Imitation, Local Interaction, and Coordination

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Abstract

This paper analyzes players' long run behavior in evolutionary coordination games with one-dimensional local interaction and imitation. Different from Alós-Ferrer and Weidenholzer's study (JET, 2008), players in our model are assumed to extract valuable information from their interaction neighbors only. It is found that the payoff-dominant equilibrium could survive in the long run with a positive less-than-one probability. We derive the conditions under which both risk-dominant-strategy and payoff-dominant-strategy takers would coexist in the long run. And the risk-dominant equilibrium is the unique long run equilibrium for the rest cases. These results supplement the findings of Alós-Ferrer and Weidenholzer. Finally, the convergence rates to all equilibria are reported.

Keywords: coordination game, imitation, local interaction

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1. Introduction

Multiplicity of the Nash equilibria has weakened the prediction power and application potentials of the game theory on human behavior. The coordination game (hereafter CG) with risk- and payoff-dominant equilibria is a typical example. And the evolutionary learning process is a dynamic method to refine the Nash equilibria. Various hypotheses have been given in the literature to characterize players' boundedly rational behavior, thus resulting in distinct refinements of the Nash equilibria. For instance, under the best-response dynamics, Kandori et al. (1993), Young (1993), Ellison (1993, 2000), Blume (1993, 1995), and Morri (2000) show that players will eventually coordinate at a risk-dominant equilibrium in the long run. However, the equilibrium could be changed to a payoff-dominant one when state-dependent (e.g., Bergin and Lipman (1996)) or time-dependent mutations (e.g., Robles (1998), and Chen and Chow (2001)) are introduced, players' strategy space (e.g., Kandori and Rob (1995)) is enlarged, or all players participate in the CG simultaneously (e.g., Robles (1997) and Hansen and Kaarboe (2002)).

In addition to the best-response dynamics, imitation is commonly observed conduct. Robson and Vega-Redondo (1996) analyze players' long run behavior under an imitation and global interaction setup. In their model, players are randomly and independently matched in pairs to play the CG at each time period, and they imitate actions yielding the higher average payoffs. Robson and Vega-Redondo show that players will eventually coordinate at payoff-dominant equilibrium in the long run because the uncertainty caused by random matching provides chances for payoff-dominant equilibrium to outperform risk-dominant equilibrium. In contrast, Vega-Redondo (1996) show that risk-dominant equilibrium is the unique long run equilibrium under the local interaction setting. Nevertheless, Eshel et al. (1998) conjecture that both payoff-dominant and risk-dominant strategies could coexist in the long run. By allowing players collecting information from other than their interaction neighbors, Alós-Ferrer and Weidenholzer (2008) show that payoff-dominant equilibrium would be the unique long run equilibrium when players' interactions are not "too global" under arbitrary network systems.

Through the rapid development of internet and media, people nowadays can obtain information from the outside world, and interact with their information providers easily. Thus, players' information and interaction neighbors are often the same. On the other hand, searching and screening huge amount of information from internet could be very time-consuming, people may just talk to their friends and relatives. Therefore, this paper tries to examine the robustness of Alós-Ferrer and Weidenholzer's (2008) outcomes under the setting that players' information sources are confined to their interaction neighbors only. In contrast with Alós-Ferrer and Weidenholzer's general interaction scheme, we will focus on a one-dimensional local interaction structure for mathematical tractability, that is, all players are assumed to sit sequentially around a circle and interact with their two immediate neighbors only. Players would imitate successful actions taken by their neighbors or themselves. Then, we can compare our results with those of Vega-Redondo (1996) and Eshel et al. (1998) directly.

It is discovered that players' long run behavior depends on game's payoff structure and population size as mutation rates (or experimental probabilities) tend to zero. When population size and gain of payoff-dominant-strategy takers are large, payoff-dominant equilibrium could emerge in the long run with a positive less-than-one probability. This is also true for some mixed stationary states, in which players taking risk-dominant and payoff-dominant strategies coexist peacefully, with payoff-dominant-strategy takers being the majority. In the remaining cases, risk-dominant equilibrium survives uniquely in the long run. These results suggest that payoff-dominant equilibrium and mixed stationary states cannot survive uniquely in the long run, but risk-dominant equilibrium can, when players gather information from their interaction neighbors only. The findings differ from Alós-Ferrer and Weidenholzer's (2008), thus could provide a supplement to theirs.

Like Eshel et al. (1998), we assume that players use average strategy payoffs as the imitation criterion.[1] However, Eshel et al. (1998) do not conduct stability analyses and characterize the equilibria. Vega-Redondo (1996) imposes an extra condition, in addition to conventional ones, on the payoff structure to enhance chances of risk-dominant equilibrium emerging in the long run.[2] In contrast, we impose no such limit.

The issue of converging to the long run equilibria is also addressed in this study. As we know, the convergence rate to the equilibrium is one important factor in model selection. Here we measure the convergence rate by the expected waiting time for the first long-run-equilibrium visit. Under the global interaction setup, Robson and Vega-Redondo (1996) show that the convergence rate to the payoff-dominant equilibrium would depend on the game's payoff structure, but is independent of its population size. Nevertheless, our results demonstrate that the convergence rates to the long run equilibria depend on both the game's payoff structure and its population size under the local interaction setup. Moreover, we discover that the convergence rate to the payoff-dominant equilibrium under local interaction could be slower than that under global interaction. This outcome is opposite to Ellison's (1993) finding under the best-reply dynamics. It is also found that much more time is needed for players to coordinate at the payoff-dominant equilibrium than at the risk-dominant equilibrium.

[1]In Eshel et al.'s (1998) study players' payoffs are determined by the risk-dominant-strategy number adopted by their neighbors only, while in our model players' payoffs are decided by the strategies employed by their neighbors. This difference will cause dissimilar conditions for the existence of long run equilibria but will not affect the equilibria's contents in the two studies.

[2]In Vega-Redondo's study (1996), players are assumed to meet either of their two neighbors infinite rounds at each time period, and to imitate actions yielding the higher random average payoff. Under the circumstance, the strong law of large numbers implies that players have the same probability to meet each of the two neighbors. Accordingly, the expected strategy payoffs in Vega-Redondo's (1996) cases equal halves of the average strategy payoffs in corresponding cases under our setup. An example is provided in footnote 3.

Our results are derived using the minimum-cost spanning tree method. This method is introduced in Appendix B. Similar approaches have been employed in previous works, such as in Young (1993) and Ellison (2000). Our method differs from these in that it can fully characterize the invariant distribution of a finite Markov chain, and derive the associated convergence rate at the same time. Interested readers could refer to Vega-Redondo (2003) for comparison among different approaches.

Finally, the rest of this paper is organized as follows. In Section 2, our model is presented. The associated outcomes are demonstrated in Section 3. The conclusions are drawn in Section 4. And all proofs for our findings are displayed in Appendix C.

2. The Model

Let $N = \{1,\ 2, \ldots, n\}$, $n \geq 5$, be the set of players. Players are assumed to sit sequentially and equally spaced around a circle. Each individual has exactly two neighbors. For $i \in N$, let $N_i = \{i-1,\ i+1\}$ be the set of player i's neighbors. Note that player $n+i$ is the same as player i by modulo n. At each time period $t \geq 1$, players meet with each of their two neighbors once to play the coordination game below.

	A	B
A	$a,\ a$	$b,\ c$
B	$c,\ b$	$d,\ d$

Here $\{A,\ B\}$ is the strategy set for all players. And $a,\ b,\ c$, and d are payoffs with $a > c,\ d > b,\ d > a$, and $a+b > c+d$. Accordingly, the state space, S, of our dynamic system is a set containing strategy profiles of all players. That is, $S \equiv \{A,\ B\}^N$ with element $\vec{s} = (s_1,\ s_2, \ldots, s_n)$, where s_i is the strategy adopted by player $i,\ i \in N$. For simplicity, labels $\vec{A} = (A,\ A, \ldots, A)$ and $\vec{B} = (B,\ B, \ldots, B)$ represent states where all players choose A and B, respectively. Under the conditions of $a+b > c+d$ and $d > a$, $\vec{A}$ is called the risk-dominant equilibrium and $\vec{B}$ the payoff-dominant equilibrium. In the beginning of each period, players' actions and payoffs occurred (after mutation) in the last period are observable to their neighbors.

Our local-interaction dynamic system consists of the imitation and mutation parts in order. In the imitation process, each player is assumed to imitate the successful action, i.e., the action yielding the highest average payoff which was adopted among his neighbors and himself. Let $\vec{s} = (s_1,\ s_2, \ldots, s_n) \in S$ be the state in the beginning of time t, and $\pi_i^A(\vec{s})$ and $\pi_i^B(\vec{s})$ be the respective average payoffs of strategies A and B among player i and his two neighbors after the game is played at time t. Here we take the convention that for strategy $E \in \{A,\ B\}$, $\pi_i^E(\vec{s}) = -\infty$ if $s_j \neq E$ for $j \in \{i-1, i, i+1\}$. Accordingly, player i's next-period boundedly rational choices, $r_i(\vec{s})$, is required to satisfy

$$r_i(\vec{s}) \in M_i(\vec{s}) \stackrel{\text{def}}{=} \{E \in \{A,\ B\} : \pi_i^E(\vec{s}) = \max(\pi_i^A(\vec{s}), \pi_i^B(\vec{s}))\ \}. \tag{2.1}$$

The computation of $M_i(\vec{s})$ in (2.1) depends on the strategies taken by five consecutive players from $i-2$ to $i+2$. There are 32 cases in total to be reckoned in order to determine $r_i(\vec{s})$ from (2.1). Since

$$r_i(\vec{s}) = s_i \text{ if } s_{i-1} = s_i = s_{i+1}, \tag{2.2}$$

14 cases are left by symmetry. These cases are classified into four categories: (i) only player i in $N_i \cup \{i\}$ taking strategy A, (ii) only player i in $N_i \cup \{i\}$ taking strategy B, (iii) two players including player i in $N_i \cup \{i\}$ taking strategy A, and (iv) two players including player i in $N_i \cup \{i\}$ taking strategy B. The corresponding values of π_i^A and π_i^B are presented in Figures A, B, AA, and BB of Appendix A. For example, if $(s_{i-2}, s_{i-1}, s_i, s_{i+1}, s_{i+2}) = (A, A, A, B, B)$ then $\pi_i^A(\vec{s}) = \frac{1}{2}[2a + (a+b)] = \frac{3a+b}{2}$ and $\pi_i^B(\vec{s}) = c + d$.[3]

[3]Under Vega-Redondo's (1996) setup, the probability of each player meeting either of his two neighbors is 1/2. Then, the random average payoffs of strategies A and B are $\frac{1}{2}[a + \frac{a+b}{2}] = \frac{3a+b}{4}$ and $\frac{c+d}{2}$, respectively. They are halves of $\pi_i^A(\vec{s})$ and $\pi_i^B(\vec{s})$, respectively. The same results occur in all other cases. Thus, our setting is equivalent to Vega-Redondo's (1996).

When $M_i(\vec{s})$ is a singleton, no ambiguity occurs. However, if $M_i(\vec{s}) = \{A,\ B\}$, selection rules should be considered. For brevity, we assume throughout this paper that players will stick to their original strategies due to inertia when $M_i(\vec{s})$ is not a singleton.[4] Thus, players' boundedly rational strategy profile, $\vec{r}(\vec{s}) = (r_1(\vec{s}), r_2(\vec{s}), ..., r_n(\vec{s}))$, satisfies

$$r_i(\vec{s}) = s_i \text{ iff } \ s_i \in M_i(\vec{s}) \text{ for all } i \in N. \tag{2.3}$$

The above imitation process induces a time-homogeneous Markov chain on S. Its probability transition matrix $Q_0 : S \times S \to [0,\ 1]$ is given by

$$Q_0(\vec{s}, \vec{r}(\vec{s})) = 1 \text{ and } Q_0(\vec{s}, \vec{u}) = 0 \text{ for } \vec{u} \neq \vec{r}(\vec{s}). \tag{2.4}$$

In particular, we have from (2.2) that $\vec{r}(\vec{A}) = \vec{A}$ and $\vec{r}(\vec{B}) = \vec{B}$. Hence,

$$Q_0(\vec{A}, \vec{A}) = Q_0(\vec{B}, \vec{B}) = 1. \tag{2.5}$$

In the Markov chain terminology, each state in S either is transient or lies in a closed connected component $\mathcal{F}$. In the latter case, it is called a stationary state. By definition, all states in any $\mathcal{F}$ can reach each other. A state is absorbing iff it lies in some $\mathcal{F}$ with $|\mathcal{F}| = 1$. Therefore, $\mathcal{F}$ is sometimes called an absorbing set. Owning to $|S| < \infty$, set S_0 containing all stationary states is nonempty and can be characterized by

$$S_0 = \{\vec{s} \in S \ : \ \lim_{t\to\infty} \nu(Q_0)^t(\vec{s}) > 0 \text{ for some initial distribution } \nu \text{ on } S\}. \tag{2.6}$$

Since $\vec{A}$ and $\vec{B}$ are absorbing states under Q_0 by (2.5), we have

$$\{\vec{A}, \vec{B}\} \subseteq S_0. \tag{2.7}$$

After completing the imitation process, players will independently alter their revised rational choices with identical probability $\epsilon > 0$, which is called the mutation

[4] Actually, we have tried another tie-breaking rule assuming that both A and B will be selected in the next period with arbitrary positive probabilities when $M_i(\vec{s})$ is not a singleton. Under the circumstance, the outcomes will change merely under boundary cases, and $\vec{B}$ will become less favorable.

rate, at the end of each period. One can regard the mutation rate as the probability of players' experimenting with new strategies. All together, our local-interaction dynamic system defines a Markov chain $\{X_t : t = 0, 1, ...\}$ on S with probability transition matrix Q_ϵ, which is a perturbation of Q_0 in (2.4) and given by

$$Q_\epsilon(\vec{s}, \vec{u}) = \epsilon^{d(\vec{r}(\vec{s}),\, \vec{u})} \cdot (1-\epsilon)^{n-d(\vec{r}(\vec{s}),\, \vec{u})} \text{ for any } \vec{s}, \vec{u} \in S, \tag{2.8}$$

where $d(\vec{r}(\vec{s}), \vec{u}) = |\{i \in N : r_i(\vec{s}) \neq u_i\}|$ is the number of mismatches between the truly-adopted strategy $\vec{u}$ and the rational choice $\vec{r}(\vec{s})$ at state $\vec{s}$.

Because $Q_\epsilon(\vec{s},\ \vec{u}) > 0$ for all $\vec{s}$ and $\vec{u} \in S$, planting mutation has the advantage of making our dynamic system $\{X_t\}$ ergodic. Let μ_ϵ be the associated unique invariant distribution, which is independent of the initial distribution and characterized by

$$\mu_\epsilon = \mu_\epsilon \cdot Q_\epsilon. \tag{2.9}$$

We are interested in the limit probability distribution $\mu_* \stackrel{\text{def}}{=} \lim_{\epsilon \to 0} \mu_\epsilon$ and its support

$$S_* \stackrel{\text{def}}{=} \{\vec{s} \in S \ : \ \mu_*(\vec{s}) > 0\}. \tag{2.10}$$

Each element in S_* is called the stochastically stable state or long run equilibrium (hereafter LRE). Moreover, we are interested in the order estimate of $E(T_\epsilon)$, i.e., the expected waiting time of first visiting S_*, where

$$T_\epsilon = \inf\{t \geq 0 : X_t \in S_*\} \tag{2.11}$$

is the first time that $\{X_t\}$ hits S_* with, say, the initial X_0 uniformly distributed on S.

By letting $\epsilon \downarrow 0$ in (2.9), Vega-Redondo (2003) shows $\mu_* = \mu_* Q_0$ and thus

$$S_* \subseteq S_0 \tag{2.12}$$

by (2.6). Employing a method in Freidlin and Wentzell (1984; pp.177-199), which is explained in Appendix B, we can find S_* and $E(T_\epsilon)$. These outcomes are given below.

3. The Results

Before presenting the results, some notations are introduced. Any state in

$$M \overset{\text{def}}{=} S_0 \setminus \{\vec{A}, \vec{B}\} \tag{3.1}$$

is called a mixed stationary state. If $M \neq \emptyset$, any $\vec{s} \in M$ consists of some A-strings alternating with equal number of B-strings as all players sit around a circle. Decompose M as $\cup_{k\geq 1} M_k$, where any $\vec{s} \in M_k$ has k A-strings and can be represented as

$$\cdots \underbrace{A\cdots A}_{a_k}\underbrace{B\cdots B}_{b_k}\underbrace{A\cdots A}_{a_1}\underbrace{B\cdots B}_{b_1}\underbrace{A\cdots A}_{a_2}\underbrace{B\cdots B}_{b_2}\cdots. \tag{3.2}$$

Here a_i and b_i are the lengths of its i-th A-string and B-string, respectively. To display our results neatly, we introduce the following notations. For positive integers m and p,

$$M_{m,\,p} \overset{\text{def}}{=} \{\vec{s} \in S \ : \ \text{all } a_i = m,\ b_j = p \text{ in } (3.2)\},$$
$$M_{m,\,\geq p} \overset{\text{def}}{=} \{\vec{s} \in S \ : \ \text{all } a_i = m,\ b_j \geq p \text{ in } (3.2)\}.$$

Similarly, we can define $M_{\geq m,\,\geq p}$, $M_{\leq m,\,\geq p}$ and so on. Furthermore,

$$\bar{M}_{\geq m,\,\geq p} = \{\vec{s} \in M_{\geq m,\,\geq p} \ : a_j = a_{j+1} = m \text{ if any } b_j = p\},$$

and

$$\begin{aligned}\tilde{M} \ &= \ \{\vec{s} \in M_{\leq 3,\,\geq 3} \ : \ b_i \geq 4 \ \text{ if } (a_i,\ a_{i+1}) = (1,2) \text{ or } (2,1), \text{ and} \\ &\qquad b_i \geq 5 \text{ if } (a_i,\ a_{i+1}) = (1,1) \text{ in } (3.2)\}.\end{aligned}$$

Throughout the paper we assume that

$$|N| = n \geq 5,\ a > c,\ d > b,\ d > a, \text{ and } a + b > c + d. \tag{3.3}$$

As a consequence, $b > c$ and then $c = \min\{a, b, c, d\}$. The notation

$$f(\epsilon) \approx \epsilon^{\alpha} \text{ means that } \lim_{\epsilon \downarrow 0} f(\epsilon)/\epsilon^{\alpha} \text{ exists and is positive,}$$

$\lceil x \rceil$ represents the least integer no less than x, while $\lfloor x \rfloor$ means the greatest integer no greater than x. The main result, which will be proved in Appendix C, is as follows.

Theorem 3.1. *Assume imitating-successful-action rule* (2.1) *and* (3.3).
(a) *If* $\frac{3a+b}{2} \geq c+d$ *and* $a+b > \frac{c+3d}{2}$, *then* $S_0 = \{\vec{A},\ \vec{B}\} \cup M$ *with*

$$M = \begin{cases} \emptyset & \text{for } \frac{c+3d}{2} < 2b \text{ or } 2b < c+d, \\ M_{1,\, \geq 3} & \text{for } c+d < 2b \leq \frac{c+3d}{2}, \\ M_{1,\, \geq 2} & \text{for } 2b = c+d. \end{cases}$$

Moreover, as $\epsilon \downarrow 0$, $S_* = \{\vec{A}\}$ *and*

$$E(T_\epsilon) \approx \epsilon^{-1} \text{if } 2b \geq c+d, \text{ while } E(T_\epsilon) \approx \epsilon^{-2} \text{ if } 2b < c+d. \tag{3.4}$$

(b) *If* $\frac{3a+b}{2} \geq c+d$ *and* $a+b \leq \frac{c+3d}{2}$, *then* $S_0 = \{\vec{A},\ \vec{B}\} \cup M$ *with*

$$M = \begin{cases} M_{\geq 2,\, \geq 3} & \text{for } \frac{c+3d}{2} < 2b \text{ or } 2b < c+d, \\ M_{\geq 1,\, \geq 3} & \text{for } c+d < 2b \leq \frac{c+3d}{2}, \\ \bar{M}_{\geq 1,\, \geq 2} & \text{for } 2b = c+d. \end{cases}$$

Moreover, $S_* = \{\vec{A}\}$ *as* $\epsilon \downarrow 0$ *and* (3.4) *holds.*
(c) *If* $\frac{3a+b}{2} < c+d$ *and* $2b \leq \frac{c+3d}{2}$, *then* $S_0 = \{\vec{A},\ \vec{B}\} \cup M_{\leq 2,\, \geq 3}$. *Moreover, as* $\epsilon \downarrow 0$,

$$\begin{cases} S_* = \{\vec{A}\} \text{ and } E(T_\epsilon) \approx \epsilon^{-\lceil \frac{n}{7} \rceil} & \text{for } 5 \leq n \leq 14, \\ S_* = S_0 \text{ and } E(T_\epsilon) \approx \epsilon^{0} & \text{for } 15 \leq n \leq 21, \\ S_* = S_0 \backslash \{\vec{A}\} \text{ and } E(T_\epsilon) \approx \epsilon^{-3} & \text{for } n \geq 22. \end{cases}$$

(d) *If* $\frac{3a+b}{2} < c+d$ *and* $2b > \frac{c+3d}{2}$, *then* $S_0 = \{\vec{A},\ \vec{B}\} \cup \tilde{M}$. *Moreover, as* $\epsilon \downarrow 0$,

$$\begin{cases} S_* = \{\vec{A}\} \text{ and } E(T_\epsilon) \approx \epsilon^{-1} & \text{for } n = 5, \\ S_* = \{\vec{A}\} \text{ and } E(T_\epsilon) \approx \epsilon^{-\lceil \frac{n}{10} \rceil} & \text{for } 6 \leq n \leq 20, \\ S_* = S_0 \text{ and } E(T_\epsilon) \approx \epsilon^{0} & \text{for } 21 \leq n < 30,\ n \neq 25, \\ S_* = S_0 \backslash M_{2,\, 3} \text{ and } E(T_\epsilon) \approx \epsilon^{-1} & \text{for } n = 25 \text{ or } 30, \\ S_* = (S_0 \backslash M_{2,\, 3}) \backslash \{\vec{A}\} \text{ and } E(T_\epsilon) \approx \epsilon^{-3} & \text{for } n \geq 31. \end{cases} \tag{3.5}$$

Note that $M_{2,\,3} \neq \emptyset$ iff $5|n$, where $5|n$ means that n is a multiple of 5.

Theorem 3.1 demonstrates that players' long run behavior and convergence rates to the LREs would depend on both the game's payoff structure and its population size. Risk-dominant, payoff-dominant, as well as mixed stationary states could be the LREs. If gain of payoff-dominant-strategy takers, d, is small with $c+d \leq \frac{3a+b}{2}$, Theorem 3.1(a)-(b) show that risk-dominant equilibrium is always the unique LRE. In contrast, if d is large enough with $c+d > \frac{3a+b}{2}$, payoff-dominant equilibrium and some mixed stationary states could emerge in the long run with a positive less-than-one probability for large population, which are discussed below.

In the case of $c+d > \frac{3a+b}{2}$ and $2b \leq \frac{c+3d}{2}$, Theorem 3.1(c) shows that, as population grows, $\vec{A}$ changes from being the unique LRE for $5 \leq n \leq 14$ to coexisting with $M_{\leq 2,\,\geq 3} \cup \{\vec{B}\}$ for $15 \leq n \leq 21$, and finally to being dominated by $M_{\leq 2,\,\geq 3} \cup \{\vec{B}\}$ for $n \geq 22$. In sum, risk-dominant equilibrium is the unique LRE when population size is small. Oppositely, for large population, both payoff-dominant equilibrium and some mixed stationary states with the ratio of B-player equal to at least 0.6 emerge, but they cannot survive alone in the long run.

In the most interesting case of $c+d > \frac{3a+b}{2}$ and $2b > \frac{c+3d}{2}$, Theorem 3.1(d) shows that, as population grows, $\vec{A}$ changes from being the unique LRE for $5 \leq n \leq 20$ to coexisting with $(M \setminus M_{2,\,3}) \cup \{\vec{B}\}$ for $21 \leq n \leq 30$, and finally to being dominated by $(M \setminus M_{2,\,3}) \cup \{\vec{B}\}$ for $n \geq 31$. As in Theorem 3.1(c), risk-dominant equilibrium is the unique LRE for small population, and payoff-dominant equilibrium and some mixed stationary states with B-player being the majority could be the LREs with a less-than-one probability for large population. However, unlike in Theorem 3.1(c), not all mixed stationary states could be the LREs. For instance, mixed stationary states in $M_{2,\,3}$ with B-player being the majority can never be the LREs.

Outcomes of Theorem 3.1(a)-(b) and Theorem 3.1(c)-(d) differ because of the following reasons. Consider that an A-string having length ≥ 3 confronts a B-string having length ≥ 3. Players' total payoffs after finishing playing are

$$\begin{array}{lcccccc} \text{state} & \dots A & A & A & B & B & B \quad \dots \\ \text{total payoff} & \dots & 2a & (a+b) & (c+d) & 2d & \dots \end{array}$$

Thus, the average payoffs of strategies A and B in the neighbors of the A-player on the boundary are $\pi^A = \frac{2a+(a+b)}{2} = \frac{3a+b}{2}$ and $\pi^B = c+d$, respectively. When $\frac{3a+b}{2} \geq c+d$, the A-players will retain their strategy, while the B-player on the boundary will switch to A. In the next step, the B-player on the new boundary will also switch to A, and so forth. Accordingly, A-string will continue to grow until $\vec{A}$ is reached. Thus, $\vec{A}$ survives uniquely as shown by Theorem 3.1(a)-(b). In contrast, for $\frac{3a+b}{2} < c+d$ and $2b > \frac{c+3d}{2}$, the A-player on the boundary will change to B, and the B-players will retain their strategy. In the following periods, B-string will continue to grow until the states in set $\tilde{M}$, which is defined before (3.3), are reached. In these states, the lengths of A-strings equal 1, 2, or 3. Since A-players and B-players in these states will be sustained, they are mixed stationary states. In particular, the states with a single A,

$$\cdots BBB \overset{\bullet}{A} BBB \cdots,$$

are mixed stationary state. Through them, other limit states can reach $\vec{B}$ at smaller cost (or fewer mutations), hence $\vec{B}$ could coexist with some mixed stationary states in the long run.

Next we compare our results with those of relevant studies. Under the condition of $a+b > \frac{c+3d}{2}$, Vega-Redondo (1996) demonstrates that risk-dominant equilibrium is the unique LRE and its convergence rate has order ϵ^{-2}. This finding is a special case of our Theorem 3.1(a) with $2b < c+d$. Theorem 3.1 also verifies Eshel et al.'s (1998) conjecture saying that mixed stationary states could be stochastically stable. Furthermore, we display that only mixed stationary states with payoff-dominant strategy being the majority could be the LREs. However, not all mixed stationary states satisfying this condition, such as states in $M_{2,\,3}$, could be the LREs. The conditions for risk-dominant and payoff-dominant equilibria to be the LREs presented here are absent in Eshel et al.'s (1998) study.

Finally, Theorem 3.1 provides the convergence rate to the LRE in all cases. Basically, the convergence rate to the LRE depends on the game's payoff structure and population size. Under global interaction, Robson and Vega-Redondo (1996) show that the convergence rate to the payoff-dominant equilibrium is of order ϵ^{-k}, where k is the smallest integer satisfying

$$\frac{d(k-1)+c}{k} \geq \max\{a,\ b\}.$$

Actually, one can illustrate $k = 2$. In contrast, Theorem 3.1(c)-(d) show that the convergence rate to the payoff-dominant equilibrium could have order ϵ^{-3}. This means that the convergence rate to the payoff-dominant equilibrium under local interaction could be slower than that under the global interaction. This outcome is different from Ellison's (1993), which shows that the convergence rate to the LRE under local interaction is faster than that under global interaction in the best-reply dynamics. Moreover, Theorem 3.1(d) displays that the convergence rates to $(M \setminus M_{2,\,3}) \cup \{\vec{B}\}$ have order ϵ^{-3}, while the convergence rate to $\vec{A}$ is either of order ϵ^{-1} or ϵ^{-2}. It suggests that more time is needed for players to coordinate at the payoff-dominant equilibrium than at the risk-dominant equilibrium.

4. Conclusions

This paper investigates players' long run behavior in evolutionary coordination games with local interaction and imitation. Players are assumed to imitate successful actions taken by their neighbors or themselves, and to collect information from their interaction neighbors only. Which limit states are the long run equilibria and convergence rates to the long run equilibria depend on the game's payoff structure and population size. We discover that possible long run equilibria include:(a)risk-dominant equilibrium alone, (b)payoff-dominant equilibrium and some mixed stationary states with payoff-dominant-strategy takers being the majority, and (c)combination of (a) and (b). This result implies that payoff-dominant equilibrium cannot survive alone in the long run, neither can mixed stationary states. However, risk-dominant equilibrium can emerge alone. Moreover, the convergence rate to payoff-dominant equilibrium under local interaction could be slower than that under global interaction. More time is needed for players to coordinate at payoff-dominant equilibrium than at risk-dominant equilibrium.

By introducing state-independent and time-independent mutations, Kandori et al. (1993), Young (1993), and Ellison (1993) show the powerful refining effect of mutation under the best-reply dynamics. The similar effect is found by Robson and Vega-Redondo (1996) under the global interaction and imitation setup. Nevertheless, such refining effect seems to fail under the local interaction and imitation setup. Alós-Ferrer and Weidenholzer (2006) show that risk-dominant equilibrium will survive uniquely when players imitate successful players yielding the highest total payoff. Alós-Ferrer and Weidenholzer's (2006) and our outcomes suggest that how players imitate will decisively affect the refining effect of mutation. Moreover, the number of neighbors with information is also an important factor in influencing the refining effect of mutation revealed by Alós-Ferrer and Weidenholzer (2008) and our outcomes.

In the future, we would like to examine whether our results here hold under more general frameworks. Our another study (Chen and Chow (2009)) shows that imitation rules are crucial in determining players' long run behavior in evolutionary prisoner's dilemma games with identical information and interaction neighbors. Thus, it will be interesting to find out whether the results would change if players' information neighbors are more than their interaction neighbors in repeated prisoner's dilemma games. Finally, other matching rules could be investigated under the local interaction setting, such as the finite-round matching for players at each period proposed by Robson and Vega-Redondo (1996).

Appendix A

In Figures A, B, AA, BB, the states depict the strategies adopted by players $i-2,\ i-1,\ i,\ i+1$, and $i+2$.

Figure A

States $\vec{s}$	Average payoffs for strategies A and B
$\cdots ABABA \cdots$	$\pi_i^A = 2b,\ \pi_i^B = 2c.$
$\cdots ABABB \cdots$	$\pi_i^A = 2b,\ \pi_i^B = \frac{3c+d}{2}.$
$\cdots BBABB \cdots$	$\pi_i^A = 2b,\ \pi_i^B = c+d.$

Figure B

States $\vec{s}$	Average payoffs for strategies A and B
$\cdots AABAA \cdots$	$\pi_i^A = a+b,\ \pi_i^B = 2c.$
$\cdots AABAB \cdots$	$\pi_i^A = \frac{a+3b}{2},\ \pi_i^B = 2c.$
$\cdots BABAB \cdots$	$\pi_i^A = 2b,\ \pi_i^B = 2c.$

Figure AA

States $\vec{s}$	Average payoffs for strategies A and B
$\cdots AAABA \cdots$	$\pi_i^A = \frac{3a+b}{2},\ \pi_i^B = 2c.$
$\cdots AAABB \cdots$	$\pi_i^A = \frac{3a+b}{2},\ \pi_i^B = c+d.$
$\cdots BAABA \cdots$	$\pi_i^A = a+b,\ \pi_i^B = 2c.$
$\cdots BAABB \cdots$	$\pi_i^A = a+b,\ \pi_i^B = c+d.$

Figure BB

States $\vec{s}$	Average payoffs for strategies A and B
$\cdots AABBA \cdots$	$\pi_i^A = a + b,\ \pi_i^B = c + d.$
$\cdots AABBB \cdots$	$\pi_i^A = a + b,\ \pi_i^B = \frac{c+3d}{2}.$
$\cdots BABBA \cdots$	$\pi_i^A = 2b,\ \pi_i^B = c + d.$
$\cdots BABBB \cdots$	$\pi_i^A = 2b,\ \pi_i^B = \frac{c+3d}{2}.$

Appendix B. The Minimum Cost Spanning Tree Method

Some terminologies are needed in order to describe the invariant distribution μ_ϵ in (2.9). Let W be a subset of S. A graph g consisting of arrows $\vec{u} \to \vec{v}$, where $\vec{u} \in S \backslash W$ and $\vec{v} \in S$, is called a W-graph if it satisfies the following conditions:

(1) every state in $S \backslash W$ is the initial of exactly one arrow;

(2) there exists a sequence of arrows leading from any state in $S \backslash W$ to W. Or equivalently, there are no cycles in the graph g.

Denote by $G(W)$ the set of all W-graphs. For any state $\vec{s} \in S$, define

$$\alpha_{\vec{s}} = \sum_{g \in G(\{\vec{s}\})} \prod_{(\vec{u} \to \vec{v}) \in g} Q_\epsilon(\vec{u}, \vec{v}). \tag{B.1}$$

It is shown in Freidlin and Wentzell (1984; p.177) that $(\alpha_{\vec{s}} : \vec{s} \in S) = (\alpha_{\vec{s}} : \vec{s} \in S) \cdot Q_\epsilon$. Since μ_ϵ is the unique probability distribution satisfying (2.9), it follows that

$$\mu_\epsilon = \frac{1}{\sum_{\vec{s} \in S} \alpha_{\vec{s}}} (\alpha_{\vec{s}} : \vec{s} \in S). \tag{B.2}$$

Equation (B.2) holds for any time-homogeneous, irreducible and aperiodic finite-state Markov chain. In general, it is not useful as computing $\alpha_{\vec{s}}$ from (B.1) is difficult. However, we have from (2.8) that for any $\vec{s}, \vec{u} \in S$,

$$Q_\epsilon(\vec{s}, \vec{u}) = \epsilon^{U(\vec{s}, \vec{u})} \cdot (1 + O(\epsilon)), \text{ where } U(\vec{s}, \vec{u}) = d(\vec{r}(\vec{s}), \vec{u}) = |\{i \in N : u_i \neq r_i(\vec{s})\}| \tag{B.3}$$

and $\vec{r}(\vec{s})$ is the rational choice uniquely determined under the rule (2.3). Hence

$$\alpha_{\vec{s}} = \sum_{g \in G(\{\vec{s}\})} \epsilon^{v(g)}(1 + O(\epsilon)) \approx \epsilon^{v(\{\vec{s}\})} \text{ for } \epsilon \text{ small,} \tag{B.4}$$

where $v(g) = \sum_{(\vec{u} \to \vec{v}) \in g} U(\vec{u}, \vec{v})$ and $v(\{\vec{s}\}) = \min_{g \in G(\{\vec{s}\})} v(g)$ are constants independent of ϵ. Define

$$v_1 = \min_{\vec{s} \in S} v(\{\vec{s}\}). \tag{B.5}$$

By (B.2) and (B.4), $\mu_* = \lim_{\epsilon \to 0} \mu_\epsilon$ does exist and the following holds.

Theorem B1. *The support* S_* *of* μ_* *defined in (2.10) is given by*

$$S_* = \{\vec{s} \in S \mid v(\{\vec{s}\}) = v_1\} \textit{ and } \mu_\epsilon(\vec{u}) \approx \epsilon^{v(\{\vec{u}\}) - v_1} \textit{for any } \vec{u} \in S. \tag{B.6}$$

In order to find the order estimate of $E(T_\epsilon)$ defined in (2.11), we need to generalize (B.5). For $k \geq 1$ define

$$v_k = \min_{|W|=k} v(W), \quad \text{where } v(W) = \min_{g \in G(W)} v(g). \tag{B.7}$$

Let W_k be any solution to v_k above. Note that $W_1 \subseteq S_*$. Since $W_k = S$ when $k = |S|$, that $W_k \subseteq S_*$ cannot always be valid unless $S_* = S$. This exceptional case will not occur in our models as it is clear from the statements of Theorem 3.1 that $S_* \subseteq S_0 \neq S$.

Theorem B2. (Chiang and Chow (2007)) *Let* T_ϵ *be given in* (2.11). *Then*

$$E(T_\epsilon) \approx \epsilon^{-\delta} \text{ as } \epsilon \downarrow 0, \quad \text{where } \delta = v_{k_0-1} - v_{k_0} \tag{B.8}$$

and $k_0 = \min\{k \geq 2 :$ there exists $W \subseteq S$ with $|W| = k, v(W) = v_k$ and $W \not\subseteq S_*\}$.

The constant δ above means "escape energy" in simulated annealing, which is a probabilistic algorithm aiming at finding the global minima of combinatorial optimization problems. With this amount of energy, any state outside S_*, the so-called "global minima" set, could reach S_*.

Regarding $U(\vec{u}, \vec{v})$ as the cost for going from $\vec{u}$ to $\vec{v}$, then $v(\{\vec{s}\})$ means the minimum cost of all spanning trees with root at $\vec{s}$. By (B.6), the set S_* consists of those states which attain the minimum cost v_1 when treated as a root. Similarly, any solution W_k to (B.7) represents an optimal choice for attaining v_k, the minimum cost of all spanning forests with k roots. The quantity $(v_k - v_{k+1})$ means the cost saved from having k roots to having $k+1$ roots in constructing optimal spanning forests on S.

Appendix C. Proof of Theorem 3.1.

In view of Theorems B1 and B2, the minimum-cost spanning tree method can be used to find the support S_* and the order estimate of $E(T_\epsilon)$. The first step is to find S_0 as $S_* \subseteq S_0$ by (2.12). This is the set of limit states under Q_0. Let $\vec{s} \in S \setminus S_0$ be any transient state. By definition, there exist $\vec{s_0} = \vec{s}, \vec{s_1}, ..., \vec{s_j}$ such that $\vec{s_j} \in S_0$ and $Q_0(\vec{s_k}, \vec{s}_{k+1}) > 0$ for $0 \leq k < j$. By (2.4) and (B.3), $\sum_{k=0}^{k=j-1} U(\vec{s_k}, \vec{s}_{k+1}) = 0$. The converse does not hold. Otherwise, $\vec{s} \in S_0$. In words,

$$\vec{s} \text{ is transient iff } \exists \text{ a zero cost path from } \vec{s} \text{ to } S_0, \text{ but not the converse.}$$

The next step is to compute $v(\{\vec{s}\})$, the minimum cost spanning tree rooted at $\vec{s} \in S$ and solutions W_k to (B.7). Then S_*, $E(T_\epsilon)$ can be obtained via (B.6)-(B.8). It suffices to consider $\vec{s} \in S_0$ as the formula above implies it saves none by taking any transient $\vec{u}$ as a root. Hence $\vec{u} \notin S_*$. By the same reason, $\vec{u} \notin W$ for any $v(W) = v_k$ in (B.7) unless no ergodic state in S_0 remains available. Note that $S_0 \neq \emptyset$ by (2.7).

Let F be any connected component in S_0. Then any two states in F can be connected by a path in F with zero total cost. Hence, when constructing a minimum cost spanning forest, all states in F should first converge to a certain state of F and then reach out from there in case the roots of the forest lie outside F. Only the last reach-out move will cost some price ≥ 1. In fact, the price is

$$\min\{U(\vec{w}, \vec{u}) : \vec{w} \in F \text{ and } \exists \text{ a zero cost path from } \vec{u} \text{ to } S_0 \setminus F\}$$

More importantly, it will be shown later that if $M \neq \emptyset$ then

$$\text{all states in } M \text{ can reach } \{\vec{A}, \vec{B}\} \text{ at minimum cost 1 per connected component.}$$

This is the key ingredient that makes the method of minimum cost spanning forests efficient for determining S_* and $E(T_\epsilon)$ via (B.6)-(B.8).

For convenience, we introduce the following notations:

$$\vec{s} \xrightarrow{k} \vec{u} \text{ means } U(\vec{s},\vec{u}) = k \text{ and } \vec{s} \overset{k}{\leftrightarrow} \vec{u} \text{ means } U(\vec{u},\vec{s}) = k \text{ as well.}$$

Note that $r_i(\vec{s})$ depends only on the strategies $(s_{i-2}, s_{i-1}, s_i, s_{i+1}, s_{i+2})$ adopted by five consecutive players from $i-2$ to $i+2$ and are, in fact, independent of the time t and the label of player i. For brevity, we define

$$r(s_{i-2}, s_{i-1}, s_i, s_{i+1}, s_{i+2}) \overset{\text{def}}{=} r_i(\vec{s}).$$

By (3.3), $d > b > c$ and $d > a > c$. That is $c = \min\{a, b, c, d\}$. By Figure B,

$$r(*, A, B, A, *) = A, \tag{C.1}$$

Here and after, $*$ means a wild card and can be A or B independently. It shows the strength of A against B. Thus, it is expected that $\{\vec{A}\} \subseteq S_*$ or even $\{\vec{A}\} = S_*$. As will be shown below, $\{\vec{A}\} = S_*$ indeed when $(3a+b)/2 \geq c+d$. Otherwise, it can happen that $\vec{B} \in S_*$, but $\vec{A} \notin S_*$ as there are three other figures in order to determine the transitions under Q_0 or Q_ϵ. Using $a+b > c+d > 2c$, Figure AA implies that $r(*, A, A, B, A) = A$ and $r(B, A, A, B, B) = A$. As to $r(A, A, A, B, B)$, it depends on the relative size of $(3a+b)/2$ and $(c+d)$. And the configuration rules in Figure A depend on whether $2b \geq (c+3d)/2$ and $2b \geq (c+d)$. Similarly, the relative size of $(a+b)$ and $(c+3d)/2$ determines the configuration rules in Figure BB. That is how we get those classifications in Theorem 3.1.

Case (a) $\frac{3a+b}{2} \geq c+d$ and $a+b > \frac{c+3d}{2}$. By (3.3), we have from Figure AA that

$$r(*, B, A, A, *) = A. \tag{C.2}$$

Of course, $r(*, A, A, B, *) = A$ holds by symmetry. In order to determine the configuration rules from the remaining Figures A and BB, the following subcases are considered.

Subcase (a1) $2b > \frac{c+3d}{2}$. Since $c+3d > 2(c+d) > 3c+d$, Figure BB implies that

$$r(*, A, B, B, *) = A, \tag{C.3}$$

while we have from Figure A that

$$r(*, B, A, B, *) = A. \tag{C.4}$$

By (C.1)-(C.4), any A-string with length ≥ 1 in a state $\vec{s}$ will expand, at zero cost, by absorbing its two neighboring Bs every period under Q_0 until $\vec{A}$ is reached. Thus, $M \stackrel{\text{def}}{=} S_0 \setminus \{\vec{A}, \vec{B}\} = \emptyset$, and $S_0 = \{\vec{A}, \vec{B}\}$. Accordingly, the most economic way under Q_ϵ from $\vec{A}$ to $\vec{B}$ and vice versa are shown as follows :

$$\vec{B} \xrightarrow{1} \cdots BB \overset{\bullet}{A} BB \cdots \xrightarrow{0} \cdots BA \overset{\bullet}{A} AB \xrightarrow{0} \cdots \xrightarrow{0} \vec{A} \text{ and } \vec{A} \xrightarrow{n} \vec{B}.$$

In particular, it saves for $\vec{A}$ to jump to $\vec{B}$ directly. Thus $v(\{\vec{A}\}) = 1$ and $v(\{\vec{B}\}) = n$. By (B.5) and (B.8), $v_1 = 1$, $v_2 = v(\{\vec{A}, \vec{B}\}) = 0$ and $\delta = v_1 - v_2 = 1$. It follows from Theorems B1 and B2 that $S_* = \{\vec{A}\}$, $\mu_\epsilon(\vec{B}) \approx \epsilon^{n-1}$ and $E(T_\epsilon) \approx \epsilon^{-1}$.

Subcase (a2) $c + d < 2b \leq \frac{c+3d}{2}$. While (C.4) still holds, Figure BB implies that

$$r(B, A, B, B, B) = B \quad \text{ and } r(A, A, B, B, B) = r(*, A, B, B, A) = A. \tag{C.5}$$

We first claim that the set M of mixed stationary states is $M_{1,\geq 3}$. By (C.1), (C.2), (C.4) and (C.5), this follows easily from the following observations:

(S1) Any A-string in a state can hold under Q_0 in the next period.

(S2) Any A-string with length ≥ 2 in a state will expand by absorbing its two neighboring Bs every period under Q_0 until $\vec{A}$ is reached.

Hence, all $a_i = 1$ for any possible $\vec{s} \in M$ with the representation in (3.2). Then

(S3) any B-string with length ≤ 2 in a state will be eliminated in the next period under Q_0 and form an A-string with length ≥ 2 which can hold by (**S1**).

Hence, all $b_i \geq 3$ for any possible $\vec{s} \in M$. This shows $M \subseteq M_{1,\geq 3}$.

(S4) $M_{1,\geq 3} \subseteq M$ by the first equation in (C.5). Thus $M = M_{1,\geq 3}$ as claimed.

As explained at the beginning of this section, we need first to find $v(\{\vec{s}\})$ for all $\vec{s} \in S_0$ in order to get S_* and $E(T_\epsilon)$ via Theorems B1 and B2. Decompose $M = M_{1,\geq 3}$ as $M = \cup_{k\geq 1} M_k$, where k is the number of A-strings in (3.2) for $\vec{s} \in M$.

Step 1. For convenience, define $M_0 = \{\vec{B}\}$. The following diagram shows any $\vec{s} \in M_k$ with $k \geq 1$ can reach some state in M_{k-1} at cost 1 and vice versa :

$$\cdots \underbrace{A}_{1} \underbrace{B\cdots B}_{b_i \geq 3} \underbrace{\dot{A}}_{1} \underbrace{B\cdots B}_{b_{i+1}\geq 3} \underbrace{A}_{1} \cdots \overset{1}{\leftrightarrow} \cdots \underbrace{A}_{1} \underbrace{B\cdots B\,\dot{B}\,B\cdots B}_{b_i+1+b_{i+1}} \underbrace{A}_{1} \cdots. \tag{C.6}$$

Since $|M_0| = 1$, the diagram above implies, by varying k successively from one on, that all states in $M \cup \{\vec{B}\}$ can reach any fixed state in it at total cost $|M|$.

Step 2. By Step 1 and (**S2**) above, all states in $M \cup \{\vec{B}\}$ can reach $\vec{A}$ by first merging to a certain state $\vec{s} \in M_1$ and then moving to $\vec{A}$ at cost 1 as shown below.

$$\cdots BB \underbrace{A}_{1} \dot{B}\,BB \cdots \xrightarrow{1} \cdots BB \underbrace{A\,\dot{A}}_{2} BB \cdots \xrightarrow{0} \cdots B \underbrace{AA\,\dot{A}\,A}_{4} B \cdots \xrightarrow{0} \cdots \cdot \xrightarrow{0} \vec{A}.$$

Step 3. By (**S2**) and (**S3**), the most economic path for $\vec{A}$ to reach $M \cup \{\vec{B}\}$ is

$$\vec{A} \xrightarrow{n-\lfloor \frac{n}{4} \rfloor} \underbrace{ABBB}_{\text{repeat } \lfloor \frac{n}{4} \rfloor \text{ times}} \cdots \underbrace{ABBB}\underbrace{B\cdots B}_{r} \in M_{\lfloor \frac{n}{4} \rfloor}, \text{ where } r = n - 4\left\lfloor \frac{n}{4} \right\rfloor. \tag{C.7}$$

For any $\vec{s} \in M \cup \{\vec{B}\}$, we have $v(\{\vec{s}\}) = |M| + n - \left\lfloor \frac{n}{4} \right\rfloor$ by Step 1 and $v(\{\vec{A}\}) = |M| + 1$ by Step 2. Because $n - \left\lfloor \frac{n}{4} \right\rfloor > 1$ for $n \geq 5$, it follows from Theorem B1 that

$$v_1 = |M| + 1, S_* = \{\vec{A}\} \text{ and } \mu_\epsilon(\vec{B}) \approx \epsilon^{n-\lfloor \frac{n}{4} \rfloor - 1}. \tag{C.8}$$

By (B.7), $v_2 = |M|$ and is attained at $\{\vec{A}, \vec{s}\}$ for any $\vec{s} \in M \cup \{\vec{B}\}$. It follows from Theorem B2 that

$$\delta = v_1 - v_2 = 1 \text{ and } E(T_\epsilon) \approx \epsilon^{-1}. \tag{C.9}$$

Subcase (a3) $2b = c + d$. While (C.4) still holds, Figure BB implies that

$$r(B, A, B, B, *) = B \text{ and } r(A, A, B, B, *) = A. \tag{C.10}$$

In comparison with (C.5), it means that when surrounded by singleton A-strings, any B-string with length ≥ 2 in a state can hold in the next period under Q_0, while the length of such a B-string has to be ≥ 3 in Subcase (a2) as indicated by the first equation of (C.5). This is the only difference between subcases (a2) and (a3) on the configuration rules under Q_0. Thus, $M = M_{1,\geq 2}$ can be verified similarly. Likewise, (C.8) and (C.9) hold still except $\mu_\epsilon(\vec{B}) \approx \epsilon^{n-\lfloor \frac{n}{3} \rfloor -1}$ as the diagram (C.7) is replaced by

$$\vec{A} \xrightarrow{n-\lfloor \frac{n}{3} \rfloor} \underbrace{ABB}_{\text{repeat } \lfloor \frac{n}{3} \rfloor \text{ times}} \cdots \underbrace{ABB}\underbrace{B\cdots B}_{r} \in M_{\lfloor \frac{n}{3} \rfloor}, \text{ where } r = n - 3\left\lfloor \frac{n}{3} \right\rfloor. \qquad (C.11)$$

Subcase (a4) $\frac{3c+d}{2} \leq 2b < c+d$. Instead of (C.4), we now have from Figure A that

$$r(B,B,A,B,B) = B \text{ and } r(A,B,A,B,*) = r(*,B,A,B,A) = A, \qquad (C.12)$$

while (C.10) remains valid from Figure BB.

We first claim that $M = \emptyset$. Since (**S2**) above still holds, we have for any possible $\vec{s} \in M$ that all $a_i = 1$ and then $b_i \geq 2$ by (C.1) and the second equation in (C.12). Yet, the first equations in (C.12) and (C.10) imply that such $\vec{s}$ will go to $\vec{B}$ in the next period under Q_0 and then stays there as $\vec{B}$ is an absorbing state. Hence, $M = \emptyset$.

It remains to find the minimum cost path from $\vec{A}$ to $\vec{B}$ and vice versa. By (C.12) and (**S2**) above, the diagram (C.11) remains optimal for $\vec{A}$ to reach $\vec{B}$ except now

$$\vec{A} \xrightarrow{n-\lfloor \frac{n}{3} \rfloor} \underbrace{ABB}_{\text{repeat } \lfloor \frac{n}{3} \rfloor \text{ times}} \cdots \underbrace{ABB}\underbrace{B\cdots B}_{r} \xrightarrow{0} \vec{B}, \text{ where } r = n - 3\left\lfloor \frac{n}{3} \right\rfloor. \qquad (C.13)$$

Due to the first equation in (C.12), the following path is optimal for $\vec{B}$ to reach $\vec{A}$:

$$\vec{B} \xrightarrow{2} \cdots BB\underbrace{\dot{A}\dot{A}}_{2}BB\cdots \xrightarrow{0} \cdots B\underbrace{A\,\dot{A}\dot{A}\,A}_{4}B\cdots \xrightarrow{0} \cdots \underbrace{AA\,\dot{A}\dot{A}\,AA}_{6}\cdots \xrightarrow{0} \cdots\cdots \xrightarrow{0} \vec{A}. \qquad (C.14)$$

Hence, $v(\{\vec{B}\}) = n - \left\lfloor \frac{n}{3} \right\rfloor > v(\{\vec{A}\}) = 2$ as $n \geq 5$ by assumption (3.3). Because $|S_0| = 2$, we have $v_1 = v(\{\vec{A}\}) = 2, v_2 = v(\{\vec{A}, \vec{B}\}) = 0$ and $\delta = v_1 - v_2 = 2 - 0 = 2$. Finally, $S_* = \{\vec{A}\}$, $\mu_\epsilon(\vec{B}) \approx \epsilon^{n-\lfloor \frac{n}{3} \rfloor -2}$ and $E(T_\epsilon) \approx \epsilon^{-2}$ by Theorems B1 and B2.

Subcase (a5) $2b < \frac{3c+d}{2}$. While (C.10) still holds, strategy A becomes even weaker as, instead of (C.12), we now have from Figure A that

$$r(A,B,A,B,A) = A \text{ and } r(B,B,A,B,*) = r(*,B,A,B,B) = B. \qquad (C.15)$$

We first show $M = \emptyset$. Since (**S2**) above still holds, all $a_i = 1$ for any possible $\vec{s} \in M$. We claim that $b_i \geq 2$ for such $\vec{s} \in M$. This is verified by observing that any string $\cdots * BABAB * \cdots$ in $\vec{s}$ will either produce an A-string of length ≥ 3 in the next period under Q_0 if any $*$ above is A, or become an all B-string at the second period if both $*$ are B. In either case it will be unable to go back to itself under Q_0, which violates the ergodic property in (2.6). But then (C.10) and (C.15) imply that such $\vec{s}$ will go to $\vec{B}$ in the next period under Q_0, which is an absorbing state under Q_0. Hence, $M = \emptyset$.

It remains to find the minimum cost path from $\vec{A}$ to $\vec{B}$ and vice versa. By (C.10) and (C.15), the diagram (C.14) remains optimal for $\vec{B}$ to reach $\vec{A}$. By (C.1), (C.15) and (**S2**) above, any AA or $ABABA$ string has to be avoided in order to reach $\vec{B}$ from $\vec{A}$. This leads to the following modification to diagram (C.11):

$$\vec{A} \xrightarrow{n-2\lfloor\frac{n}{5}\rfloor-\lfloor\frac{r}{3}\rfloor} \underbrace{ABBAB}_{\text{repeat }\lfloor\frac{n}{5}\rfloor\text{ times}} \cdots \underbrace{ABBAB} \underbrace{A}_{\lfloor\frac{r}{3}\rfloor} \underbrace{B\cdot\cdot B}_{r-\lfloor\frac{r}{3}\rfloor} \xrightarrow{0} \vec{t} \xrightarrow{0} \vec{B},$$

where $r = n - 5\left\lfloor\frac{n}{5}\right\rfloor$, and $\vec{t}$ is some intermediate state. As did in Subcase (a4), $v_1 = v(\{\vec{A}\}) = 2$, $\delta = v_1 - v_2 = 2 - 0 = 2$. Then $S_* = \{\vec{A}\}$, $\mu_\epsilon(\vec{B}) \approx \epsilon^{n-2\lfloor\frac{n}{5}\rfloor-\lfloor\frac{r}{3}\rfloor-2}$ and $E(T_\epsilon) \approx \epsilon^{-2}$ by Theorems B1 and B2.

Case (b) $\frac{3a+b}{2} \geq c+d$ and $a+b \leq \frac{c+3d}{2}$. Since $c = \min\{a,b,c,d\}$, (C.1) and (C.2) still hold as in Case (a) above. We follow the classification in Case (a) to determine the configuration rules from the remaining Figures A and BB. As a result, the configuration rules from Figure A for each subcase below will remain the same as in the corresponding one in Case (a). But the configuration rules from Figure BB will be different due to $a+b \leq \frac{c+3d}{2}$ now.

Subcase (b1) $2b > \frac{c+3d}{2}$. Using $c = \min\{a,b,c,d\}$, we still have (C.4) from Figure

A, while Figure BB implies that

$$r(A,A,B,B,B) = B \text{ and } r(B,A,B,B,B) = r(*,A,B,B,A) = A. \qquad (C.16)$$

This, together with (C.1), means that a B-string in any state can hold under Q_0 only if its length is ≥ 3 and is surrounded by A-strings with length ≥ 2. By (C.2), any A-string with length ≥ 2 can hold under Q_0. Hence, $M_{\geq 2,\geq 3} \subseteq M$. In fact, it can be shown that $M = M_{\geq 2,\geq 3}$. Since all the configuration rules, and thus the results, are the same as in Theorem 3.2 (iii) of Chen and Chow (2007) except C, D there need to be renamed as B, A respectively. The proof is omitted. In particular, $S_* = \{\vec{A}\}, \mu_\epsilon(\vec{B}) \approx \epsilon^3$ and $E(T_\epsilon) \approx \epsilon^{-1}$.

Subcase (b2) $c + d < 2b \leq \frac{c+3d}{2}$. While we still have (C.4) from Figure A, strategy B becomes stronger as Figure BB now shows

$$r(*,A,B,B,B) = B \text{ and } r(*,A,B,B,A) = A. \qquad (C.17)$$

We claim that $M = M_{\geq 1,\geq 3}$. This is easily verified as (**S1**) holds by (C.2) and (C.4), (**S3**) holds by (C.1) and (C.17), and the first equation in (C.17) shows

(T1) any B-string with length ≥ 3 in a state can hold under Q_0.

Next, we will examine in the following four steps the minimum cost to any state in S_0 from all the other states in S_0. Write $M = M_{\geq 1,\geq 3} = \cup_{k\geq 1} M_k$, where k is the number of A-strings in the representation (3.2) for $\vec{s} \in M$.

Step 1. For convenience, define $M_0 = \{\vec{B}\}$. The following diagram shows that any $\vec{s} \in M_k$ can move within M_k, with the minimum cost 1 for each move along the path, to reach some state in $M_k^* = \{\vec{s} \in M_k :$ all $a_i = 1$ in (3.2) $\}$ and vice versa :

$$\cdots \underbrace{B\cdots B}_{b_{i-1}\geq 3}\underbrace{\overset{\circ}{A}\cdots A\,\overset{\bullet}{A}}_{a_i}\underbrace{B\cdots B}_{b_i\geq 3}\cdots \overset{1}{\leftrightarrow} \cdots \underbrace{B\cdots B}_{b_{i-1}}\underbrace{\overset{\circ}{A}\cdots A}_{a_i-1}\underbrace{\overset{\bullet}{B}\,B\cdots B}_{b_i+1}\cdots \overset{1}{\leftrightarrow}\cdots\overset{1}{\leftrightarrow}\cdots\underbrace{B\cdots B}_{b_{i-1}}\underbrace{\overset{\circ}{A}}_{1}\underbrace{B\cdots B}_{b_i+a_i-1}\cdots$$

Step 2. For $k \geq 1$, (C.6) shows any state in M_k^* can reach some state in M_{k-1}^* at cost

1 and vice versa. Since $|M_0| = 1$, this and Step 1 imply that all states in $M \cup \{\vec{B}\}$ can reach any fixed state in $M \cup \{\vec{B}\}$ at cost 1 each. That is at total cost $|M|$.

Step 3. By Step 2 and (**S3**), all states in $M \cup \{\vec{B}\}$ can reach $\vec{A}$ by first attaining the following state in M_1 and then going to $\vec{A}$ at cost 1 as follows.

$$BB\ \dot{B}\ \underbrace{A \cdots A}_{n-3} \xrightarrow{1} BB\ \underbrace{\dot{A}\ A \cdots A}_{n-2} \xrightarrow{0} \vec{A}. \tag{C.18}$$

Step 4. By (**S3**) and (**T1**), the most economic path for $\vec{A}$ to reach $M \cup \{\vec{B}\}$ is

$$\vec{A} \xrightarrow{3} \dot{B}\dot{B}\dot{B}\ \underbrace{A \cdots A}_{n-3} \in M_1. \tag{C.19}$$

By Step 3, $v(\{\vec{A}\}) = |M| + 1$. By Steps 2 and 4, $v(\{\vec{s}\}) = |M| + 3$ for any $\vec{s} \in M \cup \{\vec{B}\}$. Then $v_1 = |M| + 1$ and $v_2 = |M| = v(\{\vec{A}, \vec{s}\})$. Hence, $\delta = v_1 - v_2 = 1$ by (B.8). It follows from Theorems B1 and B2 that

$$S_* = \{\vec{A}\}, \quad \mu_\epsilon(\vec{B}) \approx \epsilon^2 \text{ and } E(T_\epsilon) \approx \epsilon^{-1}. \tag{C.20}$$

Subcase (b3) $2b = c + d$. While (C.1), (C.2) and (C.4) remain valid as in Subcase (b2), strategy B becomes even stronger as Figure BB now shows

$$r(A, A, B, B, A) = A \text{ and } r(*, A, B, B, B) = r(B, A, B, B, *) = B. \tag{C.21}$$

We first examine the set M of all mixed stationary states. Note that (**S1**) and (**T1**) hold as in Subcase (b2). Because of (**S1**), we have from (C.1) that

(T2) no singleton B-string can hold under Q_0. Hence, all $b_j \geq 2$ for any $\vec{s} \in M$.

In case some $b_i = 2$ for $\vec{s} \in M$, (C.21) implies that $a_i = a_{i+1} = 1$. Otherwise, this B-string will soon be eliminated under Q_0. In summary, $M = \bar{M}_{\geq 1, \geq 2}$ as desired.

The conclusion is the same as (C.20) because all the arguments in Steps 1-4 of Subcase (b2) can be repeated except the optimality of the path in (C.19) is based on (C.1) and the first equation in (C.21).

Subcase (b4) $\frac{3c+d}{2} \leq 2b < c + d$. While (C.21) from Figure BB remains valid as above, strategy A becomes weaker as we now have (C.12) from Figure A as in Subcase (a4). As a result, (**T1**) still holds and (**S1**) is weakened to

(T3) any A-string with length ≥ 2 in a state can hold under Q_0.

Thus, $M \supseteq M_{\geq 2, \geq 3}$ by (**T1**) and (**T3**). In fact, we can verify $M = M_{\geq 2,\geq 3}$ as follows. Because of (C.1), (C.12) and (**T3**), we still have (**T2**). In turn, (C.12), (C.21) and (**T1**) imply that all $a_i \geq 2$ for any $\vec{s} \in M$ as a singleton A-string cannot hold under Q_0 at the present situation. Finally, all $b_j \geq 3$ by the first equation in (C.21).

Next, we will mimic the steps in Subcase (b2) to find the minimum cost to any state in S_0 from all the other states in S_0. Write $M = M_{\geq 2,\geq 3} = \cup_{k\geq 1} M_k$, where k is the number of A-strings in the representation (3.2) for $\vec{s} \in M$.

Step 1. For convenience, define $M_0 = \{\vec{B}\}$. For $k \geq 1$, any $\vec{s} \in M_k$ can move within M_k, with the minimum cost 1 for each move along the path, to reach some state in $M_k^\dagger = \{\vec{s} \in M_k :\ \text{all } a_i = 2 \text{ in } (3.2)\ \}$ and vice versa.

Step 2. For $k \geq 1$, any state in $M_k^\dagger$ can reach some state in $M_{k-1}^\dagger$ at cost 1 as follows.

$$\cdots \underbrace{B\cdots B}_{b_i \geq 3} \underbrace{A\,\dot{A}}_{2} \underbrace{B\cdots B}_{b_{i+1}\geq 3} \cdots \xrightarrow{1} \cdots \underbrace{B\cdots B}_{b_i} \underbrace{A}_{1} \underbrace{\dot{B}\,B\cdots B}_{b_{i+1}+1} \cdots \xrightarrow{0} \cdots \underbrace{AA}_{2} \underbrace{B\cdots B}_{b_i+2+b_{i+1}} \underbrace{A}_{2} \cdots .$$

Note that the reverse can be done at a minimum cost 2. However, it is irrelevant to the desired conclusion. Since $|M_0| = 1$, this and Step 1 imply that all states in M can reach $\{\vec{B}\}$ at cost 1 each by first merging to some state in $M_1^\dagger$ as shown below :

$$\underbrace{A\,\dot{A}}_{2} \underbrace{B\cdots B}_{n-2} \xrightarrow{1} \underbrace{A}_{1} \underbrace{\dot{B}\,B\cdots B}_{n-1} \xrightarrow{0} \vec{B} \text{ and } \vec{B} \xrightarrow{2} \underbrace{\dot{A}\dot{A}}_{2} \underbrace{B\cdots B}_{n-2}.$$

By the first equation in (C.12), the latter path above is optimal for $\vec{B}$ to reach out.

Since the diagrams (C.18) and (C.19) remain optimal, we have for any $\vec{s} \in M_{\geq 2,\geq 3}$,

$$v(\{\vec{A}\}) = |M| + 2, v(\{\vec{B}\}) = |M| + 3 \text{ and } v(\{\vec{s}\}) \geq |M| - 1 + 2 + 3 = |M| + 4.$$

Thus, $v_1 = |M|+2$, $v_2 = |M| = v(\{\vec{A}, \vec{B}\})$ and $\delta = v_1 - v_2 = 2$ by (B.8). In conclusion,

$$S_* = \{\vec{A}\},\ \mu_\epsilon(\vec{B}) \approx \epsilon^1 \text{ and } E(T_\epsilon) \approx \epsilon^{-2}. \tag{C.22}$$

Subcase (b5) $2b < \frac{3c+d}{2}$. While (C.21) from Figure BB remains valid as in Subcase (b4), strategy A is further weakened as we now have (C.15) from Figure A. So we still have (**T1**) and (**T3**). In fact, all the arguments in Subcase (b4) can be repeated exactly to get $M = M_{\geq 2,\geq 3}$ and the conclusion (C.22), except some minor modification is needed to verify (**T2**). If there is a singleton B-string in some $\vec{s} \in M$, then $\vec{s}$ must be $\cdots BBA\ \dot{B}\ ABB\cdots$. That is because any A-string with length ≥ 2 can hold forever under Q_0 by (**T3**). Using (**T1**), the following diagram shows a contradiction to the fact that any $\vec{s} \in M$ should be an ergodic state under Q_0 :

$$\cdots BB\underbrace{A\ \dot{B}\ A}_{3}BB\cdots \xrightarrow{0} \cdots * B\underbrace{B\ \dot{A}\ B}_{3}B*\cdots \xrightarrow{0} \cdots **\underbrace{B\ \dot{B}\ B}_{3}**\cdots.$$

Now it remains to consider the case that $(3a+b)/2 < c+d$, which implies $2a+c+d < 3a+b < 2(c+d) < 2(a+b)$ by (3.3). It follows easily that $2a < c+d$, $a < b$, thus $c < a < b < d$. Moreover,

$$3c+d < 2(c+d) < 2(a+b) < 4b \text{ and } 2(a+b) < c+d+2b < c+3d. \tag{C.23}$$

By (3.3) and (C.23), we still have (C.1) and (C.4) from Figures B and A, respectively. Because $(3a+b)/2 < c+d$, we have, instead of (C.2), the following from Figure AA

$$r(B,B,A,A,A) = B \text{ and } r(A,B,A,A,*) = r(*,B,A,A,B) = A, \tag{C.24}$$

which reflects the fact that strategy A is weaker than before. In order to determine the configuration rules from Figure BB, we have to compare $2b$ with $(c+3d)/2$. This leads to the remaining two cases of the theorem.

Case (c) $\frac{3a+b}{2} < c+d$ and $2b \leq \frac{c+3d}{2}$. By (C.23), the configuration rules from Figure BB is the same as shown in (C.17).

As before, we need to characterize M first. By (C.4), (C.24) and (C.17),

(U1) any A-string with length ≤ 2 and B-string with length ≥ 3 can hold under Q_0.

In particular, $M_{\leq 2,\geq 3} \subseteq M$. By (C.1) and (C.24), we have the following

$$\cdots A\ \dot{B}\ A \cdots \xrightarrow{0} \cdots A\ \dot{A}\ A \cdots.$$

The newly formed A-string above could shrink under Q_0, but there is no way to grow any B in its interior later on. Hence, the length of any B-string in a state $\vec{s} \in M$ must be ≥ 2. In fact, we can show that

(U2) all $b_i \geq 3$ for any state $\vec{s} \in M$.

By the second equation in (C.17), any BB string in a state will be eliminated in the next period under Q_0. Because of (**U1**), the only possible case for it to be recovered is to be surrounded by two A-strings each with length ≥ 3. Yet, the following diagram

$$\cdots AAA\ \dot{B}\dot{B}\ AAA \cdots \xrightarrow{0} \cdots * AB\ \dot{A}\dot{A}\ BA * \cdots \xrightarrow{0} \cdots * AA\ \dot{A}\dot{A}\ AA * \cdots$$

indicates that the original BB string cannot be recovered under Q_0, thus such a state cannot be in M. This verifies (**U2**).

Using (**U2**) and the first equation in (C.24), any A-string with length ≥ 3 will shrink successively under Q_0 and then becomes invariant by (**U1**) when its length drops to 1 or 2. Hence, all $a_i \leq 2$ for any state $\vec{s} \in M$. Combining together with (**U1**) and (**U2**), we have $M = M_{\leq 2,\geq 3}$.

Next, we mimic the steps in Subcase (b2) to find the minimum cost to any state in S_0 from all the other states in S_0. Define $M_0 = \{\vec{B}\}$. Steps 1 and 2 there can be repeated exactly. Hence, all states in $M \cup \{\vec{B}\}$ can reach any fixed state in it at total cost $|M|$. By using (C.1), (C.24) and (C.17), the minimum cost for $\vec{A}$ to reach $M \cup \{\vec{B}\}$ is at least 3 and is so achieved as in Step 4 there except the diagram (C.19) is replaced by

$$\vec{A} \xrightarrow{3} \dot{B}\dot{B}\dot{B}\ \underbrace{A \cdots A}_{n-3} \xrightarrow{0} B\ \dot{B}\dot{B}\dot{B}\ B \underbrace{A \cdots A}_{n-5} \xrightarrow{0} \cdots \xrightarrow{0} \underbrace{B \cdots B}_{n-r} \underbrace{A \cdot A}_{r} \in M_1,$$

where $r = 2$ or 1 depending on whether n is odd or not.

It remains to find an optimal path from $M \cup \{\vec{B}\}$ to $\vec{A}$. Because of (**U2**), Step 3 in Subcase (b2) no longer works. In order to avoid any B-string with length ≥ 3, it saves to start from some state $\vec{s} \in M_{2,\geq 3} \subseteq M$ which has as many As as possible. Moreover, it takes at least ℓ mutations under Q_ϵ to eliminate the ith B-string of $\vec{s}$ if $b_i \geq 3\ell$. Since any B-string in $\vec{s} \in M$ needs at least one mutation to be eliminated under Q_ϵ, it is most economic, in order to reach $\vec{A}$, to have the block $AABBBBB$ duplicated in $\vec{s}$ up to the maximum allowed $\left\lfloor \frac{n}{7} \right\rfloor$ times and one mutation is enough to eliminate those five Bs in any such block. As to the remained block of length $r = n - 7\left\lfloor \frac{n}{7} \right\rfloor$, an optimal choice for being both in $M = M_{\leq 2,\geq 3}$ and economic is $\emptyset$, B, BB, BBB, $ABBB$, $AABBB$, and $AABBBB$ for $r = 0,\ 1,\ 2,\ 3,\ 4,\ 5$ and 6, respectively. It takes 1 mutation to eliminate the extra block if $r \geq 1$. In summary, an optimal path from $M \cup \{\vec{B}\}$ to $\vec{A}$ is as follows :

$$\underbrace{AABB\ \overset{\bullet}{B}\ BB}_{\text{repeat} \left\lfloor \frac{n}{7} \right\rfloor \text{times}} \cdots (\emptyset, \overset{\bullet}{B}, \overset{\bullet}{B}\ B, \overset{\bullet}{B}\ BB, ABB\ \overset{\bullet}{B}, AABB\ \overset{\bullet}{B}, AABB\ \overset{\bullet}{B}\ B) \xrightarrow{\left\lceil \frac{n}{7} \right\rceil} \vec{t} \xrightarrow{0} \vec{A},$$

where $\vec{t}$ is some transient state. Putting everything together, we have that for any different $\vec{s}$ and $\vec{t}$ in $M \cup \{\vec{B}\}$,

$$v(\{\vec{s}\}) = |M| + 3, v(\{\vec{A}\}) = |M| + \left\lceil \frac{n}{7} \right\rceil, v(\{\vec{A}, \vec{s}\}) = |M| \text{ and } v(\{\vec{s}, \vec{t}\}) = |M| + 2.$$

The conclusion follows by comparing $\left\lceil \frac{n}{7} \right\rceil$ with 3. For example, $\left\lceil \frac{n}{7} \right\rceil = 3$ iff $15 \leq n \leq 21$. In that case, $v(\{\vec{s}\}) = |M| + 3$ for any $\vec{s} \in S_0$. Hence, $S_* = S_0$ and then $E(T_\epsilon) \approx \epsilon^0$ as $k_0 = |S_0| + 1$ and $\delta = 0$ in (B.8). Moreover, $\left\lceil \frac{n}{7} \right\rceil > 3$ iff $n \geq 22$. In that case, $v_1 = |M| + 3$ and $v_2 = |M|$. Hence, $S_* = S_0 \setminus \{\vec{A}\}$ and $E(T_\epsilon) \approx \epsilon^{-3}$.

Case (d) $\frac{3a+b}{2} < c + d$ and $2b > \frac{c+3d}{2}$. Using (C.23), we have (C.16) from Figure BB. This is the most complicated case in this paper as $M = \tilde{M}$ which was defined before the statement of the theorem. While each state in M is an absorbing state under Q_0 in all the previous cases, it is not valid here. Since all the configuration rules from those four figures are the same as Theorem 3.2 (ii) in Chen and Chow (2009) except strategies B, A were named there as C, D respectively, the conclusions remain the same. The proof is omitted here.

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