimal otherwise.*

The optimal exit strategies are also summarized in Table 3. We will find the lower bound of the discount factor δ to sustain repeated mutual cooperation as long as possible, using the simple trigger strategy combination described in Section 3. We consider two cases separately: $c \geqq v + \epsilon$ where the eternal cooperation is possible, and $v + \epsilon > c$ where only the stochastic cooperation is possible.

First, when $c \geqq v + \epsilon$, the eternal cooperation turned out to be more difficult to sustain under the stochastic outside options than under the deterministic outside option.

Proposition 3. *Suppose that $c \geqq v + \epsilon$. Let $\delta^S(v, \epsilon)$ be the lower bound to δ that sustains the eternal cooperation under the stochastic outside options. Then $\delta^S(v, \epsilon) \geqq \delta^D(v)$.*

Proof: See Appendix B.



2(a): $\epsilon \leqq (v-d)(c-v)/(g-v)$ 2(b): $(v-d)(c-v)/(g-v) < \epsilon$

Figure 2: Less Cooperation under Stochastic Outside Options ($c \geqq v+\epsilon$)

The intuition is as follows. As Figure 2 shows, the optimal deviation value for Player 1 is

$$\max\{g+\delta W, g+\delta\frac{v}{1-\delta}\} = \begin{cases} g+\delta\frac{v}{1-\delta} & \text{if } 0<\delta\leqq\delta^P(v,\epsilon) \\ g+\delta W & \text{if } \delta^P(v,\epsilon)\leqq\delta. \end{cases}$$

This is because both $g+\delta W$ and $g+\delta\frac{v}{1-\delta}$ are increasing in δ and the former is more "convex" than the latter. Figure 2 also shows that the cooperation phase value $c/(1-\delta)$ is smaller than the deviation value at $\delta=0$ but increases more rapidly than the deviation value as δ increases. There are two cases of how $c/(1-\delta)$ intersects with the deviation value.

If $\delta^D(v)\leqq\delta^P(v,\epsilon)$, that is,

$$\delta^D(v)\leqq\delta^P(v,\epsilon) \iff \epsilon\leqq\frac{(v-d)(c-v)}{g-v}, \tag{7}$$

then $c/(1-\delta)$ intersects with the deviation value when the latter is $g+\delta v/(1-\delta)$, as shown in Figure 2(a). In this case, to sustain the eternal cooperation, the lower bound of δ that satisfy

$$\frac{c}{1-\delta}\geqq\max\{g+\delta W, g+\delta\frac{v}{1-\delta}\} = g+\delta\frac{v}{1-\delta}$$

is $\delta^D(v)$ as in the case of deterministic outside option.

By contrast, if ϵ is large, so that $\delta^D(v)>\delta^P(v,\epsilon)$ (but still $c\geqq v+\epsilon$), then $c/(1-\delta)$ intersects with the deviation value when the latter is $g+\delta W$, as shown in Figure 2(b). In this case the lower bound of δ that satisfy

$$\frac{c}{1-\delta}\geqq\max\{g+\delta W, g+\delta\frac{v}{1-\delta}\} = g+\delta W$$

is strictly greater than $\delta^D(v)$. Hence the lower bound of δ is never smaller than $\delta^D(v)$.

Let us consider comparative statics when the mean v and/or the spread ϵ changes. Note that W is strictly increasing in v and ϵ. Hence, both $g+\delta W$ and $g+\delta v/(1-\delta)$ are increasing in v and nondecreasing in ϵ. This implies that the lower bound $\delta^S(v,\epsilon)$ is increasing in v and nondecreasing in ϵ, since the option value is increased in the punishment phase, while the cooperation phase value $c/(1-\delta)$ is unchanged as v and/or ϵ increases.

Corollary 2. *When $c \geqq v+\epsilon$, the lower bound $\delta^S(v,\epsilon)$ that sustains the eternal cooperation is increasing in v and nondecreasing in ϵ, i.e., as the mean and/or spread of the outside options increase it is more difficult to sustain the eternal cooperation.*

The increase in ϵ can be interpreted as the increased volatility of the outside options and the volatility can affect the value in the punishment phase only, as above. This is different from the volatility of discount factor (Dal Bó, 2007), which affects both the cooperation phase value and the punishment phase value, and the volatility of payoffs in Rotemberg and Saloner (1986). As we discussed in the Introduction, their results can be interpreted as full cooperation being more difficult under volatility. We have provided a third source of volatility via the outside options, and in the current case the eternal cooperation is negatively affected, which is consistent with the above literature.

Let us consider the case that $v+\epsilon > c$. In this case the volatility of the outside options may enhance cooperation, thanks to the increased value of the cooperation phase by the good option. However, it is optimal for Player 1 to exit at some point when $v+\epsilon$ becomes available. Hence the best we can sustain is the stochastic cooperation.

Proposition 4. *Suppose that $v+\epsilon > c$ and $2v > g+d$. Let $\delta^S(v,\epsilon)$ be the lower bound to δ that sustains the stochastic cooperation under the stochastic outside options. Then, $\delta^S(v,\epsilon) < \delta^D(v)$, i.e., for any $\delta \in [\delta^S(v,\epsilon), \delta^D(v))$, repeated cooperation for some periods is sustained under the stochastic outside options while no repeated cooperation is sustained under the deterministic option v.*

Proof: See Appendix B.

The intuition behind Proposition 4 is as follows. When $v+\epsilon > c$, Player 1 has an increased option value $c+\delta V$ in the cooperation phase by taking the good outside option $v+\epsilon$ when it becomes available. If the cooperation value $c+\delta V$ intersects with the optimal deviation value

$$\max\{g+\delta W, g+\delta\frac{v}{1-\delta}\}$$

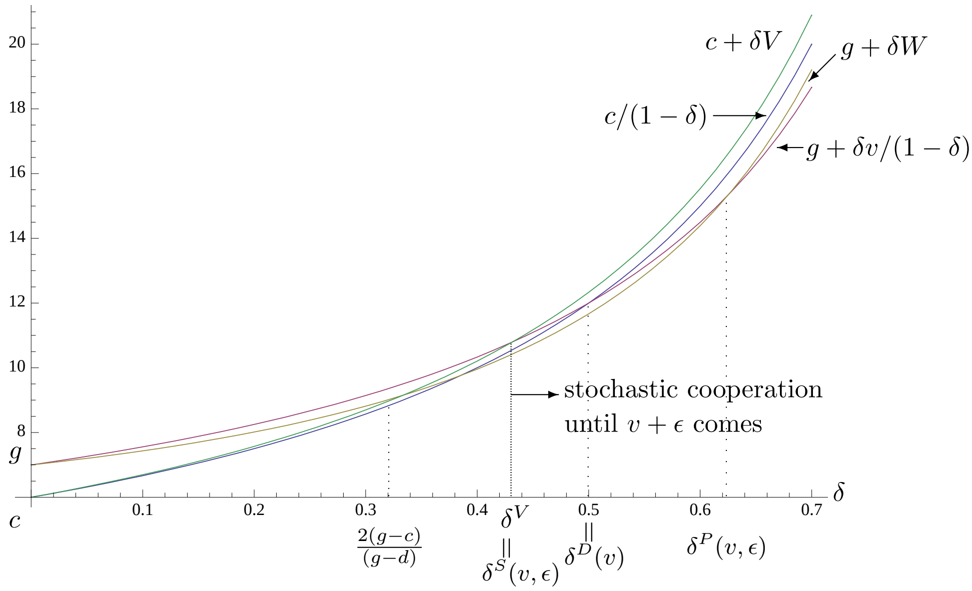


Figure 3: More Cooperation under Stochastic Outside Options ($v + \epsilon > c$)

when this is $g + \delta v/(1 - \delta)$, then by the increase of the cooperation value, the lower bound δ^V above which $c + \delta V \geqq g + \delta v/(1 - \delta)$ holds is lower than the deterministic case. (See Figure 3.[9]) If the cooperation phase value intersects with the optimal deviation value when the latter is $g + \delta W$, the option value in the punishment phase is also greater than the one in the deterministic case. By computation, cooperation is sustained if

$$c + \delta V \geqq g + \delta W \iff \delta(V - W) \geqq g - c \iff \delta \geqq \frac{2(g - c)}{g - d}. \tag{8}$$

When $2v > g + d$, this critical value of δ is also smaller than $\delta^D(v)$.

From the above argument, it is easy to see that when the mean v and/or the spread ϵ increases, $c + \delta V$ increases, making $\delta^S(v, \epsilon)$ smaller but $2(g - c)/(g - d)$ is unchanged. Therefore, it becomes weakly easier to sustain cooperation.

Corollary 3. *When $v + \epsilon > c$ and $2v > g + d$, the bound $\delta^S(v, \epsilon)$ weakly decreases as v and/or ϵ increases, i.e., as the mean and/or the spread of the outside options increase, it is weakly easier to sustain the stochastic cooperation.*

In sum, when the outside options are stochastic and one of the options is very attractive, then Player 1 with a mid-range δ who would not cooperate under the deterministic option would cooperate under the stochastic outside options, as long as she is in the game. This is

[9] The parameter values are $(g, c, d, \ell, v, \epsilon) = (7, 6, 1, 0.1, 5, 1.5)$.

because she wants to wait for the good option and a deviation in the stage game would not make it easy to wait for the good option because of the punishment.

Note, however, that even though $\delta^S(v,\epsilon) < \delta^D(v)$, still $\delta^S(v,\epsilon) \geqq \frac{2(g-c)}{g-d} > \frac{g-c}{g-d} = \underline{\delta}$ holds. Therefore, the existence of outside options in any form makes it more difficult to achieve mutual cooperation than in the ordinary repeated game. The "locked-in" feature of repeated games is a strong device to enforce mutual cooperation.

To conclude the analysis of the simple binary distributions, let us look at the effect of ϵ in more details, given a δ. As we argued in the Introduction, the increase of ϵ increases the value of the cooperation phase so that even when cooperation is not possible under the deterministic case, it may become possible. To be precise there are two cases: when no cooperation becomes stochastic cooperation (as shown in the Introduction) as ϵ increases, and when eternal cooperation becomes stochastic cooperation as ϵ increases. We clarify these cases.

Recall that $c + \delta V$ and $g + \delta W$ are linearly increasing functions of ϵ, from (4) and (5). Hence, as ϵ increases, at some point $c + \delta V$ exceeds $\frac{c}{1-\delta}$ and $g + \delta W$ exceeds $g + \delta \frac{v}{1-\delta}$. Moreover, the relationship between $c + \delta V$ and $g + \delta W$ is independent of ϵ:

$$c + \delta V \geqq g + \delta W \iff \delta \geqq \frac{2(g-c)}{g-d}, \tag{8}$$

and the relationship between $\frac{c}{1-\delta}$ and $g + \delta \frac{v}{1-\delta}$ is also independent of ϵ:

$$\frac{c}{1-\delta} \geqq g + \delta \frac{v}{1-\delta} \iff \delta \geqq \frac{g-c}{g-v}. \tag{2}$$

We first consider the case assumed in Proposition 4.

Case 1: $\frac{2(g-c)}{g-d} < \frac{g-c}{g-v}$ (i.e., $2v > g + d$).

In this case, when δ is so small that $\delta < \frac{2(g-c)}{g-d}$, then for any ϵ,

$$\frac{c}{1-\delta} < g + \delta \frac{v}{1-\delta} \qquad \text{and} \qquad c + \delta V < g + \delta W.$$

Therefore for any ϵ,

$$\max\{\frac{c}{1-\delta}, c + \delta V\} < \max\{g + \delta \frac{v}{1-\delta}, g + \delta W\},$$

so that no cooperation is sustained.

As δ increases so that $\frac{2(g-c)}{g-d} < \delta < \frac{g-c}{g-v}$, $c + \delta V > g + \delta W$ holds. Then, when ϵ is sufficiently large,

$$\max\{\frac{c}{1-\delta}, c + \delta V\} = c + \delta V \geqq \max\{g + \delta \frac{v}{1-\delta}, g + \delta W\}$$

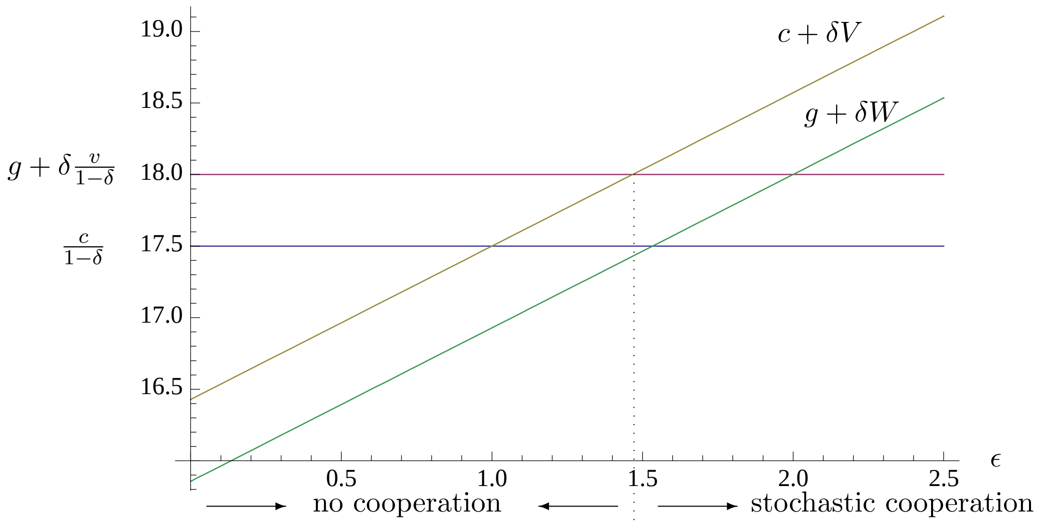


Figure 4: Perturbation induces stochastic cooperation for mid-range δ

holds so that stochastic cooperation is sustained. See Figure 4. Therefore, in this case more perturbation is good for cooperation.

When δ is even larger so that $\delta > \frac{g-c}{g-v}$, then both $c/(1-\delta) > g + \delta v/(1-\delta)$ and $c + \delta V > g + \delta W$ hold for any ϵ. In this case, without perturbation, the eternal cooperation is sustained. However, as ϵ increases, $c+\delta V$ exceeds $c/(1-\delta)$ so that the eternal cooperation breaks down, and only stochastic cooperation is sustained. Therefore when δ is large, more perturbation is not good for cooperation.

Case 2: $\frac{g-c}{g-v} \leqq \frac{2(g-c)}{g-d}$ (i.e., $g + d \leqq 2v$).

When δ is small so that $\delta < \frac{g-c}{g-v}$, then as in the Case 1 for low δ, for any ϵ, $\frac{c}{1-\delta} < g+\delta\frac{v}{1-\delta}$ and $c + \delta V < g + \delta W$ hold simultaneously. Hence

$$\max\{\frac{c}{1-\delta}, c+\delta V\} < \max\{g+\delta\frac{v}{1-\delta}, g+\delta W\},$$

so that no cooperation is sustained.

As δ increases so that $\frac{g-c}{g-v} < \delta < \frac{2(g-c)}{g-d}$, $\frac{c}{1-\delta} > g + \delta\frac{v}{1-\delta}$ holds, but $c + \delta V < g + \delta W$. In this case, although the eternal cooperation is sustained without perturbation, when ϵ becomes large, $g + \delta W$ exceeds $c/(1-\delta)$ so that the eternal cooperation collapses, and for larger ϵ, no repetition of cooperation is sustained. Therefore, in this case perturbation is bad for cooperation.

Finally, when δ is large so that both $\frac{c}{1-\delta} > g+\delta\frac{v}{1-\delta}$ and $c+\delta V > g+\delta W$ hold, the situation is the same as the large δ in Case 1. Thus increase in ϵ changes the eternal cooperation into the stochastic cooperation.

This analysis shows that perturbation makes a difference and it is good for cooperation for some range of parameters, namely, for not so large g's and mid-range δ's. Otherwise, more perturbation makes cooperation more difficult.

4.2 General Binary Distributions with a Preserved Mean

We extend the analysis to a general binary distribution to incorporate more realistic situations. For example, in employment relationships, most of the time the outside option is not so good, but once in a while a very attractive outside option may arrive. Thus we keep the binary structure but vary the probability of the attractive option greater than c. If the probability of the arrival of the good outside option is very small, there is not much discrepancy between the stochastic cooperation and the eternal cooperation, and the change in the lower bound of the discount factor has a significant meaning.

Although there are many ways to formulate a general binary distribution with a fixed mean v, we use the following formulation. Let

Suppose that there are two outside options $v^+ > v^-$, which obtain with probability p and $1-p$ respectively at the end of each period. As before, the option v^+ (resp. v^-) indicates that a stationary payoff sequence $\{v^+, v^+, \ldots\}$ (resp. $\{v^-, v^-, \ldots\}$) is given, or a one-shot payoff of $\frac{v^+}{1-\delta}$ (resp. $\frac{v^-}{1-\delta}$) is given. To keep the mean $v = pv^+ + (1-p)v^-$ between d and c, we fix v^- (a value less than v) and $v \in (d, c)$ and let $v^+(p) = (v - v^-)/p + v^-$. Note that $v^+(p)$ becomes a decreasing function of $p > 0$. For notational simplicity we often write v^+ when there is no danger of confusion. As before we find conditions for the simple trigger strategy combination to be a SPE.

Let $V(p)$ (given v and v^-) be the value in the cooperation phase, measured at the end of a period before a stochastic option arrives, when Player 1 takes only the good outside option v^+ in any period during the cooperation phase. It has the following recursive structure.

$$V(p) = p\frac{v^+}{1-\delta} + (1-p)\{c + \delta V(p)\}.$$

Therefore

$$V(p) = \frac{pv^+ + (1-p)(1-\delta)c}{(1-\delta)\{1-(1-p)\delta\}}. \tag{9}$$

Using this, we characterize the optimal exit strategy in the cooperation phase. The proof is essentially the same as that of Lemma 2.

Lemma 4. *When (C, C) is expected as long as the game continues, not taking any outside option is the optimal exit strategy for Player 1 if $c \geqq v^+$, and taking only v^+ is optimal otherwise.*

Proof: See Appendix B.

Therefore, the only condition to determine the optimal exit behavior in the cooperation phase is whether the best option exceeds c, regardless of its probability. This is a generalization of Lemma 2. In fact, this can be generalized for continuous distributions as well. See Lemma 9 below.

If (D, D) is expected forever after, the optimal exit strategy for Player 1 depends on the outside option distribution as follows. Let $W(p)$ be the value when Player 1 takes only the better outside option v^+ during the punishment phase (given v and v^-). It satisfies

$$W(p) = p\frac{v^+}{1-\delta} + (1-p)\{d + \delta W(p)\}.$$

Hence

$$W(p) = \frac{pv^+ + (1-p)(1-\delta)d}{(1-\delta)\{1-(1-p)\delta\}}. \tag{10}$$

By the same logic as the one in the proof of Lemma 3, the optimal exit strategy in the punishment phase is characterized as follows.

Lemma 5. *When (D, D) is expected as long as the game continues, waiting for the better outside option v^+ (and therefore exiting with probability p) is the optimal exit strategy for Player 1 if*

$$\delta v + (1-\delta)d \geqq v^-, \quad \text{or} \quad \delta \geqq \frac{v^- - d}{v - d}, \tag{11}$$

and taking any outside option is optimal otherwise.

Proof: See Appendix B.

Note that (11) does not depend on the probability of the good option p and is a generalization of (6). For any binary distribution, the optimal exit strategy in the punishment phase is determined by the value differences ($v^- - d$ and $v - d$), and δ.

As in the previous subsection, we divide the analysis into two cases: $v^+ \leqq c$ so that the eternal cooperation is possible and $v^+ > c$ so that only the stochastic cooperation is possible.

First, suppose that $v^+ \leqq c$, i.e., $p \geqq (v - v^-)/(c - v^-)$, so that the eternal cooperation is possible on the play path. The optimal value function on the play path is $c/(1-\delta)$. The value of a one-step deviation is $g + \delta W(p)$ if (11) holds, and it is $g + \delta v/(1-\delta)$ otherwise. Clearly, the lower bound of δ that satsify

$$\frac{c}{1-\delta} \geqq \max\{g + \delta W(p), g + \delta \frac{v}{1-\delta}\} \tag{12}$$

is not smaller than $\delta^D(v)$ which satisfy $\frac{c}{1-\delta} = g + \delta \frac{v}{1-\delta}$. Therefore, we have a generalization of Proposition 3.

Proposition 5. *Suppose that $v^+ \leqq c$. Let $\delta^S(p)$ be the lower bound of δ that sustains the eternal cooperation. Then $\delta^S(p) \geqq \delta^D(v)$, i.e., it is weakly more difficult to sustain the eternal cooperation than in the deterministic case.*

Proof: See Appendix B.

For the comparative statics, let us check whether $W(p)$ is increasing in p.

Lemma 6.

$$\frac{\partial(1-\delta)W(p)}{\partial p} \gtreqless 0 \iff \frac{v^- - d}{v - d} \gtreqless \delta.$$

Proof: First, plug in $v^+ = v^- + (v - v^-)/p$ so that

$$W(p) = \frac{(v - v^-) + pv^- + (1-p)(1-\delta)d}{(1-\delta)\{1 - \delta + p\delta\}}.$$

$$\begin{aligned}
\frac{\partial(1-\delta)W(p)}{\partial p} &= \frac{1}{\{1-\delta+p\delta\}^2}\Big[\{v^- - (1-\delta)d\}\{1-\delta+p\delta\} \\
&\qquad -\delta\{(v - v^-) + pv^- + (1-p)(1-\delta)d\}\Big], \\
&= \frac{1}{\{1-\delta+p\delta\}^2}\Big[v^- - \delta v - (1-\delta)d\Big].
\end{aligned}$$

□

Hence, precisely when it is optimal to wait for the better outside option, $W(p)$ is nonincreasing in p. That is, although the probability p of the better outside option increases, the decrease in the value $v^+(p)$ (to keep the mean as v) dominates so that the option value in the punishment phase $W(p)$ decreases. Since the on-path value $c/(1-\delta)$ is constant over p, the decrease of $W(p)$ makes the lower bound of δ that satisfy (12) weakly decreasing.

Corollary 4. *When $v^+ \leqq c$, the eternal cooperation becomes weakly easier to sustain as the probability p of the better option increases.*

Second, suppose that $v^+ > c$, or $p < (v - v^-)/(c - v^-)$. We investigate the lower bound of δ that sustains the stochastic cooperation, i.e., that satisfy

$$c + \delta V(p) \geqq \max\{g + \delta W(p), g + \delta \frac{v}{1-\delta}\}. \quad (13)$$

For the range of δ such that $v/(1-\delta) \geqq W(p)$, the increase in the on-path value implies that the lower bound of δ is less than $\delta^D(v)$. For the range of δ such that $W(p) > v/(1-\delta)$, both on-path value and the punishment value increased, as compared to the deterministic case. Let us find the smallest δ that satisfy

$$c + \delta V(p) \geqq g + \delta W(p).$$

By computation,

$$\begin{aligned}
& c + \delta V(p) \geqq g + \delta W(p), \\
\iff \quad & \delta\{V(p) - W(p)\} \geqq g - c, \\
\iff \quad & \delta \frac{\{pv^+ + (1-p)(1-\delta)c\} - \{pv^+ + (1-p)(1-\delta)d\}}{(1-\delta)\{1-(1-p)\delta\}} \geqq g - c, \\
\iff \quad & \delta(1-p)(c-d) \geqq \{1-(1-p)\delta\}(g-c), \\
\iff \quad & \delta \geqq \frac{g-c}{(1-p)(g-d)}.
\end{aligned}$$

Therefore

$$\frac{g-c}{(1-p)(g-d)} < \delta^D(v) \iff v > pg + (1-p)d. \quad (14)$$

The condition (14) is a generalization of $2v > g + d$ in Proposition 4. Since $g > c > v > d$, it is easier to be satisfied for small p, namely when $p < (v-d)/(g-d)$. (Note that this condition is compatible with $v^+ > c$.) In summary we have the following generalization of Proposition 4.

Proposition 6. *Suppose that $v^+ > c$ and $p < (v-d)/(g-d)$. Let $\delta^S(p)$ be the lower bound of δ that sustains the stochastic cooperation. Then $\delta^S(p) < \delta^D(v)$, i.e., for any $\delta \in [\delta^S(p), \delta^D(v))$, repeated cooperation is sustained under the stochastic outside options while no repeated cooperation is sustained under the deterministic option v.*

Proof: See Appendix B.

Clearly, for any $p \in (0,1)$, $\delta^S(p) \geqq \frac{g-c}{(1-p)(g-d)} > \underline{\delta}$. Thus compared to the ordinary repeated game, cooperation is still more difficult. As p decreases, so that the probability of Player 1's exit decreases, the value $V(p)$ increases since

$$\frac{\partial(1-\delta)V(p)}{\partial p} = \frac{1}{\{1-\delta+p\delta\}^2}\Big[v^- - \delta v - (1-\delta)c\Big] < 0,$$

by a similar computation as in Lemma 6 but with no restriction on the parameters. This means that the increase in the payoff $v^+(p)$ dominates the reduced probability of receiving the option. When p is close to 0, we can sustain almost the eternal cooperation under δ lower than the deterministic case.Therefore, if Player 1 has a very high option v^+, even though its probability is small, it makes Player 1 effectively more patient and cooperative than under the deterministic case. This is a quite interesting implication.

The binary distribution models illustrate well the essence of the effect of stochastic outside options on the cooperation within the repeated game. However, it is of some theoretical interest how the model and results extend to a case with a continuum of outside options, which is more standard in some economic models such as search models. In Appendix C, we show that the stochastic cooperation is sustained under lower discount factors than those of the deterministic model even under a continuum of outside options.

5 Two-Sided Outside Options

In this section we extend the model so that Player 2 also has non-negligible outside options. When both players can choose to take outside options, the rule of termination of a repeated game becomes relevant. The unilateral ending rule assumed in the one-sided option model (Table 4(a)) has a specific meaning in the two-sided option model that the repeated game ends if and only if *at least* one player chooses to exit (Table 4(b)). There is an intermediate case of two-sided option model in which both players must agree to end the game, but in that case it is straightforward to prove that any equilibrium outcome of ordinary repeated game can be sustained.[10] Therefore the essentially different models from ordinary repeated games are the one-sided option model and two-sided option model with the unilateral ending rule. Moreover, the unilateral ending rule is the most commonly analyzed rule (e.g., Gosh and Ray, 1996, Kranton, 1996a,b, Fujiwara-Greve, 2002, and Fujiwara-Greve and Okuno-Fujiwara, 2009) and describes well situations such as joint ventures and lender-borrower relationships.

First, consider the deterministic option model. Let $v_1, v_2 \in (d, c)$ be the outside options for Player 1 and Player 2 respectively. By the same argument as in Section 3, the maximal equilibrium punishment is to exit immediately after the observation of D. Under this

[10]For example, repeated (C, C) can be achieved by the following strategy combination if two players must agree to end the game: Play C and do not take outside options as long as no one played D. If someone played D in the past, play D and do not take outside options. Since one player cannot unilaterally end the game to escape, the strategy combination is a subgame perfect equilibrium if and only if the usual grim-trigger strategy combination is a subgame perfect equilibrium in the ordinary repeated Prisoner's Dilemma.

P1 \ P2	
Stay	Continue
Exit	End

4(a): One-sided Option for P1

P1 \ P2	Stay	Exit
Stay	Continue	End
Exit	End	End

4(b): Two-sided Option with Unilateral Ending Rule

Table 4: Game Continuation Patterns

punishment, Player i would not play C if $\delta < \delta_i^D(v_i)$. The range of discount factors that sustains mutual cooperation is $\delta \geqq \max\{\delta_1^D(v_1), \delta_2^D(v_2)\}$, which is weakly narrower than the one in the one-sided outside option model, since $\delta_i^D(\cdot)$ is increasing. Therefore it becomes more difficult to sustain cooperation when both players have deterministic outside options, since both players must be patient enough to stay and cooperate.

Second, let us consider the case that both Player 1 and Player 2 have fluctuating but deterministic outside options. By the same argument as in Section 3, mutual cooperation falls apart if there is a known time period at which one of the players receives an outside option greater than $v^*(\delta)$. Hence we can interpret that cooperation becomes more difficult in the sense that there are more cases of fluctuating outside options that includes $v > v^*(\delta)$ for at least one player.

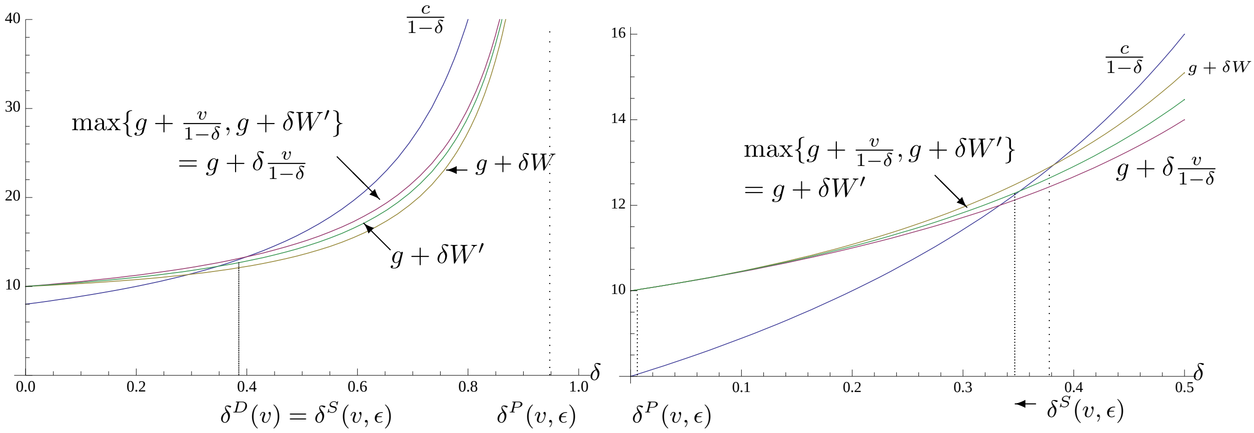
Third, suppose that Player 1 and Player 2 independently draw outside options from the same i.i.d. distribution. Since the qualitative results are the same, we focus on the simplest distribution such that $v + \epsilon$ obtains with probability 0.5 and $v - \epsilon$ obtains with probability 0.5, independently to each player. Under the independent draws, a player may take an outside option when the other player does not want to, so that the game ends with a different probability and the payoff becomes different from the one in the one-sided outside option case. Specifically, if both players want to take only $v+\epsilon$ in the punishment phase, the continuation value W', measured at the end of a period, satisfies the following recursive structure.

$$W' = \frac{1}{2} \cdot \frac{v+\epsilon}{1-\delta} + \frac{1}{4} \cdot \frac{v-\epsilon}{1-\delta} + \frac{1}{4}(d + \delta W'). \tag{15}$$

This is because with probability 1/4, one's option turns out to be $v - \epsilon$ but the partner's turned out to be $v + \epsilon$, in which case the game ends and one ends up with the low option.

Lemma 7. *For any (v, ϵ), the one-shot deviation values are ordered as follows.*

$$\delta \leqq \delta^P(v,\epsilon) \quad \Rightarrow \quad g + \delta\frac{v}{1-\delta} \geqq g + \delta W' \geqq g + \delta W;$$
$$\delta^P(v,\epsilon) \leqq \delta \quad \Rightarrow \quad g + \delta W \geqq g + \delta W' \geqq g + \delta\frac{v}{1-\delta}.$$



5(a): $\epsilon \leqq (v-d)(c-v)/(g-v)$ 5(b): $(v-d)(c-v)/(g-v) < \epsilon$

Figure 5: More Cooperation under Two-Sided Outside Options ($c > v + \epsilon$)

Proof: See Appendix B. (See also Figure 5[11].)

Proposition 7. *Suppose that Player 1 and Player 2 receive outside options (independently) from the same i.i.d. distribution such that* $v+\epsilon$ *obtains with probability* 0.5*, while* $v-\epsilon$ *obtains with probability* 0.5 *at the end of each period, where* $c > v + \epsilon$*.*

Then the lower bound of the discount factors that sustains the eternal cooperation is weakly smaller than the one for the one-sided outside option model.

Proof: See Appendix B.

The intuition is as follows. Since $c > v + \epsilon$, the optimal exit strategy in the cooperation phase is not to exit. Therefore the on-path value $c/(1-\delta)$ is the same as in the one-sided option model. In order for a player to cooperate, we need

$$\frac{c}{1-\delta} \geqq \max\{g + \delta W', g + \delta \frac{v}{1-\delta}\}.$$

From Lemma 7, when $\epsilon \leqq (v-d)(c-v)/(g-v)$, the function $c/(1-\delta)$ intersects with $\max\{g + \delta W', g + \delta v/(1-\delta)\}$ when the latter is $g + \delta v/(1-\delta)$. Thus the lower bound of δ that sustains cooperation is the same as in the one-sided option model (Figure 5(a)). For $(v-d)(c-v)/(g-v) < \epsilon$, it is strictly smaller (Figure 5(b)) since $c/(1-\delta)$ crosses with $\max\{g + \delta W', g + \delta v/(1-\delta)\}$ when the latter is $g + \delta W'$ which is strictly lower than $g + \delta W$.

[11]The parameter combination is $(g, c, d, \ell, v, \epsilon) = (10, 8, 0.5, 0.1, 5, 0.1)$ for 5(a) and $(g, c, d, \ell, v, \epsilon) = (10, 8, 0.5, 0.1, 4, 3.4)$ for 5(b).

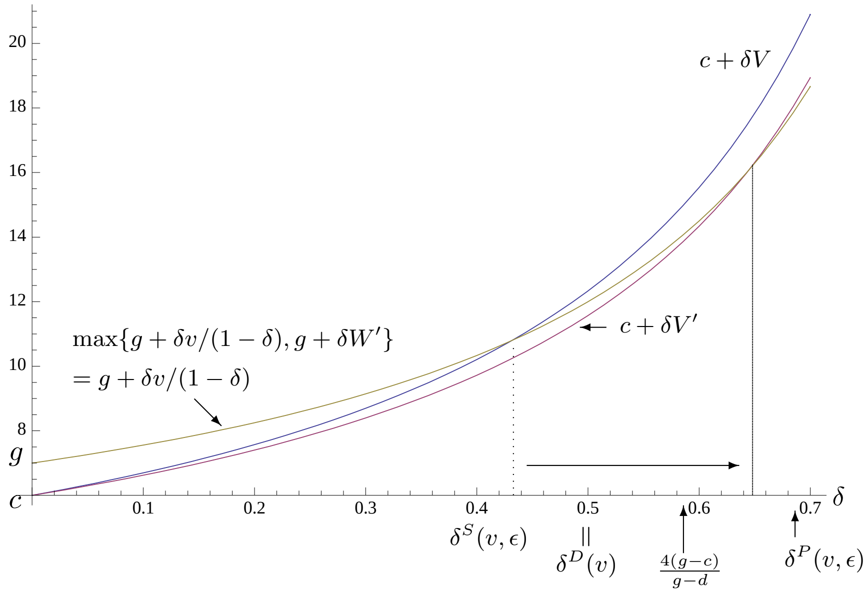


Figure 6: Less Cooperation under Two-Sided Outside Options ($v + \epsilon > c$)

By contrast, when $v + \epsilon > c$, both the value in the cooperation phase and the value in the punishment phase may decrease. The continuation value in the cooperation phase, measured at the end of a period, satisfies

$$V' = \frac{1}{2} \cdot \frac{v + \epsilon}{1 - \delta} + \frac{1}{4} \cdot \frac{v - \epsilon}{1 - \delta} + \frac{1}{4}(c + \delta V'), \tag{16}$$

while the one-sided case value can be decomposed as

$$V = \frac{1}{2} \cdot \frac{v + \epsilon}{1 - \delta} + \frac{1}{4}(c + \delta V) + \frac{1}{4}(c + \delta V).$$

Thus if $c + \delta V > (v - \epsilon)/(1 - \delta)$, then $V > V'$. Since $c > v$, the former holds.

If only the value in the cooperation phase decreases to $c + \delta V'$, while $\max\{g + \delta v/(1 - \delta), g + \delta W'\} = g + \delta v/(1 - \delta)$, then clearly the lower bound of δ that sustains Player 1's cooperation increases (Figure 6[12]). Moreover, even if both values decrease, it can be shown that the decrease in the cooperation phase is greater than the decrease in the punishment phase so that still the lower bound of δ increases.

Proposition 8. *Suppose that $v + \epsilon > c$. Then the lower bound of the discount factors that sustains the stochastic cooperation is larger than the one for the one-sided outside option model.*

[12] The parameter combination is $(g, c, d, \ell, v, \epsilon) = (7, 6, 0.2, 0.1, 5, 1.5)$.

Proof: See Appendix B.

In summary, under two-sided independent stochastic outside options, the effects of perturbation are weakened relative to the one-sided case. Cooperation is enhanced if the perturbation reduces the option value in the punishment phase, while cooperation becomes more difficult if the perturbation makes players less patient on the play path as compared to the one-sided case.

6 Concluding Remarks

In this paper we show that the presence and the *structure* of outside options in a repeated Prisoner's Dilemma can seriously affect the existence of the cooperative equilibrium. In the deterministic one-sided outside option model, the range of discount factors that sustains cooperative equilibrium is strictly narrower than the one under the ordinary infinitely repeated Prisoner's Dilemma without outside options. This is easy to understand since the player with the outside option can defect and take the outside option, instead of suffering from the mutual defection punishment.

When outside options take different values across time, whether the timing of an attractive option is certain or not can make a difference in sustaining mutual cooperation. If it is known, mutual cooperation falls apart by backward induction, while if it is stochastic, even if some outside options give higher payoff than mutual cooperation, it is possible to sustain cooperation as long as the attractive option does not come. When the probability of the attractive option decreases, almost the eternal cooperation becomes attainable.

We also found that one-sided and two-sided outside options have different effects. If both players have stochastic outside options, the relative difficulty of cooperation is *weakened* as compared to the one-sided option case. If both players can terminate the game unilaterally, the game ends more frequently and the option value is reduced, since the partner may end the game when one does not want to. This makes the cooperation easier if the punishment phase payoff is reduced but more difficult if the cooperation phase payoff is reduced.

The positive effect of perturbation on cooperation is a new finding. The possibility of good outside options makes players effectively more patient in some cases. This idea has not been derived in the literature where outside options are treated as only ending probability or a constraint.

Although the main concern in the present paper is to analyze the sustainability of mutual cooperation, it should also be of interest to characterize the set of equilibrium payoffs.

Especially, comparative static of the equilibrium payoff sets with respect to the mean value and/or the spread of the outside options has great importance. As we showed in Corollary 3 for the case of $v + \epsilon > c$, increasing any of v and ϵ can make Player 1's cooperation easier, which implies that the set of equilibrium payoffs need not be monotonically decreasing (in the sense of set inclusion) in the mean value and/or the spread of the outside options. This non-monotonicity of equilibrium payoffs as the outside options change may have significant implications to applications, for example in policy effects.[13]

Finally, we would like to point out that there is a wide scope of important applications from our analysis. For instance, repeated contracts should be designed more carefully when the agent's outside utilities are stochastic, collusion may become harder or easier if firms have outside options, and so forth. We believe that our simple model can provide meaningful insights and implications for these real-life applications.

Appendix A: (C, C) can be the easiest profile to sustain

In this Appendix, we give a sufficient condition under which repetition of (C, C) is the easiest stationary action profile to sustain.

For a set X, let $\Delta(X)$ be the set of all probability distributions over X. The set of feasible payoff combinations is

$$F := \{(u_1, u_2) \in \Re^2 \mid \exists \sigma \in \Delta(\{C, D\} \times \{C, D\}) \text{ such that } u_i = Eu_i(\sigma) \ \forall i = 1, 2\}.$$

For any feasible payoff combination $\mathbf{u} = (u_1, u_2)$, let $\underline{\delta}(\mathbf{u})$ be the lower bound of δ such that there exists a correlated action profile $\sigma \in \Delta(\{C, D\} \times \{C, D\})$ such that $\mathbf{u} = (Eu_1(\sigma), Eu_2(\sigma))$ and the following strategy combination is a SPE:
Play Path: If the history is empty or there was no deviation from σ in the past, play σ and Player 1 uses an optimal exit strategy given that σ is repeated as long as the game continues;
Punishment Phase: If there was a deviation from σ, play (D, D) and Player 1 uses an optimal exit strategy given that (D, D) is repeated as long as the game continues.

Lemma 8. *If $g - c < d - \ell$ and $c(g - c) \leqq (d - \ell)(c - v)$, then $\underline{\delta}(c, c) \leqq \underline{\delta}(\mathbf{u})$ for any $\mathbf{u} \in F$ such that $u_1 > v$ and $u_2 > 0$.*

Proof: Fix any $u \in F$. Depending on how u locates in F, the *necessary* action profiles to be played in σ are different. From Figure 7 we can see that:
(i) To attain a payoff combination in Area (i) on average, a correlated action profile must include (C, D) and (D, D) and either (C, C) or (D, C).

[13] There is a different non-monotonicity result. In a class of games called exhaustible resource games, Dutta (1995) showed that the first-best outcome is sustainable under a mid-range discount factor but not under high discount factors.

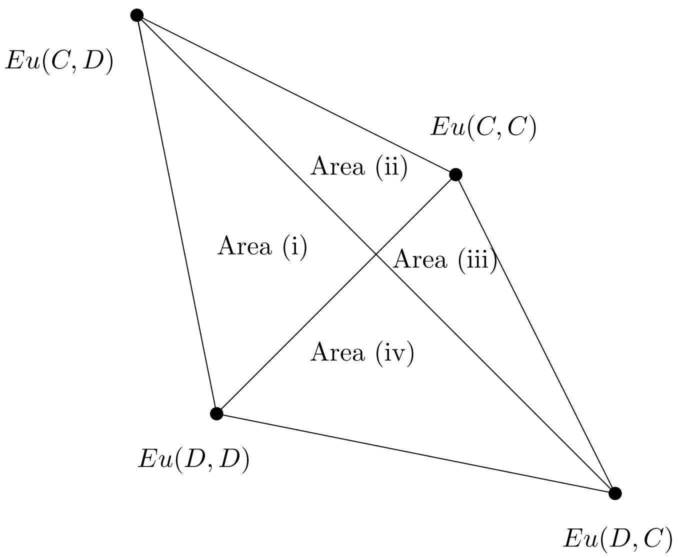


Figure 7: Areas in F

(ii) To attain a payoff combination in Area (ii) on average, a correlated action profile must include (C,D) and (C,C) and either (D,D) or (D,C).

(iii) To attain a payoff combination in Area (iii) on average, a correlated action profile must include (C,C) and (D,C) and either (C,D) or (D,D).

(iv) To attain a payoff combination in Area (iv) on average, a correlated action profile must include (D,D) and (D,C) and either (C,C) or (C,D).

We thus derive sufficient conditions for (C,C), (D,C), and (C,D) to be followed and then apply them for each Area to determine the minimum sufficient δ.

Let (u_1, u_2) be the one-shot average payoff of a correlated action profile on the play path. In each period, one of the pure action profiles in the support gets to be realized. If (C,C) is supposed to be played in this period, and if Player 1 deviates to D, her long-run payoff is $g + \delta v/(1-\delta)$ since they move to the punishment phase. If she follows (C,C), the long-run payoff is $c + \delta u_1/(1-\delta)$, since the expected one-shot payoff from tomorrow on is u_1. Hence Player 1 does not deviate from (C,C) if and only if

$$\begin{aligned} & c + \delta \frac{u_1}{1-\delta} \geqq g + \delta \frac{v}{1-\delta} \\ \iff \quad & \delta \geqq \frac{g-c}{(g-c)+(u_1-v)} =: \delta_1^{CC}(u_1). \end{aligned} \tag{17}$$

Similarly, Player 2 does not deviate from (C,C) if and only if

$$\begin{aligned} & c + \delta \frac{u_2}{1-\delta} \geqq g + \delta \cdot 0 \\ \iff \quad & \delta \geqq \frac{g-c}{(g-c)+u_2} =: \delta_2^{CC}(u_2). \end{aligned} \tag{18}$$

When (D,C) is supposed to be played, only Player 2 has an incentive to deviate. He does not deviate from (D,C) if and only if

$$\begin{aligned} & \ell + \delta \frac{u_2}{1-\delta} \geqq d + \delta \cdot 0 \\ \iff \quad & \delta \geqq \frac{d-\ell}{(d-\ell)+u_2} =: \delta^{DC}(u_2). \end{aligned} \tag{19}$$

When (C,D) is supposed to be played, only Player 1 has an incentive to deviate. She does not deviate from (C,D) if and only if

$$\begin{aligned} & \ell + \delta \frac{u_1}{1-\delta} \geqq d + \delta \frac{v}{1-\delta} \\ \iff \quad & \delta \geqq \frac{d-\ell}{(d-\ell)+(u_1-v)} =: \delta^{CD}(u_1). \end{aligned} \tag{20}$$

In order to make these lower bounds less than 1, clearly we need $u_1 > v$ and $u_2 > 0$.

Note that in general, $f(x) = x/(A+x)$ is an increasing function of x if and only if $A > 0$.[14] Therefore, if $g - c < d - \ell$, then $\delta_1^{CC}(u_1) < \delta^{CD}(u_1)$ and $\delta_2^{CC}(u_2) < \delta^{DC}(u_2)$ hold simultaneously. Note also that $v_2 < v_1$ implies that if $u_1 = u_2$, then $\delta_1^{CC}(u_1) > \delta_2^{CC}(u_2)$.

In order to make the players play a pure action profile (C,C), we need $\delta \geqq \max\{\delta_1^{CC}(u_1), \delta_2^{CC}(u_2)\} =: \delta^{CC}(u)$. For players to play (D,C) (resp. (C,D)), we only need $\delta \geqq \delta^{DC}(u_2)$ (resp. $\delta \geqq \delta^{CD}(u_1)$). Similarly, for correlated action profiles, we can classify the lower bound of δ as follows.

(i) In order to sustain (u_1,u_2) in Area (i) with as small δ as possible, we must have at least $\delta \geqq \delta^{CD}(u_1)$ but also can use either (C,C) or (D,C) in the support of the correlated action profile. Hence the lowest δ is $\max\{\delta^{CD}(u_1), \min\{\delta^{CC}(u), \delta^{DC}(u_2)\}\}$. Under the assumption of $g - c < d - \ell$, this lower bound is equal to $\delta^{CD}(u_1)$.

Now, in order to lower $\delta^{CD}(u_1)$ as much as possible in this area, we must increase u_1 as large as possible, which hits the boundary with all other areas (see Figure 7). From Figure 7 it is easy to see that we cannot increase u_1 as much in Area (i) as in Area (ii), i.e.,

$$\min_{u_1 \in Area\ (i)} \delta^{CD}(u_1) > \min_{u_1 \in Area\ (ii)} \delta^{CD}(u_1).$$

(ii) In order to sustain (u_1,u_2) in Area (ii), we need $\delta \geqq \max\{\delta^{CD}(u_1), \delta^{CC}(u)\}$, since for this we can ignore (D,C) and use (D,D) instead in the support, which does not require a high δ. Under the assumption of $g - c < d - \ell$, this lower bound is $\delta^{CD}(u_1)$.

Again, in order to reduce $\delta^{CD}(u_1)$ as much as possible, we hit the boundary, which is (C,C).

In summary so far, among the payoff combinations in the Area (i) and (ii), (C,C) is the easiest to sustain. This is because in these areas, only Player 1's deviation must be prevented and (C,C) gives the highest on-path average payoff for Player 1 in these areas.

By contrast, in Areas (iii) and (iv), we need to prevent Player 2's deviation so that the sufficient δs are as follows.

(iii) For (u_1,u_2) in Area (iii), we need $\delta \geqq \max\{\delta^{DC}(u_2), \delta^{CC}(u)\} = \delta^{DC}(u_2)$.

(iv) For (u_1,u_2) in Area (iv), we need $\delta \geqq \max\{\delta^{DC}(u_2), \min\{\delta^{CC}(u), \delta^{CD}(u_1)\}\} = \delta^{DC}(u_2)$.

[14] By differentiation, $f'(x) = A/(A+x)^2$.

To reduce $\delta^{DC}(u_2)$ as much as possible, we should increase u_2. Hence the minimum $\delta^{DC}(u_2)$ is attained in Area (iii) where $u_2 = c$. Recall that $\delta_1^{CC}(c) > \delta_2^{CC}(c)$ since $v_2 < v_1$. Hence (C, C) is the easiest to sustain in Area (iii) and (iv) if $\delta^{DC}(c) \geqq \delta_1^{CC}(c)$. This is equivalent to

$$\frac{d-\ell}{(d-\ell)+c} \geqq \frac{g-c}{(g-c)+(c-v)} \iff (d-\ell)(c-v) \geqq c(g-c).$$

Therefore, we have that if $g - c < d - \ell$ and $(d-\ell)(c-v) \geqq c(g-c)$, then $\delta^D(c,c) \leqq \delta^D(u)$ for any $u \in F$ such that $u_1 > v$ and $u_2 > 0$. □

Appendix B: Proofs

Proof of Proposition 3: Recall that

$$\frac{c}{1-\delta} \geqq g + \delta\frac{v}{1-\delta} \iff \delta \geqq \delta^D(v) \tag{2}$$
$$g + \delta W \geqq g + \delta\frac{v}{1-\delta} \iff \delta \geqq \delta^P(v,\epsilon) \tag{6}$$

We also show that the on-path value function $c/(1-\delta)$ exceeds the deviation value $g + \delta W$ for any δ above some critical δ. By computation,

$$\begin{aligned}
& \frac{c}{1-\delta} \geqq g + \delta W, \\
& \frac{c}{1-\delta} \geqq g + \delta\frac{\frac{v+\epsilon}{1-\delta} + d}{2-\delta}, \\
\iff\quad & (2-\delta)c \geqq (1-\delta)(2-\delta)g + \delta(v+\epsilon) + \delta(1-\delta)d, \\
\iff\quad & h(\delta) := -\delta^2(g-d) + \delta\{3g - (v+\epsilon) - c - d\} - 2(g-c) \geqq 0.
\end{aligned}$$

Notice that h is quadratic in δ, $h(0) = -2(g-c) < 0$ and $h(1) = c - (v+\epsilon) \geqq 0$. Therefore there exists $\delta^S(v,\epsilon) \in (0,1]$ such that for any $\delta \geqq \delta^S(v,\epsilon)$, $h(\delta) \geqq 0$ holds. Thus,

$$\frac{c}{1-\delta} \geqq g + \delta W \iff \delta \geqq \tilde{\delta}(v,\epsilon). \tag{21}$$

Note also that $\delta^D(v) \leqq \delta^P(v,\epsilon)$ if and only if $\epsilon \leqq (v-d)(c-v)/(g-v)$. Now we divide the analysis into two cases.

Case a: $0 < \epsilon \leqq (v-d)(c-v)/(g-v)$, i.e., $\delta^D(v) \leqq \delta^P(v,\epsilon)$.

In this case, the on-path value function $\frac{c}{1-\delta}$ intersects with $g + \delta\frac{v}{1-\delta}$ at $\delta^D(v)$ and at that point $\frac{v}{1-\delta} > W$. Hence (21) implies that $c/(1-\delta)$ exceeds $g + \delta W$ for any $\delta \geqq \delta^D(v)$. Therefore $\frac{c}{1-\delta} \geqq \max\{g + \delta W, g + \delta\frac{v}{1-\delta}\}$ if and only if $\delta \geqq \delta^D(v)$. See Figure 2a.

Player 2's deviation value changes depending on whether Player 1 exits immediately or not after seeing a deviation. If Player 1 exits immediately, i.e., if $\max\{W, \frac{v}{1-\delta}\} = \frac{v}{1-\delta}$, Player 2's deviation value is $g + \delta \cdot 0$. In this case $\delta \geqq \delta^D(v)$ implies that $c/(1-\delta) > g$ so that Player 2's deviation is prevented.

If Player 1 waits for the good option in the punishment phase, i.e., if $\max\{W, \frac{v}{1-\delta}\} = W$, then Player 2's deviation value is increased to

$$g + \frac{\delta}{2}d + (\frac{\delta}{2})^2 d + \cdots = g + \frac{\delta d}{2-\delta}.$$

In this case Player 2 does not deviate in the cooperation phase if and only if

$$\begin{aligned} & \frac{c}{1-\delta} \geqq g + \frac{\delta d}{2-\delta} \\ \iff \quad & h'(\delta) := -\delta^2(g-c) + \delta(3g-c-d) - 2(g-c) \geqq 0. \end{aligned} \tag{22}$$

This h' has the property that once it exceeds 0 at some δ, $h'(\delta) \geqq 0$ for all larger δ. Plug in $\delta^D(v)$ and we get

$$h'(\delta^D(v)) = \frac{(g-c)}{(g-v)^2}\{(c-v)(v-d) + v(g-v)\} > 0.$$

Therefore for any $\delta \geqq \delta^D(v)$, (22) is satisfied. Note that this argument for Player 2 does not rely on the assumption that $\delta^D(v) \leqq \delta^P(v, \epsilon)$.

In sum, when $\delta^D(v) \leqq \delta^P(v, \epsilon)$, the eternal cooperation is sustained if and only if $\delta \geqq \delta^D(v)$.

Case b: $(v-d)(c-v)/(g-v) < \epsilon$, i.e., $\delta^P(v, \epsilon) < \delta^D(v)$.

In this case, when the on-path value $c/(1-\delta)$ intersects with $g + \delta\frac{v}{1-\delta}$ (at $\delta^D(v)$), the optimal one-shot deviation value is in fact $g + \delta W$. Thus the on-path value function intersects with the optimal one-shot deviation value $\max\{g + \delta W, g + \delta\frac{v}{1-\delta}\}$ when the latter is $g + \delta W$ (see Figure 2b), at $\delta^S(v, \epsilon)$. At this point $v/(1-\delta) < W$. Therefore from (2) and (21),

$$\frac{c}{1-\delta} \geqq \max\{g + \delta W, g + \delta\frac{v}{1-\delta}\} \iff \delta \geqq \delta^S(v, \epsilon).$$

We show that $\delta^S(v, \epsilon) > \delta^D(v)$. As we have seen,

$$h(\delta^D(v)) = \frac{(g-c)}{(g-v)^2}\{(c-v)(v-d) - \epsilon(g-v)\}.$$

Therefore for $(v-d)(c-v)/(g-v) < \epsilon$, $h(\delta^D(v)) < 0$. This means that $\delta^S(v, \epsilon)$, above which $h(\delta) \geqq 0$, must be strictly greater than $\delta^D(v)$.

Since Player 2 does not deviate in the cooperation phase if $\delta \geqq \delta^D(v)$, we conclude that when $\delta^P(v, \epsilon) < \delta^D(v)$, the eternal cooperation is sustained if and only if $\delta \geqq \bar{\delta}(v, \epsilon)$ and this lower bound is strictly greater than $\delta^D(v)$. □

Proof of Proposition 4: We show that there exists a unique $0 < \delta^V < \delta^D(v)$ such that for any $\delta \geqq \delta^V$ (see Figure 3),

$$c + \delta V \geqq g + \delta\frac{v}{1-\delta}.$$

Let

$$\begin{aligned} h(\delta, v, \epsilon) \quad &:= \quad (1-\delta)(2-\delta)\{c + \delta V - g - \delta\frac{v}{1-\delta}\} \\ &= \quad -(g-v)\delta^2 + \{3g - 2c - (v-\epsilon)\}\delta - 2(g-c). \end{aligned}$$

Then

$$c + \delta V \geqq g + \delta\frac{v}{1-\delta} \iff h(\delta, v, \epsilon) \geqq 0.$$

Since $h(\delta, v, \epsilon)$ is a concave, quadratic function of δ, $h(0, v, \epsilon) = -2(g-c) < 0$, and $h(1, v, \epsilon) = \epsilon > 0$, there exists a unique $\delta^V \in (0, 1)$ such that for any $\delta \geqq \delta^V$, $h(\delta, v, \epsilon) \geqq 0$ holds. To show that $\delta^V < \delta^D(v)$, plug in $\delta = \delta^D(v)$ into h and we get

$$h(\delta^D(v), v, \epsilon) = \frac{(g-c)(v+\epsilon-c)}{g-v} > 0.$$

Thus $\delta^D(v) > \delta^V$.

Recall that from (8), we have that

$$c + \delta V \geqq g + \delta W \iff \delta \geqq \frac{2(g-c)}{g-d}.$$

Note that

$$\frac{2(g-c)}{g-d} < \delta^D(v) \iff g + d < 2v. \tag{23}$$

Therefore

$$c + \delta V \geqq \max\{g + \delta W, g + \delta \frac{v}{1-\delta}\}$$

for any $\delta \geqq \max\{\delta^V, \frac{2(g-c)}{g-d}\}$ and this lower bound is strictly smaller than $\delta^D(v)$ under the assumption $g + d < 2v$.

Next, consider Player 2. Let V_2 be the continuation payoff during the cooperation phase for Player 2. Since Player 1 exits with probability 1/2, it satisfies

$$V_2 = \frac{1}{2}\{c + \delta V_2\} + \frac{1}{2} \cdot 0.$$

Thus $V_2 = c/(2-\delta)$ and the on-path value for Player 2 is

$$c + \delta V_2 = \frac{c}{1 - \delta/2}.$$

If he deviates, Player 1 exits immediately if $v/(1-\delta) \geqq W$ or equivalently $\delta \leqq \delta^P(v, \epsilon)$, and Player 1 waits for the good option otherwise. Let W_2 be the continuation payoff during the punishment phase for Player 2, when Player 1 waits for the good option. It satisfies

$$W_2 = \frac{1}{2}\{d + \delta W_2\} + \frac{1}{2} \cdot 0,$$

so that $W_2 = d/(2-\delta)$. Hence the one-shot deviation value for Player 2 is

$$\begin{cases} g + \delta \cdot 0 & \text{if } \delta \leqq \delta^P(v, \epsilon) \\ g + \delta W_2 & \text{if } \delta^P(v, \epsilon) \leqq \delta. \end{cases}$$

Since $d \geqq 0$, it suffices to show that the lower bound of δ that satisfies

$$c + \delta V_2 \geqq g + \delta W_2$$

is less than $\delta^D(v)$. Note that the payoff structure is similar for Player 2 and Player 1;

$$V_2 - W_2 = \frac{1}{2}\{c + \delta V_2\} + \frac{1}{2} \cdot 0 - \frac{1}{2}\{d + \delta W_2\} - \frac{1}{2} \cdot 0,$$

and

$$V - W = \frac{1}{2}\{c + \delta V\} + \frac{1}{2} \cdot \frac{v+\epsilon}{1-\delta} - \frac{1}{2}\{d + \delta W\} - \frac{1}{2} \cdot \frac{v+\epsilon}{1-\delta}.$$

Hence $V_2 - W_2 = V - W$ and since

$$c + \delta V \geqq g + \delta W \iff \delta \geqq \frac{2(g-c)}{g-d},$$

Player 2 does not deviate if and only if $\delta \geqq \frac{2(g-c)}{g-d}$.

Therefore $\delta^S(v,\epsilon) = \max\{\delta^V, \frac{2(g-c)}{g-d}\}$ is the lower bound of the discount factor that sustains the stochastic cooperation and this is less than $\delta^D(v)$. □

Proof of Lemma 4: At a decision node where Player 1 expects (C,C) as long as the game continues and chooses between staying and taking the good option, she compares $c/(1-\delta)$ with $V(p)$. By computation,

$$
\begin{aligned}
& \frac{c}{1-\delta} \geqq V(p), \\
\iff \quad & \frac{c}{1-\delta} \geqq \frac{pv^+ + (1-p)(1-\delta)c}{(1-\delta)\{1-(1-p)\delta\}}, \\
\iff \quad & \{1-(1-p)\delta\}c \geqq pv^+ + (1-p)(1-\delta)c, \\
\iff \quad & c \geqq v^+.
\end{aligned}
$$

□

Proof of Lemma 5: At a decision node where Player 1 expects (D,D) as long as the game continues and chooses between taking only the good option v^+ or taking any option to leave immediately, she compares $W(p)$ with $v/(1-\delta)$. By computation,

$$
\begin{aligned}
& W(p) \geqq \frac{v}{1-\delta} \\
\iff \quad & pv^+ + (1-p)(1-\delta)d \geqq \{1-(1-p)\delta\}v \\
\iff \quad & pv^+ + (1-p)(1-\delta)d \geqq \{1-(1-p)\delta\}\{pv^+ + (1-p)v^-\} \\
\iff \quad & (1-\delta)d \geqq -\delta pv^+ + v^- - (1-p)\delta v^- \\
\iff \quad & \delta v + (1-\delta)d \geqq v^-.
\end{aligned}
$$

□

Proof of Proposition 5: The proof is essentially analogous to that of Proposition 3. First we show that there exists $\delta^S(p) \in (0,1]$ such that

$$\frac{c}{1-\delta} \geqq g + \delta W(p) \iff \delta \geqq \delta^S(p). \tag{24}$$

By computation,

$$
\begin{aligned}
& \frac{c}{1-\delta} \geqq g + \delta W(p) \\
\iff \quad & \{c - g(1-\delta)\}\{1-(1-p)\delta\} \geqq \delta pv^+ + \delta(1-p)(1-\delta)d \\
\iff \quad & h_p(\delta) := -(1-p)(g-d)\delta^2 + \delta\{(1-p)(g-c) + g - (1-p)d - pv^+\} - (g-c) \geqq 0.
\end{aligned}
$$

Again $h_p(\cdot)$ is a concave function of δ, $h_p(0) = -(g-c) < 0$, and $h_p(1) = p(c - v^+) \geqq 0$ for any $p \in (0,1)$. Therefore there exists a unique $\delta^S(p) \in (0,1]$ such that (24) holds.

From (11), we also have

$$W(p) \geqq \frac{v}{1-\delta} \iff \delta \geqq \frac{v^- - d}{v-d}.$$

Thus, depending on whether $\delta^D(v) \leqq \frac{v^- - d}{v-d}$ or $\frac{v^- - d}{v-d} < \delta^D(v)$, we have slightly different arguments. When $\delta^D(v) \leqq \frac{v^- - d}{v-d}$, the on-path value function $c/(1-\delta)$ intersects with the optimal one-shot deviation value at $\delta^D(v)$, since $v/(1-\delta) > W(p)$ at $\delta^D(v)$. Hence in this case Player 1 does not deviate if and only if $\delta \geqq \delta^D(v)$. When $\frac{v^- - d}{v-d} < \delta^D(v)$, the on-path value function intersects with the optimal one-shot deviation value at $\delta^S(p)$. Let us show that $\delta^S(p) > \delta^D(v)$. Using $v = pv^+ + (1-p)v^-$, we have

$$h_p(\delta^D(v)) = \frac{(1-p)(g-c)}{(g-v)^2}[(g-v)(v^- - d) - (g-c)(v-d)] < 0$$

since $\frac{v^- - d}{v-d} < \frac{g-c}{g-v} = \delta^D(v)$. Thus $\delta^D(v) < \delta^S(p)$.

For Player 2, his deviation value is either $g + \delta \cdot 0$ when $\delta < \frac{v^- - d}{v-d}$ so that Player 1 exits immediately after a deviation or $g + \delta W_2(p)$ where $W_2(p)$ satisfies

$$W_2(p) = p \cdot 0 + (1-p)\{d + \delta W_2(p)\},$$

when $\delta \geqq \frac{v^- - d}{v-d}$ so that Player 1 waits for v^+ in the punishment phase. In the former case, $\delta \geqq \delta^D(v)$ implies that $c/(1-\delta) > g$, hence Player 2 does not deviate. In the latter case, notice that $W(p) > W_2(p)$ since $v^+ > d \geqq 0$. Therefore Player 1's condition $c/(1-\delta) \geqq g + \delta W(p)$ implies that Player 2 does not deviate either.

In summary, the eternal cooperation is sustained if and only if $\delta \geqq \delta^D(v)$ when $\delta^D(v) \leqq \frac{v^- - d}{v-d}$ and it is sustained if and only if $\delta \geqq \delta^S(p) (> \delta^D(v))$ when $\frac{v^- - d}{v-d} < \delta^D(v)$. □

Proof of Proposition 6: This proof is analogous to the proof of Proposition 4. Let

$$\begin{aligned} h'_p(\delta) &:= (1-\delta)\{1-(1-p)\delta\}\{c + \delta V(p) - g - \delta \frac{v}{1-\delta}\} \\ &= -(1-p)(g-v)\delta^2 + \{-p(g-v^+) + 2g - c - v\}\delta - (g-c). \end{aligned}$$

Then

$$c + \delta V(p) \geqq g + \delta \frac{v}{1-\delta} \iff h'_p(\delta) \geqq 0.$$

Since $h'_p(\delta)$ is a concave, quadratic function of δ, $h'_p(0) = -(g-c) < 0$, and $h'_p(1) = p(v^+ - v) > 0$, there exists a unique $\delta^V(p) \in (0,1)$ such that for any $\delta \geqq \delta^V(p)$, $h'_p(\delta) \geqq 0$ holds. To show that $\delta^V(p) < \delta^D(v)$, plug in $\delta^D(v)$ into h'_p and we obtain

$$h'_p(\delta^D(v)) = \frac{(g-c)(v^+ - c)}{g-v} > 0.$$

Hence $\delta^V(p) < \delta^D(v)$.

Recall that $c + \delta V(p) \geqq g + \delta W(p)$ if and only if $\delta \geqq \frac{g-c}{(1-p)(g-d)}$ and this is less than $\delta^D(v)$ under the assumption of (14). Therefore Player 1 does not deviate if and only if $\delta \geqq \max\{\delta^V(p), \frac{g-c}{(1-p)(g-d)}\} =: \delta^S(p)$ and this bound is less than $\delta^D(v)$.

Next consider Player 2. Let $V_2(p)$ be the continuation value during the cooperation phase for Player 2. Since Player 1 exits with probability p, it satisfies

$$V_2(p) = (1-p)\{c + \delta V_2(p)\} + p \cdot 0.$$

The on-path value for Player 2 is $c + \delta V_2(p)$. Similarly, let $W_2(p)$ be the continuation payoff during the punishment phase for Player 2, when Player 1 waits for the good option. It satisfies

$$W_2(p) = (1-p)\{d + \delta W_2(p)\} + p \cdot 0.$$

Thus the one-shot deviation value for Player 2 is

$$\begin{cases} g + \delta \cdot 0 & \text{if } \frac{v}{1-\delta} \geqq V(p) \\ g + \delta W_2(p) & \text{if } V(p) \geqq \frac{v}{1-\delta}. \end{cases}$$

Since $d \geqq 0$, it suffices to show that the lower bound of δ that satisfies

$$c + \delta V_2(p) \geqq g + \delta W_2(p)$$

is less than $\delta^D(v)$. As in the case of the 1/2-binary distribution, the payoff structure is similar for Player 2 and Player 1;

$$V(p) - W(p) = (1-p)\{c + \delta V(p)\} + p\frac{v^+}{1-\delta} - (1-p)\{d + \delta W(p)\} - p\frac{v^+}{1-\delta}$$

while

$$V_2(p) - W_2(p) = (1-p)\{c + \delta V_2(p)\} - (1-p)\{d + \delta W_2(p)\}.$$

Hence

$$V(p) - W(p) = V_2(p) - W_2(p)$$

and thus

$$c + V_2(p) \geqq g + \delta W_2(p) \iff \delta \geqq \frac{g-c}{(1-p)(g-d)}.$$

In summary, both players do not deviate if and only if $\delta \geqq \delta^S(p) = \max\{\delta^V(p), \frac{g-c}{(1-p)(g-d)}\}$. □

Proof of Lemma 7: From (15), we have

$$W' = \frac{2(v+\epsilon) + (v-\epsilon) + (1-\delta)d}{(1-\delta)(4-\delta)}.$$

By computation

$$\begin{aligned}(\frac{v}{1-\delta} - W')(1-\delta)(4-\delta) &= (4-\delta)v - 2(v+\epsilon) - (v-\epsilon) - (1-\delta)d \\ &= -\delta(v-d) + v - d - \epsilon,\end{aligned}$$

so that

$$\frac{v}{1-\delta} \geqq W' \iff \delta \leqq \delta^P(v,\epsilon). \tag{25}$$

Moreover, by comparing (5) and (15);

$$\begin{aligned} W &= \frac{1}{2}\cdot\frac{v+\epsilon}{1-\delta} + \frac{1}{4}(d+\delta W) + \frac{1}{4}(d+\delta W) \\ W' &= \frac{1}{2}\cdot\frac{v+\epsilon}{1-\delta} + \frac{1}{4}\cdot\frac{v-\epsilon}{1-\delta} + \frac{1}{4}(d+\delta W') \\ \Rightarrow W - W' &= \frac{d+\delta W - \frac{v-\epsilon}{1-\delta}}{4-\delta}. \end{aligned}$$

Therefore $W \geqq W'$ if and only if

$$\begin{aligned} & d + W \geqq \frac{v-\epsilon}{1-\delta} \\ \iff\quad & W = \frac{1}{2}\cdot\frac{v+\epsilon}{1-\delta} + \frac{1}{2}(d+\delta W) \geqq \frac{1}{2}\cdot\frac{v+\epsilon}{1-\delta} + \frac{1}{2}\frac{v-\epsilon}{1-\delta} = \frac{v}{1-\delta} \\ \iff\quad & \delta \geqq \delta^P(v,\epsilon). \end{aligned}$$

Combined with (25), we have that

$$\begin{aligned} \delta \leqq \delta^P(v,\epsilon) \quad &\Rightarrow \quad g + \delta\frac{v}{1-\delta} \geqq g + \delta W' \geqq g + \delta W; \\ \delta^P(v,\epsilon) \leqq \delta \quad &\Rightarrow \quad g + \delta W \geqq g + \delta W' \geqq g + \delta\frac{v}{1-\delta}. \end{aligned}$$

$\square$

Proof of Proposition 7: Note that now we only need to check one player since they are symmetric.

Since $c > v+\epsilon$, the optimal exit strategy in the cooperation phase is not to exit. Thus the on-path value is $\frac{c}{1-\delta}$. From Lemma 7, the one-shot deviation value is

$$\max\{g+\delta\frac{v}{1-\delta}, g+\delta W'\} = \begin{cases} g+\delta\frac{v}{1-\delta} & \text{if } \delta \leqq \delta^P(v,\epsilon) \\ g+\delta W' & \text{if } \delta \geqq \delta^P(v,\epsilon) \end{cases}$$

As we have shown in the proof of Proposition 3, for case a ($0 < \epsilon \leqq (v-d)(c-v)/(g-v)$), $\delta^D(v) \leqq \delta^P(v,\epsilon)$ implies that for $\delta^D(v) \leqq \delta \leqq \delta^P(v,\epsilon)$, $\frac{c}{1-\delta} \geqq \max\{g+\delta\frac{v}{1-\delta}, g+\delta W'\} = g+\delta\frac{v}{1-\delta}$ so that a player cooperates. For $\delta \geqq \delta^P(v,\epsilon)$, we want to prove that

$$\frac{c}{1-\delta} \geqq g+\delta W'.$$

By computation, this is equivalent to

$$H(\delta) := -\delta^2(g-d)+\delta(5g-c-d-3v-\epsilon)-4(g-c) \geqq 0.$$

By plugging in $\delta^P(v,\epsilon)$, we have

$$H(\delta^P(v,\epsilon)) = \frac{3(v-d)+\epsilon}{(v-d)^2}\{(v-d)(c-v)-\epsilon(g-v)\}.$$

Hence $\epsilon \leqq (v-d)(c-v)/(g-v)$ implies that $H(\delta^P(v,\epsilon)) \geqq 0$ holds, i.e., for any $\delta \geqq \delta^P(v,\epsilon)$, a player does not deviate from (C,C).

Consider case b: $\epsilon > (v-d)(c-v)/(g-v)$ so that $\delta^P(v,\epsilon) < \delta^D(v)$. We focus on $\delta \geqq \delta^D(v)$. Then $\max\{g+\delta\frac{v}{1-\delta}, g+\delta W'\} = g+\delta W'$. Recall that H is quadratic in δ, $H(0) = -4(g-c) < 0$, and $H(1) = 3c-3v-\epsilon > 0$. Hence there exists a unique $0 < \delta^{S'}(v,\epsilon) < 1$ such that for any $\delta \geqq \delta^{S'}(v,\epsilon)$, $H(\delta) \geqq 0$, i.e., $c/(1-\delta) \geqq g+\delta W'$. By pluging in δ^D to $H(\delta)$, we get

$$H(\delta^D(v)) = \frac{g-c}{(g-v)^2}\{(v-d)(c-v)-\epsilon(g-v)\} < 0$$

since $\epsilon > (v-d)(c-v)/(g-v)$. Thus $\delta^{S'}(v,\epsilon) > \delta^D(v)$.

Since $g+\delta W' < g+\delta W$ for any $\delta \geqq \delta^D(v) (> \delta^P(v,\epsilon))$ and $c/(1-\delta)$ is strictly increasing in δ, it must be that $\delta^{S'}(v,\epsilon) < \delta^S(v,\epsilon)$ (Figure 3b). Therefore the lower bound is strictly smaller than the one-sided option case. $\square$

Proof of Proposition 8: First, by comparing (4) and (16),

$$\begin{aligned} V &= \frac{1}{2}\cdot\frac{v+\epsilon}{1-\delta}+\frac{1}{4}(c+\delta V)+\frac{1}{4}(c+\delta V) \\ V' &= \frac{1}{2}\cdot\frac{v+\epsilon}{1-\delta}+\frac{1}{4}\cdot\frac{v-\epsilon}{1-\delta}+\frac{1}{4}(c+\delta V') \\ \Rightarrow V-V' &= \frac{1}{4}(c+\delta V-\frac{v-\epsilon}{1-\delta})+\frac{1}{4}\delta(V-V'). \end{aligned}$$

Hence

$$V \geqq V' \iff c+\delta V \geqq \frac{v-\epsilon}{1-\delta} \iff V \geqq \frac{v}{1-\delta}$$

which is eqiuvalent to

$$(v+\epsilon)+(1-\delta)c \geqq (2-\delta)v.$$

This holds since $v + \epsilon > v$ and $c > v$. Thus for any $0 < \delta < 1$, we have

$$c + \delta V > c + \delta V'.$$

Therefore there exists a unique $\delta'^V < \delta^V$ such that for any $\delta \geqq \delta'^V$,

$$c + \delta V' \geqq g + \delta \frac{v}{1-\delta}.$$

Notice that

$$\begin{aligned} V' &= \frac{1}{2} \cdot \frac{v+\epsilon}{1-\delta} + \frac{1}{4} \cdot \frac{v-\epsilon}{1-\delta} + \frac{1}{4}(c + \delta V') \\ W' &= \frac{1}{2} \cdot \frac{v+\epsilon}{1-\delta} + \frac{1}{4} \cdot \frac{v-\epsilon}{1-\delta} + \frac{1}{4}(d + \delta W') \\ \Rightarrow V' - W' &= \frac{c-d}{4-\delta}. \end{aligned}$$

Therefore

$$c + \delta V' \geqq g + \delta W' \iff \delta(V' - W') \geqq g - c \iff \delta \geqq \frac{4(g-c)}{g-d} > \frac{2(g-c)}{g-d}.$$

In sum,

$$c + \delta V' \geqq \max\{g + \delta \frac{v}{1-\delta}, g + \delta W'\}$$

for any $\delta \geqq \max\{\delta'^V, 4(g-c)/(g-d)\}$ and this lower bound is strictly greater than $\hat{\delta}(v, \epsilon) = \max\{\delta^V, 2(g-c)/(g-d)\}$ for the one-sided option case. □

Appendix C: Continuum of one-sided outside options

Let us assume that Player 1 has a continuum of outside options with the support $[\underline{v}, \overline{v}]$. That is, at the end of each period, an option $x \in [\underline{v}, \overline{v}]$ realizes for Player 1 and if she takes this option, she receives the payoff x forever after. Let F be the (differentiable) cumulative distribution function of the outside options and f be its density function. Assume, as before, that the mean outside option $v := \int_{\underline{v}}^{\overline{v}} x f(x) dx$ is strictly between d and c.

If Player 1 takes an option of value x, then she would also take any option greater than x. Hence the optimal exit strategy is a *reservation strategy*: Player 1 takes any outside option not less than a certain level r, where r is called the reservation level. Let $V(c, r)$ be the value in the cooperation phase, at the end of a period before a stochastic outside option realizes, and when Player 1 takes any option not less than $r \in [\underline{v}, \overline{v}]$. It satisfies the following recursive equation:

$$V(c, r) = \int_r^{\overline{v}} \frac{x}{1-\delta} f(x) dx + F(r)\{c + \delta V(c, r)\}. \tag{26}$$

By differentiation of (26), we have

$$\begin{aligned} \frac{\partial V(c,r)}{\partial r} &= -\frac{r}{1-\delta} f(r) + f(r)c + f(r)\delta V(c,r) + F(r)\frac{\partial V(c,r)}{\partial r}, \\ \iff \quad \frac{\partial V(c,r)}{\partial r} &= \frac{f(r)}{1-\delta F(r)}\big[-\frac{r}{1-\delta} + c + V(c,r)\big]. \end{aligned}$$

Thus the optimal reservation level $r^*(c,\delta)$ during the cooperation phase satisfies[15]

$$\frac{r^*(c,\delta)}{1-\delta} = c + \delta V(c, r^*(c,\delta)). \tag{27}$$

This means that the optimal reservation level of the outside options is exactly where Player 1 is indifferent between taking it and not taking it. From these, we can prove a generalization of Lemma 2.

Lemma 9. *When (C,C) is expected as long as the game continues, the optimal exit strategy for Player 1 is to take only the options not smaller than $r^*(c,\delta)$, i.e., $V(c,r^*(c,\delta)) > c/(1-\delta)$, if and only if $\overline{v} > c$.*

Proof: The optimal exit strategy is to use $r^*(c)$ if and only if $V(c,r^*(c)) > \frac{c}{1-\delta}$. From (26) and (27), we have that

$$V(c, r^*(c)) > \frac{c}{1-\delta} \iff r^*(c) > c.$$

We will prove $r^*(c) > c$. For any $r \in [\underline{v}, \overline{v}]$, consider

$$V(r) := \frac{1}{1-\delta}\{\int_r^{\overline{v}} xf(x)dx + F(r)r\}.$$

By computation, $V(\underline{v}) = \frac{\underline{v}}{1-\delta}$, $V(\overline{v}) = \frac{\overline{v}}{1-\delta}$, and $V'(r) = \frac{F(r)}{1-\delta} > 0$ for any $r > \underline{v}$.

Moreover, $c > v > \underline{v}$ implies that at $r = \underline{v}$,

$$c + \delta V(\underline{v}) = c + \frac{\delta v}{1-\delta} > \frac{\underline{v}}{1-\delta},$$

and $\overline{v} > c$ implies that at $r = \overline{v}$,

$$c + \delta V(\overline{v}) = c + \frac{\delta \overline{v}}{1-\delta} < \frac{\overline{v}}{1-\delta}.$$

Together with the first order condition (26), we have that

$$c + \delta V(r) \gtreqless \frac{r}{1-\delta} \iff r^*(c) \gtreqless r. \tag{28}$$

(See Figure 7.) Notice that $\overline{v} > c$ is equivalent to

$$\begin{aligned}
& \int_c^{\overline{v}} (x-c)f(x)dx > 0 \\
\iff \quad & \int_c^{\overline{v}} xf(x)dx + F(c)c > c \\
\iff \quad & c + \frac{\delta}{1-\delta}\{\int_c^{\overline{v}} \frac{x}{1-\delta} f(x)dx + F(c)c\} > \frac{c}{1-\delta} \\
\iff \quad & c + \frac{\delta}{1-\delta} V(c) > \frac{c}{1-\delta}.
\end{aligned}$$

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[15]It is straightforward to check the second order condition:

$$\frac{\partial^2 V(c,r)}{\partial r^2} = -\frac{1}{1-\delta} \times \frac{f(r)}{1-\delta F(r)} < 0.$$

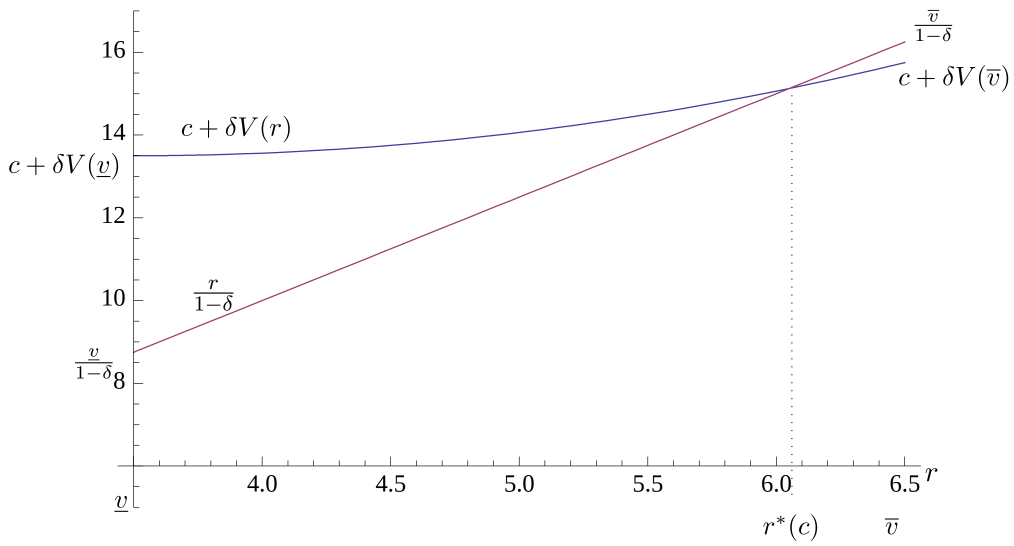

Figure 7: $c + \delta V(r) \gtreqless \frac{r}{1-\delta} \iff r^*(c) \gtreqless r$

Thus, from (28), we established that $r^*(c) > c$. □

Thus it has become clear that, when the *best* outside option exceeds the mutual cooperation payoff c, the eternal cooperation cannot be sustained. In the following we focus on the stochastic cooperation, i.e., we assume that $\overline{v} > c$.

If Player 1 deviates, she expects (D, D) as long as the game continues. Therefore the continuation value, using r as the reservation level, is expressed as

$$V(d, r) = \int_r^{\overline{v}} \frac{x}{1-\delta} f(x)dx + F(r)\{d + \delta V(d, r)\}.$$

By the first order condition, the optimal reservation level after deviation $r^*(d, \delta)$ satisfies

$$\frac{r^*(d, \delta)}{1-\delta} = d + \delta V(d, r^*(d, \delta)). \tag{29}$$

In general, from (27) and (29) and the definition of V, for any $u = c, d$,

$$\frac{r^*(u, \delta)}{1-\delta} = u + \delta V(u, r^*(u, \delta))$$

$$\iff \quad r^*(u, \delta) = (1-\delta)u + \delta(1-\delta)\Big\{\int_{r^*(u,\delta)}^{\overline{v}} \frac{x}{1-\delta} f(x)dx + F(r^*(u, \delta))\frac{r^*(u, \delta)}{1-\delta}\Big\}.$$

Hence, for any $\delta \in (0, 1)$ and any $u = c, d$, the optimal reservation level $r^*(u, \delta)$ is the solution to the following equation:

$$r = (1-\delta)u + \delta \int_r^{\overline{v}} x f(x)dx + \delta F(r)r. \tag{30}$$

The RHS of (30) is a monotone increasing function of r, taking value from $(1-\delta)u + \delta v$ to $(1-\delta)u + \delta\overline{v}$. See Figure 8.

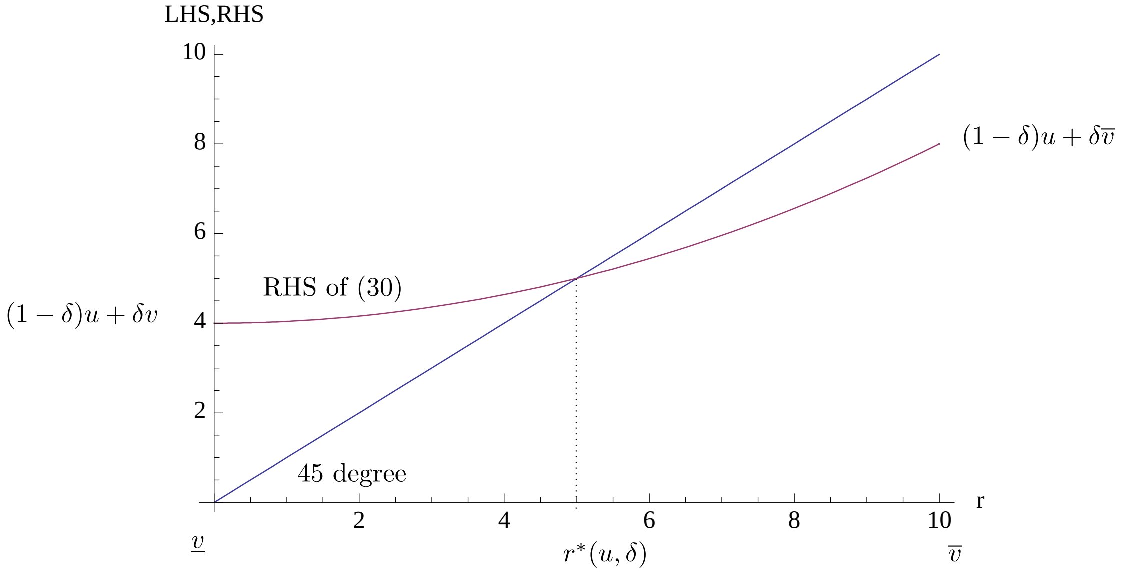

Figure 8: Optimal Reservation Level (F =UNIF$[0, 10]$, $\delta = 0.8$, $u = 0$)

From this observation, we have that if $(1-\delta)d+\delta v \leqq \underline{v}$, i.e., $\delta \leqq \frac{\underline{v}-d}{v-d}$, then the optimal reservation level in the punishment phase is $\underline{v}$. This $\frac{\underline{v}-d}{v-d}$ is the counterpart of $\delta^P(v, \epsilon)$ and $\frac{v^- - d}{v-d}$ of the binary distribution models. Therefore, if

$$\frac{\underline{v}-d}{v-d} \geqq \delta_1^D(v) \iff (\underline{v}-d)(g-v) \geqq (g-c)(v-d), \tag{31}$$

then, from Lemma 9, the cooperation phase value is greater than the deterministic case while the deviation value is the same as the deterministic case, so that the smallest δ that deters Player 1's deviation is less than $\delta^D(v)$, as in the case of Figure 3.

Proposition 9. *Assume that $c < \overline{v}$, (31), and $g - v \leqq gF(c)$. Let δ^F be the lower bound of δ that sustains the stochastic cooperation under the continuum of outside options. Then $\delta^F < \delta^D(v)$.*

Proof: It suffices to prove that Player 2 does not deviate under $\delta \geqq \delta^F$. Recall that Player 1 exits with probability $1 - F(r^*(d, \delta))$ if the optimal reservation level is $r^*(d, \delta)$. Hence Player 2's deviation value is

$$\begin{cases} g + \delta \cdot 0 & \text{if } \delta \leqq \frac{\underline{v}-d}{v-d} \\ g + \delta \frac{d}{1-\delta F(r^*(d,\delta))} & \text{if } \frac{\underline{v}-d}{v-d} \leqq \delta. \end{cases}$$

Player 2's total expected payoff in the cooperation phase is

$$\frac{c}{1 - \delta F(r^*(c, \delta))}.$$

Since we have assumed that $\delta_1^D(v) \leqq \frac{\underline{v}-d}{v-d}$, it suffices to show that the smallest δ that satisfies

$$\frac{c}{1 - \delta F(r^*(c, \delta))} \geqq g \tag{32}$$

is not more than $\delta_1^D(v)$. By rearrangement, (32) is equivalent to

$$\delta \geqq \frac{g-c}{gF(r^*(c, \delta))}.$$

From Lemma 9, $c < r^*(c,\delta)$ so that $F(c) < F(r^*(c,\delta))$. Together with the assumption that $g - v \leqq gF(c)$, we have that

$$g - v \leqq gF(c) < gF(r^*(c,\delta)).$$

This in turn implies that

$$\frac{g-c}{gF(r^*(c,\delta))} < \frac{g-c}{g-v}.$$

□

We have shown that there is a case of continuum outside options in which the stochastic cooperation is sustained under lower discount factors than those of the deterministic model.

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