On the Impossibility of an Exact Imperfect Monitoring Folk Theorem*

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Abstract

It is shown that, for almost every two-player game with imperfect monitoring, the conclusions of the classical folk theorem are false. So, even though these games admit a well-known approximate folk theorem, an exact folk theorem may only be obtained for a zero measure set of games.

A complete characterization of the efficient equilibria of almost every such game is also given, along with an inefficiency result on the imperfect monitoring prisoner's dilemma.

1 Introduction

It is well known that repeated games with imperfect monitoring admit an approximate folk theorem (Fudenberg, Levine and Maskin 1994). For a large class of such games, any feasible individually rational payoff may be approximated by perfect public equilibria (PPE), as players become infinitely patient. However, the question of whether such payoffs can be achieved exactly in equilibrium is still open.

In this paper it will be shown that the answer for this question is that, generically, no. Formally, for almost every game with imperfect monitoring, there are feasible, strictly individually rational payoffs which are not PPEs. That is, the conclusions of the classical folk theorem are false for almost every game with imperfect monitoring. Therefore an exact folk theorem may only be obtained for a zero measure class of such games.

In proving this result, a complete characterization of the efficient equilibria of generic two player games with imperfect monitoring will be given. This is also of independent interest, as it may be useful to the study of particular games. It

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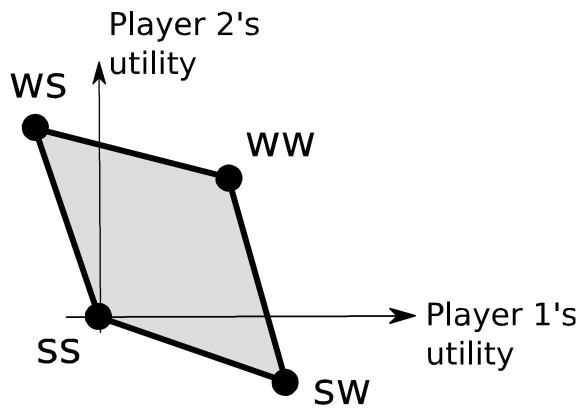

Figure 1: The feasible payoffs in the prisoner's dilemma.

will be shown that, although determining perfect public equilibria in general is a very hard problem, for almost every two player game they may be found by solving a static mechanism design problem.

The first description of an efficient PPE for a repeated game with imperfect monitoring was given by Athey and Bagwell (2001) in a repeated duopoly game. Lemma 3 extends their techniques to get a complete description of the set of efficient PPE for almost every two player game.

Another related work is Fudenberg, Levine and Takahashi (2006). They present an algorithm for determining the limit set of efficient PPEs. Although their techniques apply to the n-player case, they only characterize the limit set as players become infinitely patient. In contrast, our characterization, while being limited to the 2-player case, gives the set of efficient PPE for fixed values of impatience factor less than one.

The characterization will also be used to obtain an inefficiency result for the repeated imperfect monitoring prisoner's dilemma.

These points can be illustrated by the following classical example which will be revisited along the exposition of the paper.

1.1 The prisoner's dilemma

Consider a repeated prisoner's dilemma with imperfect monitoring, usually described as a partnership problem. Two players own a firm and operate it. Each period, they may either cooperate (c) or defect (d), i.e., their effort levels are $s_i \in \{c, d\}$. Although the firm's profit $y \in Y$ is observed, it is randomly distributed according to $\pi(\cdot|s)$. Players' payoffs are $r_i = y/2 - s_i$. Therefore, while effort is not directly observable, it affects the distribution of y.

The average payoff to player i is defined as

$$g_i(s) = E(r_i|s) = \int r_i(y,s)d\pi(y|s).$$

For the game to be interesting, they should be distributed as in a prisoners' dilemma, as indicated on Figure 1. That is,

$$\begin{aligned} cd_1 &< dd_1 < cc_1 < dc_1 \\ dc_2 &< dd_2 < cc_2 < cd_2 \end{aligned}$$

where we have abused notation by using pairs such as cd to denote both strategy profiles and average payoffs as $g(cd)$.

This stage game is repeated every period and players discount future utility by some impatience factor $0 < \delta < 1$.

It is well known that, in this example, the approximate folk theorem holds generically if Y has at least three elements (Fudenberg, Levine and Maskin, 1994). That is, that any feasible individually rational payoff may be approximated by PPEs, as δ approaches to 1. Yet, we will show that generically the prisoner's dilemma has no efficient equilibria,[1] for any discount factor (see Theorem 1).

Also, this means that the game has feasible strictly individually rational payoffs which are not equilibrium values. So, the conclusions of the classic folk theorem fail. We will show that this phenomenon is very general. Except for some trivial cases, an exact folk theorem is false for almost every game with imperfect monitoring (see Theorem 2).

However, the example is somewhat misleading, as the anti-folk theorem is not simply an inefficiency result. Indeed, there are open sets of games which have efficient equilibria. Thus, the general case will be somewhat more complicated.

About the prisoner's dilemma inefficiency, Radner, Myerson and Maskin (1986) showed that in some cases where the approximate folk theorem fails, the equilibria are bounded away from the efficient frontier. However, we are not aware of inefficiency results in cases where the approximate folk theorem holds in the literature.[2]

We now lay down the model, give the characterization of the efficient equilibria and prove the results.

2 The Framework

2.1 The stage game

Two players play a stage game at $t = 0, 1, 2, \cdots$. Player i takes actions s_i in a *finite* set S_i. Actions are not directly observable, but they induce probabilities

[1] Throughout the paper, we will use equilibria interchanging with PPEs.

[2] In the recent book on this topic, Mailath and Samuelson (2006) have not mentioned any result in this line.

$\pi(\cdot|s)$ on a *finite* set of public outcomes Y. The actual payoff $r_i(s_i, y)$ to a player depends on his own action and on the public outcome. But not directly on the other player's action (although it does affect the distribution of y). We also allow for mixed actions α_i in ΔS_i, the space of probabilities on S_i. Probabilities $\pi(\cdot|\alpha)$ and payoffs $r_i(\alpha_i, y)$ are defined as usual, for any mixed strategy profile α.[3]

We may collect these elements into the following:

Definition 1 *A stage game* Γ *is a list* $(S_1, S_2, Y, \pi : S \longrightarrow \Delta Y, r_1 : (S_1, Y) \longrightarrow \mathbb{R}, r_2 : (S_2, Y) \longrightarrow \mathbb{R})$*, where* S_1*,* S_2 *and* Y *are finite sets.*

Let $g_i(\alpha) = E(r_i(\alpha_i, y)|\alpha)$ be the average gain of player i from playing a profile α.

2.2 The supergame

The public history at time t is defined as $h^t = (y^1, y^2, \cdots y^{t-1})$. Players also observe their own actions, and player i's private history is $h_t^i = (a_i^1, a_i^2, \cdots a_i^{t-1})$. A strategy for player i is a sequence of maps $\{\sigma_i^t : h^t \times h_i^t \longrightarrow \Delta S_i\}_{t=0}^{\infty}$.

In the repeated game players maximize

$$v_i = (1 - \delta) \sum \delta^t E g_i^t$$

for $0 \leq \delta < 1$. The factor $(1 - \delta)$ normalizes supergame payoffs as average stage game payoffs.

Our solution concept is the perfect public equilibrium, henceforth PPE. That is, a subgame perfect equilibria in which players condition their actions only on the public history. Abreu Pearce and Stachetti (1990) show that, as long as other players play public strategies, there is no gain in conditioning on private history.[4] Let $PPE(\delta)$ denote the set of perfect public equilibrium values for a given discount factor δ.

A typical stage game has a strategic form such as

$$\begin{array}{cc} & \begin{array}{cc} l & r \end{array} \\ \begin{array}{c} u \\ m \\ d \end{array} & \begin{pmatrix} A & B \\ C & D \\ E & F \end{pmatrix} \end{array}$$

Throughout this paper we will use letters $A, B, C, \cdots$ to denote both action profiles, such as ll, and their payoff vectors, $g(ll)$.

We admit correlated equilibria, as in Aumann (1987): every period players can condition their action on a public randomization device. This makes the set of feasible payoffs a convex polygon - the convex hull of the pure strategy payoffs. Figure 2 shows the set of feasible payoffs of a typical game.

[3] In the prisoner's dilemma game, $S_i = \{\text{cooperate,defect}\}$, y is the firm's profit and $r_i = y/2 - s_i$.

[4] Yet, Kandori and Obara (2006) show that efficiency may be improved if all players use private strategies.

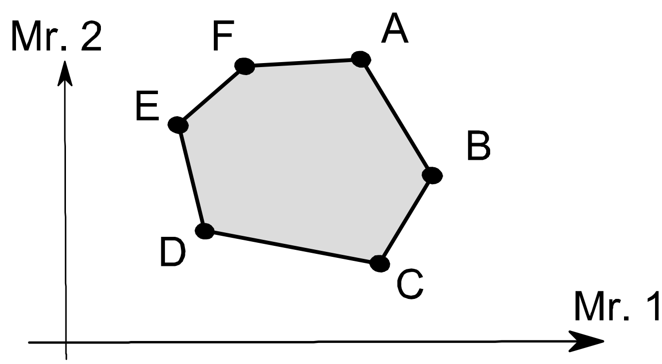

Figure 2: The set of feasible payoffs for a generic game.

For simplicity, we also make the following assumption, which holds for almost every game.

A1. *The set of feasible payoffs of the supergame, V, is a polygon such that each side contains only two pure action profiles.*[5]

2.3 The recursive method

We rely heavily on recursive methods, so we now state some definitions and known results which will be useful. Abreu, Pearce and Stachetti (1990), Fudenberg and Tirole (1991), Mailath and Samuelson (2006) and Fudenberg, Levine and Maskin (1994) are good references.

The key idea of the recursive approach is to factor equilibria into an action profile to be played on the current period and continuation values to be played conditional on the public outcome y. The continuation values for each players are expressed by reward functions $u : Y \longrightarrow \mathbb{R}^2$.

Definition 2 *A reward function u δ-enforces α if for $i = 1, 2$ and every s in S_i.*

$$\begin{aligned} v_i &= (1-\delta)g_i(\alpha) + \delta E(u_i(y)|\alpha) \\ &\geq (1-\delta)g_i(s_i, \alpha_{-i}) + \delta E(u_i(y)|s_i, \alpha_{-i}) \end{aligned} \tag{1}$$

that is,

$$E(u_i(y)|\alpha) - E(u_i(y)|s_i, \alpha_{-i}) \geq \frac{1-\delta}{\delta}(g_i(s_i, \alpha_{-i}) - g_i(\alpha)).$$

A key element of the recursive approach is the Bellman map T. The Bellman map of a set $W \subset \mathbb{R}^2$ is the set of values that may be achieved using promises on W.

[5]More precisely, only payoff vectors of two pure action profiles.

Definition 3 *The Bellman map T is defined for compact subsets of $\mathbb{R}^2$ by*

$$T(W) = \text{co}\{g(\alpha) + E(u|\pi(\cdot|\alpha)) : u \textit{ takes values in } W \textit{ and } \delta\textit{-enforces } \alpha\},$$

where co is the convex hull. A set W in $\mathbb{R}^2$ is self-generating if $W \subset T(W)$.

The key facts we will use are summarized in the next:

Proposition 1 (Known results) *The set $PPE(\delta)$ is compact and convex, and is the largest fixed point of T. All self-generating sets are contained in $PPE(\delta)$. T takes compact sets on compact sets.*

3 The efficient PPEs

3.1 Mixing property

We will now give, for almost every game, a complete description of the efficient equilibria. Since the set of feasible payoffs is a polygon, we just need a method to determine which are the equilibria on each side of the polygon. Then, by applying this method to each side of the polygon, we may find all efficient equilibria on a finite number of steps.

Consider, from now on, a side $\overline{AB}$ of the set of feasible values. For simplicity, assume $\overline{AB}$ to be negatively inclined with slope m, as in Figure 2. Also, suppose $\overline{AB}$ contains no static equilibria. Our strategy will be to somehow restrict the difficult problem of determining the game's efficient PPEs to the one-dimensional segment $\overline{AB}$.

In general this cannot be done, because for the equilibrium payoffs in $\overline{AB}$ one may use equilibrium values outside $\overline{AB}$ as punishments. So, we need a generic condition which we call mixing. A strategy profile is locally mixing if deviations from it cannot be detected with certainty. The stage game is said to be mixing if every profile is locally mixing.

Definition 4 *The strategy profile s is locally mixing if $\pi(\cdot|\widetilde{s}_i, s_{-i}) \ll \pi(\cdot|s)$[6] for $i = 1, 2$ and every $\widetilde{s}_i$ in S_i. The stage game Γ is mixing if all $\pi(\cdot|s)$ are equivalent, for every s in S. Or equivalently, if every profile s in S is locally mixing.*

Mixing is the key property that allows us to work out the equilibria on any given side $\overline{AB}$ of feasible values polygon without having to know all the equilibria of the game. Because mixing implies that any public history may happen with positive probability, any punishments prescribed in equilibrium may actually be carried out with positive probability. Thus, equilibria with values on $\overline{AB}$ may only use continuation values and strategy profiles on $\overline{AB}$.

Let $I(\delta) = \overline{AB} \cap PPE(\delta)$. In terms of the Bellman map these facts translate into the following:

[6] That is, $\pi(\cdot|\widetilde{s}_i, s_{-i})$ is absolutely continuous with respect to $\pi(\cdot|s)$.

Lemma 1 *If A and B are locally mixing, the closed interval $I(\delta)$ is the largest self-generating closed interval in $\overline{AB}$.*

Proof. $I(\delta)$ is a closed interval, once it is the intersection of $\overline{AB}$ with $PPE(\delta)$, which is know to be closed and convex. By Proposition 1 every self generating interval in $\overline{AB}$ is contained in $I(\delta)$. Because of mixing, $T(W) \cap \overline{AB} \subset T(W \cap \overline{AB})$. But since $T(PPE(\delta)) = PPE(\delta)$, we have

$$\begin{aligned} I(\delta) &= PPE(\delta) \cap \overline{AB} = T(PPE(\delta)) \cap \overline{AB} \\ &\subset T(PPE(\delta) \cap \overline{AB}), \end{aligned}$$

that is, $I(\delta)$ is self-generating. ∎

3.2 The characterization

In this subsection we will work out what the set of equilibria in $\overline{AB}$ is. This result is recorded as Lemma 3.

Lemma 1 shows that to have the characterization, all that we have to do is to find the largest self-generating interval on $\overline{AB}$. This will be done now. Take some interval $[a, b]$ in $\overline{AB}$. We check what are the conditions for $[a, b]$ to be self-generating. Let $[a', b']$ be $T([a, b]) \cap \overline{AB}$. By the definition of T,

$$a' = (1 - \delta)A + \delta E(\overline{u}) = (1 - \delta)A + \delta E(\overline{u} - a), \tag{2}$$

where the expectation is conditional on A and $\overline{u}$ is the promise function enforcing A with minimal Eu_1. Let ε_δ be the horizontal distance between $E(\overline{u})$ and a. By rearranging equation (2), we see that $a' \leq a$ if and only if

$$\frac{\delta}{1 - \delta}\varepsilon_\delta \leq a_1 - A_1, \tag{3}$$

that is, $[a, b]$ is self-generating if and only if (3) holds.

We now calculate ε_δ. By its definition, we know that

$$\begin{aligned} \varepsilon_\delta &= \min_u Eu_1 - a_1 \\ &\quad \text{s.t.} \quad u \ \delta\text{-enforces } A \\ &\quad\quad\quad\ \ u(y) \in [a, b], \text{ for all } y \in Y \end{aligned} \tag{4}$$

and that u is the promise function that solves program (4). Since δ-enforcing is invariant by translation, we will show that, at least for δ large enough, it is possible to find ε_δ by a much simpler program.

Consider the program

$$\begin{aligned} \widetilde{\varepsilon}_\delta &= \min_u \max_y -u_1 \\ &\quad \text{s.t.} \quad u \ \delta\text{-enforces } A \\ &\quad\quad\quad\ \ Eu = 0 \\ &\quad\quad\quad\ \ u_2 = mu_1 \end{aligned} \tag{5}$$

in which u is normalized to have 0 average and m is the slope of segment $\overline{AB}$. This program is in fact very simple, because if u satisfies its constraints for some δ, then $\frac{\delta}{1-\delta}\frac{1-\delta'}{\delta'}u$ satisfies the constraints for δ'. Therefore, $\widetilde{\varepsilon}_\delta = \frac{1-\delta}{\delta}\widetilde{\varepsilon}_{1/2}$.

The next lemma shows formally that the programs are equivalent for δ large enough.

Lemma 2 *If δ is large enough, then programs (4) and (5) are equivalent.*

Proof. First let us prove that $\widetilde{\varepsilon}_\delta \leq \varepsilon_\delta$. If u satisfies the constraints in (4), then $u - Eu$ satisfies the constraints in (5). Now, take $\widetilde{u}_{1/2}$ solving program (5) for $\delta = 1/2$ and let $u_\delta = \frac{\delta}{1-\delta}u_{1/2}$ be the solutions for other values of δ. Then, $u = \widetilde{u}_\delta + a + \varepsilon_\delta(1, m)$ satisfies all restrictions of program (4), except for having all $u_1(y) \leq b_1$. But if we take δ high enough, this restriction is also satisfied, so $\varepsilon_\delta \leq \widetilde{\varepsilon}_\delta$.[7] ■

By (3), $[a, b]$ is self-generating if and only if

$$\widetilde{\varepsilon}_{1/2} \leq a_1 - A_1. \tag{6}$$

Condition (6) allows us to describe the conditions for a segment to be self-generating in very simple terms. Since the set of equilibria on $\overline{AB}$ is the largest self-generating interval, this also allows us to describe this set. We now sum up the previous discussion in a lemma that describes all the efficient equilibria on $\overline{AB}$.

Define P^i_α be defined as

$$\begin{aligned} &\min_u \max_y -u_i && \quad (7)\\ &\text{s.t.} \quad u \text{ 1/2-enforces } \alpha \\ &\qquad Eu = 0 && \text{(balanced expected payment)}\\ &\qquad u_2 = mu_1 && \text{(budget balance).} \end{aligned}$$

which is the amount of punishment that has to be inflicted on player i for him to play his unfavorable action α. Therefore, P^1_A is exactly what we called $\widetilde{\varepsilon}_{1/2}$.

From (6) we have the following:

Lemma 3 *Suppose that A and B are locally mixing. Let a and b be points in $\overline{AB}$ with $a_1 = A_1 + P^1_A$ and $b_2 = B_2 + P^2_B$ (see Figure 3). Then, for δ large enough the set of equilibria in $\overline{AB}$ is*

- *$[a, b]$ if $a_1 < b_1$;*
- *empty, otherwise.*

Hence, the only restrictions on the equilibria are that players have to get at least what they would get playing their least favored action plus the amount of punishment necessary to enforce it.

[7] This proof and some computer simulations show that, in practice, δ does not have to be very high. For interesting cases we find δ around 2/3 enough.

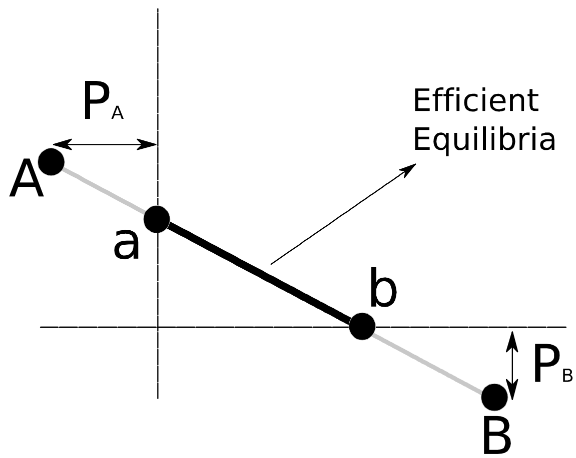

Figure 3: The set of equilibria on $\overline{AB}$.

Note that, to enforce profile A (resp. B), player 1 (resp. 2) must receive punishment strictly greater than his gain from deviating to his best response to A (resp. B). We have the following useful:

Lemma 4 *Suppose that the stage game is mixing. At any equilibrium payoff in $\overline{AB}$, player 1 (resp. 2) receives strictly more than on his best response to A (resp. B).*

Proof. Let $\widehat{A}$ be the profile where player 2 plays A_2 and player 1 plays his best response to A_2. For program (7) applied to A, since u enforces A and $E(u|A) = 0$, we must have

$$E(-u_1|\widehat{A}) \geq \widehat{A}_1 - A_1.$$

Therefore, $\max_y - u_i > \widehat{A}_1 - A_1$ and $A_1 + P_A^1 > \widehat{A}_1$. ■

4 The anti-folk theorem

The anti-folk theorem and the inefficiency result for the prisoner's dilemma follow easily from the previous lemmata. We start with the inefficiency in the prisoner's dilemma:

Theorem 1 *Let Γ be a mixing stage game and $\overline{AB}$ be a segment negatively inclined contained in the Pareto frontier (as in Figure 3). If any player plays the same action on profiles A and B, then there is no PPE payoff in $\overline{AB}$.*

Proof. For simplicity suppose again that $\overline{AB}$ is negatively inclined and that A is player 1's least favored action. Suppose also it is player 2 that plays the same action on profiles A and B. Therefore, player 1's best response to A_2 dominates B_1. By Lemma 4 there is no efficient equilibrium in $\overline{AB}$. ■

For the prisoner's dilemma example of section 1.1, since the Pareto frontier is made up of segments $\overline{cd, cc}$ and $\overline{cc, dc}$, this theorem applies immediately. Hence, for almost every prisoner's dilemma there are no efficient equilibria.

We now turn to the anti-folk theorem. The additional assumptions used are necessary because there are pathological examples in which the folk theorem is true for the stage game, so these cases have to be excluded.

Theorem 2 (Anti-folk) *The folk theorem is false generically for games with feasible payoffs strictly dominating the minimax point and without efficient static equilibria. That is, almost every such game has feasible individually rational payoffs which are not equilibrium payoffs.*

Proof. Consider a game with mixing and only two pure strategy profiles on each edge of the feasible set (this is the generic situation). Take a segment $\overline{AB}$ with a point strictly dominating the minimax value. Suppose that it is negatively inclined, and vertices labeled as in the Lemma 3.

In any equilibrium payoffs in $\overline{AB}$ player 1 gets strictly more than by playing his best response to A. But the best response to A yields at least as much utility as his minimax value. The positively inclined case is similar. This completes the proof. ■

5 Conclusions

Our main contribution is to show that, even though games with imperfect monitoring admit an approximate folk theorem (e.g., Fudenberg, Levine and Maskin, 1994), an exact folk theorem is valid only for a zero measure set of such games.

This finding also shows that the classical folk theorem is extremely unstable with respect to imperfect monitoring. Given a game with perfect monitoring, if its informational structure suffers a random perturbation, the classical folk theorem will be false with probability one.

We must point out that our anti-folk theorem is not simply an inefficiency result, since there are open sets of games with efficient PPEs. Yet, the message that the prisoner's dilemma and other similar examples gives is that exact efficiency is difficult to achieve for repeated games.

Finally, we believe that the characterization and the method presented here may be also useful to studying particular games as well.

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