Hierarchical cheap talk

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Abstract

We investigate situations in which agents can only communicate to each other through a chain of intermediators, for example because they have to obey institutionalized communication protocols. We assume that all involved in the communication are strategic, and might want to influence the action taken by the final receiver. The set of outcomes that can be induced in pure strategy perfect Bayesian Nash equilibrium is a subset of the equilibrium outcomes that can be induced in direct communication, characterized by Crawford and Sobel (1982). Moreover, the set of supportable outcomes in pure equilibria is monotonic in each intermediator's bias, and the intermediator with the largest bias serves as a bottleneck for the information flow. On the other hand, there can be mixed strategy equilibria of intermediated communication that ex ante Pareto-dominate all equilibria in direct communication, as mixing by an intermediator can relax the incentive compatibility constraints on the sender. We provide a partial characterization of mixed strategy equilibria, and show that the order of intermediators matters with respect to mixed equilibria, as opposed to pure strategy ones.

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1 Introduction

In many settings, physical, social, or institutional constraints prevent people from communicating directly. In the US army, companies report to battalions, which in turn report to brigades (companies are not allowed to report directly to brigades). Similarly, in many organizations, there is a rigid hierarchical structure for communication flow within the organization. Even without explicit regulations, there are time and resource constraints preventing all communication to be direct. The managing director of a large company cannot give instructions to all workers of the company directly. Instead, she typically only talks directly to high level managers, who further communicate with lower level managers, who in turn talk to the workers. Finally, in traditional societies, the social network and various conventions might prevent direct communication between two members of the society. For example, a man might not be allowed to talk directly to a non-relative woman; instead, he has to approach the woman's parents or husband, and ask them to transfer a piece of information.

There is a line of literature in organizational economics, starting with Sah and Stiglitz (1986) and Radner (1992), investigating information transmission within organizations.[1] However, all the papers in this literature assume homogeneity of preferences and hence abstract away from strategic issues in communication. As opposed to this, in this paper we analyze information transmission through agents who are strategic and interested in influencing the outcome of the communication.

To achieve this, we extend the classic model of Crawford and Sobel (1982; from now on CS), to investigate intermediated communication.[2] We investigate communication along a given chain: player 1 privately observes the realization of a continuous random variable and sends a message to player 2, who then sends a message to player 3, and so on, until communication reaches player n. We refer to player 1 as the *sender*, player n as the *receiver*, and players $2, .., n-1$ as the *intermediators*. The receiver, after receiving a message from the last intermediator, chooses an action on the real line,

[1] Radner (1992), Bolton and Dewatripont (1994), and van Zandt (1999) examine organizations in which different pieces of information have to get to the same member, but any member can potentially process a task once having all pieces of information. Sah and Stiglitz (1986) and Visser (2000) study the contrast between the performance of hierarchic and polyarchic organizations in a related setting. Garricano (2000) and Arenas et al. (2008) consider networks in which individuals specialize to solve certain tasks, and it takes a search procedure (through communication among agents) to find the right individual for the right problem.

[2] For a more general class of sender-receiver games than the CS framework, see Green and Stokey (2007).

which affects the well-being of all players. We assume that all intermediators are strategic, and have preferences from the same class of preferences that CS considers for senders.

First, we consider pure strategy perfect Bayesian Nash equilibria (PBNE) of such indirect communication games, and show that any outcome that can be induced in such equilibria can also be induced in the direct communication game between the sender and the receiver (the equilibria of which are characterized in CS). Hence, if one restricts attention to pure strategy equilibria, intermediators can only filter out information, as opposed to facilitating more efficient information transmission. We present a simple condition for checking if a given equilibrium outcome of the direct communication game can be achieved with a certain chain of intermediators. This condition reveals that the order of intermediators does not matter in pure strategy PBNE. We also show that the set of pure strategy PBNE outcomes is monotonic in each of the intermediators' biases: increasing an intermediator's bias (in absolute value) weakly decreases the set of PBNE outcomes. In the standard context of state-independent biases and symmetric loss functions, only the intermediator with the largest bias (in absolute terms) matters: this intermediator becomes a bottleneck in information transmission.

More surprisingly, we show that when allowing for mixed strategies, there can be equilibria of the indirect communication game that can strictly improve communication (resulting in higher ex ante expected payoff for both the sender and the receiver) relative to all equilibria of the game with direct communication. This has implications for organizational design, as the result shows that hierarchical communication protocols can increase information transmission in the organization, if communication is strategic.

At the core of this result is the observation first made by Myerson (1991, p285-288), that noise can improve communication in sender-receiver games. Myerson provides an example with two states of the world in which there is no informative equilibrium with noiseless communication. However, when player 1 has access to a messenger pigeon that only reaches its target with probability 1/2, then there is an equilibrium of the game with communication in which the sender sends the pigeon in one state but not the other one, and the receiver takes different actions depending on whether the pigeon arrives or not. Obviously, the same equilibrium can be induced with a strategic intermediator (instead of a noisy communication device) if conditional on the first state, the intermediator happens to be exactly indifferent between inducing either of the two equilibrium actions. What we show in the context of the CS model is that such indifferences, which are necessary to induce strategic intermediators to randomize, can be created endogenously in equilibrium, for an open set of environments. The intuition why such induced mixing can improve information revelation by the sender is similar to the one in Myerson (1991) and in Blume et al. (2007): the induced noise can relax the incentive compatibility constraints on the sender, by making certain messages (low messages for a positively biased sender, high messages for a negatively biased sender) relatively more attractive.

As a motivation for studying such mixed equilibria, we think that the idea of purification (Harsányi (1973)) is particularly appealing in communication games. In particular, one can view mixed equilibria in indirect communication games in which all players have a fixed known bias function as limits of pure strategy equilibria of communication games in which players' ex post preferences have a small random component. This assumption makes the model more realistic, as it is typically a strong assumption that the bias of each player is perfectly known by others.

We provide a characterization of all mixed strategy equilibria of indirect communication games. In particular, we show that there is a positive lower bound on how close two actions induced in a PBNE can be to each other, which depends on the last intermediator's bias. This implies that there is a finite upper bound on the number of actions induced in a PBNE. Furthermore, we show that all PBNE are outcome-equivalent to an equilibrium in which: (i) almost all types of the sender send a pure message; (ii) the receiver plays a pure action after any message; (iii) for the sender and for all intermediators, the distribution of actions induced by different equilibrium messages can be ranked with respect to first-order stochastic dominance. We establish additional results for the case of one intermediator, including that the number of distinct messages sent by the sender is exactly equal to the number of actions induced in equilibrium, and that the intermediator can mix between at most two distinct messages. By example we show that, in contrast to pure strategy PBNE, the order of multiple intermediators can matter with respect to the set of possible mixed strategy PBNE outcomes.

We analytically solve for two types of mixed strategy PBNE in the broadly studied uniform-quadratic specification of the model, for one intermediator, and characterize the set of bias pairs for which these equilibria exist. We show that both of these types of equilibria exist in a full-dimensional set of parameters. As opposed to the case of pure strategies, the set of equilibria is not monotonic in the intermediator's bias. For certain specifications of the sender bias, the only PBNE with an intermediator whose bias is close enough to 0 is babbling, while there are informative equilibria with intermediators with larger biases. However, once the bias of the intermediator is too large, again only the babbling equilibrium prevails. We find PBNE involving nontrivial mixing both when the intermediator is biased in the opposite and when she is biased in the same direction as the sender. Interestingly, the latter requires the intermediator to be strictly more biased than the sender.

We conclude the paper by providing a simple sufficient condition for the existence of an intermediator that can facilitate nontrivial information transmission in cases when the only equilibrium in the direct communication game between the sender and the receiver is babbling.

Our paper is complementary to two recent working papers. Ivanov (2009) considers a setting similar to ours, but with only one intermediator, and focusing on the uniform-quadratic specification of the CS framework. Moreover, the set of questions investigated by Ivanov differs from ours: the paper does not investigate the set of equilibria of an indirect communication game for a given set of players. Instead, the paper shows that when the intermediator can be freely selected by the receiver, there exists a strategic mediator and a mixed strategy PBNE of the resulting indirect communication game in which the receiver's ex ante payoff is as high as the maximum payoff attainable by any mechanism. This is a sharp welfare improvement result, but it only applies to the uniform-quadratic specification of the model. Galeotti et al. (2009), a working paper concurrent with the first version of our paper, examines strategic communication on general networks, but in a setting where the state space and therefore communication is much simpler than in the CS model: players receive binary signals. Moreover, Galeotti et al. (2009) restricts attention to equilibria in pure strategies.

There are other recent papers investigating the effect of nonstrategic noise in communication in the CS framework. Blume et al. (2007) examine communication in the CS model with an exogenously specified noise: with a certain probability the receiver gets not the sender's original message, but a random message. Although some of the intuition for improved communication is similar to ours, the equilibria achieving welfare improvement are very different than in our model: some sender types are required to mix among all possible messages. Goltsman et al. (2008) characterize equilibrium payoffs in the uniform-quadratic specification of the CS framework if players have access to an impartial mediator.[3] They also investigate delegation, that is when the receiver can ex ante commit to a message-contingent action plan.[4]

Also related to our paper is Krishna and Morgan (2004), which shows that there exist mixed equilibria in multiple rounds of two-sided communication (the uninformed player is required to be able to talk as well) that can improve information transmission relative to the best equilibrium with only one round of communication.[5] Again, the structure of equilibria improving information

[3] See Forges (1986) and Myerson (1986) for earlier papers on general communication devices.

[4] See also Kovac and Mylovanov (2008) for stochastic delegation.

[5] See also Aumann and Hart (2003) for cheap talk with multiple rounds of communica-

transmission relative to the CS equilibria is very different than in our setting. In particular, in Krishna and Morgan, high and low types of senders might pool at some stage of the communication game in sending the same message, while intermediate types send a separate message. Such non-monotonicities cannot occur in equilibrium in our setting.

There is a line of literature investigating the possibility that the sender's preferences are private information, so that from the point of view of the receiver, the sender's action is a random variable even conditioning on the state - analogously to the noise that intermediators can introduce in our model. Olszewski (2004) and Chen et al. (2006) consider the possibility that with some probability, the receiver is a nonstrategic type always telling the truth.[6] Li and Madarász (2008) consider a model in which all potential types of the sender are from the original CS framework.

Niehaus (2008) considers chains of communication, as in our paper, but in a setting with no conflict of interest among agents, and hence non-strategic communication. Instead, Niehaus assumes an exogenous cost of communication and examines the welfare loss arising from agents not taking into account positive externalities generated by communication.

2 The model

Here we formally extend the model in CS to chains of communication. In particular, we impose the same assumptions as CS for all players involved in the communication chain.

We consider the following sequential-move game with $n \geq 3$ players. In stage 1, player 1 (the *sender*) observes the realization of a random variable $\theta \in \Theta = [0,1]$, and sends a message $m_1 \in M_1$ to player 2 (which only player 2 observes, not the other players). From now on we will refer to θ as the state. The c.d.f. of θ is $F(\theta)$, and we assume it has a density function f that is strictly positive and absolutely continuous on $[0,1]$. In stage $k \in \{2,...,n-1\}$ player k sends a message $m_k \in M_k$ to player $k+1$ (which only player $k+1$ observes). Note that the message choice of player k in stage k can be conditional on the message she received in the previous stage from player $k-1$ (but not on the messages sent in earlier stages, since she did not observe those). We assume that M_k is a Borel set that has the cardinality of the continuum, for every $k \in \{1,...,n-1\}$. We refer to players $2,...,n-1$ as *intermediators*. In stage n of the game, player n (the *receiver*) chooses an action $y \in R$. This action choice can be conditional on

tion.

[6] In Chen et al. (2006) the receiver has multiple types, as well.

the message she received from player $n-1$ in stage $n-1$ (but none of the messages sent in earlier stages).

More formally, the strategy of player 1 is defined by a probability distribution μ^1 on the Borel-measurable subsets of $[0,1]\times M_1$ for which $\mu^1(A\times M_1) = \int\limits_A f$ for all measurable sets A, the strategy of player $k \in \{2,...,n-1\}$ is a probability distribution μ^k on the Borel-measurable subsets of $M_{k-1}\times M_k$, and the strategy of player n is a probability distribution μ^n on the Borel-measurable subsets of $M_{n-1}\times R$. Given the above probability distributions, there exist regular conditional distributions $p^1(\cdot|\theta)$ for every $\theta \in \Theta$, and $p^k(\cdot|m_{k-1})$ for every $k \in \{2,...,n\}$ and $m_{k-1} \in M_{k-1}$.[7]

The payoff of player $k \in \{1,...,n\}$ is given by $u^k(\theta, y)$, which we assume to be twice continuously differentiable and strictly concave in y. Note that the messages $m_1,...,m_{n-1}$ sent during the game do not enter the payoff functions directly; hence the communication we assume is cheap talk.

We assume that for fixed θ, $u^n(\theta, y)$ reaches its maximum value 0 at $y^k(\theta) = \theta$, while $u^k(\theta, y)$ reaches its maximum value 0 at $y^k(\theta) = \theta + b^k(\theta)$ for some $b^k(\theta) \in R$. We refer to $y^k(\theta)$ as the ideal point of player k at state θ, and to $b^k(\theta)$ as the bias of agent k at state θ. Note that we normalize the receiver's bias to be 0 in every state.

As opposed to the original CS game, intermediators in our model might need to condition their messages on a nondegenerate probability distribution over states. For this reason, it will be convenient for us to extend the definition of a player's bias from single states to probability distributions over states. Let Ω be the set of probability distributions over Θ. Let $b^k(\mu) = \arg\max\limits_{y\in R} \int\limits_{\theta\in\Theta} u^k(\theta, y)\, d\mu - \arg\max\limits_{y\in R} \int\limits_{\theta\in\Theta} u^n(\theta, y)\, d\mu$, for every probability distribution $\mu \in \Omega$, and every $k \in \{1,...,n-1\}$. In words, $b^k(\mu)$ is the difference between the optimal actions of player k and the receiver, conditional on belief μ. Note that the term is well-defined, since our assumptions imply that both $\int\limits_{\theta\in\Theta} u^k(\theta, y)\, d\mu$ and $\int\limits_{\theta\in\Theta} u^n(\theta, y)\, d\mu$ are strictly concave in y.

We adopt two more assumptions of CS into our context. The first is the single-crossing condition $\frac{\partial^2 u^k(\theta,y)}{\partial\theta\partial y} > 0$, for every $k \in \{1,...,n\}$. This in particular implies that all players in the game would like to induce a higher action at a higher state. The second one is that either $b^1(\theta) > 0$ at every $\theta \in \Theta$ or $b^1(\theta) < 0$ at every $\theta \in \Theta$, and that either $b^k(\mu) > 0$ at every $\mu \in \Omega$ or $b^k(\mu) < 0$ for every $k \in \{2,...,n-1\}$ and $\mu \in \Omega$. In words, players $1,...,n-1$ have well-defined directions of biases (either positive or negative). The condition imposed on the sender is the same as in CS, while

[7] See Loeve (1955, p137-138).

the condition imposed on the intermediators is stronger in that their biases with respect to any belief (as opposed to only single states) are required to be of the same sign.

Finally, we assume that all parameters of the model are commonly known to the players.

We refer to the above game as the *indirect communication game*. Occasionally we will also refer to the direct communication game between the sender and the receiver. This differs from the above game in that there are only two stages, and two active players. In stage 1 the sender observes the realization of $\theta \in \Theta$ and sends message $m_1 \in M_1$ to the receiver. In stage 2 the receiver chooses an action $y \in R$.

The solution concept we use is perfect Bayesian Nash equilibrium (PBNE). For the formal definition of PBNE we use in our context, see Appendix A.

Both in the context of the indirect and the direct communication game, we will refer to the probability distribution on $\Theta \times R$ induced by the PBNE strategy profile as the *outcome* induced by a PBNE. Two PBNE are *outcome-equivalent* if the above outcome distributions are the same.

3 Pure strategy equilibria

Our first result establishes that every pure strategy PBNE in the game of indirect communication is outcome-equivalent to a PBNE of the direct communication game between the sender and the receiver. That is, the set of possible pure strategy equilibrium outcomes in any indirect communication game is a subset of the set of possible equilibrium outcomes in the direct communication game obtained by eliminating the intermediators. This makes characterizing pure strategy PBNE in indirect communication games fairly straightforward, since the characterization of PBNE in the direct communication game is well-known from CS. In particular, whenever the sender has a nonzero bias, there is a finite number of distinct equilibrium outcomes.

For the formal proofs of all propositions, see Appendix B. The intuition behind the proof is that in a pure strategy PBNE, every message of the sender induces a message of the final intermediator and an action by the receiver deterministically. Hence, the sender can effectively choose which action to induce, among the ones that can be induced in equilibrium.

Proposition 1: For every pure strategy PBNE of the indirect communication game, there is an outcome-equivalent PBNE of the direct communication game.

Next we give a necessary and sufficient condition for a given PBNE of the direct communication game to have an outcome-equivalent PBNE in an indirect communication game.

Let $\Theta(y)$ be the set of states at which the induced outcome is y, for every $y \in Y$. Furthermore, for ease of exposition, we introduce the convention that whenever $\Theta(y)$ is a singleton consisting of state θ', then $\int\limits_{\theta \in \Theta(y)} u^k(y,\theta) f(\theta) d\theta \geq \int\limits_{\theta \in \Theta(y)} u^k(y',\theta) f(\theta) d\theta$ iff $u^k(y,\theta') \geq u^k(y',\theta')$, although formally both integrals above are 0.

Proposition 2: Fix a PBNE of the direct communication game, and let Y be the set of actions induced in equilibrium. Then there is an outcome-equivalent PBNE of the indirect communication game if and only if

$$\int\limits_{\theta \in \Theta(y)} u^k(y,\theta) f(\theta) d\theta \geq \int\limits_{\theta \in \Theta(y)} u^k(y',\theta) f(\theta) d\theta \qquad (1)$$

for every $y, y' \in Y$ and $k \in \{2, ..., n-1\}$.

In words, the condition in the proposition requires that conditional on the set of states in which a given equilibrium action is induced, none of the intermediators would rather induce any of the other equilibrium actions. The intuition behind the result is straightforward: if conditional on states in which equilibrium action y is induced, an intermediator strictly prefers a different equilibrium action y', then there has to be at least one equilibrium message after which the equilibrium strategy prescribes the intermediator inducing y even though given his conditional belief he prefers y' - a contradiction. The condition in Proposition 2 is convenient, since it can be checked for all intermediators one by one.

An immediate corollary of the result is that the order of intermediators is irrelevant with respect to the set of pure strategy PBNE outcomes, since the necessary and sufficient condition in Proposition 2 is independent of the sequencing of intermediators.

Another corollary of the result, stated formally below, is that the set of equilibrium outcomes is monotonically decreasing in the bias of any intermediator. For the intuition behind this, consider an intermediator with a positive bias (the negative bias case is perfectly symmetric). Conditional on the set of states inducing equilibrium action y, the set of actions that the intermediator strictly prefers to y is an open interval with the left-endpoint at y. Moreover, this interval gets strictly larger if the intermediator's bias increases, making it less likely that the condition in Proposition 2 holds for a given PBNE of the direct communication game.

In order to define increased bias for some player formally in the general specification of the model, we need to introduce some new notation (the definition is much simpler for state-independent biases, see below). Let $k \in \{2, ..., n-1\}$. Fix player k's two payoff functions, u^k and v^k. We say that v^k is *more positively (resp. negatively) biased than* u^k, if there exist affine transformations of u^k and v^k, u^{k*} and v^{k*} respectively, such that

$$\frac{\partial v^{k*}(\theta, y)}{\partial y} > \frac{\partial u^{k*}(\theta, y)}{\partial y} \quad \left(\text{resp.} \quad \frac{\partial v^{k*}(\theta, y)}{\partial y} < \frac{\partial u^{k*}(\theta, y)}{\partial y}\right) \tag{2}$$

for every θ and y.

An example of v^k being more positively biased than u^k (equivalently, u^k being more positively biased than v^k) is when v^k is obtained by shifting u^k to the right, that is if there exists $\delta > 0$ such that $u^k(y, \theta) = v^k(y + \delta, \theta)$ for every y and θ.

Proposition 3: Let $k \in \{2, ..., n-1\}$ and fix the preferences of all players other than k. Let u^k be a payoff function implying positive (respectively, negative) bias. If v^k is more positively (resp. negatively) biased than u^k, then for every pure strategy PBNE of the indirect communication game in which player k's payoff function is v^k, there is an outcome-equivalent pure-strategy PBNE of the indirect communication game in which player k's payoff function is u^k.

The above results simplify considerably for the case where players have state-independent biases and symmetric loss functions, that is when there exist $b^1, ..., b^{n-1} \in R$ and $l : R \to R_+$ with $l(0) = 0$ such that $u^k(\theta, y) = -l(|y - \theta - b^k|)$ for every $k \in \{1, ..., n\}$. In this context, conditional on the set of states that induce an equilibrium action y, the set of actions player k (for $k \in \{2, ..., n-1\}$) strictly prefers to y is $(y, y+2b^k)$. Therefore, the condition in Proposition 2 simplifies to $|y - y'| \geq 2|b^k|$ for every two actions $y \neq y'$ induced in equilibrium, while Proposition 3 simplifies to stating that the set of outcomes that can be supported in pure strategy PBNE is monotonically decreasing in $|b^k|$, for every $k \in \{2, ..., n-1\}$. Note also that if for some $k^* \in \{2, ..., n-1\}$, we have $|b^{k^*}| \geq |b^k|$ for every $k \in \{2, ..., n-1\}$ then $|y - y'| \geq 2|b^{k^*}|$ implies that $|y - y'| \geq 2|b^k|$ for every $k \in \{2, ..., n-1\}$, and hence the condition in Proposition 2 holds. This means that only the intermediator with the largest absolute value bias matters in determining which pure strategy PBNE outcomes of the direct communication game can be supported as a PBNE outcome in indirect communication, as the intermediator becomes a *bottleneck* in the strategic transmission flow of information.

We conclude the section with a brief discussion of a limit case of our model when player 1 is unbiased (from the point of view of the receiver), since this case is not considered explicitly in CS. Propositions 1-3 above can easily modified to cover this case. In particular, the direct communication game in this case has an equilibrium with full information revelation, and for every $m \in Z_{++}$ at least one partition equilibrium with m partition cells (in the uniform-quadratic specification of CS, the unique such equilibrium partition is when all cells have length $\frac{1}{m}$). With intermediators involved, in pure strategy PBNE only a subset of the above outcomes can be supported, the ones in which induced equilibrium actions are far enough from each other, relative to the biases of the intermediators.

4 Mixed strategy equilibria

In this section, we analyze mixed strategy PBNE of indirect communication games. We show that for some parameter values there can be mixed equilibria that ex ante Pareto-dominate all PBNE of the direct communication game. That is, intermediators might facilitate better information transmission. However, as opposed to pure strategy PBNE, the set of mixed strategy PBNE is a complicated nonmonotonic function of the intermediators' biases.

In Subsection 4.1, we derive general properties of mixed strategy PBNE. In Subsection 4.2, we provide two classes of examples of mixed strategy PBNE in the uniform-quadratic specification of the model. In Subsection 4.3, we show that the order of intermediators matters for the set of possible PBNE outcomes. Finally, in Subsection 4.4, we provide a condition for an intermediator to be able to facilitate information transmission, in cases where no information can be transmitted in a direct communication game.

4.1 General properties of mixed equilibria

Below we show that although there might be many different mixed strategy PBNE of an indirect communication game, all of them are outcome-equivalent to some equilibrium in which the following properties hold: (i) the state space is partitioned to a finite number of intervals such that in the interior of each partition cell, player 1 sends the same (pure) message; (ii) the receiver plays a pure strategy after any message; (iii) the probability distribution over actions that different messages of a player $i \in \{1, ..., n-1\}$ induce can be ordered with respect to first-order stochastic dominance. Moreover, we show that there is a finite upper bound on the number of actions that can be induced in a PBNE of a given indirect communication game.

Before we state the above results formally, we first establish that the assumptions we imposed on the preferences of players imply that for every intermediator, there exists a minimal bias, that is there exists a belief over states such that the intermediator's bias is weakly smaller given this belief than given any other belief.

Claim 1: There exists $\underline{b}^k > 0$ such that $\min\limits_{\mu \in \Omega} |b^k(\mu)| = \underline{b}^k$, for every $k \in \{2, ..., n-1\}$.

We refer to $\underline{b}^k$ as the minimum absolute bias of player k. Next we show that in any PBNE, the receiver plays a pure strategy, and any distinct actions induced in equilibrium cannot be closer to each other than the minimum absolute bias of player $n-1$. The first result is an immediate consequence of the strict concavity of the receiver's payoff in the action choice, implying that for any belief, there is a unique payoff-maximizing action. The intuition behind the second result is that given that the receiver always chooses the action maximizing his expected payoff conditional on the message he receives, if two equilibrium actions y and y' are closer to each other than the minimum absolute bias of player $n-1$, then along the equilibrium path, it has to be that player $n-1$ either sends a message inducing y although conditional on his beliefs y' would yield him a higher expected payoff, or the other way around.

Proposition 4: After any message, the receiver plays a pure strategy, and if $y, y' \in R$ are two distinct actions that are induced in a PBNE, then the distance between them is weakly larger than $\underline{b}^{n-1}$.

The result implies that $\frac{1}{\underline{b}^{n-1}}+1$ serves as an upper bound on the number of distinct actions that can be induced in a PBNE. Note also that if player $n-1$ has a state-independent loss function and constant bias b_{n-1} (as assumed for players in most of the literature) then Proposition 4 implies that equilibrium actions have to be at least $|b_{n-1}|$ away from each other, since in this case $b^{n-1}(\mu) = b_{n-1}$ for every $\mu \in \Omega$.

The next result shows that just like in a direct communication game a la CS, in every PBNE of an indirect communication game, the state space is partitioned to a finite number of intervals such that at all states within the interior of an interval, the sender sends essentially the same message. Moreover, the distribution of actions induced by equilibrium messages of both the sender and the intermediators can be ranked with respect to first-order stochastic dominance. To get an intuition for this result, first note that given the strict concavity of the receiver's utility function, given any belief, he has a unique optimal action choice. Therefore, the distributions of actions induced by equilibrium messages of player $n-1$ can be trivially ordered with respect to first-order stochastic dominance. Then strict concavity of the utility function of player $n-1$ implies that in equilibrium he can mix between at most two messages, and that if he mixes between two different messages, then there cannot be a third message inducing an in-between action. This in turn implies that the distributions of actions induced by equilibrium messages of player $n-1$ can be ranked with respect to first-order stochastic dominance. By an iterative argument we show that this result extends to players $n-2$, ..., 1. Then the single-crossing property holding for player 1's utility function can be used to establish that the set of states from which player 1 sends a given equilibrium message form an interval.

Proposition 5: Every PBNE is outcome-equivalent to a PBNE in which Θ is partitioned into a finite number of intervals such that in the interior of any interval, player 1 takes a pure action. Moreover, the distribution of outcomes induced by different messages player $i \in \{1, ..., n-1\}$ sends in a PBNE can be ranked with respect to first-order stochastic dominance.

If there is a single intermediator in the game, then some additional features can be established for all PBNE. In particular, any PBNE is outcome-equivalent to one in which the intermediator mixes between at most two messages along any path of play. Moreover, PBNE are of a particular structure in that the state space is partitioned into components, where play within a component is connected through mixing by the intermediator. More precisely, components are the smallest events such that, along the equilibrium path, if the state is in the interior of a given component, then it is common knowledge among players that the state is in the closure of the component, while if the state is outside the closure of the given component, then it is common knowledge among players that the state is outside the interior of the component. The next subsection provides some examples of how strategic interaction can be within a component.

Proposition 6: If $n = 3$, then for every PBNE there is an outcome-equivalent PBNE, such that Θ can be partitioned into a finite number of intervals $B_1, ..., B_K$, referred below as components, such that for any component B_k the following hold: (i) The interior of B_k can be partitioned into a finite number of intervals $I_1^k, ..., I_{j_k}^k$ such that player 1 sends message $m_1^{j,k}$ with probability 1 at any $\theta \in int(I_j^k)$ and message $m_1^{j,k}$ is not sent from any state $\theta \notin cl(I_j^k)$; (ii) If the intermediator is positively biased, then for

$j \in \{1, ..., j_k - 1\}$ after message $m_1^{j,k}$ he mixes between messages $m_2^{j,k}$ and $m_2^{j+1,k}$, and after message $m_1^{j_k,k}$ he sends message $m_2^{j_k,k}$ with probability 1; (iii) If the intermediator is negatively biased, then for $j \in \{2, ..., j_k\}$ after message $m_1^{j,k}$ he mixes between messages $m_2^{j,k}$ and $m_2^{j-1,k}$, and after message $m_1^{1,k}$ he sends message $m_2^{1,k}$ with probability 1; (iv) The receiver chooses a different action after every message sent in equilibrium.

Proposition 6 implies that in a game with a single intermediator, the number of distinct messages (messages that induce different distributions of actions) sent in equilibrium, both by the sender and by the intermediator, is equal to the number of actions induced in equilibrium.

Another special feature of an indirect communication game with a single intermediator is that if the intermediator is biased in the same direction as the sender, and his bias is more moderate than the sender's, then all PBNE are essentially in pure strategies. The only caveat is that at a finite number of states, the sender might mix between two distinct messages. Hence, by the results obtained in the previous section, such an intermediator cannot improve the efficiency of communication. This point was also made by Ivanov (2009), in the uniform-quadratic specification of the model.[8]

Proposition 7: Suppose $n = 3$ and both the sender and the intermediator have positive (respectively, negative) bias. If the sender is more positively (respectively, negatively) biased than the intermediator, then every PBNE is outcome-equivalent to a PBNE in which players play pure actions at almost every state.

4.2 Examples of mixed equilibria

Below we investigate two types of single-component mixed PBNE with a single intermediator, in the frequently studied uniform-quadratic specification of the model: the one with two equilibrium actions, and the one with three equilibrium actions in the component. We show that both types of mixed equilibria exist for a full-dimensional set of biases, hence the existence of nondegenerate mixed equilibria is not a knife-edge case in the indirect communication game. The examples also demonstrate that, as opposed to pure-strategy PBNE, the outcomes that can be supported in mixed-strategy PBNE are not monotonic in the magnitude of the intermediator's bias.

Throughout this subsection, we assume that $n = 3$. Moreover, we assume that the state is distributed uniformly on $[0, 1]$, and the utility functions are

[8] See Proposition 1 on p13.

given by:

$$u^i(\theta, y) = -(\theta + b^i - y)^2$$

for every $i \in \{1, 2, 3\}$ with $b^3 = 0$. In words, players have fixed biases and their loss functions are quadratic.

Note that due to the quadratic utility, player 3 must play the conditional expectation of θ given the message she receives. That is,

$$y = E(\theta|m_2)..$$

4.2.1 2-action mixed equilibria

From Proposition 6, we know that there is an outcome-equivalent PBNE in which (i) player 1 sends message m_1^1 at any $\theta \in [0, x_1)$ and message m_1^2 at any $\theta \in (x_1, 1]$ (at state x_1, he can choose any mixture of m_1^1 and m_1^2); (ii) if the intermediator is negatively biased, then after receiving m_1^1, he sends m_2^1, and after receiving m_1^2, he mixes between m_2^1 and m_2^2; (iii) if the intermediator is positively biased, then after receiving m_1^1, he mixes between m_2^1 and m_2^2, and after receiving m_1^2, he sends m_2^2. Figure 1 depicts such a mixed PBNE for $b^1 = \frac{3}{10}$ and $b^2 = -\frac{2}{15}$.

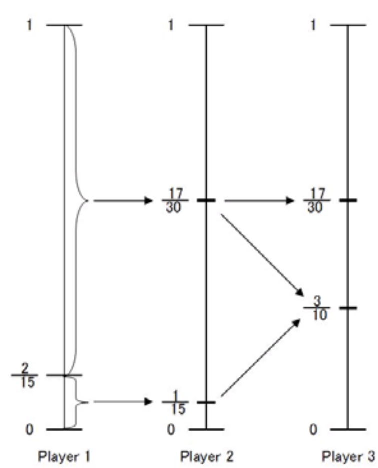

Figure 1

In Appendix A we characterize the region where such equilibrium exist and analytically compute equilibrium strategies. Figure 2 illustrates the range of parameter values for which a 2-action mixed equilibrium exists, for $b^1 > 0$. The horizontal axis represents the sender's bias, while the vertical axis represents the intermediator's bias. Notice that the region with $b^1 > 0$

and $b^2 > 0$ can be obtained by rotating the region with $b^1 < 0$ and $b^2 < 0$ in a point-symmetric manner with respect to the origin.

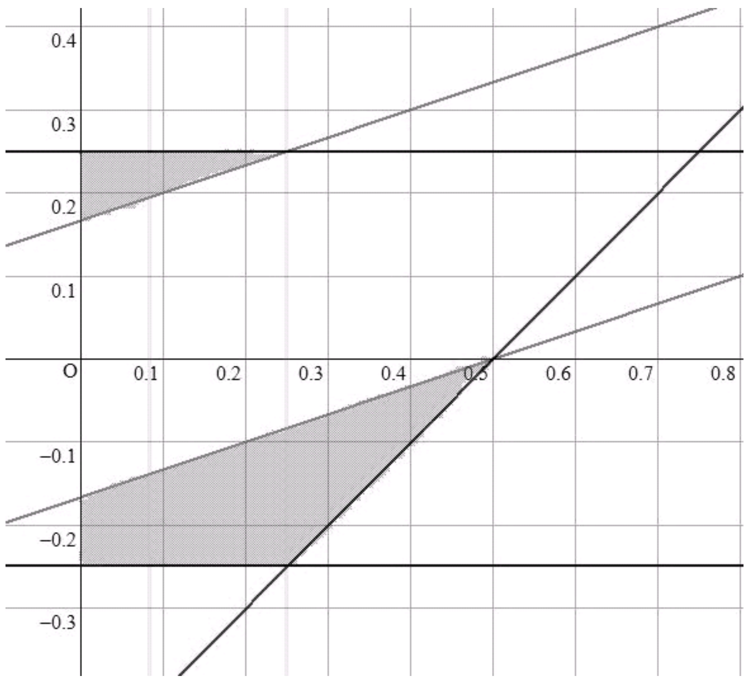

Figure 2

The upper triangular region depicts the cases when the sender and the intermediator are both positively biased and a 2-action mixed PBNE exists. Note that in all these cases the intermediator is more biased than the sender. The lower four-sided region represents the cases when the intermediator's bias is of the opposite sign of the sender and a 2-action mixed PBNE exists. Recall from CS that if $b^1 \in (0.25, 0.5)$, then the only PBNE in the game of direct communication is babbling, while for each such b^1 there is a range of b^2 (in the negative domain) such that there exists a 2-action mixed PBNE.

Notice that for any fixed b^1, if b^2 is small enough in absolute value, then there is no 2-action mixed PBNE. Hence, the set of mixed PBNE outcomes, as opposed to pure strategy ones, is not monotonic in the magnitude of the intermediator's bias.

4.2.2 3-action mixed equilibria

Here, we show that within a component, mixed strategies can have a more complicated structure than in a 2-action mixed equilibria above. In particular, there is a region of parameter values for which there is a single component mixed PBNE with three actions induced in equilibrium. Figure 3 illustrates such an equilibrium when the sender's bias is $1/48$, while the intermediator's bias is $1/16$.

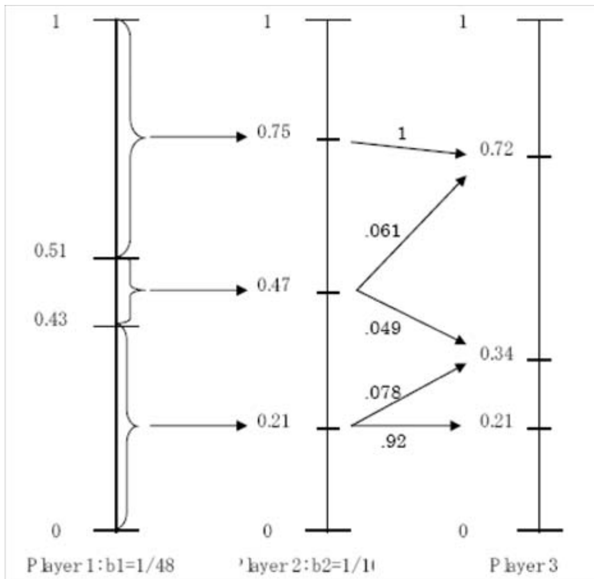

Figure 3

Suppose that $b^2 > 0$. Then Proposition 6 implies that any 3-action mixed PBNE is outcome-equivalent to a PBNE in which (i) if $\theta \in [0, x_1)$ then player 1 sends message m_1^1, if $\theta \in (x_1, x_2)$ then player 1 sends message m_1^2, while if $\theta \in (x_2, 1]$ then player 1 sends message m_1^3; (ii) after receiving m_1^1, player 2 mixes between m_2^1 and m_2^2, after receiving m_1^2, player 2 mixes between m_2^2 and m_2^3, and after receiving m_1^3, player 2 sends m_2^3.

Like 2-action mixed PBNE, 3-action mixed PBNE are also unique for any given pair of biases. But they are much more complicated to solve for than 2-cell equilibria. The reason is that one of the equilibrium conditions requires player 1 to be indifferent at state x_1 between two nontrivial lotteries. Still, we are able to characterize 3-action mixed PBNE. In Appendix A we derive a closed form solution for all the variables of interest describing a 3-action mixed equilibrium, as a function of x_1. The value x_1 is the solution of a complicated cubic equation. The complexity of analytically solving for this type of equilibrium, together with the result below that such equilibrium does exist for a full-dimensional set of parameter values, suggests that a sharp characterization of all mixed PBNE outcomes might be infeasible.

Figure 4 illustrates the regions of parameter values for which this type of equilibrium exists, for $b^1 > 0$.

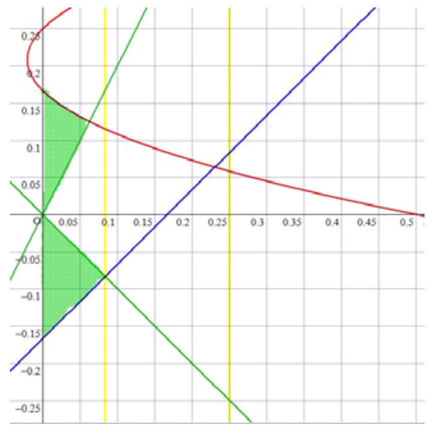

Figure 4

As opposed to 2-action mixed equilibria, 3-action mixed equilibria can exist when both b^1 and b^2 are close to 0. Hence, the existence of mixed PBNE with k number of actions for a given pair of biases is not monotonic in k.

4.3 The order of intermediators in mixed equilibrium

In the previous section, we showed that the order of intermediators does not matter in pure strategy PBNE. We conclude this section with an example showing that the order of intermediators does matter with respect to mixed strategy PBNE.

Consider 4 players, i, j, k, and h, with biases $b_i = \frac{3}{10}$, $b_j = \frac{3}{10}$, $b_k = -\frac{2}{15}$, and $b_h = 0$, respectively. We consider two indirect communication games: In game A, $i = 1$, $j = 2$, $k = 3$, and $h = 4$. In game B, $i = 1$, $j = 3$, $k = 2$, and $h = 4$. Notice that in both cases i is the sender and h is the receiver. The only difference is the order of the intermediators, j and k.

In game A, it is straightforward to see that the outcome that puts probability 1 on $\frac{3}{10}$ conditional on states $[0, \frac{2}{15})$, probability $\frac{7}{52}$ on $\frac{3}{10}$ conditional on states $[\frac{2}{15}, 1]$, and probability $\frac{45}{52}$ on $\frac{17}{30}$ conditional on states $[\frac{2}{15}, 1]$, is a PBNE outcome. To see this, notice that i and j has exactly the same bias. Thus, conditional on any event, if i were j, he would like to induce whatever probability distribution on the state that j wants to induce for player k. Hence, i fully revealing the state to j is always compatible with PBNE, and analyzing such equilibria is equivalent to analyzing a 3-player indirect communication game, where j is the sender, k is the intermediator, and h is the receiver. In the previous subsection, we showed that in this game it is a PBNE that player j partitions the states into two components, $[0, \frac{2}{15})$, in which case k advises h to play $\frac{3}{10}$ with probability 1, and $[\frac{2}{15}, 1]$, in which case k advises h to play $\frac{3}{10}$ with probability $\frac{7}{52}$ and to play $\frac{17}{30}$ with probability $\frac{45}{52}$.

Now we show that this outcome cannot be achieved in any PBNE of game B. Suppose the contrary, i.e. that this outcome is generated by some PBNE in game B. Since $\left(\frac{3}{10} + \frac{17}{30}\right)/2 - \frac{3}{10} = \frac{2}{15}$, j advises h to play $\frac{3}{10}$ with probability one if the expectation of the state is strictly less than $\frac{2}{15}$, and j advises h to play $\frac{17}{30}$ with probability one if the expectation of the state is strictly more than $\frac{2}{15}$. If the expectation of the state is $\frac{2}{15}$, j is indifferent between two choices. Now, suppose wlog (by Proposition 6) that in the PBNE player i does not randomize, and that player j has the expectation of $\frac{2}{15}$ conditional on some player i's message, m_i', that is sent with positive probability in the PBNE. This implies that m_i' is sent by a positive measure of types on $[0, \frac{2}{15})$. Since action $\frac{3}{10}$ is taken with probability one conditional on the state lying in $[0, \frac{2}{15})$ in the PBNE in consideration, this implies that player j induces action $\frac{3}{10}$ with probability one conditional on message m_i'. Thus, j cannot randomize in the PBNE of game B with positive probability. But then the outcome of the PBNE in game B is no longer identical to that of the original PBNE in game A, leading to a contradiction.

4.4 When can an intermediator facilitate information transmission?

In the previous subsections, we presented examples showing that there can be nontrivial information transmission in an indirect communication game, improving the ex ante welfare of the receiver, even if in the corresponding direct communication game, all equilibria involves babbling. A natural question to ask is when this is the case. The following result provides a simple sufficient condition for the existence of an intermediator being able to facilitate information transmission in equilibrium. We focus on the case when the sender is positively biased (the case of a negatively biased sender is perfectly symmetric).

Let $y_a^b = \arg\max \int_a^b u^3(\theta, y) f(\theta) d\theta$.

Proposition 8: Let $b^1(\theta) > 0$ for every $\theta \in \Theta$. If $u^1(a, a) < u^1(a, y_a^1)$ for each $a \in [0, 1)$ and $b^1(0) < y_0^1$, then all PBNE of the direct communication game involve babbling, while there exists an intermediator such that in the resulting indirect communication game, there is a PBNE in which the ex ante payoff of the receiver is higher than in a babbling PBNE.

It is easy to see that the condition $u^1(a, a) < u^1(a, y_a^1)$ for each $a \in [0, 1)$ is necessary and sufficient for the direct communication game not to have any informative equilibria. Condition $b^1(0) < y_0^1$ is an easy to check sufficient condition for the existence of an informative equilibrium in an indirect communication game, if the intermediator can be freely selected.

5 Discussion: Ex ante welfare

At the end of the previous section, we provided a sufficient condition for indirect communication to be able to improve the ex ante welfare of the receiver, in cases when direct communication cannot facilitate information transmission. Ivanov (2008) establishes a much stronger welfare-improvement result for the uniform-quadratic specification of the model: he shows that whenever $b_1 \in (0, \frac{1}{2})$, that is whenever there is a mechanism that can improve the ex ante welfare of the parties, there is a strategic intermediator and a PBNE of the resulting indirect communication game which attains the same ex ante welfare as the optimal mechanism.[9] The ex ante welfare gains when using an intermediator can be quite large, as Ivanov points out.

For general preferences and prior distributions on the state, welfare comparisons between pure and mixed PBNE in indirect communication games is a hard problem. One difficulty is that unlike in the uniform-quadratic case, the interests of the sender and the receiver are not aligned anymore ex ante. Below we demonstrate this through an example of a mixed PBNE in which ex ante the sender is strictly worse off than in the babbling equilibrium, that is in the worst pure strategy PBNE.

Consider again the example in Figure 1, with the only modification that the sender's utility function is now $u^1(\theta, y) = -(\theta + \frac{3}{10} - y)^{2r}$, where $r \in \mathbb{Z}$. By symmetry, the very same construction of the strategy profile as in Figure 1 constitutes a PBNE. It can be shown that, whenever $r \geq 3$, the mixed PBNE gives a strictly lower payoff to the sender than in the babbling PBNE. This is because in high states, in the mixed PBNE, there is a significant chance that the induced action is very far from the ideal point of the sender, relative to the maximal distance between the sender's ideal point and the action induced in babbling equilibrium. If the sender is risk-averse enough, this makes his over all ex ante welfare worse than in the babbling equilibrium. Hence, in an indirect communication game, the sender can be worse off than in the worst PBNE of the corresponding direct communication game. By contrast, it is easy to see that the receiver can never be worse off in any PBNE than in a babbling equilibrium: one strategy that is always feasible for the sender is choosing the babbling action independently of the messages received.

[9] The benchmark that is the maximum welfare that can be attained through a mechanism was derived in the uniform-quadratic case by Goltsman et al. (2007).

6 Conclusion

Our analysis of intermediated communication yields simple implications for organizational design if one restricts attention to pure-strategy equilibria: intermediators cannot facilitate transmission of information that cannot be transmitted in equilibrium in direct communication between a sender and a receiver, but they can invalidate informative equilibria of direct communication. The information loss relative to direct communication is smaller the less intermediators are involved in the chain, and the less biased they are relative to the receiver. We also show that the order of intermediators does not matter for what information can be transmitted through the chain.

At the same time, our findings reveal that the implications are much more complex with respect to mixed strategy equilibria. Different types of nontrivial mixed equilibria exist for an open set of parameter values of the model, and the existence of a given type of equilibrium is nonmonotonic in the intermediators' biases. By introducing noise in the information transmission, intermediators in a mixed strategy equilibrium can improve information transmission relative to direct communication. This can provide a rationale for establishing hierarchical communication protocols in an organization, even if such protocols are not necessitated by capacity constraints. Our investigations in the uniform-quadratic specification of the model suggest that involving an intermediator can improve information transmission if the intermediator's bias (relative to the receiver) is more moderate than the sender's, and it is in the opposite direction.

7 Appendix A

7.1 Formal definition of Perfect Bayesian Nash Equilibrium

In order to define PBNE formally in our context, we need to introduce beliefs of different players at different histories. We define a collection of beliefs through a probability distribution β^k on the Borel-measurable subsets of $M_{k-1} \times \Omega$ for every $k \in \{2, ..., n\}$, as a collection of regular conditional distributions $\beta^k(m_{k-1})$ for every $m_{k-1} \in M_{k-1}$ and $k = \{2, ..., n\}$ that are consistent with the above probability distributions.

Definition: A strategy profile $(p^k())_{k=1,...,n}$ and a collection of beliefs $(\beta^k())_{k=2,...,n}$ constitute a PBNE if:

(i) [optimality of strategies given beliefs]

For every $\theta \in \Theta$ and $m_1 \in supp(p^1(\cdot|\theta))$, we have:

$$m_1 \in \arg\max_{m_1' \in M_1}$$

$$\int_{m_2 \in M_2} \cdots \int_{m_{n-1} \in M_{n-1}} \int_{y \in R} u^1(\theta, y) dp^n(y|m_{n-1}) dp^{n-1}(m_{n-1}|m_{n-2}) ... dp^2(m_2|m_1').$$

For every $k \in \{2, ..., n-1\}$, $m_{k-1} \in M_{k-1}$ and $m_k \in supp(p^k(\cdot|m_{k-1}))$, we have:

$$m_k \in \arg\max_{m_k' \in M_k}$$

$$\int_{\theta \in \Theta} E(u^k(\theta, y)|m_k') d\beta^k(\theta|m_{k-1})$$

where

$$E(u^k(\theta, y)|m_k') =$$

$$\int_{m_{k+1} \in M_{k+1}} \cdots \int_{m_{n-1} \in M_{n-1}} \int_{y \in R} u^k(\theta, y) dp^n(y|m_{n-1}) dp^{n-1}(m_{n-1}|m_{n-2}) ... dp^{k+1}(m_{k+1}|m_k').$$

And for every $m_{n-1} \in M_{n-1}$ and $y \in supp(p^n(\cdot|m_{n-1}))$, we have:

$$y \in \arg\max_{y' \in R} \int_{\theta \in \Theta} u^n(\theta, y') d\beta^n(\theta|m_{n-1}).$$

(ii) [consistency of beliefs with actions]

$\beta^k()$ constitutes a conditional distribution of the probability distribution on $\Theta \times M_{k-1}$ generated by strategies $p^1(), ..., p^{k-1}()$, for every $k \in \{2, ..., n\}$.

(iii) [consistency of beliefs across players]

For any $k \in \{2, ..., n-1\}$, if $m_k \in M_k$ is sent along some path of play consistent with $(p^k())_{k=1,...,n}$, then $\beta^{k+1}(m_k)$ is in $co(\{\beta^k(m_{k-1})|m_{k-1} \in \widehat{M}_{k-1}(m_k)\})$, where $\widehat{M}_{k-1}(m_k)$ is the set of messages m_{k-1} in M_{k-1} such that there is a path of play consistent with $(p^k())_{k=1,...,n}$ in which player $k-1$ sends message m_{k-1} and player k sends message m_k. Similarly, if $m_1 \in M_1$ is sent along some path of play consistent with $(p^k())_{k=1,...,n}$ then $\beta^2(m_1)$ is in $co(\{\beta^1(\theta)|\theta \in \widehat{\Theta}(m_1)\})$, where $\widehat{\Theta}(m_1)$ is the set of states at which player 1 sends m_1.

7.2 Complete characterization of 2-action and 3-action single-component equilibria in the uniform-quadratic case

To simplify notation, we will label each message such that

$$m_k^j = E(\theta|m_k = m_k^j).$$

With this notation we have that player 3's strategy is just $y = m_2$, while the set of messages sent by player 1 correspond to the midpoints of the partition cells of the given equilibrium.

Wlog assume that $b^2 < 0$ (the case of $b^2 > 0$ is perfectly symmetric).It is convenient to do an analysis with a fixed signed b^2, because the sign of b^2 determines that after which messages player 2 mixes in a given type of equilibrium.

7.2.1 2-action mixed equilibria

By Bayes' rule, m_1 only depends on x_1: we must have $m_1^1 = x_1/2$ and $m_1^2 = \frac{1+x_1}{2}$. Also by Bayes' rule, $m_2^2 = m_1^2 = \frac{1+x_1}{2}$. Because player 2 must be indifferent between her messages after receiving m_1^2, we must also have $m_2^1 = m_2^2 + 2b^2 = \frac{1+x_1}{2} + 2b^2$. Then, for player 1 to be indifferent between both messages in state x_1 we must have

$$x_1 = 1 + 2\Delta$$

where $\Delta = b^2 - b^1$.

So it must be that $-\frac{1}{2} \leq \Delta \leq 0$. Substituting this value of x_1 we can solve for the messages $m_1^1 = \Delta + \frac{1}{2}$, $m_2^1 = 1 + \Delta + 2b^2$, and $m_1^2 = m_2^2 = 1 + \Delta$. For the probability $p(m_2^1|m_1^2)$, which we denote simply by p, by Bayes' rule we have

$$p = \frac{1}{8} \cdot \frac{(1+4b^2)(1+2\Delta)}{b^2\Delta}$$

For this to be feasible, $p \geq 0$, so $-\frac{1}{4} \leq b^2$ (as $1 + 2\Delta = x_1$ has to be nonnegative). From $p \leq 1$ we get $\Delta \leq -2b^2 - \frac{1}{2}$. It is trivial to check that these conditions together with the condition $0 \leq x_1 \leq 1$ are also sufficient for equilibrium. In terms of b^1 and b^2 the constraints become $\max\{-\frac{1}{4}, b^1 - \frac{1}{2}\} \leq b^2 \leq \frac{1}{3}b^1 - \frac{1}{6}$.

7.2.2 3-action mixed equilibria

As in the case of 2-action mixed PBNE, the messages sent in equilibrium by player 1 are determined by x_1 and x_2: $m_1^j = (x_{j-1} + x_j)/2$ for every $j \in \{1, 2, 3\}$. By Bayes' rule $m_1^1 = m_2^1 = x_1/2$. Using player 2's indifferences between messages in which she mixes, we get that $m_2^2 = x_1/2 + 2b^2$, $m_2^3 = x_1/2 + x_2$.

Player 1's indifference, when the state is x_2, is equivalent to

$$x_2 = x_1 + 2\Delta.$$

Denote the probabilities $p(m_2^{j+1}|m_1^j)$ by p_j. From Bayes' rule applied to m_2^3, we get

$$p_2 = \frac{(1 - x_2)(1 - x_2 - x_1)}{x_2\Delta}.$$

And using Bayes' rule for m_2^2 we get

$$p_1 = \frac{1}{4}\frac{(x_2 - 4b^2)(2x_2 - 1)(1 - x_1)}{x_1 x_2 b^2}.$$

This defines the equilibrium in terms of x_1. Now, to actually calculate x_1, it is necessary to work with player 1's indifferences between two non trivial lotteries.

Assuming that $0 \leq x_1 \leq x_2$ we must have that

- $p_2 \geq 0$ iff $(x_1 + x_2)/2 = x_1 + \Delta \leq 1/2$.
- $p_2 \leq 1$ iff $x_2 = x_1 + 2\Delta \geq 1/2$.

Notice that (assuming $x_2 \geq 1/2$, which follows from $p_2 \leq 1$)

- $p_1 \geq 0$ iff $x_2 \geq 4b_2$, or $b_2 \leq x_2/4 = x_1/4 + \Delta/2$.
- $p_1 \leq 1$ iff

  $$2\,x_1^3 + (-3 + 4\,b_2 - 8\,b_1)\,x_1^2 + \left(2\,b_2 + 1 + 10\,b_1 - 8\,b_1\,b_2 + 8\,b_1^2\right)x_1 + 8\,b_2^2 - 2\,b_2 - 2\,b_1 - 8\,b_1^2 \geq 0$$

  this sounds complicated for now, but we will simplify it below.

The final equation we need is P1s indifference constraint when her type is x_1. This reduces to

$$6\,x_1^3 + (12\,b_2 - 9 - 24\,b_1)\,x_1^2 + \left(-24\,b_1\,b_2 + 3 + 18\,b_1 - 6\,b_2 + 24\,b_1^2\right)x_1 + 8\,b_2^2 - 8\,b_1^2 - 2\,b_2 - 2\,b_1 = 0 \tag{3}$$

Unfortunately, the closed form Cardano solution of this equation is very complicated and not very helpful. But we may use it to simplify the condition that $p_1 \leq 1$ to

$$(3x_1 + 4\Delta - 1)\Sigma \geq 0,$$

where $\Sigma = b_1 + b_2$. Assuming $p_2 \leq 1$, this reduces to

- $p_2 \leq 1$ iff $\Sigma \geq 0$.

Summing up, there is a solution iff we can find x_1 solving equation (3) with:

- $1/2 - 2\Delta \leq x_1 \leq 1/2 - \Delta$ from $0 \leq p_2 \leq 1$.
- $0 \leq 2\Sigma \leq x_1$ from from $p_1 \leq 1$ and $p_1 \geq 0$.

Note that these imply $0 \leq x_1 \leq x_1 + 2\Delta = x_2 \leq 1$.

Now we have to consider two cases.

Case 1: $b_2 \le 1/8$

In this case, the binding constraints are:

- $1/2 - 2\Delta \le x_1 \le 1/2 - \Delta$ from $0 \le p_2 \le 1$.
- $0 \le \Sigma$ from from $p_1 \le 1$.

Notice that we must have $\Delta \le 1/4$.
Let $f(x_1)$ be the function defined by the left-hand side of (3). We have

$$f(1/2 - 2\Delta) = (1 - 4\Delta)(2\Delta - b_2)$$

$$f'(1/2 - 2\Delta) = (6 + 24b_2)\Delta - 3/2 \le 0$$

and

$$f(1/2 - \Delta) = -\frac{1}{2}(1 + 2\Delta)(1 - 6\Delta)\Sigma$$

$$f'(1/2 - \Delta) = -\frac{3}{2}(1 - 2\Delta)^2 \le 0.$$

First notice that the leading coefficient of the cubic $f(x)$ is positive. Because f' is negative in both endpoints of the interval, there is no solution in the interval if f has the same sign in the extremes. So there are two possible cases:
(i) $2\Delta - b_2 \le 0$ and $1 - 6\Delta \le 0$. This leads to a contradiction, as $1/6 \le \Delta \le b_2/2 \le 1/16$.
(ii) The other case is $2\Delta - b_2 \ge 0$ and $1 - 6\Delta \ge 0$. There are equilibria in this case iff:

- $0 \le \Delta \le 1/6$
- $b_2 \le 1/8$
- $\Sigma \ge 0$
- $2\Delta \ge b_2$ or $b_1 \le b_2/2$ (notice that this implies $0 \le \Delta$).

Case 2: $b_2 \geq 1/8$

In this case, the binding constraints are:

- $0 \leq 2\Sigma \leq x_1 \leq 1/2 - \Delta$.

Note that the binding ones are $b_2 \geq 1/8$, $b_1 + b_2 \geq 0$ and $3b_2 + b_1 \leq 1/2$. And this implies that $b_2 \leq 1/4$ and $-1/4 \leq b_1 \leq 1/8$. Also, $\Delta \leq 1/2$ and $\Sigma \leq 1/4$. We have

$$f(2(b_1 + b_2)) = 4(1 + 24b_2^2 - 10b_2 - 2b_1)\Sigma$$

$$f'(2\Sigma) = 3(1 - 4b_2)(1 - 10b_2 - 6b_1) < 0.$$

If $\Delta \leq 1/6$, then the right endpoint is negative, from the previous subsection. So there will be solutions in the area that satisfies the additional constraints

- $\Delta \leq 1/6$
- $1 + 24b_2^2 - 10b_2 - 2b_1 \geq 0$

If $\Delta \geq 1/6$, f is positive on the right endpoint. It can be shown that the left endpoint has positive f, so that there can be no solutions.

To summarize the above findings, there are two regions where the equilibrium under consideration exists. These are:

- First region:

1. $b^2 \leq b^1 + 1/6$.
2. $b^2 \leq 1/8$.
3. $b^2 \geq -b^1$.
4. $b^2 \geq 2b^1$.

- Second region:

1. $\Delta \leq 1/6$.
2. $1 + 24(b^2)^2 - 10b^2 - 2b^1 \geq 0$.
3. $b^2 \geq 1/8$.

These regions are depicted in Figure 4.

7.3 Appendix B: Proofs

Proof of Proposition 1:

Fix a pure strategy PBNE of the indirect communication game. Let $m_1(\theta)$ be the message choice of player 1 in this equilibrium at state θ, and let $\overline{M_1} = \{m_1 \in M_1 | \exists\ \theta \in \Theta \text{ st } m_1(\theta) = m_1\}$. Similarly, let $\overline{M_{n-1}}$ be the set of messages sent along the equilibrium path by player $n-1$. Since the PBNE at hand is in pure strategies, there is a partition $\{\overline{M_1}^{m_{n-1}}\}_{m_{n-1}\in\overline{M_{n-1}}}$ of $\overline{M_1}$ such that whenever player 1 sends message $m_1 \in \overline{M_1}^{m_{n-1}}$ in stage 1, player $n-1$ sends message m_{n-1} in stage $n-1$. Let $\Theta(\overline{M_1}^{m_{n-1}})$ be the set of states at which player 1 sends a message from $\overline{M_1}^{m_{n-1}}$.

Construct now the following strategy profile in the direct communication game: choose exactly one $m_1 \in \overline{M_1}^{m_{n-1}}$ for every $m_{n-1} \in \overline{M_{n-1}}$, and let the sender send message m_1 at every $\theta \in \Theta(\overline{M_1}^{m_{n-1}})$. Furthermore, let the action choice of the receiver after $m_1 \in \overline{M_1}^{m_{n-1}}$ be the same as her action choice after m_{n-1} in the PBNE of the indirect communication game, for every $m_{n-1} \in \overline{M_{n-1}}$. After any other message $m_1 \in M_1$ (which are not used along the induced play path), assume that the receiver chooses one of the actions along the above-defined play path.

To show that this is a PBNE, first we point out that given the receiver's strategy, the sender does not have a profitable deviation at any state. This is because in the given profile, at any state, the sender can induce the same action choices as she can in the PBNE of the indirect communication game. Second, the receiver gets equilibrium message $m_1 \in \overline{M_1}^{m_{n-1}}$ in the above direct communication profile at exactly the same states as she receives $m_{n-1} \in \overline{M_{n-1}}$ in the PBNE of the indirect communication game. Hence, after any message sent along the induced play path, the action prescribed for the receiver is sequentially rational, given the updated belief of the receiver regarding the state after receiving m_1. ■

Proof of Proposition 2:

("If" part)

Supposing that (1) holds, we construct a PBNE of an indirect communication game that is outcome-equivalent to the original equilibrium in the direct communication game: For each y and k, choose exactly one message from M_k, $m_k(y)$, so that $m_k(y) \neq m_k(y')$ if $y \neq y'$, where $y, y' \in Y$. Let player 1 send message $m_1(y)$ conditional on $\Theta(y)$ and let player $k \in \{2, \dots, n-1\}$ send message $m_k(y)$ conditional on player $k-1$'s message $m_{k-1}(y)$. In the off-path event that player $k-1$ sends a message not in $\bigcup_{y\in Y}\{m_{k-1}(y)\}$, let player k send an arbitrary message in $\bigcup_{y\in Y}\{m_k(y)\}$. Finally, let player n take an action y conditional on player $n-1$'s message $m_{n-1}(y)$. Again, in the off-path event that player $n-1$ sends a message not in $\bigcup_{y\in Y}\{m_{n-1}(y)\}$, let player n take an arbitrary action in Y.

Note that players 1 and n don't have an incentive to deviate because the constructed strategy profile specifies the same correspondence of messages/actions to states as in the original PBNE of the direct communication game. Moreover, condition (1) implies that players $k \in \{2,\ldots,n-1\}$ don't have an incentive to deviate, given the beliefs induced by the strategy profile described above. This concludes that the strategy profile constitutes a PBNE.

("Only if" part)

Suppose $\int_{\theta\in\Theta(y)} u^k(y,\theta)f(\theta)d\theta < \int_{\theta\in\Theta(y)} u^k(y',\theta)f(\theta)d\theta$ for some $y,y' \in Y$ and $k \in \{2,...,n-1\}$, and that there exists a PBNE of the indirect communication game that is outcome-equivalent to the given PBNE of the direct communication game. Let $\overline{M_{k-1}}(y)$ be the set of messages of player $k-1$ along the equilibrium path that induce player k to send a message from M_k that eventually induces y. Let $\Theta(m_{k-1})$ be the set of states at which message m_{k-1} is sent by player $k-1$, for every $m_{k-1} \in \overline{M_{k-1}}(y)$. By optimality of strategies given beliefs in PBNE (see Appendix A for the formal definition of PBNE),

$$\int_{\theta\in\Theta} u^k(y,\theta)\beta^k(m_{k-1})(\theta)d\theta \geq \int_{\theta\in\Theta} u^k(y',\theta)\beta^k(m_{k-1})(\theta)d\theta. \qquad (4)$$

By consistency of beliefs with actions in PBNE, $\beta^k()$ constitutes a conditional distribution of the probability distribution on $\Theta \times M_{k-1}$ generated by the PBNE strategies, which together with (4) implies

$$\int_{\theta\in\Theta(y)} u^k(y,\theta)f(\theta)d\theta \geq \int_{\theta\in\Theta(y)} u^k(y',\theta)f(\theta)d\theta,$$

contradicting the starting assumption. ■

Proof of Proposition 3: We will provide a proof for the case of positive biases. The case of negative biases is perfectly symmetric.

Let G^u and G^v stand for the games where the utility function of player k is u^k and v^k, respectively. Let s^* constitute a PBNE G^v. Then Proposition 1 implies that $\Theta(y)$ is an interval (possibly degenerate) for every $y \in Y$, where Y is the set of actions induced by s^*. Proposition 2 implies that there is an outcome-equivalent PBNE to s^* in G^u iff

$$\int\limits_{\theta\in\Theta(y)} u^k(y,\theta)f(\theta)d\theta \geq \int\limits_{\theta\in\Theta(y)} u^k(y',\theta)f(\theta)d\theta \tag{5}$$

for every $y, y' \in Y$ (recall our convention for the above inequality if $\Theta(y)$ is a singleton). Also by Proposition 2, since s^* constitutes a PBNE G^v, we have:

$$\int\limits_{\theta\in\Theta(y)} v^k(y,\theta)f(\theta)d\theta \geq \int\limits_{\theta\in\Theta(y)} v^k(y',\theta)f(\theta)d\theta \tag{6}$$

for every $y, y' \in Y$.

Fix now $y, y' \in Y$. Since u^k implies positive bias, (5) holds trivially if $y' < y$. Suppose now that $y' > y$.

Since v^k is more positively biased than u^k, condition (2), together with $f(\theta) > 0$ for every $\theta \in \Theta$, imply:

$$\frac{\partial \int_{\theta\in\Theta'} v^k(\theta,\widehat{y})f(\theta)d\theta}{\partial \widehat{y}} > \frac{\partial \int_{\theta\in\Theta'} u^k(\theta,\widehat{y})f(\theta)d\theta}{\partial \widehat{y}} \tag{7}$$

for all $\widehat{y} \in [y, y']$. This implies $\int\limits_{\theta\in\Theta(y)} v^k(y,\theta)f(\theta)d\theta - \int\limits_{\theta\in\Theta(y)} v^k(y',\theta)f(\theta)d\theta$ is strictly larger than $\int\limits_{\theta\in\Theta(y)} u^k(y,\theta)f(\theta)d\theta - \int\limits_{\theta\in\Theta(y)} u^k(y',\theta)f(\theta)d\theta$. Then (6) implies (5). ■

Proof of Claim 1: Wlog assume that player k has a positive bias. Then by assumption $b^k(\mu) > 0$ for every $\mu \in \Omega$, that is $\arg\max\limits_{y\in R} \int\limits_{\theta\in\Theta} u^k(\theta,y)d\mu > \arg\max\limits_{y\in R} \int\limits_{\theta\in\Theta} u^n(\theta,y)d\mu$ for every $\mu \in \Omega$. Since u^k and u^n are continuous in y and θ and, $\int\limits_{\theta\in\Theta} u^k(\theta,y)d\mu$ and $\int\limits_{\theta\in\Theta} u^n(\theta,y)d\mu$ are continuous in y and in μ (the latter wrt the weak topology). Then $\arg\max\limits_{y\in R} \int\limits_{\theta\in\Theta} u^k(\theta,y)d\mu - \arg\max\limits_{y\in R} \int\limits_{\theta\in\Theta} u^n(\theta,y)d\mu$ is continuous in μ. Moreover, Ω is compact, hence there are $\underline{\mu} \in \Omega$ and $\underline{b}^k > 0$ such that

$$\arg\max_{y\in R} \int\limits_{\theta\in\Theta} u^k(\theta,y)d\mu - \arg\max_{y\in R} \int\limits_{\theta\in\Theta} u^n(\theta,y)d\mu \geq$$
$$\arg\max_{y\in R} \int\limits_{\theta\in\Theta} u^k(\theta,y)d\underline{\mu} - \arg\max_{y\in R} \int\limits_{\theta\in\Theta} u^n(\theta,y)d\underline{\mu} = \underline{b}^k.$$

■

Proof of Proposition 4: In PBNE, after any message $m_{n-1} \in M_{n-1}$, the receiver plays a best response to belief $\beta^n(m_{n-1})$. Since the receiver's payoff is strictly concave in y and takes its maximum in $[0,1]$ for every $\theta \in \Theta$, the expected payoff is strictly concave in y and takes its maximum in $[0,1]$ for any belief. Hence, there is a unique best response action for the receiver for $\beta^n(m_{n-1})$.

Fix now a PBNE, let $m_{n-1} \in M_{n-1}$ be a message sent in equilibrium, and let $y(m_{n-1})$ be the action chosen by the receiver after receiving message m_{n-1}. Wlog assume that player $n-1$ has a positive bias (the negative bias case is perfectly symmetric). Note that by our definition of PBNE, $\beta^n(m_{n-1})$ is a convex combination of beliefs $\beta^{n-1}(m_{n-2})$ for which $m_{n-1} \in supp(p^{n-1}(m_{n-2}))$. It cannot be that for every such belief $\beta^{n-1}(m_{n-2})$,

$$\arg\max_{y \in R} \int_{\theta \in \Theta} u^n(\theta, y) d\beta^{n-1}(m_{n-2}) < y(m_{n-1}),$$

since this would imply that $\arg\max_{y \in R} \int_{\theta \in \Theta} u^n(\theta, y) d\beta^n(m_{n-1}) < y(m_{n-1})$, contradicting that $y(m_{n-1})$ is an optimal choice for player n after receiving m_{n-1}. Therefore, there is $m_{n-2} \in M_{n-2}$ such that $\arg\max_{y \in R} \int_{\theta \in \Theta} u^n(\theta, y) d\beta^{n-1}(m_{n-2}) \geq y(m_{n-1})$. By Lemma 1,

$$\arg\max_{y \in R} \int_{\theta \in \Theta} u^{n-1}(\theta, y) d\beta^{n-1}(m_{n-2}) \geq$$
$$\arg\max_{y \in R} \int_{\theta \in \Theta} u^n(\theta, y) d\beta^{n-1}(m_{n-2}) + \underline{b}^{n-1} \geq y(m_{n-1}) + \underline{b}^{n-1}.$$

Thus, given belief $\beta^{n-1}(m_{n-2})$, player $n-1$ strictly prefers inducing any action from $(y(m_{n-1}), y(m_{n-1}) + \underline{b}^{n-1})$ to inducing $y(m_{n-1})$. Therefore, there cannot be any message $m'_{n-1} \in M_{n-1}$ that induces an action from $(y(m_{n-1}), y(m_{n-1}) + \underline{b}^{n-1})$, since this would contradict the optimality of m_{n-1} given m_{n-2}. This implies that the distance between any two equilibrium actions has to be at least $\underline{b}^{n-1}$. ■

Proof of Proposition 5: Fix a PBNE, and consider an outcome-equivalent PBNE in which if two messages $m_i, m'_i \in M_i$ used in equilibrium induce the same probability distribution over actions, then $m_i = m'_i$, for every $i \in \{1, ..., n-1\}$.

For ease of exposition, if $m_i \in M_i$ first-order stochastically dominates $m_i' \in M_i$ for some $i \in \{1, ..., n-1\}$, then we will simply say that m_i is higher than m_i'.

Proposition 4 implies that every m_{n-1} induces a pure action by the receiver. Since different equilibrium messages induce different actions, it trivially holds that the distribution of outcomes that different messages that player $n-1$ sends in PBNE can be ranked with respect to first-order stochastic dominance. Moreover, since $u_{n-1}(\theta, y)$ is strictly concave in y for every $\theta \in \Theta$, there can be at most two optimal messages for player $n-1$, and in this case they have to induce actions such that there is no other equilibrium action in between them (otherwise inducing the latter action would be strictly better than inducing the originally considered actions). This establishes that the distribution of actions induced by the equilibrium messages of player $n-2$ can be ranked with respect to first-order stochastic dominance: they can be either degenerate distributions corresponding to one of the finite number of actions induced in equilibrium (which in turn corresponds to one of the equilibrium messages of player $n-1$), or mixtures between two neighboring equilibrium actions (which correspond to mixtures between two equilibrium messages of player $n-1$). Hence, we can partition the equilibrium messages of player $n-2$ into a finite number of sets $S_1^{n-2}, ..., S_{k_{n-2}}^{n-2}$ such that each set consists of messages inducing a distribution of actions with the same support, and the distribution of actions induced by messages in a set with a higher index first-order stochastically dominates the distribution of actions induced by messages in a set with a lower index. Moreover, the distribution of outcomes induced by messages within set S_j^{n-2}, for any $j \in \{1, ..., k_{n-2}\}$, can be ranked with respect to first-order stochastic dominance, too.

We will now make an inductive argument. Suppose that for some $l \in \{2, ..., n-2\}$, it holds that the equilibrium messages of every player $l' \in \{l, ..., n-2\}$ can be partitioned into a finite number of sets $S_1^{l'}, ..., S_{k_{l'}}^{l'}$ such that each set consists of messages inducing a distribution of actions with the same support, the distribution of actions induced by messages in a set with a higher index first-order stochastically dominates the distribution of actions induced by messages in a set with a lower index, and the distributions of outcomes induced by messages in each set can be ordered with respect to first-order stochastic dominance. Let $\overline{m}_j^{l'}$ and $\underline{m}_j^{l'}$ stand for the highest and lowest message from $S_j^{l'}$, whenever it exists.

Given that $\int_{\theta \in \Theta} u^l(\theta, y) d\beta$ is strictly concave in y for any belief β, at any history in which player l moves, the set of optimal messages for player l is either: (i) all elements of S_j^l for some $j \in \{1, ..., k_l\}$; (ii) $\overline{m}_j^l$ for some $j \in \{1, ..., k_l\}$; (iii) $\underline{m}_j^l$ for some $j \in \{1, ..., k_l\}$; or (iv) $\overline{m}_j^l$ and $\underline{m}_{j+1}^l$ for some

$j \in \{1, ..., k_l - 1\}$. Therefore, any equilibrium message of player $l-1$ can be partitioned into a finite number of sets $S_1^{l-1}, ..., S_{k_{l-1}}^{l-1}$ such that each set consists of messages inducing a distribution of actions with the same support, the distribution of actions induced by messages in a set with a higher index first-order stochastically dominates the distribution of actions induced by messages in a set with a lower index, and the distributions of outcomes induced by messages in each set can be ordered with respect to first-order stochastic dominance. By induction, for every $l' \in \{1, ..., n-2\}$, it holds that the equilibrium messages of player l' can be partitioned into a finite number of sets $S_1^{l'}, ..., S_{k_{l'}}^{l'}$ such that each set consists of messages inducing a distribution of messages of player $l'+1$ with the same support, the distribution of actions induced by messages in a set with a higher index first-order stochastically dominates the distribution of actions induced by messages in a set with a lower index, and the distributions of outcomes induced by messages in each set can be ordered with respect to first-order stochastic dominance.

Given the above result, the single-crossing condition imposed on u_1 implies that there is a finite set of states $\theta_1, ..., \theta_t$ such that $\theta, \theta' \in (\theta_j, \theta_{j+1})$ implies that there is a message that is uniquely optimal for player 1 at both θ and θ', for every $j \in \{1, ..., t-1\}$. Moreover, if there is a message sent in equilibrium by player 1 at θ_j that differs from the uniquely optimal message for player 1 at states in (θ_{j-1}, θ_j) and from the uniquely optimal message for player 1 at states in (θ_j, θ_{j+1}), then the above message is only sent in equilibrium at state θ_j. ■

Proof of Proposition 6: Given a PBNE, construct an outcome-equivalent PBNE as we did in the proof of Proposition 5. This allows us to have a partition of Θ, $P = (P_1, \ldots, P_J)$. Any partition of Θ, $B = (B_1, \ldots, B_K)$, which is coarser than this partition, $P = (P_1, \ldots, P_J)$, is consistent with part (i) of Proposition 6, with the use of messages as specified in the proof of Proposition 5. This construction, together with the fact that player 3 always has a unique optimal action, implies part (iv). Set $m_1 = j$ if $\theta \in I_j$, and $m_2 = y$ if $p^3(y|m_2) > 0$ (so y is given probability 1), without loss of generality.

Let partition B be such that each cell B_k is minimal with the property that, if $P_j \subseteq B_k$ and $p^2(y|j) > 0$ where y is a PBNE action taken by player 3, then $P_{j'} \subseteq B_k$ for all j' with $p^2(y|j') > 0$. Fixing k, consider B_k and the partition of B_k, $I^k = (I_1^k, \ldots, I_{j_k}^k)$, whose cells are also the cells of the original partition, $P = (P_1, \ldots, P_J)$. By the construction of B, part (v) is automatically satisfied. In the proof of Proposition 5, we have shown that player 2 can mix between at most two messages in the outcome-equivalent PBNE constructed there. This implies, together with the construction of B, that the number of messages that player 2 sends in the PBNE is equal to or larger by 1 than that of player 1. The case where the number of player 2's messages exceeds that of player 1 occurs only if player 2 mixes both conditional on m_1^1 and conditional on $m_1^{j_k}$. But this cannot be the case by (ii) and (iii), which we show below.

Now we prove part (iii). The proof for (ii) is perfectly symmetric, so we provide a proof only for part (iii). To prove the claim, renumber the cells in partition $I^k = (I_1^k, \dots, I_{j_k}^k)$ from left to right, i.e. if $\theta \in I_j^k$ and $\theta' \in I_{j'}^k$ with $\theta < \theta'$, then $j < j'$. Let player 1 send a message m_1^j conditional on the state lying in I_j^k. Also, let $y^1, \dots, y^{j_k}(, y^{j_k+1})$ be the PBNE actions induced by states in B_k (We already know that there are j_k (or $j_k + 1$) actions induced). Rename them so that $y^j < y^{j'}$ if $j < j'$, and set $m_2^j = y^j$.

Now suppose the contrary, i.e. that after message m_1^1 player 2 sends two messages, m_2^1 and m_2^2 with positive probabilities. (We know from the proof of Proposition 5 that player 2 mixes over at most two messages, and that there is no message induced in equilibrium between these two messages; finally, m_2^1 must have positive probability after m_1^1 by the construction of B; hence it suffices to rule out the case in consideration.) For this to be an equilibrium, conditional on m_1^1, player 2 has to be indifferent between m_2^1 and m_2^2:

$$\int_{I_1^k} u^2(\theta, m_2^1) f(\theta) d\theta = \int_{I_1^k} u^2(\theta, m_2^2) f(\theta) d\theta. \tag{8}$$

Next, note that player 3 is maximizing his payoff at m_2^1, which is induced only by m_1^1, so we have:

$$m_2^1 = \arg\max_y \int_{I_1^k} u^3(\theta, y) f(\theta) d\theta.$$

Since player 2 is negatively biased, this implies:

$$m_2^1 \geq \arg\max_y \int_{I_1^k} u^2(\theta, y) f(\theta) d\theta.$$

Hence, the first-order condition and the concavity of u^2 implies:

$$\int_{I_1^k} \frac{\partial u^2(\theta, m_2^1)}{\partial y} f(\theta) d\theta \leq 0.$$

Now, recall that u^2 is strictly concave. This implies:

$$\int_{I_1^k} \frac{\partial u^2(\theta, \bar{y})}{\partial y} f(\theta) d\theta < 0 \quad \text{for all} \quad \bar{y} \in (m_2^1, m_2^2],$$

which implies:

$$\int_{I_1^k} u^2(\theta, m_2^1) f(\theta) d\theta > \int_{I_1^k} u^2(\theta, m_2^2) f(\theta) d\theta.$$

This contradicts Equation (8), which completes the proof. ■

Proof of Proposition 7: Wlog assume that both u^1 and u^2 imply negative bias, and fix a PBNE. Suppose there is a component B_k (as defined in Proposition 6) in which player 2 mixes (Remember that this is the only case in which players mix in the PBNE, except player 1's possible mixing at a finite number of states (the boundaries of the cells) that has zero probability (hence never affects the outcome equivalence result)). Let $(I_1^k, \ldots, I_{j_k}^k)$ be the partition of the states within the component corresponding to distinct messages from player 1.

By Proposition 6, after receiving m_1^1, player 2 puts probability 1 on message m_2^1. Since $j_k \geq 2$, the following two claims are true: Conditional on the state lying in I_1^k, player 1 weakly prefers the action that player 3 plays after receiving m_2^1, call it y_1, to the action that player 3 plays after receiving m_2^2, call it y_2 (Recall from Proposition 4 that the receiver always plays a pure strategy). Also, he weakly prefers y_2 to y_1 conditional on I_2^k.

Thus, the continuity of u^1 implies that he is indifferent between y_1 and y_2 at the state $\bar{\theta} \in cl(I_1^k) \cap cl(I_2^k)$:

$$u^1(\bar{\theta}, y_1) = u^1(\bar{\theta}, y_2). \tag{9}$$

But since player 2 is mixing between m_2^1 and m_2^2, he is indifferent between y_1 and y_2 conditional on the state lying in I_2^k:

$$\int_{I_2^k} u^2(\theta, y_1) f(\theta) d\theta = \int_{I_2^k} u^2(\theta, y_2) f(\theta) d\theta. \tag{10}$$

By definition, $\bar{\theta} < \theta$ for all $\theta \in I_2^k \setminus \{\bar{\theta}\}$. Thus the single-crossing condition $\frac{\partial^2 u^2}{\partial \theta \partial y} > 0$ implies that for all $\theta \in I_2^k \setminus \{\bar{\theta}\}$,

$$u^2(\theta, y_2) - u^2(\bar{\theta}, y_2) > u^2(\theta, y_1) - u^2(\bar{\theta}, y_1).$$

This and Equation (10) imply:

$$\int_{I_2^k} u^2(\bar{\theta}, y_1) f(\theta) d\theta > \int_{I_2^k} u^2(\bar{\theta}, y_2) f(\theta) d\theta,$$

which implies:

$$u^2(\bar{\theta}, y_1) > u^2(\bar{\theta}, y_2). \tag{11}$$

The assumption that player 1 is more negatively biased than player 2 implies that there exist affine transformations of u^1 and u^2, u^{1*} and u^{2*} respectively, such that

$$\frac{\partial u^{1*}(\bar{\theta}, y)}{\partial y} < \frac{\partial u^{2*}(\bar{\theta}, y)}{\partial y} \tag{12}$$

for all y. Equation (9) implies:

$$\int_{y_1}^{y_2} \frac{\partial u^{1*}(\bar{\theta}, y)}{\partial y} dy = 0.$$

This and Equation (12) implies:

$$\int_{y_1}^{y_2} \frac{\partial u^{2*}(\bar{\theta}, y)}{\partial y} dy > 0,$$

which implies: $u^{2*}(\bar{\theta}, y_1) < u^{2*}(\bar{\theta}, y_2)$, hence $u^2(\bar{\theta}, y_1) < u^2(\bar{\theta}, y_2)$. But this contradicts condition (11). This concludes that there cannot be an equilibrium component with more than 1 induced actions, which completes the proof. ■

Proof of Proposition 8: First, note that, by definition, y_0^1 is the optimal choice of player 3 in the babbling PBNE. Next, note that the only pure strategy PBNE is babbling. To see this, suppose that there are exactly two actions induced in a pure PBNE. Then there exists $x \in (0, 1)$ such that these two actions are y_0^x and y_x^1, and $u^1(x, y_0^x) = u^1(x, y_x^1)$. Thus, strict concavity of u^1 and $x \in (y_0^x, y_x^1)$ (by $b^3(x) = 0$) implies $u^1(x, x) > u^1(x, y_x^1)$. But this contradicts our presumption that $u^1(a, a) < u^1(a, y_a^1)$ for each $a \in [0, 1]$. Thus, there exists no pure PBNE with two actions. But CS proved that if there exists a PBNE with N actions, there also exists a PBNE with n actions for $1 \leq n \leq N$. Proposition 1 then implies that there cannot exist pure PBNE with 2 or more actions.

Now we show that there exists a mixed PBNE with 2 actions. Consider the following strategy profile with parameter $\epsilon > 0$: Player 1 sends message $m_1 \in M_1$ if $\theta \in [0, \epsilon)$ and $m_1' \in M_1$ if $\theta \in [\epsilon, 1]$. If player 2 receives m_1, he sends message m_2, and if he receives m_1', he sends m_2 with probability p and m_2' with probability $1 - p$. Conditional on off-path messages $M_1 \setminus \{m_1, m_1'\}$, he sends a message from $\{m_2, m_2'\}$. Player 3 plays action y^* if he receives m_2 and y_ϵ^1 if he receives m_2'. Conditional on off-path messages $M_2 \setminus \{m_2, m_2'\}$, he plays an action from $\{y^*, y_\epsilon^1\}$.

It is easy to see that there exists an intermediator indifferent between inducing y^* or inducing y_ϵ^1, conditional on $\theta \in [\epsilon, 1]$. This would hold for example for an intermediator with quadratic loss function and state-independent bias $\frac{y^*+y^1_\epsilon}{2} - E(\theta|\theta \in [\epsilon, 1])$. Hence, below we only need to check whether the strategies of the sender and the receiver are compatible with PBNE.

We show that there exists $\bar{\epsilon}$ such that if $\epsilon < \bar{\epsilon}$, there exists $p \in (0,1)$ and y^* such that this is indeed a PBNE.

To see this, we need to establish the followings: If $\epsilon < \bar{\epsilon}$, (i) $u^1(\epsilon, y^*) = u^1(\epsilon, y^1_\epsilon)$, (ii) $y^\epsilon_0 < y^* < y^1_0$, and (iii) y^* is a best response conditional on players 1 and 2's strategies. (i) ensures that player 1 takes a best response to the opponents' strategies. (ii) ensures that $p \in (0,1)$. Player 3 takes a best response to the opponents' strategies conditional on m'_2, by definition of y^1_ϵ. Since given any on-path messages players put probability 0 on off-path events, and conditional on any off-path messages, players can have arbitrary beliefs that make their choices optimal, (i), (ii), and (iii) are enough to establish that the strategy profile constitutes a PBNE.

Note that if we have $u^1(\epsilon, y^\epsilon_0) < u^1(\epsilon, y^1_\epsilon) < u^1(\epsilon, y^1_0)$, then we have (i) and (ii), ignoring (iii). To see this, notice that this inequality ensures that there exists $y' \in (y^\epsilon_0, y^1_\epsilon)$ such that $u^1(\epsilon, y') = u^1(\epsilon, y^1_\epsilon)$. To see that $y' \notin [y^1_0, y^1_\epsilon)$, recall that we have assumed that $b^1(0) < y^1_0$, which implies that $\frac{\partial u^1(0, y^1_0)}{\partial y} < 0$. By continuity, there exists ϵ' such that for all $\epsilon < \epsilon'$, $\frac{\partial u^1(\epsilon, y^1_0)}{\partial y} < 0$. Hence y' cannot be contained in $[y^1_0, y^1_\epsilon)$ if $\epsilon < \epsilon'$.

Now we show that $u^1(\epsilon, y^\epsilon_0) < u^1(\epsilon, y^1_\epsilon) < u^1(\epsilon, y^1_0)$. Since $u^1(a,a) < u^1(a, y^1_a)$ for each $a \in [0,1]$, in particular we have $u^1(0,0) < u^1(0, y^1_0)$. By continuity, there exists ϵ'' such that for all $\epsilon < \epsilon''$, $u^1(\epsilon, y^\epsilon_0) < u^1(\epsilon, y^1_\epsilon)$. Also, we have shown that for all $\epsilon < \epsilon'$, $\frac{\partial u^1(\epsilon, y^1_0)}{\partial y} < 0$. Thus we have $u^1(\epsilon, y^1_\epsilon) < u^1(\epsilon, y^1_0)$ if $\epsilon < \epsilon'$.

Finally we prove (iii). Fix $\epsilon < \min\{\epsilon', \epsilon''\} := \bar{\epsilon}$. Then y^* is uniquely determined by condition (i). We prove that there exists p such that player 3 takes a best response at y^*. Let $\tilde{y}(p)$ be the best response when the mixing probability is p. Notice that $\tilde{y}(p)$ is continuous in p. Because of strict concavity, the best response is uniquely determined conditional on any event, hence $\tilde{y}$ is a function. Note that $\tilde{y}(0) = y^\epsilon_0$ and $\tilde{y}(1) = y^1_0$. This implies that there exists $p \in (0,1)$ such that $\tilde{y}(p) = y^*$, since we know that $y^\epsilon_0 < y^* < y^1_0$. Thus we have proved (iii).

Overall, we have found that there exists $\bar{\epsilon}$ such that if $\epsilon < \bar{\epsilon}$, there exists $p \in (0,1)$ and y^* such that this is indeed a PBNE.

Notice that player 3 has an option to play y^1_0, conditioning on any messages. Since strict concavity implies the uniqueness of the best response conditional on any event, this implies that, given the conditions in this proposition, the PBNE we constructed gives a strictly higher ex ante payoff to player 3 than in any pure strategy PBNE. This completes the proof. ■

8 References

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