Approximate Implementability with Ex Post Budget Balance

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Abstract

This paper characterizes public and private monitoring technologies with respect to which the efficient outcome is approximately implementable in team production by way of ex post budget-balanced linear transfers.

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1 Introduction

It is well known that a team may fail to produce efficiently when its revenue is shared within the team. In a seminal paper, Holmström (1982) demonstrates this possibility when a defection of one member cannot be distinguished from another's. When it is not possible to identify a shirker (non-shirker, to be more precise), one may need to punish every member of the team for a poor team performance for the sake of incentive. Thus a conflict arises between incentive and budget-balance. One way to recover efficiency is to introduce a principal who does not contribute to production at all. The principal can help the members of the team achieve the efficient outcome by serving as a residual claimant. This is a very important insight because it suggests a novel interpretation of principal's role in team production: a budget-breaker.

However there are also other ways to achieve efficient production even if there is no principal in the usual sense. This paper provides a clean characterization of (public and private) monitoring technologies with respect to which the most efficient outcome can be *approximately* attained while preserving ex post budget-balance. We find a subtle notion of distinguishability which plays a crucial role in determining whether or not the efficient production is possible.

Monitoring activity is central in our analysis. As Alchian and Demsetz (1972) emphasize, monitoring is a basic ingredient of team production. One reason why a production process is not completely decentralized in the market economy is that there is substantial asymmetric and/or unverifiable information in the very process of production. Such information is generated by explicit or implicit monitoring activities of team members. This is exactly the source of advantage of team production over decentralized production. Working as a team generates useful information otherwise unavailable, which are then collected and used to discipline each member of the team. For this reason, we pay special attention to the case of private monitoring as well as the more conventional public monitoring setting.

Since we wish to emphasize monitoring activity, we are interested in *approximate efficiency* rather than *exact efficiency*. To motivate this position, consider the following simple team consisting of two players. Player 1 is a monitor, and can only monitor or shirk. She does not contribute to production directly. Player 2 is a worker who either works or shirks. Suppose also that no information regarding player 2's behavior is available when player 1 does not monitor. The efficient outcome is clearly (shirk,work) whenever monitoring entails some cost. However, it cannot be achieved by any contract. On the other hand, as we show later, we may obtain an approximately efficient outcome where player 1 monitors with arbitrarily small probability.

This example is also useful to convey another point of departure of this paper from the traditional approach. To achieve approximate efficiency, we use a particular type of contract, namely one that assigns transfers contingent on both realized signals and recommended actions. Suppose that the cost of monitoring is $c > 0$ and the cost of working is $d > 0$. Suppose also that there are two public signals $\{X, Y\}$. The probability of X is always $1/2$ when player 1 does not monitor. The probability of X is $3/4$ when player 1 monitors and player 2 works, and 1 when player 1 monitors and player 2 shirks. For this example, we can show that the efficient outcome cannot be approximated by any *standard contract* (signal-contingent compensation schemes) with ex post budget balance. With any standard contract, player 1 needs to be indifferent between monitoring and shirking to approximate efficiency (it can be shown that player 2's randomization does not help). This implies that $c = 1/4\,(w_1\,(X) - w_1\,(Y))$, where $w_1\,(\omega)$ is the transfer to player 1 when $\omega \in \{X, Y\}$ realizes. Ex post budget balance requires $c = 1/4\,(w_2\,(Y) - w_2\,(X))$. Since player 2's incentive constraint is $d \leq \Pr(\text{monitor}) \times 1/4\,(w_2\,(Y) - w_2\,(X))$, it follows that $\Pr(\text{monitor})$ cannot be smaller than d/c.

Nonetheless, there exist more general contracts that approximate efficiency with budget-balance. In order to implement such general contracts, suppose that the team has access to:

– a disinterested mediator or a machine that makes confidential, non-binding recommendations to each player over what to play, and

– transfer schemes that depend on both recommended actions and realized signals.

Then we can obtain an approximately efficient equilibrium, where player 1 is recommended to monitor with small probability. Player 1 is rewarded when X is observed only if monitoring was recommended (See Example 3.7). In this paper, we focus on this type of contracts.

With such contracts in hand, there are many ways to approximate the most efficient outcome. It is sometimes possible to choose a different team member as its contingent "budget-breaker" for a different recommended action profile. The standard case in which one player (= principal) always serves as a budget-breaker is a special case.

As is clear from the above description of contacts, the notion of equilibrium we employ is *correlated equilibrium*. In addition to the economic advantage discussed above, there is a well-known technical advantage of correlated equilibrium over Nash equilibrium. Correlated equilibrium is easier to handle than Nash equilibrium because it is characterized by a set of linear inequalities. Our case is no exception. With correlated equilibrium, the problem of approximate implementation of any team outcome reduces to a simple linear programming program, whose dual provides necessary and sufficient conditions on the team's monitoring technology for approximately implementation for any profile of utility functions.

Our necessary and sufficient conditions consist of two separate conditions: *Convex Independence*(CI) and *Non-Intersecting Cones*(NIC). The first condition is about *detectability*. It means that any individual's deviation from the recommended action can be detected from realized signals *and* realized action profile of the other members. Note that signals may not generate enough information for effective monitoring by themselves. They become more informative when combined with the realized action profile which contains information about the quality of the signals.

The second condition is about *distinguishability*. NOC makes it possible to identify a non-deviator for any pattern of outcomes generated by unilateral deviations. In another word, it guarantees us to avoid a situation where every individual's unilateral deviation (which can be contingent on recommended actions) looks similar. This problem can be alleviated by introducing a principal because it is always possible to distinguish one particular individual as a non-deviator: namely, principal. Our condition allows for such principal, but recognizes that less is required for budget balanced implementation. This weaker (indeed, weakest) condition allows for the possibility of a *random principal*. That is, the team's budget breaker(s) may be appointed at random, without even the appointees necessarily knowing about it.

We use duality throughout this paper. The primal and dual linear programs studied below provids two sides of the same coin, two different points of view of the same team problem. As we already mentioned, the dual of the original problem provides a necessary and sufficient condition for approximate implementation. In addition, we obtain dual representations of the above two conditions: CO and NOC, which allow more economically meaningful interpretations. The dual of convex independence corresponds to *Enforceability*. It means that convex independence is equivalent to the existence of compensation schemes to approximately implement any action profile for any profile of utility functions. On the other hand, NOC is closely related to budget-balance. The dual representation of NOC is called CABC(clear-all-budget-constraints). It means that any incentive compatible contract can replaced by an incentive compatible and budget-balancing contract. This clarifies separate roles each condition plays for approximate implementation with budget balance. CI guarantees that every action profile is approximately implementable without budget-balance and NOC turns any incentive compatible contract into a budget-balancing one.

1.1 Literature Review

Holmström (1982) is the seminal paper on team production. In the same deterministic setting, Legros and Matthews (1993) showed that approximate ineffficinecy can be obtained for a general class of environments even when exact efficiency is not obtained. Legros and Matsushima (1991) studied team problems with finite stochastic signals. They identified a necessary and sufficient condition for exact efficiency. Our paper is different from their paper in that we focus on approximate efficiency and allow both discrete and continuous signals.

Rahman (2005) identified *convex independence* as necessary and sufficient for approximate implementability without ex post budget balance. Obara (2003) showed that a similar, but stronger condition is necessary and sufficient for full surplus extraction for multiagent mechanism design problems where the distribution of the private types are determined endogenously by agents' actions. Neither paper did not consider budget-balancing constraint.

The study of correlated equilibrium with linear programming techniques originated with Nau and McCardle (1990), who used duality to demonstrate existence of correlated equilibrium. Myerson (1997) used their linear programming formulation to obtain a refinement of correlated equilibrium based on duality. Rahman (2005) applied duality to the theory of contracts by formulating a linear program that characterizes optimum contracts with expected budget balance, using correlated equilibrium as a solution concept. In that paper, This paper extends the techniques of Rahman (2005) by defining an ex post budget balanced linear programming problem that characterizes approximate implementability.

2 Model

Let $I = \{1, \ldots, n\}$ be a finite set of players (with $n > 2$), A_i the set of actions available to player $i \in I$, and $A = \prod_i A_i$ the space of action profiles. Actions are not verifiable, or not observable until the game is over. A *correlated strategy* is any probability measure $\sigma \in \Delta(A)$. The profile of individual utilities over action profiles is captured by the map $\mathbf{v} : I \times A \to \mathbb{R}$. We denote by $v_i(a)$ the utility to a player $i \in I$ from action profile $a \in A$.

The team's *monitoring technology* is given by a pair $(S, \Pr)$ where $S = S_1 \times ... \times S_n$ is a finite set and $\Pr : A \to \Delta(S)$ is a conditional probability system. S_i is the set of private signals for player i. *Public monitoring* is a special class of monitoring technologies where $S_1 =, ..., = S_n$ and $\Pr(s|a) > 0$ if and only if $s_1 =, ..., = s_n$.

The team has access to linear transfers $\zeta : I \times A \times S \to \mathbb{R} \cup \{\pm\infty\}$. This is interpreted as a contract which assigns payments contingent on individuals, recommended actions, and reported signals. This formulation assumes that *recommended* actions are verifiable. This is without loss of generality. Even if recommended actions are not directly verifiable, we can let the players announce their personal recommendation as a verifiable message. As we will see, there exists a budget-balancing mechanism, which is similar to a mechanism by Cremer and McLean (1988), which extracts recommendded actions with 0 cost for almost all correlated strategy. For simplicity, we just assume that recommended actions are verifiable.

3 Duality Approach to Approximate Implementability

3.1 Approximate Implementability

Let us begin by finding necessary and sufficient conditions for approximate implementability. We focus on *communication equilibria*, thereby ignoring sequential reporting constraints. Consider the following primal problem:

$$W(\mathbf{v}) \;:=\; \sup_{\sigma\geq 0,\xi} \sum_{(i,a)} \sigma(a) v_i(a) \quad \text{s.t.}$$

$$\sum_{a_{-i}} \sigma(a)[v_i(b_i,a_{-i}) - v_i(a)] + \sum_{(a_{-i},s)} \xi_i(a,s)\Pr(s|b_i,a_{-i},\rho_i) - \xi_i(a,s)\Pr(s|a) \leq 0$$

$$\sum_{i\in I} \xi_i(a,s) = 0$$

$$\sum_{a\in A} \sigma(a) = 1$$

where $\rho_i : S_i \to S_i$ is player i's reporting strategy and

$$\Pr(s|a,\rho_i) \;:=\; \sum_{\widehat{s}_i \in \rho_i^{-1}(s_i)} \Pr(\widehat{s}_i, s_{-i}|a).$$

is a conditional distribution on report profiles given $a \in A$, $i \in I$, and ρ_i. We denote the truth-telling strategy by τ_i.

The team chooses a correlated strategy $\sigma \in \Delta(A)$ and a function $\xi : I \times A \times S \to \mathbb{R}$ of (probability weighted) payments (i.e., $\xi_i(a,s) = \sigma(a)\zeta_i(a,s)$) to maximize welfare. Even if $\sigma(a) = 0$ for some a and $\xi_i(a,s) > 0$, we can approximate such σ by a correlated action profile with full support and an appropriate choice of $\zeta_i(a,s)$. Therefore this change of variable does not cause any problem. The last constraint makes sure that σ is a probability measure, the penultimate constraint makes sure that payments add up to 0 for every possible recommendation a and report s, and finally the first family of constraints, indexed by $i \in I$ and $a_i, b_i \in A_i$ and ρ_i, makes sure that obeying the organizer and reporting truthfully is incentive compatible.

With the Lagrangian of this primal and a little symbolic arithmetic, it is not difficult to show that the associated *dual* problem is given by the following linear program:

The dual is given by the following linear program:

$$W(\mathbf{v}) \;=\; \inf_{\lambda\geq 0,\eta,\nu} \nu \quad \text{s.t.}$$

$$\sum_{i\in I} v_i(a) + \sum_{i\in I}\sum_{(b_i,\rho_i)} \lambda_i(a_i,b_i,\rho_i)[v_i(a) - v_i(b_i,a_{-i})] \;\leq\; \nu$$

$$\sum_{(b_i,\rho_i)} \lambda_i(a_i,b_i,\rho_i)(\Pr(s|b_i,a_{-i},\rho_i) - \Pr(s|a)) \;=\; \eta(a,s),$$

The dual minimizes ν by choosing $\lambda \geq 0$, which indexes the multipliers on the primal incentive constraints, η, which indexes the multipliers associated with the team's ex post resource constraint, and ν, which is the multiplier associated with the the probability constraint on σ.

The first family of dual constraints, indexed by $a \in A$, arises from the primal when σ is interpreted as a multiplier. The second family, indexed by $i \in I$, $a \in A$, and $s \in S$, arises when ξ is interpreted as a multiplier. (That the dual value equals the primal value follows from the Fundamental Theorem of Linear Programming (FTLP).)

From this dual problem, we can identify two conditions on the information structure which are necessary and sufficient for approximate efficency. When $A = (x_1, ..., x_m)$ and $B = (x_{m+1}, ..., x_n)$ are collections of vectors in the same linear vector space V, define $conv^*\{A|B\}$ as follows:

Definition 3.1.

$$\text{conv}\ ^*\{A \mid B\} = \{v \in V : \exists\lambda_i \geq 0,\ > 0 \text{ for some } j \leq m,\ \sum_{i=1}^{n} \lambda_i = 1, v = \sum_{i=1}^{n} \lambda_i x_i\}$$

Our first condition is *convex independence.*

Definition 3.2. A monitoring technology (S, Pr) satisfies *convex independence* (CI) if for every player $i \in I$ and individual action $a_i \in A_i$,

$$\text{Pr}[a_i, \tau_i] \ \notin \ \text{conv}\ ^*\{Pr[b_i, \rho_i] : b_i \neq a_i \mid Pr[a_i, \rho_i]\}$$

where $\text{Pr}[b_i, \rho_i] : A_{-i} \to \Delta(S)$ is the family of conditional probability vectors for s given $b_i \in A_i$ and $\rho_i : S_i \to S_i$ indexed by a_{-i} such that $\text{Pr}[b_i, \rho_i](a_{-i})(s) = \text{Pr}(s|b_i, a_{-i}, \rho_i)$.

CI means that any pure or mixed deviation in action is statistically detectable by looking at both recommended actions and reported signals. Indeed, take any player i and action a_i. For any mixed strategy α_i on A_i and ρ_i, the equation

$$\text{Pr}[a_i, \tau_i] \ = \ \sum_{b_i, \rho_i} \alpha_i(b_i, \rho_i) \, \text{Pr}[b_i, \rho_i]$$

can only be satisfied when the marginal of α_i on A_i places all its mass on the pure strategy a_i. Any other α_i would violate this equality, implying that there exists a correlated strategy such that α_i could be statistically detected when a_i was recommended to player i. Note that we allow deviations purely in terms of reporting strategies. Such deviations do not have to be detected as they are not profitable.

To compare this condition to other well known conditions, suppose that monitoring is public. Since $n > 2$, we can ignore deviations regarding to report. Then the above condition is reduced to:

$$\Pr[a_i] \;\notin\; \text{conv}\ \{Pr[b_i] : b_i \neq a_i \mid Pr[a_i]\}$$

where $\Pr[b_i] : A_{-i} \to \Delta(S)$ is the family of conditional probability vectors for s given $b_i \in A_i$ indexed by a_{-i} such that $\Pr[b_i](a_{-i})(s) = \Pr(s|b_i, a_{-i})$.

Note that this simplified version of CI is different from a more standard condition of enforceability with public monitoring:

$$\Pr(s|a) \notin \text{conv}\{\Pr(s|b_i, a_{-i}) : b_i \neq a_i\},$$

which means that any deviation is detectable by using realized signals only. As is well known, this condition is sufficient (and necessary in our setting) for exact implementation of $a \in A$ without budget-balance. It is standard in the literature on repeated games. See, for example, Fudenberg et al. (1994), Compte (1998), Kandori and Matsushima (1998), or Obara (2005). CI is necessary and sufficient for approximate implementation without budget-balance, thus much weaker than this condition.

(PUT EXAMPLE FROM THE SLIDE HERE)

Next, we introduce our second condition. Fix any player $i \in I$. Let us *define* the subset $C_i \subset \mathbb{R}^{A\times S}$ to be the set of all vectors $\eta \in \mathbb{R}^{A\times S}$ for which there exists $\lambda_i \geq 0$ such that given (a, s),

$$\eta(a,s) \;=\; \sum_{b_i \in A_i, \rho_i} \lambda_i(a_i, b_i, \rho_i)(\Pr(s|b_i, a_{-i}, \rho_i) - \Pr(s|a)).$$

We will call C_i the "cone" of player i. If $\eta \in C_i$ then clearly $\alpha\eta \in C_i$ for any $\alpha > 0$, and if $\eta_0, \eta_1 \in C_i$ then $\eta_\alpha = \alpha\eta_1 + (1-\alpha)\eta_0 \in C_i$ for any $\alpha \in [0,1]$, making C_i a convex cone "pointed" at the origin of $\mathbb{R}^{A\times S}$.

Definition 3.3. A monitoring technology $(S, \Pr)$ has *non-intersecting cones* (NIC) if

$$\bigcap_{i \in I} C_i \;=\; \mathbf{0},$$

where C_i stands for the cone of player i, and $\mathbf{0}$ stands for the origin of $\mathbb{R}^{A\times S}$.

What does this condition mean? Consider the defining equation

$$\eta(a,s) \;=\; \sum_{b_i \in A_i, \rho_i} \lambda_i(a_i, b_i, \rho_i)(\Pr(s|b_i, a_{-i}, \rho_i) - \Pr(s|a)).$$

We may interpret $\alpha_i(b_i, \rho_i|a_i) = \lambda_i(a_i, b_i, \rho_i)/\sum_{b_i,\rho_i} \lambda_i(a_i, b_i, |\rho_i)$ as the conditional probability that player i plays b_i and ρ_i when recommended to play a_i, making the set $\{\alpha_i(a_i) : a_i \in A_i\}$ a deviating strategy for player i. NIC means that any profile of such unilateral deviating strategies cannot move conditional probabilities on S exactly in the same direction for every $a \in A$. Without NIC, ex post budget-balance must fail because every player's incentive would "overlap." When NIC is satisfied, it is possible to punish some while rewarding others. Holmström (1982) appointed a principal to play the role of budget-breaker (See example 3.6.). NIC allows the members of a team to share the role of principal internally. Sometimes it might be allocated stochastically, thereby leading to a notion of *random principal*.

Now we prove that CI and NIC is necessary and sufficient for approximately efficient implementation with budget-balance. We begin with one weak regurarity condition.

Assumption R If there exists $\lambda_i(a_i, \rho_i) \geq 0, i \in I$ such that

$$\sum_{\rho_i} \lambda_i(a_i \rho_i)(\Pr(s|a, \rho_i) - \Pr(s|a)) = \eta(a,s),$$

then $\eta = \mathbf{0} \in \mathbb{R}^{A \times S}$

The following lemma is the key step to our first main result.

Lemma 3.4. *Suppose that Assumption R is satisfied. Then a monitoring technology* $(S, \Pr)$ *satisfies CI and NIC if and only if*

$$\sum_{b_i \in A_i, \rho_i} \lambda_i(a_i, b_i, \rho_i)(\Pr(s|b_i, a_{-i,\rho_i}) - \Pr(s|a)) = \eta(a,s)$$

for every (i, a, s) *and* $\lambda \geq 0$ *implies that* $\lambda_i(a_i, b_i, \rho_i) = 0$ *for all* $i \in I$*,* $b_i \neq a_i$*, and* ρ_i*.*

Proof. For sufficiency, suppose $(S, \Pr)$ satisfies CI and NIC and η satisfies the condition above. By definition, η must lie in the intersection of all cones C_i, which by NIC implies that $\eta = \mathbf{0}$. Therefore, by CI any non-negative weights λ that satisfy $\sum_{b_i,\rho_i} \lambda_i(a_i, b_i, \rho_i)(\Pr(s|b_i, a_{-i}, \rho_i) - \Pr(s|a)) = 0$ for $i \in I$ also satisfies $\lambda_i(a_i, b_i, \rho_i) = 0$ for all $b_i \neq a_i$, and ρ_i.

For necessity, suppose that the second part of the lemma holds. Then by substituting $\eta = \mathbf{0}$, convex independence follows. On the other hand, suppose that, given η and the above condition, it follows that $\lambda_i(a_i, b_i, \rho_i) = 0$ for all $b_i \neq a_i$, ρ_i. Combined with assumption R, this implies that η equals the zero vector in $\mathbb{R}^{A\times S}$, so NIC is satisfied. □

Now we can prove our first main theorem.

Theorem 3.5. *Suppose that Assumption R is satisfied. A monitoring technology* $(S, \Pr)$ *satisfies both CI and NIC if and only if*

$$V(\mathbf{v}) \;=\; \max\{v(a) : a \in A\}$$

for every $\mathbf{v} \in \mathbb{R}^{nA}$*, where* $v := \sum_{i\in I} v_i$*.*

Proof. For sufficiency, if $(S, \Pr)$ satisfies CI and NIC then by lemma 3.4

$$\sum_{b_i\in A_i,\rho_i} \lambda_i(a_i, b_i, \rho_i)(\Pr(s|b_i, a_{-i}, \rho_i) - \Pr(s|a)) \;=\; \eta(a, s)$$

for every (i, a, s) and $\lambda \geq 0$ implies that $\lambda_i(a_i, b_i, \rho_i) = 0$ for all (i, a_i, b_i, ρ_-) with $a_i \neq b_i$. By the first family of dual constraints, it follows that any dual solution must satisfy

$$\sum_{i\in I} v_i(a) \leq \nu$$

for every $a \in A$. Minimizing ν with respect to this constraint with $\lambda = 0$ and $\eta = 0$ yields the claimed equality.

For necessity, if either CI or NIC fail then by lemma 3.4, there exists η and nonnegative λ such that

$$\sum_{b_i,\rho_i} \lambda_i(a_i, b_i, \rho_i)(\Pr(s|b_i, a_{-i}, \rho_i) - \Pr(s|a)) \;=\; \eta(a, s)$$

for all $a \in$ and $s \in S$ and $\lambda_j(\widehat{a}_j, \widehat{b}_j, \widehat{\rho}_j) > 0$ for some player j, $\widehat{a}_j, \widehat{b}_j \in A_i$ and $\widehat{\rho}_j$. Furthermore, $\widehat{b}_j \neq \widehat{a}_j$ by assumption R. Choose $\mathbf{v}$ as follows. For any a_{-j}, the utility to each player depending on whether or not j plays $\widehat{a}_j$ is given by (first is j then anyone else):

a_j	$\widehat{a}_j$
1, 0	0, 2

For any a with $a_j \neq \widehat{a}_j$, the first dual constraint becomes $1 + \sum_{\rho_j} \lambda(a_j, \widehat{a}_j, \rho_j) \leq \nu$. This can be made smaller than $2(n-1)$ by multiplying λ and η by a small positive number. At $\widehat{a}_j$, the constraint becomes $2(n-1) - \sum_{b_j,\rho_j} \lambda_j(\widehat{a}_j, b_j, \rho_j) \leq \nu$. Since $\sum \lambda > 0$, there is a feasible dual solution with $\nu < 2(n-1) = \max\{v(a)\}$,as required. □

Note that we did not use any particular property of efficient action profiles in the above proof. Indeed, CI and NIC is not only necessary and sufficient for approximate implentability of an efficient action profile, but also *any* action profile. Formally, we have the following corollary.

Corollary 3.6. *A monitoring technology* $(S, \Pr)$ *satisfies CI and NIC if and only if any action profile is approximately implementable with ex post budget balance for any* $\mathbf{v}$.

Proof. Replace the primal objective with the indicator function of any action profile and repeat the proof of the main theorem. □

(Comment: Necessary and sufficient conditions without Assumption R are available. We need to use a weaker NIC. Namely,

Definition 3.7. A monitoring technology $(S, \Pr)$ has *non-intersecting cones* (NIC) if

$$\eta(a,s) \;=\; \sum_{b_i \in A_i, \rho_i} \lambda_i(a_i, b_i, \rho_i)(\Pr(s|b_i, a_{-i}, \rho_i) - \Pr(s|a)).$$

for some $\eta \neq \mathbf{0}$ and λ implies that $\lambda_i(a_i, b_i, \rho_i) = 0$ for any $b_i \neq a_i$.

Basically, any situation excluded by Assumption R can be allowed and approximate implementation is still obtained. This is an honest necessary and sufficient condition, but not as nice as the current NIC. Besides, I suspect that Assumption R may be a vacuous assumption when $n > 2$ (I have an example which violates Assumption R when $n = 2$, which is already a very special, nongeneric example.) It would be nice if Assumption R is completely eliminated.)

3.1.1 When recommended actions are not verifiable

3.2 Examples

Consider the following examples to better understand our condition.

Example 3.8. Suppose there exists an individual 0 such that A_0 is a singleton. Then the dual constraints associated with player 0 are given by

$$\lambda_0(a_0, a_0)\left(\Pr(s|a) - \Pr(s|a)\right) \;=\; \eta(a, s) \;=\; 0.$$

It follows that any feasible dual solution must satisfy $\eta(a, s) = 0$ for every (a, s). Thus convex independence suffices for approximate implementability with ex post budget balance for this team. This is the case where the standard resolution of team problem applies. Since player 0 cannot be a deviator, she can become a "principal" and serve as a "budget-breaker".

Example 3.9. Here we work out a two players ($I = \{1, 2\}$) × two signals ($S = \{x, y\}$) example. The players are playing the following normal-form game (left) with the monitoring technology (right) below:

	w	s_2
m	$2, -1$	$-1, 0$
s_1	$3, -1$	$0, 0$

	w	s_2
m	$p, 1-p$	$q, 1-q$
s_1	$1/2, 1/2$	$1/2, 1/2$

Suppose that $1/2 < p < q$. As already shown, even approximate efficiency is not obtained with any standard contract. We now find budget balanced contracts that approximately implement (d, w). Specifically we will implement player 1 playing m with any probability $\sigma > 0$ and player 2 playing w with probability 1. Let $\zeta : A \times S \to \mathbb{R}$ denote the vector of monetary transfers to player 1 from player 2. Set $\zeta(a, s) = 0$ except for $((m, w), x)$. That is, any money is transferred at all only when (m, w) is recommended and x is realized. The incentive constraints given s_1 and s_2 are clearly satisfied. The other incentive constraints are:

$$m: \qquad 1 + \frac{\sigma(m,w)}{\sigma(m)}\left(-\left(p - \frac{1}{2}\right)\zeta(m,w)\right) \leq 0$$

$$w: \qquad 1 + \frac{\sigma(m,w)}{\sigma(w)}\left((q-p)(-\zeta(m,w))\right) \leq 0$$

These two inequalities can be satisfied by taking $\zeta(m, w)$ large enough.

Let's check CI and NOC for this particular example. CI is clearly satisfied for this game. It is also easy to check that NOC is also satisfied. NOC requires that the following equations with $\lambda \geq 0$ imply $\lambda = 0$.

$$
\begin{array}{lrcl}
(m,w): & \lambda_1(m,s_1)[(\frac{1}{2},\frac{1}{2})-(p,1-p)] & = & \lambda_2(w,s_2)[(q,1-q)-(p,1-p)] \\
(m,s_2): & \lambda_1(m,s_1)[(\frac{1}{2},\frac{1}{2})-(q,1-q)] & = & \lambda_2(s_2,w)[(p,1-p)-(q,1-q)] \\
(s_1,w): & \lambda_1(s_1,m)[(p,1-p)-(\frac{1}{2},\frac{1}{2})] & = & \lambda_2(w,s_2)[(\frac{1}{2},\frac{1}{2})-(\frac{1}{2},\frac{1}{2})] \\
(s_1,s_2): & \lambda_1(s_1,m)[(q,1-q)-(\frac{1}{2},\frac{1}{2})] & = & \lambda_2(s_2,w)[(\frac{1}{2},\frac{1}{2})-(\frac{1}{2},\frac{1}{2})]
\end{array}
$$

By inspection, it easily follows that $\lambda = 0$. Thus NOC is satisfied as expected.

What would happen for other p and q? We can show that NOC is satisfied if and only if $p \neq q$ and $(p-1/2)(q-1/2) > 0$ (CI is also satisfied given these conditions). In particular, if the public signal is perfectly informative about player 2's behavior (For example, $\Pr(x|m,w) = \Pr(y|m,s_2) = 1$), we do not obtain approximate implementability with ex post budget balance.

3.3 Dual Representations of CO and NIC

We can derive equivalent representations of CO and NOC using duality, which allow more economically meaningful interpretations.

Definition 3.10. A monitoring technology $(S, \Pr)$ provides *strict incentives* (SI) if given any $D_i : A_i \times A_i \to \mathbb{R}_+$ for every i, there exists $\xi : I \times A \times S \to \mathbb{R}$ such that

$$\sum_{(a_{-i},s)} \xi_i(a,s)(\Pr(s|b_i,a_{-i},\rho_i) - \Pr(s|a)) \leq -D_i(a_i,b_i)$$

for all (i, a_i, b_i, ρ_i) with $b_i \neq a_i$.

SI means that strict incentive for any full support correlated strategy can be provided with contractual payments ξ for every player, recommended action, and against any possible deviation. If $D_i(a_i, b_i)$ is interpreted as a player's deviation gain from playing b_i when recommended to play a_i, then SI means that, for any $\{D_i \mid i \in I\}$, there exists a contract such that a deviator's expected loss in payments outweighs his deviation gain given any recommended action. As it turns out, CI is equivalent to SI.

Proposition 3.11. *A monitoring technology $(S, \Pr)$ satisfies CI if and only if it satisfies SI.*

Proof. Consider the following linear program.

$$\sup_{\xi} \; 0 \;\; \text{s.t.} \quad \sum_{(a_{-i},s)} \xi_i(a,s)(\Pr(s|b_i,a_{-i},\rho_i) - \Pr(s|a)) \;\leq\; -D_i(a_i,b_i).$$

The dual of this problem is given by

$$\inf_{\lambda \geq 0} - \sum_{i,a_i,b_i \neq a_i,\rho_i} \lambda_i(a_i,b_i,\rho_i) D_i(a_i,b_i) \;\; \text{s.t.} \quad \sum_{b_i \in A_i,\rho_i} \lambda_i(a_i,b_i,\rho_i)(\Pr(s|b_i,a_{-i},\rho_i) - \Pr(s|a)) \;=\; 0.$$

If SI is satisfied then the value of the primal is zero. By FTLP, the dual value must be also zero for every $\{D_i\}$ if SI is satisfied. This implies that $\lambda_i(a_i,b_i,\rho_i) = 0$ for any $b_i \neq a_i$ for any feasible dual solution, in other words, CI. Conversely, if SI is not satisfied then there exists $\{D_i\}$ for which the feasible set is empty, and the primal value is $-\infty$. For the dual, at such $\{D_i\}$ there must exist $\lambda \geq 0$ that satisfies the dual constraint and sets the dual objective to strictly negative, which implies that $\lambda_i(a_i,b_i,\rho_i) > 0$ for some (i,a_i,b_i,ρ_i) with $b_i \neq a_i$, therefore CI fails. □

NIC can also be translated to an equivalent condition with a dual economic interpretation.

Definition 3.12. A monitoring technology $(S,\Pr)$ *clears all budget constraints* (CABC) if for any $K : A \times S \to \mathbb{R}$ there exists $\xi : I \times A \times S \to \mathbb{R}$ such that

$$\sum_{(a_{-i},s)} \xi_i(a,s)[\Pr(s|b_i,a_{-i},\rho_i) - \Pr(s|a)] \;\leq\; 0$$

and

$$\sum_{i \in I} \xi_i(a,s) \;=\; K(a,s)$$

for all $a_i, b_i \in A_i$, ρ_i and $i \in I$.

The function $K(a,s)$ may be regarded as a budget surplus or deficit for each combination of recommended action and realized signal. CABC means that any level of budgetary surplus or deficit can be achieved by a team without disrupting incentive compatibility constraints. This condition can be shown to be equivalent to NIC.

Proposition 3.13. *A monitoring technology* $(S,\Pr)$ *satisfies NIC if and only if it satisfies CABC.*

Proof. Consider the following primal problem.

$$\begin{aligned} V(\mathbf{v}) &:= \sup_{\zeta} \quad 0 \quad \text{s.t.} \\ \sum_{(a_{-i},s)} \xi_i(a,s)[\Pr(s|b_i,a_{-i},\rho_i) - \Pr(s|a)] &\leq 0 \\ \sum_{i\in I} \xi_i(a,s) &= K(a,s). \end{aligned}$$

The dual of this problem is

$$\begin{aligned} V(\mathbf{v}) &= \inf_{\lambda\geq 0,\eta} \sum_{a,s} \eta(a,s)K(a,s) \quad \text{s.t.} \\ \sum_{b_i\in A_i} \lambda_i(a_i,b_i,\rho_i)(\Pr(s|b_i,a_{-i},\rho_i) - \Pr(s|a)) &= \eta(a,s), \end{aligned}$$

If CABC is satisfied, then the value of the primal equals 0 for any $K : A \times S \to \mathbb{R}$. By FTLP, the value of the dual is also 0 for any $K : A \times S \to \mathbb{R}$. Therefore, any η satisfying the constraint for some λ must be 0 for all (a,s), so NIC is satisfied.

For necessity, if NIC is satisfied then the value of the dual is always 0 for any $K : A \times S \to \mathbb{R}$. By FTLP, the value of the primal is also 0 for any K. Therefore, given K, there exists a feasible primal solution $\xi_i(a,s)$ that satisfies all the primal constraints, and CABC is satisfied. □

This result further clarifies the relative roles of CI and NIC. As already mentioned, CI is necessary and sufficient for approximate enforceability of any action profile. However, the team's budget may not be balanced for each (a,s) ex post (it can only be balanced in expectation). This is where NIC comes in. NIC guarantees existence of a further contract that absorbs any budgetary deficit or surplus without disrupting incentive constraints. Therefore, the original contract plus this further contract can implement the same action profile with ex post budget-balance.[1]

3.4 Exact implementability

We briefly touch the issue of *exact* implementability. We will fix a given correlated strategy $\sigma \in \Delta(A)$, and ask for necessary and sufficient conditions on a monitoring

[1] A similar argument is provided by dAspremont et al. (2004) for Bayesian Incentive Compatible Mechanisms.

technology for any team with any profile of utility functions to be able to implement σ as a correlated equilibrium with linear transfers.

This section serves two purposes. One is to show that our method can be readily applied to the question of exact implementability. The other is to connect our results to other literature in this familiar setting. We show that we can replicate many known results. Furthermore, our approach is straightforward and often provides a simpler proof. We can also provide a new interpretation of the old conditions.

Fix a correrated strategy $\sigma \in \Delta(A)$ and consider the following linear program:

$$V(\mathbf{v}|\sigma) := \sup_{\zeta} \quad 0 \quad \text{s.t.}$$

$$\sum_{a_{-i}} \sigma(a)[v_i(b_i, a_{-i}) - v_i(a)] + \sum_{(a_{-i},s)} \sigma(a)\zeta_i(a,s)[\Pr(s|b_i, a_{-i}, \rho_i) - \Pr(s|a)] \leq 0$$

$$\sum_{i\in I} \zeta_i(a,s) = 0.$$

The associated dual program is given by:

$$V(\mathbf{v}|\sigma) = \inf_{\lambda\geq 0,\eta} \sum_{(i,a_i,b_i)} \lambda_i(a_i,b_i) \sum_{a_{-i}} \sigma(a)[v_i(a) - v_i(b_i,a_{-i})] \quad \text{s.t.}$$

$$\sigma(a) \sum_{b_i\in A_i,\rho_i} \lambda_i(a_i,b_i,\rho_i)(\Pr(s|b_i,a_{-i},\rho_i) - \Pr(s|a)) = \eta(a,s),$$

The value of the primal equals the value of the dual by FTLP. If σ is exactly implementable (i.e., there exists a budget balanced $\zeta : I \times A \times S \to \mathbb{R}$ that makes σ a correlated equilibrium) then $V(\mathbf{v}|\sigma) = 0$, otherwise $V(\mathbf{v}|\sigma) = -\infty$.

The dual above motivates the following definitions, which are outcome- specific versions of the previous ones.

Definition 3.14. A monitoring technology $(S, \Pr)$ satisfies *convex independence at* σ (CI[σ]) if for every player $i \in I$ and $a_i \in A_i$ such that $\sigma(a_i) > 0$,

$$\Pr[a_i|a_i] \notin \text{conv}^*\{\Pr[b_i,\rho_i|a_i] : b_i \neq a_i \mid \Pr[a_i,\rho_i|a_i]\},$$

where, for every $(i, b_i, a_{-i}, \rho_i, s)$,

$$\Pr[b_i,\rho_i|a_i](a_{-i})(s) = \sigma(a)\Pr(s|b_i,a_{-i},\rho_i).$$

Fix any player $i \in I$. Define $C_i[\sigma] \subset \mathbb{R}^{A\times S}$ to be the set of all vectors $\eta \in \mathbb{R}^{A\times S}$ for which there exists $\lambda_i \geq 0$ such that given $(a,s) \in A \times S$,

$$\eta(a,s) = \sigma(a) \sum_{b_i\in A_i} \lambda_i(a_i,b_i,\rho_i)(\Pr(s|b_i,a_{-i},\rho_i) - \Pr(s|a)).$$

Definition 3.15. A monitoring technology $(S, \Pr)$ has *non-intersecting cones at σ* (NIC[σ]) if

$$\bigcap_{i \in I} C_i[\sigma] \;=\; \mathbf{0},$$

where $C_i[\sigma]$ stands for the cone of player i, and $\mathbf{0}$ stands for the origin of $\mathbb{R}^{A\times S}$.

Let us introduce a version of Assumption R here.

Assumption R[σ] If there exists $\lambda_i(a_i, \rho_i) \geq 0$ such that

$$\sigma(a) \sum_{\rho_i} \lambda_i(a_i, \rho_i)(\Pr(s|a, \rho_i) - \Pr(s|a)) = \eta(a, s),$$

then $\eta = \mathbf{0} \in \mathbb{R}^{A\times S}$

Theorem 3.16. *Fix any $\sigma \in \Delta(A)$. Suppose that Assumption R[σ] is satisfied. A monitoring technology satisfies CI[σ] and NIC[σ] if and only if $V(\mathbf{v}|\sigma) = 0$ for every $\mathbf{v} \in \mathbb{R}^{nA}$, where $v = \sum_{i\in I} v_i$.*

The proof of this result is omitted since it is almost identical to that of Theorem 3.5. This result characterizes exact implementability with ex post budget balance for any given correlated strategy. For instance, in Example 3.9, the profile (d, w) is only approximately implementable, but not exactly implementable. However, every completely mixed correlated strategy is exactly implementable with ex post budget balance, which explains why every correlated strategy is approximately implementable with ex post budget balance.

As before, we can translate CI and NIC into their equivalent duals. Their dual representations are as follows.

Definition 3.17. A monitoring technology $(S, \Pr)$ provides *strict incentives* (SI) at σ (SI([σ])) if given any $D_i : A_i \times A_i \to \mathbb{R}_+$ for every i, there exists $\zeta : I \times A \times S \to \mathbb{R}$ such that

$$\sum_{(a_{-i}, s)} \sigma(a)\zeta_i(a, s)(\Pr(s|b_i, a_{-i}, \rho_i) - \Pr(s|a)) \;\leq\; -\sigma(a_i)D_i(a_i, b_i)$$

for all (i, a_i, b_i, ρ_i) with $b_i \neq a_i$.

Definition 3.18. A monitoring technology $(S, \Pr)$ *clears all budget constraints* (CABC) at σ (CABC([σ])) if for any $K : A \times S \to \mathbb{R}$ there exists $\zeta : I \times A \times S \to \mathbb{R}$ such that

$$\sum_{(a_{-i}, s)} \sigma(a)\zeta_i(a, s)[\Pr(s|b_i, a_{-i}, \rho_i) - \Pr(s|a)] \;\leq\; 0$$

and

$$\sigma(a) \sum_{i \in I} \zeta_i(a, s) \;=\; \sigma(a) K(a, s)$$

for all $a_i, b_i \in A_i$ and $i \in I$.

Then we can prove the following theorem.

Proposition 3.19. *A monitoring technology* (S, Pr) *satisfies CI*$([\sigma])$ *at* σ *if and only if it satisfies SI*$([\sigma])$*. A monitoring technology* (S, Pr) *satisfies NIC*$([\sigma])$ *if and only if it satisfies CABC*$([\sigma])$*.*

To compare our results to the existing results, consider a special case where $\sigma \in \Delta(A)$ puts probability mass 1 on some action profile. Denote this action profile by a^*. Also assume that monitoring is public.

Then our CI simply becomes:

$$\mathrm{Pr}(s|a^*) \notin \mathrm{conv}\{\mathrm{Pr}(s|b_i, a^*_{-i}) : b_i \neq a^*_i\},$$

Denote player i's cone corresponding to the pure strategy a^* by $C_i[a^*] \subset \mathbb{R}^S$, which consists of all vectors $\eta \in \mathbb{R}^S$ for which there exists $\lambda_i \geq 0$ such that

$$\eta(s) \;=\; \sum_{b_i \in A_i} \lambda_i(a^*_i, b_i)(\mathrm{Pr}(s|b_i, a^*_{-i}) - \mathrm{Pr}(s|a^*)).$$

Then our NIC can be stated as

$$\bigcap_{i \in I} C_i[a^*] \;=\; 0$$

Note that reporting strategy ρ_i does not appear in both expressions as truthful revelation of private signals comes for free when monitoring is public (and there are at least three players).[2]

It is clear that CI is just enforcability. Our NIC is a generalization of a famous *pairwise identifiability* condition in Fudenberg et al. (1994). Pairwise identifiability implies (but not implied by) that, for all $i \neq j$,

$$C_i[a^*] \bigcap C_j[a^*] \;=\; 0$$

[2]When there are only two players, then only Nash equilibrium profiles can be implemented even whem monitoring is public. This is because the only way to extract private information is to make ζ constant across all $a \in A$ and $s \in S$.

This means that player i's deviation and player j's deviation can be statistically distinguished at a^*. This is a sufficient condition for exact implementability of a^* with ex post budget balance.

On the other hand, our NIC allows some cones to overlap. We do not need to tell who is likely to be a deviator, what we need is to find a player who is likely to be innocent given any non-equiribrium distribution of signal profiles. This is why NIC given a^* is weaker than pairwise identifiability at a^*, indeed necessary and sufficient for exact implementability combined with CI. Of course our original NIC for approximate implementation is even weaker.

(PUT A PICTURE HERE LATER)

Legros and Matsushima (1991) found the necessary and sufficient conditon for exact implementability of a pure action profile (Proposition 3). Of cousrse CI and NIC or SI and CABC is equivalent to their condition. An advantage of our approach is twofold. First the proof (identical to the proof of thm 3.5) is very straightfoward. Second, we can decompose the necessary and sufficient condition into two conditions, each of which has a clear economic interpretation.

3.5 Approximate Implementability for Given Utility Functions

Now we derive necessary and sufficient conditions for approximate implementability for given utility functions $\mathbf{v} \in \mathbb{R}^{I\times A}$. Let α_i be a deviating strategy for player i. This time α_i is a contingent plan. That is, $\alpha_i(b_i, \rho_i|a_i)$ is a probability to play (b_i, ρ_i) when a_i is recommended. Player i's strategy α_i is a *profitable strategy* at a^* if

$$\sum_{b_i\in A_i,\rho_i} \alpha_i(b_i,\rho_i|a_i^*)\left(v_i(b_i,a_{-i}^*) - v_i(a^*)\right) > 0$$

A profile of deviation strategies $\alpha = (\alpha_1, ..., \alpha_n)$ is *β-profitable* with respect to $\beta : A \to \mathbb{R}^I_+$ given a^* if

$$\sum_i \sum_{b_i\in A_i,\rho_i} \alpha_i(b_i,\rho_i|a_i)\beta_i(a_i)\left(v_i(b_i,a_{-i}) - v_i(a)\right) > \sum_i v_i(a) - \sum_i v_i(a^*).$$

for every $a \in A$.

The dual problem is the same as before.

$$V(\mathbf{v}) \;=\; \inf_{\lambda\geq 0,\eta,\nu} \quad \nu \quad \text{s.t.}$$

$$\sum_{i=1}^{n} v_i(a) + \sum_{i=1}^{n} \sum_{b_i\in A_i,\rho_i} \lambda_i(a_i,b_i,\rho_i)(v_i(a)-v_i(b_i,a_{-i})) \;\leq\; \nu$$

$$\sum_{b_i\in A_i,\rho_i} \lambda_i(a_i,b_i,\rho_i)(\Pr(s|b_i,a_{-i},\rho_i)-\Pr(s|a)) \;=\; \eta(a,s).$$

The value of this dual problem coincides with the solution of the primal problem, i.e. the optimal team outcome, by FTLP.

The definition of CI is as follows.

Definition 3.20. A monitoring technology $(S,\Pr)$ satisfies *convex independence* (CI) given $\mathbf{v}$ at a^* if

$$\Pr(s|a^*) = \sum_{b_i\in A_i,\rho_i} \alpha_i(b_i,\rho_i|a_i^*)\Pr(s|b_i,a_{-i}^*,\rho_i)$$

for every $s\in S$ implies that α_i is not a profitable strategy at a^*

This condition can be interpreted in the same way as before.

Next we need to introduce a version of NIC. Let us *define* the subset $C(a^*)\subset\mathbb{R}^S$ given a^* to be the set of all vectors $\eta\in\mathbb{R}^S$ for which there exists a β-profitable profile α at a^* such that, for every s, a_{-i}, and every i,

$$\eta(s) \;=\; \beta_i(a_i^*)\sum_{b_i\in A_i,\rho_i} \alpha_i(b_i,\rho_i|a_i^*)(\Pr(s|b_i,a_{-i}^*,\rho_i)-\Pr(s|a^*)).$$

Definition 3.21. A monitoring technology $(S,\Pr)$ satisfies NIC given $\mathbf{v}$ at a^* if

$$C(a^*) = \mathbf{0},$$

Now we can prove the following theorem via duality.

Theorem 3.22. *Suppose that $a^*\in A$ is the unique optimal action profile given $v: A\to\mathbb{R}^n$. A monitoring technology $(S,\Pr)$ satisfies both CI and NIC at a^* if and only if $V(\mathbf{v}) \;=\; v(a^*)$.*

Proof. For sufficiency, suppose that $(S,\Pr)$ satisfies CI and NIC at a^*. If

$$\sum_{b_i\in A_i,\rho_i} \lambda_i(a_i,b_i,\rho_i)(\Pr(s|b_i,a_{-i},\rho_i)-\Pr(s|a)) \;=\; \eta(a,s)$$

for every (i, a, s) for some $\lambda \geq 0$ and $\eta(a, s)$ such that $\eta(a^*, s) \neq 0$, then $\alpha(b_i, \rho_i|a_i) = \lambda_i(a_i, b_i, \rho_i)/\sum_{b_i \in A_i, \rho_i} \lambda_i(a_i, b_i, \rho_i)$ should not be β-profitable at a^*, where $\beta(a_i) = \sum_{b_i \in A_i, \rho_i} \lambda_i(a_i, b_i, \rho_i)$. This implies that $\sum_{i=1}^n v_i(a) + \sum_{i=1}^n \sum_{b_i \in A_i, \rho_i} \lambda_i(a_i, b_i, \rho_i)(v_i(a) - v_i(b_i, a_{-i})) \geq \sum_{i=1}^n v_i(a^*)$ for some $a \in A$, thus $v(a^*) \leq \nu$.

If $\eta(a^*, s) = 0$, then CI implies that $\sum_{b_i \in A_i, \rho_i} \lambda_i(a^*, b_i, \rho_i)(v_i(a^*) - v_i(b_i, a^*_{-i})) \geq 0$ for every i. Thus again $v(a^*) \leq \nu$. In any case, the value of the dual is bounded below by $v(a^*)$, which is achieved by setting $\lambda = 0, \eta = 0$, and minimizing ν.

For necessity, if $(S, \Pr)$ does not satisfy CI for i at a^*, then we can set $\eta = 0$, $\lambda_j = 0$ for all $j \neq i$, and find λ_i such that $\sum_{b_i \in A_i, \rho_i} \lambda_i(a^*, b_i, \rho_i)(v_i(a^*) - v_i(b_i, a^*_{-i})) < 0$ and $\lambda_i(a_i, b_i, \rho_i) = 0$ for $a_i \neq a^*_i$. Since $a^* \in A$ is the unique optimal action profile, this implies that $\nu < v(a^*)$ is feasible.

Finally, if $(S, \Pr)$ does not satisfy NIC at a^*, then there exists η, β, and a β-profitable strategy α at a^* such that $\eta(a, s) = \beta_i(a_i) \sum_{b_i \in A_i, \rho_i} \alpha_i(b_i, \rho_i|a_i)(\Pr(s|b_i, a_{-i}, \rho_i) - \Pr(s|a))$. Define $\lambda_i(a_i, b_i, \rho_i) = \beta_i \alpha_i(b_i, \rho_i|a_i)$. Then η and λ satisfy the last constraint of the dual problem. Moreover,

$$\sum_i \sum_{b_i \in A_i, \rho_i} \lambda_i(a^*, b_i, \rho_i)(v_i(b_i, a_{-i}) - v_i(a)) > \sum_i v_i(a) - \sum_i v_i(a^*)$$

for every $a \in A$ by definition of β-profitable strategy α at a^*. Hence

$$\sum_i v_i(a) + \sum_i \sum_{b_i \in A_i, \rho_i} \lambda_i(a_i, b_i, \rho_i)(v_i(a) - v_i(b_i, ah_{-i})) < \sum_i v_i(a^*)$$

for every $a \in A$. This also implies that $\nu < v(a^*)$ is feasible. □

(COMMENT: Here there is no clear duality interpretation. CI is stronger than SI. NIC is weaker than CABC. So there is not much point in separating CI and NIC. I tried to get a version of CI and NIC which exactly corresponds to SI and CABC here, but the conditions necessarily become too messy. I left this section in the current draft, but probably we should cut it off.)

3.6 Liquidity Constraints

This application is still another proof of the power of duality approach. Consider situations where each agent may be liquidity constrained. That is, $\zeta_i(a, s)$ needs to be more than or equal to $L_i(a)$ for every $(a, s) \in A \times S$. This additional requirement can be taken into account simply by adding $\zeta_i(a,s) \geq L_i(a)$ to the primal problem. Rewrite this constraint as $\xi_i(a,s) - \sigma(a) L_i(a) \geq 0$. Then the dual problem becomes

$$\begin{aligned} V(\mathbf{v}) &= \inf_{\lambda, q \geq 0, \eta, \nu} \nu \quad \text{s.t.} \\ \sum_{i \in I} v_i(a) + \sum_{b_i \in A_i, \rho_i} \lambda_i(a_i, b_i, \rho_i)(v_i(a) - v_i(b_i, a_{-i})) - \sum_{i,s} q_i(a,s) L_i(a) &\leq \nu \\ \sum_{b_i \in A_i} \lambda_i(a_i, b_i, \rho_i)(\Pr(s|b_i, a_{-i}, \rho_i) - \Pr(s|a)) - q_i(a,s) &= \eta(a,s). \end{aligned}$$

where $q_i(a,s)$ is a multiplier for the liquidity constraint for agent i. Note that the last equation implies that

$$\sum_s q_i(a,s) = \sum_s \eta(a,s)$$

Therefore

$$\sum_{i,s} q_i(a,s) L_i(a) = \sum_s \eta(a,s) \sum_i L_i(a)$$

Thus we can eliminate $q_i(a,s)$ from the dual problem as follows:

$$\begin{aligned} V(\mathbf{v}) &= \inf_{\lambda, \eta, \nu} \nu \quad \text{s.t.} \\ \sum_{i \in I} v_i(a) + \sum_{b_i \in A_i, \rho_i} \lambda_i(a_i, b_i, \rho_i)(v_i(a) - v_i(b_i, a_{-i})) - \sum_s \eta(a,s) \sum_i L_i(a) &\leq \nu \\ \sum_{b_i \in A_i, \rho_i} \lambda_i(a_i, b_i, \rho_i)(\Pr(s|b_i, a_{-i}, \rho_i) - \Pr(s|a)) &\geq \eta(a,s). \end{aligned}$$

By FTLP, The value of the primal and the dual coincides. Therefore we have a result: liquidity constraints matter for the maximized efficiency level only in terms of *aggregate liquidity constraints* $\sum_{i \in I} L_i(a)$. Note that this result does not reuire any assumption (such as CI or NIC) on information structure. This result is a reminiscent of Theorem 5 in Legros and Matsushima (1991) or Theorem 4 in Legros and Matthews (1993). Our result generalizes those results to private monitoring setting and providea a very simple proof based on duality.

Finally, suppose that CI and NIC is satisfied. In this case, the liquidity constraints must bind for every agent. Otherwise we can "redistribute" liquidity constraints among the agents so that no one is liquidity constrained. Since efficiency should be achieved without liquidity constraints, this is a contradiction. The following theorem summarizes these observation.

Theorem 3.23. $V(\mathbf{v})$ *is the same for different set of liquidity constraints* $\{L_i(a), i \in I\}$ *and* $\{L'_i(a), i \in I\}$ *if* $\sum_{i\in I} L_i(a) = \sum_{i\in I} L'_i(a)$*. Furthermore, if CI, NIC is satisfied, then* $V(\mathbf{v}) < \max\{v(a) : a \in A\}$ *only if liquidity constraints are binding for every agent.*

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