Multiple Internet Auctions of Heterogenous Goods under Single-Unit Demand
–Extended Abstract–

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Introduction On platforms for online auctions usually single units of one type of good are auctioned off in several auctions at the same time. Interdependencies between these auctions should be considered and they should not be analyzed separately as in most of the existing literature on Internet auctions. We analyze a model with several auctions (English proxy auctions with soft close) that are offered by independent sellers and are run at the same time and find a Perfect Bayesian (epsilon-) equilibrium considering bidders' strategies. We assume that every bidder wants to buy exactly one unit of the good and allow for different valuations of a bidder for the goods in the different auctions. The reason might be differences in color, design, quality, brand etc. Thus, preferences are described as in the assignment game of Shapley and Shubik (1972).

In our model bidders submit bids sequentially and bids are price bids (in contrast to announcing demand to a given price). As a result the auctions considered may be run independently and over a longer time horizon. Auctions are English proxy auctions in the style of eBay or Amazon auctions. Bidders may bid several times in every auction. The second highest submitted bid determines the standing bid and, when no new bids arrive, the price. The highest submitted bid determines the high bidder and at the end winner and is hidden for the other bidders. All auctions end when no bidder wants to submit a new bid. Coordination on efficient assignments in the equilibrium is a result of the submitted price bids and does not have to be organized by a central seller. To achieve this coordination bidders may have to submit multiple bids, even in one auction. This kind of behavior is often observed in Internet auctions. The model is most closely related to that of Peters and Severinov (2004) where homogeneous goods are sold. Our model can be seen as a generalization of their model with respect to valuations.

Other models that analyze strategic bidding in this environment consider bids consisting of one price bid or several stages where bidders announce their demand at current standing bids. All the models with several bidding rounds have in common that a central auctioneer determines current high bidders or determines if there exists excess demand (or overdemanded sets of goods) and increases prices for some of the goods following certain rules (for example Demange, Gale, and Sotomayor (1986), Leonard (1983), Crawford and Knoer (1981), Kelso and Crawford (1982), Gul and Stacchetti (2000), Ausubel and Milgrom (2002)).[1] In our model we do not need such a central authority.

The Model We model the bidding game as a game Γ^b_{ext} in extensive form with imperfect and incomplete knowledge. The *finite set of players* $\mathcal{I} = 0 \cup \mathcal{N} \cup \mathcal{M}$ consists of the nature player 0,

[1] Most of those models are more general with respect to bidders valuations.

the set of n bidders $\mathcal{N} := \{B1, \dots, Bn\}$, and the set of m sellers (or, equivalently, m auctions) $\mathcal{M} := \{A1, \dots, Am\}$ who each offer one good for sale.

The vector of sellers' valuations is $v^S \in \{0, \dots, \bar{v}^S\}^m \subset \mathbb{N}^m$ with $\bar{v}^S \in \mathbb{N} < \infty$ denoting some large number. We assume that every seller sets his reserve price r_j equal to his valuation, i.e. $r_j := v_j^S$ for all j. Every bidder wants to buy exactly one unit of the offered goods, i.e. his valuation for a second good is always equal to zero. For every bidder and every auctioned good the matrix of buyers' private and independent valuations $V := (v_{ij})_{n \times m}$ with $v_{ij} \in \{0, \dots, \bar{v}\} \subset \mathbb{N}$ for all $i \in \mathcal{N}$, $j \in \mathcal{M}$, and $\bar{v}^S$, $\bar{v} \in \mathbb{N} < \infty$ are some large numbers, gives the valuation of the respective buyer for each good in case he buys this good only.

$u_j^S(\cdot)$, $u_i^B(\cdot)$: $X \times \mathbb{R}^m \to \mathbb{R}$ for $j \in \mathcal{M}$ and $i \in \mathcal{N}$ are sellers' and bidders' payoff functions. The functions are defined over *allocations* (x,p) where $x \in X \subset \{0,1\}^{n \times m}$ is a $n \times m$ matrix with $x_{ij} = 0$ if bidder i does not win auction j and $x_{ij} = 1$ if bidder i wins auction j and $p = (p_{A1}, \dots, p_{Am}) \in \mathbb{R}^m$ is a vector of prices. The functions $u_j^S(\cdot)$ and $u_i^B(\cdot)$ are given by linear von Neumann-Morgenstern utility functions $u_j^S(x,p) = p_j - v_j$ if $x_{ij} = 1$ for some i and $u_j^S(x,p) = v_j$ otherwise and $u_i^B(x,p) = \max_{j \in \mathcal{M}}\{v_{ij} \cdot x_{ij}\} - \sum_{j=1}^{m} p_j \cdot x_{ij}$.

$b^s \in [0, \bar{b}]^m \subset \mathbb{R}^m$ resp. $b^h \in (0, \bar{b}]^m \subset \mathbb{R}^m$ with $b_j^h \geq b_j^s$ for all j are the vector of standing bids resp. the vector of high bids. The possible bids are bounded above by some large number $\bar{b} \in \mathbb{R} < \infty$. The function B^h: $\mathcal{M} \to \mathcal{N} \cup \mathcal{M}$ assigns to every auction $j \in \mathcal{M}$ a bidder $i \in \mathcal{N}$ as current high bidder (i.e. $B^h(j) = i$ if i submitted the current high bid b_j^h) or, by convention $B^h(j) = j$ if no bidder has yet bid in auction j. The current standing bid b_j^s in an auction j is equal to $b_j^{(2)} := \max\{b_{kj}$: bid submitted hitherto in auction j by any bidder $k \neq B^h(j)\}$. Note that $b_j^{(2)} = b_j^h$ if two bidders submitted bids equal to b_j^h. In this case the bidder who first submitted the bid is current high bidder. If no bidder has yet bid in auction j then $b_j^s = r_j$.

The nature player selects randomly a bidder to submit a bid. Every time a bidder is selected to bid he can submit a bid in one auction of his choice or he can decide not to bid. A bidder i is free to bid an amount $b_{ij} \in (b_j^s, \bar{b}]$ in an auction j, where $b_j^s := r_j$ at the beginning. When he bids b_{ij} in j he becomes high bidder if his bid is above the current high bid. Otherwise his bid lies below b_j^h and his bid determines the new standing bid.

If a bidder decides not to bid and is not current high bidder in any auction then he will not be selected again to bid. Thus, this decision is de facto an exit decision. If every remaining bidder high bidder in at least one auction is selected once to bid and decides not to bid, i.e. all bidders are asked once to bid in the current situation without any change, then all the auctions end. The standing bids b_j^s at the end of the auctions are also referred to as prices p_j.

We assume *perfect recall.* A bidder i can always observe the vector of reserve prices r, the vector of standing bids b^s, the identities of the high bidders B^h, his own high bids b_j^h for all j with $B^h(j) = i$ and his own valuations v_{ij} for all j. He cannot distinguish between high bids $b_j^h \geq b_j^s$ for all j with $B^h(j) \neq i$. We consider only pure strategies $\sigma_i(\cdot)$ that assign an action of the set of available actions $C(H)$ to every information set $H \in \mathcal{H}_i$ of player i. Σ_i is the set of pure strategies of player i.

A relevant part of our equilibrium strategy σ_i^* of bidder i is the maximum possible payoff in an auction that bidder i can currently achieve. It is determined by the difference between his valuation for the good sold at an auction j and the lowest price at that he can win the auction: The *current maximum possible payoff* Δ_{ij} *for bidder* i in auction j is $\Delta_{ij} := v_{ij} - b_j^s$. In this definition we neglect increments. A bidder will have to overbid the current standing bid by at least ε to become high bidder. Remember, that every bidder can observe the current standing bid b_j^s and all submitted bids except for the current high bid. For the definition of strategy σ_i^* we need the current maximum possible payoff $\Delta_{i(1)}$ and the current second-highest possible payoff $\Delta_{i(2)}$ for bidder i at a stage of the bidding game:

Definition 1 *The* current maximum possible payoff $\Delta_{i(1)}$ for bidder i *is defined as* $\Delta_{i(1)} := \max_{j \in \mathcal{M}} \Delta_{ij}$ *and* $D_i := \{j : j \in \arg\max_{j \in \mathcal{M}} \Delta_{ij}\}$ *is i's demand set. A representative element of* D_i *is denoted by* $j_{i(1)}$*. Correspondingly, the* current second-highest possible payoff $\Delta_{i(2)}$ for bidder i *is defined as* $\Delta_{i(2)} := \max_{j \in \mathcal{M} \setminus D_i} \Delta_{ij}$.

Definition 2 (Strategy σ_i^*) *The* strategy σ_i^*: $\mathcal{H}_i \rightarrow \mathcal{C}(H)_{H \in \mathcal{H}_i}$ *of bidder* $i \in \mathcal{N}$ *specifies, that whenever bidder* i *is selected to bid (i.e. one of his information sets* $\mathcal{H}_i$ *is reached) he chooses the following action:*

1. *If* $B^h(j) = i$ *for any* $j \in \mathcal{M}$*, then bidder* i *does not submit any bid.*
2. *If* $B^h(j) \neq i$ *for all* $j \in \mathcal{M}$ *bidder* i *bids as follows:*
   (a) *If* $\Delta_{i(1)} < 0$*, then he does not bid.*
   (b) *If* $\Delta_{i(1)} \geq 0$ *and* $|D_i| = 1$*, then he bids in auction* $j_{i(1)} \in D_i$*. He determines his bid as* $b_{ij_{i(1)}} = v_{ij_{i(1)}} - \max\{\Delta_{i(2)}, 0\}$.
   (c) *If* $\Delta_{i(1)} \geq 0$ *and* $|D_i| > 1$*, then bidder* i *chooses randomly (with uniform probability) one of the auctions* $j_{i(1)} \in D_i$*. The selected auction is denoted by* $j'_{i(1)}$ *and his bid* $b_{j'_{i(1)}}$ *is determined as*[2] $b_{ij'_{i(1)}} = b^s_{j'} + \varepsilon$.

When a bidder i submits a bid he selects his bid $b_{ij_{i(1)}}$ in the auction with the maximum possible payoff such that the minimum payoff he might achieve by this bid is equal to the maximum payoff he might achieve in any other auction at the current stage. To specify a Perfect Bayesian equilibrium (PBE) of the bidding game Γ^b_{ext} it is necessary to describe the associated beliefs. The belief rules state that bidder i beliefs that the high bid is higher than or equal to the current standing bid and that no bidder bids above his valuation. A bidder i considers these belief rules to make Bayesian updating of the probability distributions over the nodes included in an information set. We will not need these beliefs in the proofs neglecting ε-differences in payoffs and without considering increments. With strategy σ^* and these belief rules I can now specify the PBE of the bidding game Γ^b_{ext}.

Theorem 1 *The symmetric bidding strategies* σ_i^* *and the belief rules of all bidders* $i \in \mathcal{N}$ *constitute a Perfect Bayesian (epsilon-) equilibrium of the bidding game* Γ^b_{ext}*.*[3]

We also show that all assignments x^* resulting from σ^* are efficient and that the prices in equilibrium are Vickrey prices. Following Shapley and Shubik (1972) and Roth and Sotomayor (1990) we find that equilibrium allocations are in the core and that they are the bidder optimal allocations in the core.

[2]To really have a chance to become high bidder bidder i has to increase the current standing bid by a minimum amount ε. A companion paper considers a model with increments in detail.

[3]The equilibrium is an epsilon-equilibrium as defined by Radner (1980), because payoffs lie in some ε-range around the payoffs considered here, where we neglect increments and payoff differences in the magnitude of ε.

Conclusion We find an PBE in a multiple independent auctions environment. The equilibrium strategy prescribes *multiple bidding* which is often observed in Internet auctions. Our model may explain why multiple bidding occurs. In our model it is an essential means that helps bidders coordinate on the bidder optimal core outcome.

In contrast to other related models the equilibrium strategy prescribes in many situations to increase the standing bid by more than the minimum amount (increment). This is a result of the heterogeneous valuations of a bidder and might also be an explanation for observations made in Internet auctions.

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