CONTINUOUS VALUES OF EXACT MARKET GAMES

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Abstract. We prove the uniqueness of core selection values on $\mathcal{HM}$ - the space generated by exact market games with a finite dimensional strictly convex core. Though previous works studied the uniqueness of the value on spaces of non-differentiable market games (e.g.- [3], [4],[5]) they did so under the assumption that the games were of the form $f \circ \mu$ where μ is a vector of mutually singular measures, and this assumption was essential in these studies. The present paper contributes to the study of the uniqueness of the value in the more general case, i.e. - when μ is not necessarily a vector of mutually singular measures. This is in fact the first contribution in this direction.

1. Introduction

One of the most basic solution concepts of cooperative game theory is the Shapley value. It was first introduced in the setting of n-players games, where it can be viewed as the players' expected payoffs. It has a wide range of applications in various fields of economics and political science. In many such applications it is necessary to consider games that involve a large number of players s.t. most of them are "insignificant". Among the typical examples we find voting among stockholders of a corporation and markets with perfect competition. In such cases it is fruitful to model the game as a cooperative game with a "continuum of players". Aumann and Shapley (1974) expanded the definition of the value to suit. The value was defined using the axioms of efficiency, symmetry and monotonicity. In Aumann and Shapley [1, 1974] it is shown that the value exist and is indeed unique on some spaces of games with a continuum of players (e.g. - the space of all "differentiable" market games). It was unknown for quite a long time if there is a value on spaces of "non-smooth" games. An example for such a space of games is the space of all market games. Mertens [6, 1988] introduced a value on a very large set of "non-differentiable" games. Nevertheless Mertens work did not resolve the question of the uniqueness of the value on spaces of non-differentiable games. The first contribution in this direction are the works of Haimanko [3],[4],[5]. Haimanko proved the uniqueness of the Mertens value on some important spaces of non-differentiable games of the form $f \circ \mu$ where μ is a vector of mutually singular measures. However, the more general case, i.e.- where μ is not necessarily a vector of mutually singular measures, did not subdue to his analysis, as the assumption of mutually singularity is crucial in [3],[4],[5]. In this paper we prove the uniqueness of core selection values on $\mathcal{HM}$ - the space of all exact market games with a finite dimensional strictly convex core. Thus, we contribute to the study of the uniqueness of the value on spaces of games of the form $f \circ \mu$, where μ is not necessarily a vector of mutually singular measures. This is in fact the first contribution in this direction.

2. Preliminaries

2.1. **The Mertens Value.** Mertens [6] proves the existence of a value over a large space of games Q_M.

Theorem 2.1 (Mertens [6], Section 2)**.** *Let $v \in Q_M$. Then for every $\xi \in B(I, \mathcal{C})$ $[\varphi_D]_\xi(\chi)$ exists for P-almost every χ and is P-integrable in χ. In particular the map $\varphi_M : Q_M \longrightarrow FA$ given by*

$$(\varphi_M v)(S) = \int [\varphi_D(v)]_S(\chi) dP(\chi)$$

is a value of norm 1 on Q_M, where P is a cylindric measure on $B(I, C)$ which is invariant w.r.t. each measurable automorphism of $(I, \mathcal{C})$

2.2. **Exact Market Games - The Spaces $\mathcal{HM}$ and $\mathcal{HM}_\lambda$.** We turn now to define $\mathcal{HM}$ - the space of finite dimensional exact market games with a strictly convex core.

Definition 2.2. The space $\mathcal{HM}$ is the space of games of the form $f \circ \mu$ where $f \in HM^k$ and $\mu \in \left(NA^1_+\right)^k$ for some $k \geq 1$.

For every probability measure $\lambda \in NA^1_+$ and $k \geq 1$, let

$$\mathcal{Z}^k_\lambda = \left\{ \mu \in \left(NA^1_+\right)^k \ : \ \bar{\mu} \ll \lambda, \ \frac{d\mu}{d\lambda} \in \left(L^\infty(I, \lambda)\right)^k, \frac{d\mu_1}{d\lambda}, ..., \frac{d\mu_k}{d\lambda} \text{ are linearly independent} \right\}.$$

Definition 2.3. Let $\mathcal{HM}_\lambda$ be the space of games of the form $f \circ \mu$ where $f \in HM^k$ and $\mu \in \mathcal{Z}^k_\lambda$ for some $k \geq 1$.

2.3. **The Derivative of a Game $f \circ \mu \in \mathcal{HM}_\lambda$.** For every $k \geq 1$ let $D^k = \{t\mathbf{1}_k : t \in \mathbb{R}\}$ be the diagonal of $\mathbb{R}^k$, and

$$S^k_\perp = \{x \in \mathbb{R}^k \ : \ ||x||_2 = 1, \ \sum x_i = 0, \ x \perp D^k\} \cup \{0\}. \tag{2.1}$$

For every probability measure $\lambda \in NA^1_+$ let $\mathcal{L}^k(\lambda)$ the set of all vectors of k probability measures μ having densities $(\phi_1, ..., \phi_k) \in \left(L^1(\lambda)\right)^k$ w.r.t. λ, and $\mathcal{L}(\lambda) = \bigcup_{k=1}^{\infty} \mathcal{L}^k(\lambda)$. Endow

$$Z_\lambda = \prod_{\mu \in \mathcal{L}(\lambda)} S^k_\perp \times \mathcal{R}(\mu) \tag{2.2}$$

with its product topology and let Y_λ be the topological space whose underlying space is $\{\chi \in L^\infty(\lambda) \ : \ 0 \leq \chi \leq 1 \ \lambda - a.e.\}$, and whose topology is the minimal topology s.t. the map $T : Y_\lambda \to Z_\lambda$ defined below, is continuous:

$$T(y) = \left(x_1(\mu)(y), x_2(\mu)(y)\right)_{\mu \in \mathcal{L}(\lambda)}, \tag{2.3}$$

where

$$x_1(\mu)(y) = \begin{cases} \frac{(\mu - \bar{\mu})(y)}{||(\mu - \bar{\mu})(y)||_2}, & \mu(y) \notin D^k, \\ 0, & \mu(y) \in D^k, \end{cases} \tag{2.4}$$

and

$$x_2(\mu)(y) = \mu(y). \tag{2.5}$$

Lemma 2.4. *T is a topological embedding.*

Definition 2.5. The **space of directions with perspective** λ X_λ is the closure of Y_λ in Z_λ.

Remark 2.6. By abuse of notation we identify Y_λ with $T(Y_\lambda)$. By Lemma 2.4 no confusion should result.

For each game $f \circ \mu \in \mathcal{HM}_\lambda$, where $f \in HM^k$ and $\mu \in \mathcal{Z}^k_\lambda$ define the **derivative operator** $\partial(f \circ \mu) : X_\lambda \to L^\infty(\lambda)$ by

$$\partial(f \circ \mu)(x) = df\left(\mathbf{1}_k, x_1(\mu), \frac{d\mu}{d\lambda}\right). \tag{2.6}$$

Remark 2.7. For every $y \in Y_\lambda$ we have

$$\partial(f \circ \mu)(y) = df\left(\mu(y), \frac{d\mu}{d\lambda}\right). \tag{2.7}$$

Lemma 2.8. [2] *The derivative operator ∂ is well defined. Furthermore $\partial(\mathcal{HM}_\lambda) \subset C(X_\lambda, L^\infty(\lambda)$, and for every $f \circ \mu \in \mathcal{HM}_\lambda$ $\partial(f \circ \mu)$ determines the game $f \circ \mu$ and if $\partial(f \circ \mu) \geq 0$ then $f \circ \mu \in (\mathcal{HM}_\lambda)_+$.*

2.4. **Proper Approximation of Vector Measures.** A sequence of partitions $(\Pi_n)_{n=1}^{\infty}$ of $\mathcal{C}$ is called **admissible** if it is increasing and the σ-algebra generated by $\bigcup_{n\geq 1} \Pi_n$ is $\mathcal{C}$. Let $\nu \in NA_+^1$ be a probability measure and let $\mu \in \mathcal{Z}_\nu^k$. A sequence of measures $\{\mu^n\}_{n=1}^{\infty} \subset \mathcal{Z}_\nu^k$ is a **proper approximation** of μ w.r.t. ν, if $||\mu_i^n - \mu_i||_{BV} \underset{n\to\infty}{\longrightarrow} 0$, and $||\frac{d\mu_i}{d\nu} - \frac{d\mu_i^n}{d\nu}||_\infty \underset{n\to\infty}{\longrightarrow} 0$, for every $1 \leq i \leq k$.

If $\mu \in \mathcal{Z}_\nu^k$ and $\Pi = (\Pi_n)_{n=1}^{\infty}$ is an admissible sequence of partitions we define the **Π-generated approximation** of μ w.r.t. ν as follows: for every $s \in I$ denote by $a_s^n \in \Pi_n$ the unique atom $a \in \Pi_n$ s.t. $s \in a$. For every $n \geq 0$ let $\phi_n : I \to \mathbb{R}^k$, $\phi_n(s) = \begin{cases} \frac{\mu(a_s^n)}{\nu(a_s^n)}, & \nu(a_s^n) > 0, \\ 0, & \nu(a_s^n) = 0. \end{cases}$ Let μ^n be the vector measure whose Radon-Nikodym derivative w.r.t ν is ϕ_n. The sequence $\{\mu^n\}_{n=1}^{\infty}$ is the Π-generated approximation of μ w.r.t. ν.

Lemma 2.9. *Let $\mu \in \mathcal{Z}_\nu^k$. Then, there is an admissible sequence of partitions $\Pi = (\Pi_n)_{n=1}^{\infty}$ s.t. the Π-generated approximation of μ w.r.t. ν is a proper approximation of μ w.r.t. ν.*

2.5. **Representation of values on $\mathcal{HM}$.** The following Theorem is the main result in [2], and it plays a major role in the proof of Proposition 3.1:

Theorem 2.10. *For every $\lambda \in NA_+^1$ and every value φ on $\mathcal{HM}$ there is a measure P_λ on the Borel subsets of X_λ with values in $\mathcal{L}(L^\infty(\lambda), L^2(\lambda)$ s.t. for every game $f \circ \mu \in \mathcal{HM}_\lambda$ and every coalition S we have*

$$\varphi(f \circ \mu)(S) = \left\langle \int_{X_\lambda} \partial(f \circ \mu)(x) dP_\lambda(x), \chi_S \right\rangle_\lambda, \tag{2.8}$$

where $\langle \cdot, \cdot \rangle_\lambda$ is the $L^2(\lambda)$ pairing.

3. Statement of the Main Results

Our first result is the following Proposition on proper approximations of core selection values:

Proposition 3.1. *Let $k \geq 3$ and and $f \circ \mu \in \mathcal{HM}_\lambda$, s.t. $f \in HM^k$ and $\mu \in \mathcal{Z}_\lambda^k$. For sequence $\{\mu^n\}_{n=1}^{\infty} \subset \mathcal{Z}_\lambda^k$ which is a proper approximation of $\mu \in \mathcal{Z}_\lambda^k$ w.r.t. λ, there is a subsequence $\{\mu^{n_j}\}_{j=1}$ s.t. for every coalition $S \in \mathcal{C}$ we have*

$$\varphi(f \circ \mu - f \circ \mu^{n_j})(S) \underset{n\to\infty}{\longrightarrow} 0.$$

As a corollary of 3.1 we have the main result of the paper:

Theorem 3.2. *The Mertens value is the unique core selection on $\mathcal{HM}$.*

The following is an extension of 3.2:

Theorem 3.3. *Suppose $\varphi : \mathcal{HM} \to FA$ is a value s.t. for each $f \circ \mu \in \mathcal{HM}$, where $f \in HM^k$ and $\mu \in \mathcal{Z}^k$ there is a function $g = g(f, \mu)$ which satisfies $\varphi(f \circ \mu) = g \circ \mu$. Then φ is the Mertens value.*

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