CHARACTERIZATION OF MULTIDIMENSIONAL SPATIAL MODELS OF ELECTIONS WITH A VALENCE DIMENSION

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Abstract. Spatial models of political competition are typically based on two assumptions. One is that all the voters identically perceive the platforms of the candidates and agree about their score on a "valence" dimension. The second is that each voter's preferences over policies are decreasing in the distance from that voter's ideal point, and that valence scores enter the utility function in an additively separable way.

The goal of this paper is to examine the restrictions that these two assumptions impose, starting from a more primitive (and observable) data. Specifically, we consider the case where only the ideal point in the policy space and the ranking over candidates are known for each voter. We provide necessary and sufficient conditions for this collection of preference relations to be consistent with utility maximization as in the standard models described above. That is, we characterize the case where there are policies $x_1, \ldots, x_m$ for the m candidates and numbers $v_1, \ldots, v_m$ representing valence scores, such that a voter with an ideal policy y ranks the candidates according to $v_i - ||x_i - y||^2$.

Keywords: Elections, Spatial model, Valence, Euclidean preferences.

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1. Introduction

Since the seminal works of Hotelling [15] and Downs [10], spatial models of elections are widely used in the political economy literature. Typically, these models identify the policy space with a finite dimensional Euclidean space. Each voter in the electorate is assumed to have an ideal point in the policy space. Candidates then choose their platforms and each voter votes for the candidate with the closest platform to her ideal policy. Usually, the emphasis is on equilibrium analysis of the resulting game between candidates. This literature is too enormous to mention specific works.

More recently, authors incorporated a "valence" dimension to the standard model. This additional dimension influence voters' preferences and was shown to have a dramatic effect on the outcome of the political game, both in theory and in empirical studies. This additional dimension may represent any non–policy issue on which candidates differ in the "score" they get from the electorate. Examples include charisma, experience, past success, communication skills, etc. The reader can find references to many works that discuss this point in the related literature section below.

When valence issues are present, the utility of a voter with an ideal point (in the policy space) y if candidate i wins the elections is usually taken to be of the form $v_i - ||x_i - y||^2$, where v_i and x_i are the valence score and platform of candidate i respectively. This particular form of utility function implies that each voter's utility is decreasing in the Euclidean distance between his own ideal point and the winning candidate platform, and that valence scores enter the utility function in an additively separable way. Although very natural, it is not clear why this particular functional form is more appropriate than others to represent preferences. Possibly, using a different type of utility function can make a difference for the predictions of a theoretical model and for the statistical significance of an empirical model. Furthermore, an implicit assumption made here is that all voters perceive in the same way each of the candidates' platforms, and give the same valence score to every candidate. That is, x_i and v_i are common to all the voters for each i. This is certainly a crucial assumption since allowing voters to disagree on candidates platforms and valence scores can lead to an intractable model.

Obviously, it is very hard (not to say impossible) to extract the entire preferences of each voter over the policy space and the valence score he gives to each candidate. Therefore, it is not easy to check whether the aforementioned assumptions make sense in any particular political campaign. Thus, an important matter is to identify conditions on more easily observable data that guarantee consistency with the spatial model assumptions. Introducing such necessary and sufficient conditions is the main result of this paper.

Specifically, we assume that, for each voter in the electorate, only the ideal policy and the ranking of candidates can be observed. While this may also seem quite demanding, it is much more reasonable than observing the entire utility function of the voter. We characterize the case where this data is consistent with voters having utility functions as above. That is, we characterize the case where there are policies $x_1, \ldots, x_m$ for the m candidates and numbers $v_1, \ldots, v_m$ representing valence scores, such that a voter with an ideal policy y ranks the candidates according to $v_i - ||x_i - y||^2$.

We use four conditions for the characterization. The first three are quite standard, while the last one is more technical and is required for the proof. We think of our result as "good news" since it shows that if the data satisfies rather weak assumptions then it is consistent with the standard spatial model with a valence dimension[1]. However, our model suffers from several weaknesses which make its applicability limited[2]. We therefore think that the main contribution of this paper is to introduce the theoretical "revealed preferences" approach to the literature on spatial models of elections. To the best of our knowledge, this is the first attempt in this direction, and we hope it will lead to further investigation.

1.1. **Related literature.** Papers using spatial models of elections with valence issues similar to the one studied here are numerous in recent years. Examples include Ansolabehere and Snyder [1], Aragones and Palfrey [2], Degan [8], Dix and Santore [9], Enelow and Hinich [11], Gersbach [12], Groseclose [14], Kim [16] and Schofield [19] among others. These papers study different aspects of the political game and provide various interpretations for the additive constant in the utility functions of the voters.

From a mathematical perspective, our main result is closely related to the result in Azrieli and Lehrer [6], who characterize categorization systems that are generated by proximity to a set of prototypical cases. Furthermore, there is a surprisingly close connection between the result of this paper and the axiomatic derivation of Gilboa and

[1] Our conditions are not sufficient if one doesn't allow for a valence dimension, and we do not know how to characterize data consistent with spatial models without this additive term. See subsection 3.4.

[2] See Section 3.

Schmeidler [13] in their Case-Based Decision Theory[3]. The literature on scoring rules (Myerson [18], Smith [20], Young [21]) is also closely related from a technical point of view.

Finally, the mathematical object we deal with here is known in the geometry literature as Voronoi diagram or Dirichlet tessellation. The most relevant papers in this literature are Ash and Bolker [3], [4] and Aurenhammer [5]. The book by Boots et al. [7] surveys applications of Voronoi diagrams in many different fields.

1.2. **Organization.** The next section contains the model and the main result of the paper. In Section 3 we discuss in more detail several issues related to our model. In particular, this section contains some additional results concerning the uniqueness of the representation, the failure of our result if the set of voters is finite, and the special cases of three candidates and one–dimensional policy space. We also point to some possible directions of related future research. All the proofs are in Section 4.

2. Model and Result

Let $C = \{1, 2, \dots, m\}$ be the set of candidates where $m \geq 2$. The policy space is taken to be $\mathbb{R}^d$ with[4] $d \geq 2$. Each potential voter is identified with her ideal point in the policy space and we assume that, for every $y \in \mathbb{R}^d$, there is a voter with y as her ideal policy. Thus, the set of voters is also $\mathbb{R}^d$. We will use the letters i, j, k, l to denote candidates (elements of C) and x, y, z to denote voters or policies (points in $\mathbb{R}^d$).

Our primitive is a collection of binary relations $\{\succeq_y\}_{y \in \mathbb{R}^d}$ over C, one for every voter $y \in \mathbb{R}^d$. The interpretation of $i \succeq_y j$ is that a voter with an ideal point y (weakly) prefers candidate i to candidate j. As usual, for any $i, j \in C$, we let $i \succ_y j$ if and only if both $i \succeq_y j$ and $j \not\succeq_y i$, and $i \sim_y j$ if and only if both $i \succeq_y j$ and $j \succeq_y i$. The following properties will be used for the characterization.

(A1) Weak order: For every $y \in \mathbb{R}^d$, $\succeq_y$ is complete and transitive.

(A2) Continuity: For every $i, j \in C$, the set $\{y \in \mathbb{R}^d \ : \ i \succ_y j\}$ is open.

(A3) Convexity: For every $i, j \in C$ and $y, z \in \mathbb{R}^d$, if $i \succeq_y j$ ($i \succ_y j$) and $i \succeq_z j$ then $i \succeq_{\alpha y + (1-\alpha) z} j$ ($i \succ_{\alpha y + (1-\alpha) z} j$) for every $\alpha \in (0, 1)$.

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[3]We thank Itzhak Gilboa for pointing out this connection.

[4]Our result does not hold in the case $d = 1$. We elaborate on this case in subsection 3.7.

(A4) Heterogeneity: For every three distinct candidates $\{i, j, k\} \subseteq C$ there is $y \in \mathbb{R}^d$ such that $i \succ_y j \succ_y k$, and for every four distinct candidates $\{i, j, k, l\} \subseteq C$ the sets $\{y \in \mathbb{R}^d \ : \ i \sim_y j \sim_y k\}$ and $\{y \in \mathbb{R}^d \ : \ i \sim_y j \sim_y l\}$ are not equal.

The first property is standard. The second implies that if a voter with ideal point y strictly prefers candidate i over j then any voter with ideal point sufficiently close to y also prefers i over j. *(A3)* states that the set of voters preferring candidate i over j is convex. Finally, *(A4)* requires the population of voters to be sufficiently diverse in its preferences. Namely, for any (strict) ranking of every three candidates there should be a voter who ranks these candidates according to that given order; And for every three candidates there should be a voter that is indifferent between them but strictly prefers some given fourth candidate over the three. We note that if $m = 2$ then *(A4)* is trivially satisfied, and if $m = 3$ then the second part of *(A4)* is trivially satisfied.

Before stating our result we need one more definition.

Definition 1. *Let* $\{x_1, x_2, \ldots, x_m\} \subseteq \mathbb{R}^d$ *and* $\{v_1, v_2, \ldots, v_m\} \subseteq \mathbb{R}$. *We say that the set* $\{(x_1, v_1), (x_2, v_2), \ldots, (x_m, v_m)\} \subseteq \mathbb{R}^{d+1}$ *is in a* general position *if the following two conditions hold:*
(i) For every distinct $1 \leq i, j, k \leq m$*, the vectors* x_i, x_j, x_k *are affinely independent in* $\mathbb{R}^d$ *(equivalently,* $x_j - x_i$ *and* $x_k - x_i$ *are linearly independent in* $\mathbb{R}^d$*).*
(ii) For every distinct $1 \leq i, j, k, l \leq m$*, the set* $\{y \in \mathbb{R}^d : v_i - ||x_i - y||^2 = v_j - ||x_j - y||^2 = v_k - ||x_k - y||^2 = v_l - ||x_l - y||^2\}$ *is of dimension* $d - 3$ *at most*[5].

Informally speaking, if a set of points is *not* in a general position then it has a 'degenerate structure'. We remark that if the points $\{(x_1, v_1), (x_2, v_2), \ldots, (x_m, v_m)\}$ are independently drawn from some continuous distribution over $\mathbb{R}^{d+1}$ then the resulting set will be in a general position with probability 1. The precise meaning of the term general position varies with the context in which it is used. The reader is referred to Matoušek (2002, pp. 3-5), where this concept is discussed in greater detail.

Theorem 1. *The following two statements are equivalent:*
(i) The collection of binary relations $\{\succeq_y\}_{y \in \mathbb{R}^d}$ *satisfies properties (A1) through (A4).*

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[5]The dimension of a set $A \subseteq \mathbb{R}^d$ is defined as the dimension of the affine hull of A. In our case, the set under consideration is the intersection of hyperplanes in $\mathbb{R}^d$ so it is an affine subspace. See Section 4 for details.

(ii) There are points $\{x_1, x_2, \ldots, x_m\} \subseteq \mathbb{R}^d$ *and numbers* $\{v_1, v_2, \ldots, v_m\} \subseteq \mathbb{R}$ *such that* $\{(x_1, v_1), (x_2, v_2), \ldots, (x_m, v_m)\}$ *is in a general position and, for every* $i, j \in C$ *and every* $y \in \mathbb{R}^d$, $i \succeq_y j$ *if and only if* $v_i - ||x_i - y||^2 \geq v_j - ||x_j - y||^2$.

The point x_i is interpreted as the platform of candidate i, and v_i is the score of i on the valence dimension ($1 \leq i \leq m$). Theorem 1 states that properties *(A1)* through *(A4)* are equivalent to the existence of $\{(x_1, v_1), (x_2, v_2), \ldots, (x_m, v_m)\}$ in a general position such that each voter y ranks the candidates according to their score $v_i - ||x_i - y||^2$. Note that all the voters agree on the location of the candidates in the policy space and on the their score in the valence dimension.

3. Discussion

3.1. **Uniqueness.** Examining the proof of Theorem 1, one can see that the platforms and valences derived from the properties *(A1)-(A4)* are not unique. However, we do have the following connection between any two representations of the voters' preferences.

Proposition 1. *Assume* $\{(x_1, v_1), (x_2, v_2), \ldots, (x_m, v_m)\} \subseteq \mathbb{R}^{d+1}$ *represent the preferences* $\{\succeq_y\}_{y \in \mathbb{R}^d}$ *as in Theorem 1. Then* $\{(x'_1, v'_1), (x'_2, v'_2), \ldots, (x'_m, v'_m)\} \subseteq \mathbb{R}^{d+1}$ *also represent* $\{\succeq_y\}_{y \in \mathbb{R}^d}$ *if and only if there is a positive number* $\alpha > 0$ *and a vector* $\beta \in \mathbb{R}^d$ *such that* $x'_i = \alpha x_i + \beta$ *for every* $1 \leq i \leq m$*, and such that the equation*[6]

$$\frac{1}{\alpha}(v'_i - v'_j) = v_i - v_j + (1 - \alpha)(||x_j||^2 - ||x_i||^2) + 2\beta \cdot (x_i - x_j) \tag{1}$$

holds for every $i, j \in C$*. In particular, if* $x_i = x'_i$ *for* $1 \leq i \leq m$ *(i.e.,* $\alpha = 1$ *and* $\beta = 0$*) then there is some* $\gamma \in \mathbb{R}$ *such that* $v'_i = v_i + \gamma$ *for* $1 \leq i \leq m$.

This result can be interpreted as follows. We may rescale and change the origin of the policy space to get different sets of platforms that induce the same preferences. But once the unit of measurement and the origin are fixed the platforms are uniquely determined by the preferences. Moreover, once platforms are fixed, the relative valences of the various candidates (the differences $v_i - v_j$) are also unique.

3.2. **Finite set of voters.** A shortcoming of our model is that we assume that the preferences of a voter with ideal point y are observable *for every* $y \in \mathbb{R}^d$. It is much more reasonable that the preferences of only a finite number of voters are observable. It is tempting to try to get a similar result to that of Theorem 1 with a finite set of voters.

[6]For two vectors $z, w \in \mathbb{R}^d$ we denote by $z \cdot w = \sum_{i=1}^d z_i w_i$ the standard inner product in $\mathbb{R}^d$.

A natural modification of property *(A3)* in this case is that, for any $i, j \in C$, the convex hull of the set of voters preferring candidate i over j and the convex hull of the set of voters preferring candidate j over i are disjoint[7].

However, we cannot get an analogue of Theorem 1 in this case. We demonstrate the problem with the following example. Let $d = 2$, $C = \{1, 2, 3\}$ and fix some $\epsilon > 0$. The set of voters, denoted Y, consists of six voters with the ideal points

$$Y = \{y_1 = (\epsilon, \epsilon),\ y_2 = (-\epsilon, -\epsilon),\ y_3 = (-\epsilon, -4),\ y_4 = (\epsilon, -4),\ y_5 = (4, \epsilon),\ y_6 = (4, -\epsilon)\}.$$

The preferences of these six voters are as follows. Voters $\{y_2, y_3\}$ prefer candidate 1 over candidate 2 (the rest of the voters prefer candidate 2 over candidate 1). Voters $\{y_1, y_2, y_3, y_5\}$ prefer candidate 1 over candidate 3, and voters $\{y_1, y_5\}$ prefer candidate 2 over candidate 3. Figure 1 illustrates the location of the voters' ideal points in the policy space and their rankings.

It is easy to check that the above condition of disjointness of the convex hulls is satisfied. However, we claim that it cannot be the case that the voters agree on the platforms and valences of the three candidates. Indeed, assume to the contrary that there are $\{(x_1, v_1), (x_2, v_2), (x_3, v_3)\}$ that represent these preferences as in Theorem 1.

The locations of the points y_1, y_2, y_3, y_4 and the preferences of these voters imply that the line $\{y \in \mathbb{R}^2 \ :\ v_1 - ||y - x_1||^2 = v_2 - ||y - x_2||^2\}$ should be close to both points $(0, 0)$ and $(0, -4)$. Similarly, the line $\{y \in \mathbb{R}^2 \ :\ v_1 - ||y - x_1||^2 = v_3 - ||y - x_3||^2\}$ should be close to both points $(4, 0)$ and $(0, -4)$, and the line $\{y \in \mathbb{R}^2 \ :\ v_2 - ||y - x_2||^2 = v_3 - ||y - x_3||^2\}$ should be close to both points $(0, 0)$ and $(4, 0)$. For sufficiently small ϵ, it must be the case that the point $\bar{y} = (1, -1)$ is in the triangle generated by these three lines. It means that at this point we must have

$$v_1 - ||\bar{y} - x_1||^2 < v_2 - ||\bar{y} - x_2||^2 < v_3 - ||\bar{y} - x_3||^2 < v_1 - ||\bar{y} - x_1||^2,$$

which is impossible. If there was a voter with ideal point $\bar{y}$ and transitive preferences over candidates this could not have been happening. The characterization in the case of a finite voter's set remains unresolved.

3.3. **Euclidean preferences.**

Our model does not presume any specific kind of preferences of the voters over the policy space[8]. The primitive only consists of rankings of the candidates by voters. So properties *(A1)-(A4)* imply that voters agree on the platforms

[7] Assume for simplicity that only strict preferences are allowed.

[8] We thank Jim Peck for this remark.

and valences of the candidates *and* that the preferences of voter y over candidates can be represented by $u_y(i) = v_i - ||x_i - y||^2$. It is possible to assume from the start that voters have Euclidean preferences. However, since utility is an ordinal notion, such modification does not affect our results whatsoever.

The Euclidean norm is intimately related to convexity. Other norms typically induce non-convex sets. It would be interesting to study preferences induced by other norms than the Euclidean.

3.4. **The valence dimension.** Theorem 1 fails if we require all candidates to have the same score (zero, w.l.o.g.) on the valence dimension[9]. Thus, more restrictions must be imposed on voters' preferences in order to allow a representation in the form $||x_i - y||^2$. Finding natural additional properties that distinguish this case from the more general one studied in this paper is an important direction for future research.

3.5. **The cases $m = 2$ and $m = 3$.** In contrast to the claim of subsection 3.4, if there are only two or three candidates then it is possible to represent the voters' preferences without resorting to valences. The case $m = 2$ is trivial since one only needs to choose the platforms x_1 and x_2 in equal distance from the hyperplane separating the voters that prefer candidate 1 from those preferring candidate 2. In the case $m = 3$ we state this fact as a proposition.

Proposition 2. *Assume $m = 3$. The preferences $\{\succeq_y\}_{y \in \mathbb{R}^d}$ satisfy properties (A1) through (A4) if and only if there are $x_1, x_2, x_3 \in \mathbb{R}^d$ in a general position[10] such that $i \succeq_y j$ if and only if $||x_i - y||^2 \leq ||x_j - y||^2$.*

3.6. **Observing just the first best.** Theorem 1 crucially depends on property *(A1)*. In particular this means that we must observe the entire ranking of each voter over C. In reality it is usually hard to extract this information from voters. A more plausible assumption is that only the most preferred candidate(s) is (are) observed for each voter.

A possible way to formalize this is to assume that the primitive is a function $f : \mathbb{R}^d \to 2^C$ with the interpretation that $f(y) \subseteq C$ is the set of candidates which voter y prefers the most. We do not know how to get a similar result to that of Theorem 1 in this case when the dimension of the policy space is $d \geq 2$. However, it turns out that when $d = 1$ a simple characterization is possible (see the next subsection).

[9]The reader is referred to Ash and Bolker (1985, Corollary 10) for a counter example.

[10]Here we simply mean that these three vectors are affinely independent.

3.7. **The case $d = 1$.** If the policy space is one dimensional (as is the case in many papers) then Theorem 1 is no longer true, even if appropriately modified. The reason for this failure is that the set of voters who are indifferent between some three candidates is typically empty. This set plays a major role in the proof of the main result. Nevertheless, we can get a representation similar to that of Theorem 1 if we assume that only the most preferred candidates are observed for each voter (as in the previous subsection). We will use the following properties for the characterization.

(B1) For every $i \in C$, the set $\{y \in \mathbb{R} \ : \ f(y) = \{i\}\}$ is not empty and open.

(B2) For every $i \in C$ and $y, z \in \mathbb{R}$, if $i \in f(y)$ ($\{i\} = f(y)$) and $i \in f(z)$ then $i \in f(\alpha y + (1-\alpha)z)$ ($\{i\} = f(\alpha y + (1-\alpha)z)$) for every $\alpha \in (0,1)$.

Before stating the result, we need a definition analogue to Definition 1.

Definition 2. *The set $\{(x_1, v_1), (x_2, v_2), \ldots, (x_m, v_m)\} \subseteq \mathbb{R}^2$ is* well-ordered *if there is a permutation $\pi : C \to C$ such that the following two conditions hold:*
(i) $x_{\pi(1)} < x_{\pi(2)} < \ldots < x_{\pi(m)}$.
(ii) $a_{\pi(1)\pi(2)} < a_{\pi(2)\pi(3)} < \ldots < a_{\pi(m-1)\pi(m)}$ where $a_{\pi(i)\pi(i+1)} = \frac{x^2_{\pi(i)} - x^2_{\pi(i+1)} + v_{\pi(i+1)} - v_{\pi(i)}}{2(x_{\pi(i)} - x_{\pi(i+1)})}$ for $i = 1, 2, \ldots, m-1$.

Proposition 3. *The correspondence $f : \mathbb{R} \to 2^C$ satisfies properties (B1) and (B2) if and only if there is a well-ordered set $\{(x_1, v_1), (x_2, v_2), \ldots, (x_m, v_m)\} \subseteq \mathbb{R}^2$ such that $f(y) = argmax\{v_i - (x_i - y)^2 \ : \ i \in C\}$.*

4. Proofs

4.1. **Proof of Theorem 1.** The proof of Theorem 1 is similar to the proof of the main result in Azrieli and Lehrer (2007). We therefore provide an outline of the proof and only detail those steps that did not appear in that paper.

A simple but important observation is that, for any $x_i \neq x_j \in \mathbb{R}^d$ and $v_i, v_j \in \mathbb{R}$, the set $\{y \in \mathbb{R}^d \ : \ v_i - ||x_i - y||^2 = v_j - ||x_j - y||^2\}$ is an affine subspace of dimension $d-1$ (a hyperplane), perpendicular to the direction $x_i - x_j$. Indeed, a simple computation shows that this set can be rewritten as $\{y \in \mathbb{R}^d \ : \ y \cdot (x_i - x_j) = \frac{1}{2}(v_j - v_i + ||x_i||^2 - ||x_j||^2)\}$. Similarly, the set $\{y \in \mathbb{R}^d \ : \ v_i - ||x_i - y||^2 > v_j - ||x_j - y||^2\}$ is an open half space in $\mathbb{R}^d$ (given that $x_i \neq x_j$).

(ii)* implies *(i):

Fix the sets $\{x_1, x_2, \ldots, x_m\} \subseteq \mathbb{R}^d$ and $\{v_1, v_2, \ldots, v_m\} \subseteq \mathbb{R}$. Property *(A1)* is obviously satisfied. Denote $A_{ij} = \{y \in \mathbb{R}^d \ : \ v_i - ||x_i - y||^2 = v_j - ||x_j - y||^2\}$ and $B_{ij} = \{y \in \mathbb{R}^d \ : \ v_i - ||x_i - y||^2 > v_j - ||x_j - y||^2\}$. By property *(i)* of Definition 1, $x_i \neq x_j$ for every $i \neq j \in C$. Thus, each B_{ij} is open and convex and each A_{ij} is the boundary of the closed half space $B_{ij} \cup A_{ij}$. This shows that properties *(A2)* and *(A3)* are satisfied.

Property *(A4)* is satisfied because the set $\{(x_1, v_1), (x_2, v_2), \ldots, (x_m, v_m)\}$ is in a general position. Indeed, take any distinct $i, j, k \in C$. We need to show that there is some y with $i \succ_y j \succ_y k$. If this was not true then it must be that B_{ij} and B_{jk} do not intersect. But this can only happen if $x_i - x_j$ and $x_j - x_k$ are linearly dependent, a contradiction to the assumption of general position (property *(i)*). Finally, take any distinct $i, j, k, l \in C$. We need to show that $B_{ij} \cap B_{jk}$ and $B_{ij} \cap B_{jl}$ are not equal. The set $B_{ij} \cap B_{jk}$ is the intersection of two (non-parallel) hyperplanes in $\mathbb{R}^d$ and so it is of dimension $d - 2$. If $B_{ij} \cap B_{jk} = B_{ij} \cap B_{jl}$ then $B_{ij} \cap B_{jk} \cap B_{jl} = B_{ij} \cap B_{jk}$ is also of dimension $d - 2$, contradicting the assumption of general position (property *(ii)*). $\square$

(i)* implies *(ii):

The proof is constructive. We first find the platforms $x_1, x_2, \ldots, x_m$ of the candidates, and then construct the valences $v_1, v_2, \ldots, v_m$. We need however to state some preliminary claims. The proofs of all these claims can be found in Azrieli and Lehrer (2007).

Claim 1. *For every ordered pair (i, j) of distinct candidates there is a non-zero vector $s_{ij} \in \mathbb{R}^d$ and a number $c_{ij} \in \mathbb{R}$ such that $\{y \in \mathbb{R}^d \ : \ i \succeq_y j\} = \{y \in \mathbb{R}^d \ : \ s_{ij} \cdot y \leq c_{ij}\}$. Moreover, these vectors and numbers can be chosen such that $s_{ji} = -s_{ij}$ and $c_{ji} = -c_{ij}$ for every (i, j).*

Fix a collection $\{s_{ij}, c_{ij}\}_{i,j \in C}$ as in Claim 1 until the end of the proof.

Claim 2. (A4) *implies that, for every $i, j, k \in C$, the vectors s_{ij} and s_{ik} are linearly independent.*

Claim 3. *For every $i, j, k \in C$, the vectors s_{ij}, s_{ik} and s_{jk} are* not *linearly independent.*

For $t, s \in \mathbb{R}^d$, denote by $R(t, s)$ the ray that starts at t and continues in the direction of s. That is $R(t, s) = \{t + \alpha s \ : \ \alpha \geq 0\}$.

Claim 4. *If $x_1, x_2 \in \mathbb{R}^d$ satisfy $x_2 - x_1 = \alpha s_{12}$ for some $\alpha > 0$ then, for every $3 \leq i \leq m$, the rays $R(x_1, s_{1i})$ and $R(x_2, s_{2i})$ intersect .*

We are now in the position to construct the sets $\{x_1, x_2, \ldots, x_m\}$ and $\{v_1, v_2, \ldots, v_m\}$. The point x_1 is chosen arbitrarily. Next, define $x_2 = x_1 + \alpha_{12} s_{12}$, where $\alpha_{12} > 0$ is arbitrary. For every $3 \le i \le m$, define x_i to be the unique point of intersection (by Claim 4) of the rays $R(x_1, s_{1i})$ and $R(x_2, s_{2i})$. A key point in the proof is that, when $\{x_1, x_2, \ldots, x_m\}$ are defined in this way, then, for every $1 \le i, j \le m$, $x_j - x_i = \alpha_{ij} s_{ij}$ for some $\alpha_{ij} > 0$. This fact follows from Proposition 1 (page 26) in Azrieli and Lehrer (2007). Finally, choose v_1 arbitrarily and define $v_i = v_1 - ||x_1||^2 + ||x_i||^2 - 2\alpha_{1i} c_{1i}$ for every $2 \le i \le m$.

It is useful to note that $\alpha_{ij} s_{ij} = \alpha_{1j} s_{1j} - \alpha_{1i} s_{1i}$ for every $3 \le i, j \le m$. Indeed, the left–hand side of the equality is $x_j - x_i$ while the right–hand side is $(x_j - x_1) - (x_i - x_1)$. This implies also that $\alpha_{ij} c_{ij} = \alpha_{1j} c_{1j} - \alpha_{1i} c_{1i}$. To see this, take $y \in \mathbb{R}^d$ such that $1 \sim_y i$ and $1 \sim_y j$ (the existence of such y is guaranteed by Claim 2). Transitivity implies that $i \sim_y j$. So $y \cdot s_{1i} = c_{1i}$, $y \cdot s_{1j} = c_{1j}$ and $y \cdot s_{ij} = c_{ij}$. Multiplying these equalities by α_{1i}, α_{1j} and α_{ij} correspondingly, and subtracting the first from the second we get the above equality.

To complete the proof we need to check that the set $\{(x_1, v_1), (x_2, v_2), \ldots, (x_m, v_m)\}$ is in a general position and that, for every $i, j \in C$ and $y \in \mathbb{R}^d$, $i \succeq_y j$ if and only if $v_i - ||x_i - y||^2 \ge v_j - ||x_j - y||^2$. For the latter we have

$$\begin{aligned}
& i \succeq_y j \iff s_{ij} \cdot y \le c_{ij} \iff (x_j - x_i) \cdot y \le \alpha_{ij} c_{ij} \iff (x_j - x_i) \cdot y \le \alpha_{1j} c_{1j} - \alpha_{1i} c_{1i} \\
& \iff (x_j - x_i) \cdot y \le \frac{1}{2}\left(v_1 - v_j + ||x_j||^2 - ||x_1||^2\right) - \frac{1}{2}\left(v_1 - v_i + ||x_i||^2 - ||x_1||^2\right) \\
& \iff (x_j - x_i) \cdot y \le \frac{1}{2}\left(v_i - v_j + ||x_j||^2 - ||x_i||^2\right) \iff v_i - ||x_i - y||^2 \ge v_j - ||x_j - y||^2.
\end{aligned}$$

For the former, the vectors x_i, x_j, x_k are affinely independent since $x_j - x_i = \alpha_{ij} s_{ij}$ and $x_k - x_i = \alpha_{ik} s_{ik}$, and these are linearly independent vectors by Claim 2. Finally, each of the sets $\{y \in \mathbb{R}^d \ : \ i \sim_y j \sim_y k\}$ and $\{y \in \mathbb{R}^d \ : \ i \sim_y j \sim_y l\}$ is an affine subspace of dimension $d - 2$. By *(A4)* they are not equal so their intersection is of dimension $d - 3$ at most. This proves that $\{(x_1, v_1), (x_2, v_2), \ldots, (x_m, v_m)\}$ is in a general position. □

4.2. **Proof of Proposition 1.** First, it is easy to check that if there are $\alpha > 0$ and $\beta \in \mathbb{R}^d$ such that $x'_i = \alpha x_i + \beta$ for $1 \le i \le m$, and in addition equation (1) is satisfied then $\{(x_1, v_1), (x_2, v_2), \ldots, (x_m, v_m)\}$ and $\{(x'_1, v'_1), (x'_2, v'_2), \ldots, (x'_m, v'_m)\}$ represent the same preferences.

Now, assume that $\{(x_1, v_1), (x_2, v_2), \ldots, (x_m, v_m)\}$ and $\{(x'_1, v'_1), (x'_2, v'_2), \ldots, (x'_m, v'_m)\}$ represent the same preferences $\{\succeq_y\}_{y \in \mathbb{R}^d}$. It follows from the proof of Theorem 1 that for every $i,j \in C$ there is a positive number, say $t_{ij} > 0$, such that $x_j - x_i = t_{ij}(x'_j - x'_i)$ (with the convention $t_{ij} = -t_{ji}$). Fix some three candidates $i,j,k \in C$. Sum up the equalities $x_j - x_i = t_{ij}(x'_j - x'_i)$, $x_i - x_k = t_{ki}(x'_i - x'_k)$, $x_k - x_j = t_{jk}(x'_k - x'_j)$ and rearrange the terms to obtain $(x'_i - x'_j)(t_{ki} - t_{ij}) + (x'_k - x'_j)(t_{jk} - t_{ki}) = 0$. But the vectors x'_i, x'_j, x'_k are affinely independent so $t_{ki} - t_{ij} = t_{jk} - t_{ki} = 0$. It follows that $t_{ij} = t_{ki} = t_{jk}$, so there is a number $\alpha > 0$ such that $x_j - x_i = \alpha(x'_j - x'_i)$ for every $i,j \in C$. Now, define $\beta = x_1 - \alpha x'_1$. For every $2 \leq i \leq m$ we have $x_1 - x_i = \alpha(x'_1 - x'_i)$ or $x_i - \alpha x'_i = x_1 - \alpha x'_1 = \beta$. That is, $x'_i = \alpha x_i + \beta$ for every $1 \leq i \leq m$.

Finally, we must have $\frac{1}{2}\left(v_i - v_j + ||x_j||^2 - ||x_i||^2\right) = \frac{1}{2}\left(v'_i - v'_j + ||x'_j||^2 - ||x'_i||^2\right)$ for every $i,j \in C$. Substituting $\alpha x_i + \beta$ for x'_i and $\alpha x_j + \beta$ for x'_j and rearranging we obtain equation (1). In particular, if $x'_i = x_i$ and $x'_j = x_j$ then $v'_i - v_i = v'_j - v_j$. Define $\gamma = v'_1 - v_1$. It follows that $v'_i = v_i + \gamma$ for every $1 \leq i \leq m$. $\square$

4.3. **Proof of Proposition 2.** The *if* part follows from Theorem 1, so we only need to prove the *only if* part. By Theorem 1, there are $(x_1, v_1), (x_2, v_2), (x_3, v_3)$ in a general position that represent the preferences. It follows that the vectors $x_1 - x_2$ and $x_1 - x_3$ are linearly independent. Therefore, there is $\beta \in \mathbb{R}^d$ that solves the two equations $\beta \cdot (x_1 - x_2) = \frac{v_2 - v_1}{2}$ and $\beta \cdot (x_1 - x_3) = \frac{v_3 - v_1}{2}$. Notice that the same vector β must satisfy also $\beta \cdot (x_2 - x_3) = \frac{v_3 - v_2}{2}$. Define $x'_i = x_i + \beta$ for $i = 1,2,3$.

By Proposition 1, the set $\{(x'_1, v'_1), (x'_2, v'_2), (x'_3, v'_3)\}$ represent the same preferences as $\{(x_1, v_1), (x_2, v_2), (x_3, v_3)\}$ if the equation $v'_i - v'_j = v_i - v_j + 2\beta \cdot (x_i - x_j)$ is satisfied for every $i,j \in C$. By construction, the vector β satisfies $\beta \cdot (x_i - x_j) = \frac{v_j - v_i}{2}$ for every i,j. It follows that $v'_1 = v'_2 = v'_3 = 0$ solve the above equations. That is, $\{(x'_1, 0), (x'_2, 0), (x'_3, 0)\}$ represent the preferences $\{\succeq_y\}_{y \in \mathbb{R}^d}$. $\square$

4.4. **Proof of Proposition 3.** Assume first that the correspondence f can be represented as in the proposition. We can assume w.l.o.g. that π is the identity, so $x_1 < x_2 < \ldots < x_m$ and $a_{12} < a_{23} < \ldots < a_{(m-1)m}$. It is also convenient to denote $a_{01} = -\infty$ and $a_{m(m+1)} = +\infty$. Now, for every $1 \leq i \leq m-1$, a simple computation shows that $v_i - (x_i - y)^2 \geq v_{i+1} - (x_{i+1} - y)^2$ if and only if $y \leq a_{i(i+1)}$ (the same equivalence holds when the weak inequalities are replaced by strict ones). It follows that candidate i $(1 \leq i \leq m)$ is the unique maximizer of $\{v_j - (x_j - y)^2 \ : \ j \in C\}$ if and only if $y \in (a_{(i-1)i}, a_{i(i+1)})$ and that i is a maximizer (not necessarily unique) of this expression if and only if $y \in [a_{(i-1)i}, a_{i(i+1)}]$. This shows that f satisfies properties *(B1)* and *(B2)*.

Conversely, assume that f satisfies *(B1)* and *(B2)*. These properties imply that there is a permutation of the candidates, w.l.o.g. the identity, and a sequence of numbers $a_{12} < a_{23} < \ldots < a_{(m-1)m}$ such that $f(y) = \{i\}$ if and only if $y \in (a_{(i-1)i}, a_{i(i+1)})$ and $f(y) = \{i, i+1\}$ if and only if $y = a_{i(i+1)}$ for $1 \leq i \leq m$.

Take any set of points $x_1 < x_2 < \ldots < x_m$. Define $v_1 = 0$ and, for every $1 \leq i \leq m-1$, let $v_{i+1} = 2a_{i(i+1)}(x_i - x_{i+1}) - x_i^2 + x_{i+1}^2 + v_i$. Rearranging, this gives $a_{i(i+1)} = \frac{x_i^2 - x_{i+1}^2 + v_{i+1} - v_i}{2(x_i - x_{i+1})}$ for $i = 1, 2, \ldots, m-1$. Thus, the set $\{(x_1, v_1), (x_2, v_2), \ldots, (x_m, v_m)\} \subseteq \mathbb{R}^2$ is well-ordered. Finally, we need to check that $f(y) = argmax\{v_i - (x_i - y)^2 \ : \ i \in C\}$. This is true since

$$i \in f(y) \iff y \in [a_{(i-1)i}, a_{i(i+1)}] \iff \frac{x_{i-1}^2 - x_i^2 + v_i - v_{i-1}}{2(x_i - x_{i+1})} \leq y \leq \frac{x_i^2 - x_{i+1}^2 + v_{i+1} - v_i}{2(x_{i-1} - x_i)}$$
$$\iff v_i - (y - x_i)^2 \geq v_{i-1} - (y - x_{i-1})^2 \text{ and } v_i - (y - x_i)^2 \geq v_{i+1} - (y - x_{i+1})^2$$
$$\iff v_i - (y - x_i)^2 \geq v_j - (y - x_j)^2 \text{ for all } j \neq i.$$

□

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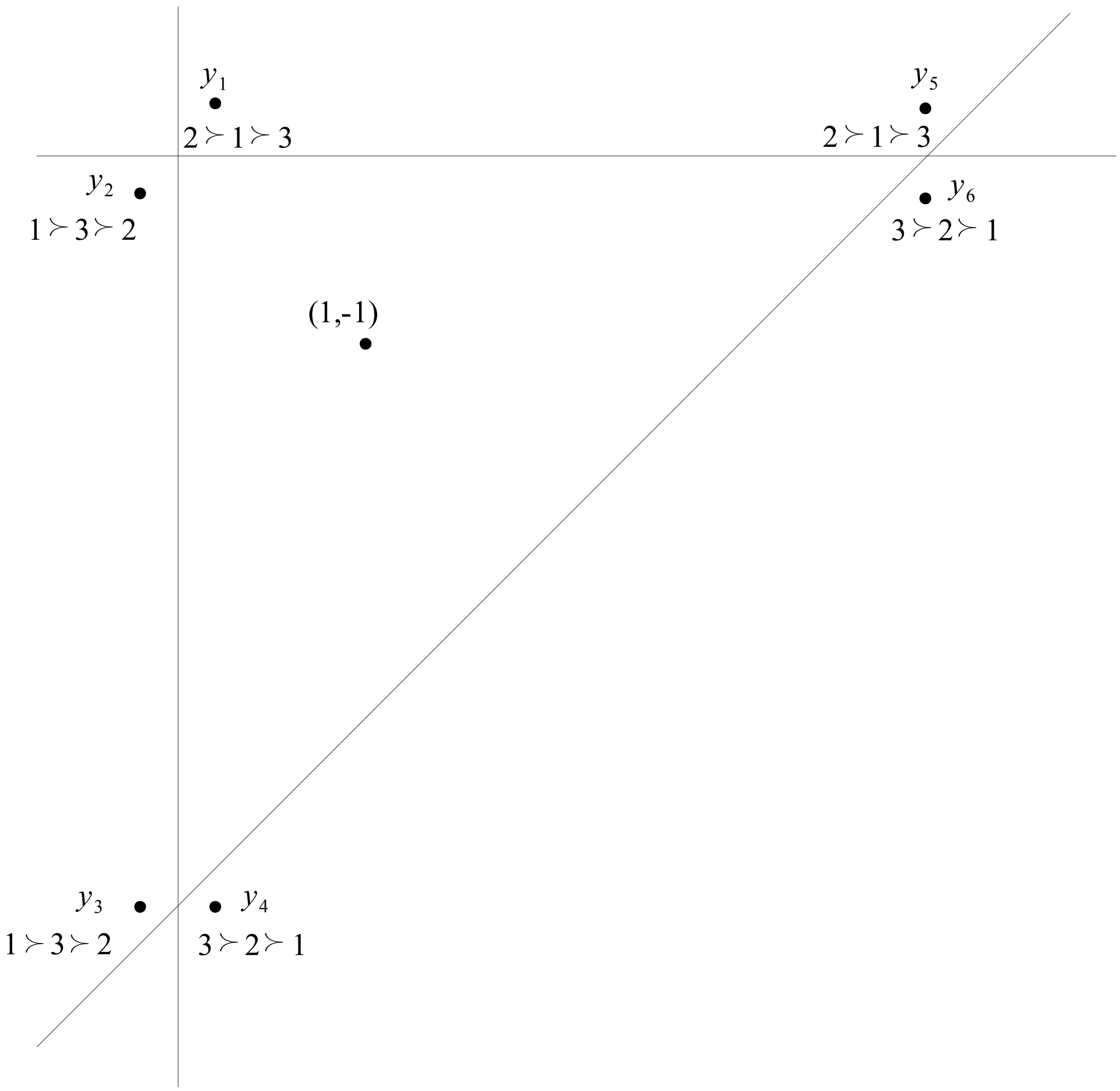

y_1
$2 \succ 1 \succ 3$
y_2
$1 \succ 3 \succ 2$
(1,-1)
y_3
$1 \succ 3 \succ 2$
y_4
$3 \succ 2 \succ 1$
y_5
$2 \succ 1 \succ 3$
y_6
$3 \succ 2 \succ 1$

Figure 1