Supermodular Bayesian Implementation: Learning and Incentive Design*

By Laurent Mathevet †

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Abstract

I develop supermodular implementation in incomplete information. Supermodular implementable social choice functions (scf) are scf that are Bayesian implementable with mechanisms that induce a supermodular game. If a mechanism induces a supermodular game, agents may learn to play some equilibrium in a dynamic setting. The paper has two parts. The first part is concerned with sufficient conditions for (truthful) supermodular implementability in quasilinear environments. There, I describe a constructive way of modifying a mechanism so that it supermodularly implements a scf. I prove that, any Bayesian implementable decision rule that satisfies a joint condition with the valuation functions, requiring their composition to produce bounded substitutes, is (truthfully) supermodular implementable. This joint condition is always satisfied on finite type spaces; it is also satisfied by C^2 decision rules and valuation functions on a compact type space. Then I show that allocation-efficient decision rules are (truthfully) supermodular implementable with balanced transfers. Third, I establish that C^2 Bayesian implementable decision rules satisfying some dimensionality condition are (truthfully) supermodular implementable with an induced game whose interval prediction is the smallest possible. The second part provides a Supermodular Revelation Principle.

Keywords: Implementation, mechanisms, learning dynamics, stability, strategic complementarities, supermodular games.

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1 Introduction

The question of how an equilibrium outcome arises in a mechanism is largely open in implementation theory and mechanism design. Theoretical and experimental works

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†Humanities and Social Sciences, MC 228-77, California Institute of Technology, Pasadena CA 91125, USA. E-mail Address: lmath@hss.caltech.edu

have revealed that learning and stability are serious issues in many existing mechanisms.[1] This is particularly troublesome, because the idea behind implementation and mechanism design is usually normative and practical in nature: Incentive design explicitly aims to construct mechanisms that achieve some socially desirable outcome in equilibrium.

This paper develops supermodular Bayesian implementation. This theory contributes to the literature on incentives by its explicit purpose and methodology. Supermodular Bayesian implementation explicitly aims to improve learning and stability in mechanism design. Supermodular implementable scf are scf that are Bayesian (weakly) implementable with a mechanism that induces a supermodular game. If we think of messages as real numbers, a supermodular game is a game in which the marginal utility that an agent receives from playing a bigger message increases as other players also play bigger messages. In such games, best-responding behaviors are always monotone which helps boundedly rational agents find their way to equilibrium. Therefore, this theory contributes to fill the important gap in the literature emphasized in Jackson [27]: "Issues such as how well various mechanisms perform when players are not at equilibrium but learning or adjusting are quite important [...] and yet have not even been touched by implementation theory. [This topic] has not been looked at from the perspective of designing mechanisms to have nice learning or dynamic properties."

Contrary to the traditional methodology of mechanism design and implementation, the mechanisms in this paper derive their learning and stability properties from the game that they induce and not from the solution concept. One striking feature of the traditional approach is that it postulates that the solution concept, used by the mechanism to implement some social choice functions (scf), captures the properties of the mechanism. But most solution concepts are subject to criticisms on the basis of learning and stability. Implementation in dominant strategies and implementation in undominated strategies are examples.[2] Yet recent economic research still concentrates on this methodology. But why focus on the solution concept? My paper proposes an alternative approach by using a weak solution concept - Bayesian Nash equilibrium - and by instead focusing on supermodular games as the class of games induced by the mechanism.

Supermodular games are theoretically appealing in mechanism design and implementation. Milgrom and Roberts [37] and Vives [52] have shown that, under adaptive learning dynamics, play in supermodular games ends up in between the least and the greatest Nash equilibrium. For example, Cournot dynamics, fictitious play and Bayesian learning are adaptive dynamics. This convergence result extends to the kind of sophisticated learning dynamics considered in Milgrom and Roberts [38]. Adaptive and sophisticated learning dynamics encompass such a wide range of backward and forward-looking behaviors that they confer robustness on supermodular games. In particular, if the equilibrium is unique, then convergence to the equilibrium is ensured.

[1] Muench and Walker [40] , Cabrales [7] and Cabrales and Ponti [8] show that learning and stability may be serious issues in (resp.) the Groves-Ledyard [22], Abreu-Matsushima [4] and Sjöström [45] mechanisms. On the experimental side, Healy [23] and Chen and Tang [12] provide evidence that convergence may fail to occur in various mechanisms, such as Proportional Tax or the paired-difference mechanism.

[2] Implementation in dominant strategies is restrictive and only gives unambiguous learning results when dominance is strict (Saijo et al. [44], Cason [10]). And implementation in undominated strategies relies on eliminating weakly dominated strategies, so it has the perverse consequence of excluding limit points of some learning dynamics (Cabrales [7] and Cabrales and Ponti [8])

But supermodular games are also attractive in an implementation framework because their mixed strategy equilibria are locally unstable under monotone adaptive dynamics like Cournot dynamics and fictitious play (Echenique and Edlin [19]). While ruling out mixed strategy equilibria is often unsatisfactory in implementation theory, it is sensible in supermodular implementation. To the contrary, many pure-strategy equilibria are stable; in a parameterized supermodular game, all those equilibria that have monotone comparative statics are stable, such as the extremal equilibria (Echenique [17]).

Supermodular games and mechanisms that induce such games are supported by strong experimental evidence. Chen and Gazzale [14] presents experiments on a parameterized game whose parameter determines the degree of complementarities. They obtain unambiguous results that in this game, convergence is significantly better when the parameter lies in the range where the game is supermodular. In mechanism design, experiments on the Groves-Ledyard mechanism have shown that convergence is far better when the punishment parameter is high than when it is low (Chen and Plott [11] and Chen and Tang [12]). It turns out that the Groves-Ledyard mechanism induces a supermodular game when the punishment parameter is high. Finally, Healy [23] tests five public goods mechanisms in a repeated game setting and observes convergence only in those mechanisms that induce a supermodular game.

The centerpiece of my analysis is Theorem 1. It establishes that in quasilinear environments with real type spaces, any Bayesian implementable scf that satisfies a joint condition with the valuation functions, requiring their composition to produce bounded substitutes, is truthfully supermodular implementable. That is, if the joint condition holds, any Bayesian incentive-compatible scf can be implemented by a direct mechanism that induces a supermodular game. This joint condition is always satisfied on finite type spaces; it is also satisfied by twice-continuously differentiable scf and valuation functions on a compact type space. So, the result is fairly general. Beyond the claim itself, this theorem describes a constructive way of modifying an existing mechanism so that it supermodularly implements a scf. The main insight is that it is always possible to add complementarities into the transfers without affecting the incentives. The technique is simple, yet powerful. I explain it formally in the next section in the context of a public goods example. The intuition is that incentives lie at the expected-value level while complementarities reside in the complete information payoffs. Therefore, adding any function with null expected value to the transfers does not alter incentives, but may change the shape of the best-responses. There are functions whose complementarities are strong but expectation is null.

In quasilinear environments, the mechanism designer is often interested in that there be no transfers into or out of the system. This is known as the budget balance condition. It is quite important, because allocation-efficiency and budget balancing imply full efficiency. Achieving full efficiency is difficult under dominant strategy implementation (Green and Laffont [21]) but possible under Bayesian implementation (Arrow [5] and d'Aspremont and Gérard-Varet). Theorem 2 shows that budget balancing is also possible under supermodular Bayesian implementation. Any allocation-efficient decision rule is supermodular implementable with balanced transfers, if the joint condition of bounded substitutes is satisfied.

Given supermodular implementation relies on weak implementation, it is as useful as the bounds represented by the greatest and the least equilibrium are tight. The truthful equilibrium indeed delivers the desired outcome, but the space between the extremal equilibria may contain undesired equilibrium outcomes that are limit points of learning dynamics. If those bounds define a small interval and the scf is continuous, then at least learning dynamics do not lead to a social outcome that is far from the desired outcome. I deal with the multiple equilibrium problem by developing optimal and unique supermodular implementation. Optimal supermodular implementation involves designing a mechanism whose induced supermodular game generates the weakest complementarities in the class of mechanisms that supermodularly implement the scf. I prove that this produces the tightest bounds among those mechanisms (Proposition 2). Therefore, an optimally supermodular implementable scf cannot admit a direct mechanism that induces a supermodular game with narrower bounds. The main result (Theorem 3) is that all twice-continuously differentiable and Bayesian implementable scf satisfying some dimensionality condition are optimally supermodular implementable on smooth domains. Unique supermodular implementation defines that situation where the truthful equilibrium is the unique equilibrium of the supermodular game induced by a mechanism. All adaptive dynamics converge (Milgrom and Roberts [37]). Theorem 4 provides sufficient conditions for unique supermodular implementation.

The paper presents traditional models where supermodular Bayesian implementation applies and examples where it gives sharp predictions. Supermodular implementation can be applied to public goods models. In a public goods example with quadratic preferences (Section 2), a designer uses the expected externality mechanism to implement an allocation-efficient scf. In the game induced by the mechanism, many learning dynamics cycle and fail to converge to the truthful equilibrium. Nevertheless, the mechanism can be modified to induce a supermodular game where all adaptive dynamics pin down the truthful equilibrium. Supermodular implementation can also be applied to the traditional principal-agent problem with hidden information. In a team-production example,[3] a principal contracts with a set of agents and monitors their contribution to maximize net profits. The scf is optimally supermodular implementable and truthtelling is the unique equilibrium of the supermodular game induced by the mechanism. But there are applications that are challenging for the present theory. I give examples of binary-choice models such as auctions and public goods that violate the condition of bounded substitutes. A possible way around this problem is approximate implementation. For those scf, I show that there exist arbitrarily close scf that are supermodular implementable.

Although quasilinear environments are common in mechanism design and implementation, it is important to consider general preferences. One of the first questions that come to mind is that of the restrictiveness of direct mechanisms. Under weak implementation, the traditional Revelation Principle says that direct mechanisms cause no loss of generality in dominant strategies or Bayesian equilibrium. Answering the same question for supermodular implementation is particularly relevant, because the space of mechanisms to consider is very large. The Supermodular Revelation Principle (Theorems 5 and 6) says that if there exists a mechanism that supermodularly implements a scf such that the range of the equilibrium strategies in the desired equilibrium is a complete lattice, then there is a direct mechanism that supermodularly implements that scf truthfully. I give an example of a supermodular implementable scf where this range is not a lattice and that cannot be supermodularly implemented by any direct mechanism. Thus, the example suggests that the condition of the theorem is somewhat minimally sufficient. Although this revelation principle is not as general as the tradi-

[3]This is a simplified version of the team production model of McAfee and McMillan [34].

tional one, it measures the restriction imposed by direct mechanisms in supermodular implementation and gives conditions that may warrant their use.

A number of other papers are related to learning and stability in the context of implementation or mechanism design. Chen [13] deserves mention because it is one of the first papers explicitly aimed at learning and stability in mechanism design. In a complete information environment with quasilinear utilities, she constructs a mechanism that Nash implements Lindahl allocations and induces a supermodular game. My paper builds the framework of supermodular Bayesian implementation and generalizes her result in incomplete information. Abreu and Matsushima [3] establishes that for any scf f and $\epsilon > 0$, there is an ϵ-close scf f_ϵ for which a mechanism exists where iterative deletion of strictly dominated strategies leads to a unique profile whose outcome is f_ϵ.[4] In their terminology, any scf is virtually implementable in iteratively undominated strategies. The result is very general and the solution concept is strong enough to predict convergence of many learning dynamics to the unique equilibrium outcome.[5] However, there are arguments questioning this result on the basis of learning and stability. Cabrales [7] argues that the concept of virtual implementation is not as innocuous as it first appears. When the mechanism implements f_ϵ, it actually implements it in iteratively strictly ϵ-undominated strategies. In other words, elimination of weakly dominated strategies is the solution concept that underlies the exact-implementation problem for f (See Abreu and Matsushima [4]); virtual implementation is a way of turning it into elimination of strictly dominated strategies for f_ϵ. Another weakness of their result is that it does not seem to extend to infinite sets of types; this issue is related to important theoretical questions (Duggan [16]) and it is not merely technical. The Abreu-Matsushima mechanism also employs a message space whose dimension increases to infinity as ϵ vanishes. In contrast to Abreu-Matsushima [3], this paper studies exact implementation with direct mechanisms on finite or infinite type sets. Cabrales [7] demonstrates that there are learning dynamics that converge to equilibria of the canonical mechanism for Nash implementation. But those dynamics require players to strictly randomize over all possible improvements on past play.[6] This rules out many natural dynamics considered here. In addition to these papers, there are general impossibility results on the stability of equilibrium outcomes in the Nash implementation of Walrasian and Lindahl allocations (Jordan [28] and Kim [30]).

2 Motivation and Intuition

This section provides an economic example of a designer who uses the expected externality mechanism (Arrow [5] and d'Aspremont and Gérard-Varet [15]) to implement a scf. The environment is simple: Two agents with smooth utilities and compact real type spaces. Yet the mechanism induces a game where learning and stability fail under many dynamics.

Then I describe a new approach which consists in modifying the existing mechanism in order to induce a supermodular game. In the example, the benefit is immediate: All adaptive and sophisticated dynamics converge to the truthful equilibrium, and the

[4] Abreu and Matsushima use the Euclidean metric.

[5] See e.g Milgrom and Roberts [38]. But note that there are games where some adaptive dynamics à la Milgrom and Roberts [37] do not converge to a uniquely rationalizable profile.

[6] This feature allows play to exit the integer game when players fallen into it.

equilibrium is stable.

2.1 A Public Goods Example

Consider a principal who needs to decide the level of a public good, such as the size of a bridge. Let $X = [0, 2]$ denote the possible values of the public good. There are two agents, 1 and 2, whose type spaces Θ_1 and Θ_2 are $[0, 1]$. Types are independently uniformly distributed. The agents' preferences are quasilinear, $u_i(x, \theta_i) = V_i(x, \theta_i) + t_i$, where $x \in X$, $\theta_i \in \Theta_i$, and $t_i \in \mathbb{R}$ is the transfer from the principal to agent i, $i = 1, 2$. The valuation functions are $V_1(x, \theta_1) = \theta_1 x - x^2$ and $V_2(x, \theta_2) = \theta_2 x + x^2/2$.

The principal wants to make an allocation-efficient decision, that is, she aims to maximize the sum of the valuation functions by choosing $x^*(\theta) = \theta_1 + \theta_2$. Since $x^*(.)$ is not directly enforceable, the principal will use a mechanism with monetary transfers to entice the agents to reveal their true type. The principal opts for the expected externality mechanism,[7] as it allows direct implementation of allocation-efficient decision rules. Therefore, she sets the transfers as follows:

$$t_1(\hat{\theta}_1, \hat{\theta}_2) = E_{\theta_2}\left[\theta_2(\hat{\theta}_1 + \theta_2) + \frac{(\hat{\theta}_1 + \theta_2)^2}{2}\right] - E_{\theta_1}[\theta_1(\theta_1 + \hat{\theta}_2) - (\theta_1 + \hat{\theta}_2)^2]$$

and

$$t_2(\hat{\theta}_1, \hat{\theta}_2) = E_{\theta_1}[\theta_1(\theta_1 + \hat{\theta}_2) - (\theta_1 + \hat{\theta}_2)^2] - E_{\theta_2}\left[\theta_2(\hat{\theta}_1 + \theta_2) + \frac{(\hat{\theta}_1 + \theta_2)^2}{2}\right].$$

I will study learning and stability in this example. Time proceeds in discrete periods $t \in \{0, 1, \ldots\}$ and agents are assumed to learn or evolve as time passes, according to some learning rule. At each time t, the two agents meet with the principal to play the Bayesian game induced by the expected externality mechanism. The principal initiates the process. The agents observe the history of play from 0 to $t-1$ and then publicly play a strategy. From the strategies played in the past, each agent updates her beliefs about her opponent's future strategy using some specified rule; then, given those updated beliefs, she plays the strategy which maximizes her expected payoffs in the mechanism. Since the focus here will be on convergence and stability rather on the speed of convergence, no stopping time is specified.

In this context, a strategy is a deception. A deception for player i at period t is a function $\hat{\theta}_i^t : \Theta_i \to \Theta_i$, that is, a player announces some type for each true type.[8]

The questions are: Will the profile played at t converge to the truthful equilibrium as $t \to \infty$? If players were in the truthful equilibrium, will they return to this equilibrium after an exogenous perturbation? The first question asks whether the agents ever learn to play truthfully and reach an agreement. The second one asks whether truthtelling is a stable equilibrium.

The players' best-replies determine the answer. For $i = 1, 2$, define the set of deceptions Σ_i as the set of measurable functions in $\Theta_i^{\Theta_i}$ and let $\mathcal{P}(\Sigma_i)$ be the set of (Borel) probability measures over Σ_i. Let $\mu_i^t \in P(\Sigma_j)$ be player i's beliefs about player

[7]See e.g Section 23.D in Mas-Colell et al. [35].

[8]Announcing a deception in the Bayesian game might seem more realistic when type sets are finite (the example would have similar conclusions in the finite case), but it will come down to choosing an intercept between -1 and 1.

j's deceptions at time t. Those beliefs will depend on the history of play and so μ_i^t defines the different learning models. Let $br_i^t : \Theta_i \to \Theta_i$ denote player i's best-reply at time t as a function of her true type.

For any beliefs μ_i^t, player i's expected utility in the mechanism are strictly concave in her own announcement. Therefore, computing the first-order condition gives

$$br_1^t(\theta_1) = \min\{\max\{\theta_1 + 1 - 2E[\, E_{\theta_2}[\hat{\theta}_2^t(\theta_2)] \,|\, \mu_1^t], 0\}, 1\} \quad (1)$$

$$br_2^t(\theta_2) = \min\{\max\{\theta_2 - \frac{1}{2} + E[\, E_{\theta_1}[\hat{\theta}_1^t(\theta_1)] \,|\, \mu_2^t], 0\}, 1\}. \quad (2)$$

where $E[.|\, \mu_i^t]$ is an expectation with respect to i's beliefs μ_i^t over Σ_j.

In the dynamical system given by (1) and (2), if player 1 believes player 2's strategy has increased on average, then 1 decreases her strategy twice as much and vice-versa; whereas 2 tries to match any average-variation in 1's strategy. This game has a flavor of "matching-pennies," and this will be the source of instability and learning deficiency.

Learning often fails to occur in this example. There are many learning dynamics for which, not only do the agents not converge to truth-revealing, but they are unable to reach a decision as the play cycles forever. Consider first fictitious play in finite versions of the example. For simplicity, let Θ_i be the finite type set $\{0, .5, 1\}$, so Σ_i is finite. Suppose everything else is unchanged. Consider the model of weighted fictitious play (See e.g Ho [24]). Deceptions are initially given arbitrary weights and beliefs are updated by depreciating all weights by $1 - \phi$ and adding one to the weight of the opponent's deception played at $t - 1$. Here, if players use an identical rule, the profile converges to the truthful equilibrium unless ϕ is high enough ($\phi > .8$), in which case cycling occurs. But there is no reason a priori for both players to use the same learning rule. For asymmetric rules, learning becomes more uncertain. The player with the highest ϕ often outweighs the other one in a non-linear fashion and prevents learning.[9]

Consider now the original model with continuous types. Cournot dynamics suffers from cycling and this conclusion holds wherever the principal sets the starting profile (except truthtelling). Besides, if the agents were to play the truthful equilibrium, the slightest belief perturbation would destabilize it.

Of course, Cournot dynamics is prone to cycling, since the past only matters through the last period. But cycling prevails for many families of dynamics with a larger memory size.[10]

Learning also fails to occur for other forms of learning dynamics. Adaptive dynamics do not encompass all rational behaviors, such as forward-looking behaviors. Unfortunately, many sophisticated learning processes à la Milgrom-Roberts [38] are also plagued with cycles in the example.

While supermodularity is not necessary for convergence, those learning failures can be interpreted as a lack of complementarities. It is clear from (1) and (2) that the game

[9]If 1 learned according to a fictitious play rule with ϕ_1 while 2 used ϕ_2, then the sequence would enter a cycle for many values of $\phi_1 \geq .9, \phi_2 \geq .55$

[10]Consider dynamics where players only remember the last T periods. They assign a probability ϕ to the deception played at $t - 1$ and $(1 - \phi)\delta^k/C$ to that played at $t - k$ where C is normalized so that the probabilities add up to one. Simulations reveal that learning fails under many values of the parameters. Let $(\hat{\theta}_1^0(.), \hat{\theta}_2^0(.))$ be the pair of zero-functions. For $T \in \{2, 3\}, \delta = .9$ and $\phi \geq .5$, the process enters a cycle even though the last few periods are weighted almost equally. This suggests that increasing the memory size may improve learning. For $T = 4, \delta = .8$ and $\phi \leq .65$, the profile converges to the truthful equilibrium, but it cycles for $\phi \geq .7$. A larger memory does not necessarily improve learning, as cycling reappears when $T = \{5, 6\}, \delta = .8$ for values of ϕ below .65.

induced by the expected externality mechanism is not supermodular, for the best-replies cannot be both increasing.

2.2 Intuition in Differentiable Environments

The theory in this paper suggests to transform an existing mechanism into one which induces a supermodular game. The general transformation technique is simple and efficient. After transforming the mechanism in the previous example, all adaptive dynamics now converge to the truthful equilibrium, and the equilibrium is stable.

Consider a twice-differentiable environment where $f = (x, t)$ is truthfully Bayesian implementable.[11] I search for transfer functions $\{t_i^{SM}\}$ that solve the following system of equations:

$$E_{\theta_{-i}}[t_i^{SM}(., \theta_{-i})] = E_{\theta_{-i}}[t_i(., \theta_{-i})], i = 1, \ldots, n. \quad (3)$$

$$\frac{\partial^2 V_i(x_i^*(\hat{\theta}_i, \hat{\theta}_{-i}), \theta_i)}{\partial\hat{\theta}_i \partial\hat{\theta}_j} + \frac{\partial^2 t_i^{SM}(\hat{\theta}_i, \hat{\theta}_{-i})}{\partial\hat{\theta}_i \partial\hat{\theta}_j} \geq 0, \text{ for all } \hat{\theta}, \theta_i, i = 1, \ldots, n, j \neq i. \quad (4)$$

Condition (3) says that t_i^{SM} and t_i have the same expected value when j plays truthfully. If i's best-reply under t_i was to tell the truth when j played truthfully, then it is still the case under t_i^{SM}. So (x^*, t^{SM}) is truthfully Bayesian implementable. Condition (4) demands that the cross-partials of t_i^{SM} compensate those of $V_i \circ x_i^*$, so the induced Bayesian game is supermodular.[12]

The main insight of the paper is that *it is always possible to add complementarities into the transfers without affecting the incentives* that appear in the expected value (3). The general transformation technique appears later in the paper, but the following transfers are an example:

$$t_i^{SM}(\hat{\theta}) = \sum_{j \neq i} \rho_i \hat{\theta}_i \hat{\theta}_j + E_{\theta_{-i}}[t_i(\hat{\theta}_i, \theta_{-i})] - \sum_{j \neq i} \rho_i \hat{\theta}_i E_{\theta_j}[\theta_j], \; i = 1, \ldots, n. \quad (5)$$

Clearly, (5) satisfies (3) for any ρ_i, $i = 1, \ldots, n$, and $\partial^2 t_i^{SM}(\hat{\theta})/\partial\hat{\theta}_i \partial\hat{\theta}_j = \rho_i$. If $\partial^2 V_i(x_i^*(\hat{\theta}_i, \hat{\theta}_j), \theta_i)/\partial\hat{\theta}_i \partial\hat{\theta}_j$ is bounded below, a property that I call *bounded substitutes*, then (4) simply requires finding a real number ρ_i that exceeds the absolute value of that lower bound.

In addition to its simplicity, this technique is powerful. The public goods model of Section 2.1 is an example. There, (x^*, t) is truthfully Bayesian implementable by virtue of the expected externality mechanism. Since the assumption of bounded substitutes holds, there exist ρ_i, $i = 1, 2$, such that (x^*, t^{SM}) is supermodular implementable. In this example, Theorem 4 of Section 5.3 implies that there are values ρ_1 and ρ_2 for which truthtelling is the unique equilibrium of the supermodular mechanism. In contrast to the expected externality mechanism, all adaptive dynamics now converge to the truthful equilibrium, and the equilibrium is stable.

[11] A scf is truthfully Bayesian implementable if $\theta_i^*(\theta_i) = \theta_i$ for all θ_i and $i = 1, \ldots, n$ is a Bayesian equilibrium of the direct mechanism $\Gamma = (\Theta, f)$.

[12] If the complete information payoffs define a supermodular game for each $\theta \in \Theta$, then the Bayesian game is supermodular.

3 Lattice-theoretic Definitions and Supermodular Games

The basic definitions of lattice theory in this section are discussed in Milgrom-Roberts [37] and Topkis [48].

A set M with a transitive, reflexive, antisymmetric binary relation $\succeq$ is a *lattice* if for any $x, y \in M$, $x \vee y = \sup_M\{x, y\}$ and $x \wedge y = \inf_M\{x, y\}$ exist. It is *complete* if for every non-empty subset A of M, $\inf_M A$ and $\sup_M A$ exist. A nonempty subset A of M is a *sublattice* if for all $x, y \in A$, $x \vee y, x \wedge y \in A$. A *closed interval* $[x, y]$ in M is the set of $m \in M$ such that $y \succeq m \succeq x$. The *order-interval* topology on a lattice is the topology whose subbasis for the closed sets is the set of closed intervals. In Euclidean spaces the order-interval topology coincides with the usual topology. Lattices are endowed with their order-interval topology.

For two nonempty subsets A, B of M, A is smaller than B in the *strong set order*, denoted $A \sqsubseteq B$, if $a \in A$ and $b \in B$ imply that $a \wedge b \in A$ and $a \vee b \in B$. Let $(\Theta, \geq)$ be a lattice. A correspondence $\phi : \Theta \twoheadrightarrow M$ is increasing if for any $\theta, \theta' \in \Theta$ with $\theta' \geq \theta$, we have $\phi(\theta) \sqsubseteq \phi(\theta')$.

Let T be a partially ordered set; $g : M \to \mathbb{R}$ is *supermodular* if, for all $m, m' \in M$, $g(m) + g(m') \leq g(m \wedge m') + g(m \vee m')$; $g : M \times T \to \mathbb{R}$ has *increasing (decreasing) differences* in (m, t) if, whenever $m \succeq m'$ and $t \succeq t'$, $g(m, t) - g(m', t) \geq (\leq) g(m, t') - g(m', t')$; $g : M \times T \to \mathbb{R}$ satisfies the *single-crossing property* in (m, t) if, whenever $m \succeq m'$ and $t \succeq t'$, $g(m'', t') \geq g(m', t')$ implies $g(m'', t'') \geq g(m', t'')$ and $g(m'', t') > g(m', t')$ implies $g(m'', t'') > g(m', t'')$. If g has decreasing differences in (m, t), then variables m and t are said to be substitutes. If g has increasing differences or satisfies the single-crossing property in (m, t), then m and t are said to be complements.

A game is described by $(N, \{(M_i, \succeq_i)\}, u)$, where N is a finite set of players, and each player $i \in N$ has a strategy space M_i with an order $\succeq_i$ and a payoff function $u_i : \prod_{i \in N} M_i \to \mathbb{R}$ such that $u = (u_i)$.

Definition 1 *A game $\mathcal{G} = (N, \{(M_i, \succeq_i)\}, u)$ is supermodular if for all $i \in N$,*

1. *$(M_i, \succeq_i)$ is a complete lattice;*
2. *u_i is bounded, supermodular in m_i for each m_{-i} and has increasing differences in (m_i, m_{-i});*
3. *u_i is upper-semicontinuous in m_i for each m_{-i}, and continuous in m_{-i} for each m_i.*

4 The Framework of Supermodular Bayesian Implementation

Let $N = \{1, \ldots n\}$ denote a collection of agents, indexed by i and j. A collective planner faces a set Y of alternatives, with generic element $y \in Y$ from which the planner must choose. Let Y be equipped with σ-algebra $\mathcal{Y}$. For each agent $i \in N$, let Θ_i be the set of i's possible types equipped with σ-algebra $\mathcal{F}_i$. Let $\Theta = \prod_{i \in N} \Theta_i$ be equipped with σ-algebra $\mathcal{F} = \times_{i \in N} \mathcal{F}_i$. Agents have a common prior ϕ on $(\Theta, \mathcal{F})$ known to the planner.

The planner's desired outcomes are represented by a social choice function $f : \Theta \to Y$ that is $\mathcal{F}$-measurable.

A mechanism is a tuple $\Gamma = (\{(M_i, \succeq_i)\}, g)$ where agent i's message space M_i is endowed with an order $\succeq_i$ and an underlying σ-algebra $\mathcal{M}_i$; letting $\mathcal{M} = \times_{i\in N}\mathcal{M}_i$, $g : M \to Y$ is an outcome function that is $\mathcal{M}$-measurable. A strategy for agent i is a measurable function $m_i : \Theta_i \to M_i$. Denote by $\Sigma_i(M_i)$ the set of equivalence classes of measurable functions from $(\Theta_i, \mathcal{F}_i)$ to M_i. This set is endowed with the pointwise order, also denoted $\succeq_i$. A direct mechanism is one for which each $M_i = \Theta_i$, $\mathcal{M}_i = \mathcal{F}_i$ and $g = f$. In this case, $\Sigma_i(\Theta_i)$ is called the set of i's deceptions and its elements are denoted $\hat{\theta}_i(.)$. Direct mechanisms vary by the order on type spaces.

Each agent i's preferences over alternatives are given by a utility function $u_i : Y\times\Theta_i \to \mathbb{R}$ that is $\mathcal{Y}\times\mathcal{F}_i$-measurable. These utility functions are uniformly bounded by some $\overline{u}$. For any $h : \Theta \to \mathbb{R}$, denote $E_\theta[h(\theta)] = \int_\Theta h(\theta)d\phi(\theta)$. For $m_{-i}(.) \in \prod_{j\neq i} \Sigma_j(\Theta_j)$, agent i's preferences over strategy profiles in $\Sigma_i(\Theta_i)$ are given by i's ex-ante payoffs, defined as

$$u_i^g(m_i(.), m_{-i}(.)) = E_\theta[u_i(g(m_i(\theta_i), m_{-i}(\theta_{-i})), \theta_i)].$$

When types are independently distributed, let $\phi = \times\phi_i$ where ϕ_i is defined on $(\Theta_i, \mathcal{F}_i)$. For any $h : \Theta_{-i} \to \mathbb{R}$, denote $E_{\theta_{-i}}[h(\theta_{-i})] = \int_{\Theta_{-i}} h(\theta_{-i})d\phi_{-i}(\theta_{-i})$. For $m_i \in M_i$ and $m_{-i}(.) \in \prod_{j\neq i} \Sigma_j(\Theta_j)$, agent i's interim payoffs at type θ_i are $E_{\theta_{-i}}[u_i(g(m_i, m_{-i}(\theta_{-i}))$ $, \theta_i)]$. Player i's ex-ante payoffs can be written

$$u_i^g(m_i(.), m_{-i}(.)) = \int_{\Theta_i} E_{\theta_{-i}}[u_i(g(m_i, m_{-i}(\theta_{-i})), \theta_i)]\, d\phi_i(\theta_i).$$

The Bayesian game induced by mechanism Γ is $\mathcal{G} = (N, \{(\Sigma_i(M_i), \succeq_i)\}, u^g)$ where $u^g = (u_i^g)$ is the vector of ex-ante payoffs. If a scf is Bayesian implementable with a mechanism that induces a supermodular game, then it is supermodular implementable in the sense defined next.

Definition 2 *The mechanism Γ supermodularly implements the scf $f(.)$ if there exists a Bayesian equilibrium $m^*(.)$ such that $g(m^*(\theta)) = f(\theta)$ for all $\theta \in \Theta$, and if the induced game $\mathcal{G}$ is supermodular. The scf f is said to be supermodular Bayesian implementable.*

Definition 3 *A scf is truthfully supermodular Bayesian implementable (TSBI) if there exists a direct mechanism that supermodularly implements the scf $f(.)$ such that $\hat{\theta}(\theta) = \theta$ for all $\theta \in \Theta$ is a Bayesian equilibrium.*

The Bayesian game $\mathcal{G}$ is formulated in its ex-ante version as opposed to interim. Beyond the traditional arguments contrasting those two versions, there are important technical differences between ex-ante and interim supermodular Bayesian games (Van Zandt [50]). In particular, the results in Milgrom-Roberts [37] can only be directly applied to the ex-ante version $\mathcal{G}$.

5 Supermodular Implementation on Quasilinear Domains

This section deals with supermodular implementation when agents have quasilinear utility functions. The objective is to give general conditions under which a scf is *TSBI* and the mechanism satisfies some further requirements. There are four main results.

The first provides general conditions for supermodular implementability. The second answers the question of supermodular implementation and budget balancing. The third gives sufficient conditions for a scf to be *TSBI* in a game form whose interval between extremal equilibria is the smallest possible. The fourth offers sufficient conditions for unique supermodular implementation.

5.1 Environment and Definitions

An alternative y is a vector $(x, t_1, \ldots, t_n)$ where x is an element of a compact set $X \subset \mathbb{R}^m$ and $t_i \in \mathbb{R}$ for all i. Each agent i has a type space $\Theta_i \subset \mathbb{R}$ (finite or infinite). Endow Θ_i with the usual order. Notice that $\Sigma_i(\Theta_i)$ is a complete lattice with the pointwise order.[13]

The set X_i is a compact subset of $\mathbb{R}^{m_i}$ such that $\prod_{i \in N} X_i = X$. For all $i \in N$, preferences are quasilinear with Bernoulli utility function $u_i(x, \theta_i) = V_i(x_i, \theta_i) + t_i$ where $x_i \in X_i$ and $t_i \in \mathbb{R}$. The function $V_i : X_i \times \Theta_i \to \mathbb{R}$ is called i's valuation function. Denote $V = (V_i)_{i \in N}$.

In this environment, a scf $f = (x, t)$ is composed of a decision rule $x : \Theta \mapsto (x_i(\theta))$ where $x_i : \Theta \to X_i$, and transfer functions $t_i : \Theta \to \mathbb{R}$. Typically, $x(.)$ represents the desired outcomes while transfers are chosen by the planner.

Say that V and $x(.)$ are C^2, if there exist open sets $O_i \supset \Theta_i$ and $U_i \supset X_i$, $i = 1, \ldots, n$, such that $V : U_i \times O_i \to \mathbb{R}$ and $x : \prod_{i \in N} O_i \to U_i$ are C^2.

Define the continuous family of decision rules and valuation functions as

$$\begin{aligned}\mathcal{F}_{\text{cont}} = \{(\boldsymbol{V}, x(.)) : \boldsymbol{V}{:}(x, \theta) \mapsto (V_i(x_i, \theta_i)), V_i \text{ is bounded, } V_i(x_i(\hat{\theta}), \theta_i) \text{ is continuous in } \\ \hat{\theta}_{-i} \text{ for fixed } \hat{\theta}_i, \theta_i \text{ and } V_i(x_i(\hat{\theta}), \theta_i) \text{ is usc in } \hat{\theta}_i \text{ for fixed } \hat{\theta}_{-i}, \theta_i, \text{ for all } i \in N\}.\end{aligned}$$

Agents' types are assumed to be independently distributed. For all $i \in N$, the distribution of i's types admits a bounded density with full support.

Here a scf f is *TSBI* if, in the direct mechanism with the usual order $\geq_i$ on $\mathbb{R}$, truthtelling is a Bayesian equilibrium.

The following definitions concern the composition of the valuation functions and the scf. For any $\theta_i', \theta_i'' \in \Theta_i$, let

$$\Delta V_i((\theta_i'', \theta_i'), \hat{\theta}_{-i}, \theta_i) = V_i(x_i(\theta_i'', \hat{\theta}_{-i}), \theta_i) - V_i(x_i(\theta_i', \hat{\theta}_{-i}), \theta_i).$$

Say that (V, x) has *bounded substitutes* or that $V_i \circ x_i$ has substitutes bounded by T_i, if for all $i \in N$, there is $T_i \in \mathbb{R}$ such that, for all $\theta_i'' \geq \theta_i'$ and $\theta_{-i}'' \geq \theta_{-i}'$, $\Delta V_i((\theta_i'', \theta_i'), \theta_{-i}'', \theta_i) - \Delta V_i((\theta_i'', \theta_i'), \theta_{-i}', \theta_i) \geq T_i(\theta_i'' - \theta_i') \sum_{j \neq i}(\theta_j'' - \theta_j')$ for all $\theta_i \in \Theta_i$. Consider the case where $N = \{1, 2\}$ as an illustration. The condition means that there exists a real number such that, as (say) agent 2 increases her announcement, the marginal valuation that agent 1 receives from increasing her announcement can decrease by no more than the product between that real number and the increase in each agent's announcement.[14] In twice-continuously differentiable environments, the condition simply means that the cross-partial derivatives, $\partial^2 V_i(x_i(\hat{\theta}), \theta_i)/\partial\hat{\theta}_i\partial\hat{\theta}_j$, are uniformly bounded below. Hence it requires that if agents' announcements are strategic substitutes in the game with no transfers,[15] then at least they are boundedly so. Notice that this assumption is always

[13] See Lemma 1 in Van Zandt [50].
[14] Recall Section 2.2.
[15] See Section 3

satisfied when type sets are finite. Moreover, it is also satisfied whenever the decision rule $x_i(.)$ and the valuation functions V_i are C^2-functions for all i on compact type sets.

The pair (V, x) has *bounded complements* if $(-V, x)$ has bounded substitutes. Likewise, say that $u_i \circ f$ has bounded complements if the previous definition is satisfied when transfers are included.

The pair (V_i, x_i) is *ω-Lipschitz* in $\hat{\theta}_i$ if there exists $\omega > 0$ such that for all $\hat{\theta}_{-i}$ and θ_i, $\Delta V_i((\theta_i'', \theta_i'), \hat{\theta}_{-i}, \theta_i) \leq \omega(\theta_i'' - \theta_i')$, for all $\theta_i'' \geq \theta_i'$. The same definition applies to transfer functions. In differentiable environments, it simply means that the corresponding first-derivatives are bounded above.

The valuation functions V and decision rule $x(.)$ have *δ-increasing differences* if for each $i \in N$, there is $\delta_i > 0$ such that for all $\hat{\theta}_i'' \geq \hat{\theta}_i'$ and $\theta_i'' \geq \theta_i'$, $E_{\theta_{-i}}[\Delta V_i((\hat{\theta}_i'', \hat{\theta}_i'), \theta_{-i}, \theta_i'')] - E_{\theta_{-i}}[\Delta V_i((\hat{\theta}_i'', \hat{\theta}_i'), \theta_{-i}, \theta_i')] \geq \delta_i(\hat{\theta}_i'' - \hat{\theta}_i')(\theta_i'' - \theta_i')$.

5.2 General Result and Implementation with Budget Balance

This subsection contains two main results. The characterization theorem deals with supermodular Bayesian implementability on quasilinear families. Basically, if the decision rule and the utility functions are relatively well-behaved, in the sense of $\mathcal{F}_{\text{cont}}$ and bounded substitutes, then a decision rule is Bayesian implementable with transfers if and only if it is supermodular Bayesian implementable with transfers. The second theorem provides sufficient conditions to satisfy budget balancing.

Theorem 1 *Let $(V, x) \in \mathcal{F}_{cont}$. If (V, x) has bounded substitutes, then there exist transfers t such that the scf $f = (x, t)$ is TBI and $E_{\theta_{-i}}[t_i(., \theta_{-i})]$ is usc, if and only if, there are transfers t^{SM} such that (x, t^{SM}) is TSBI and $E_{\theta_{-i}}[t_i^{SM}(., \theta_{-i})]$ is usc. Moreover, the transfers have the same expected value: $E_{\theta_{-i}}[t_i(., \theta_{-i})] = E_{\theta_{-i}}[t_i^{SM}(., \theta_{-i})]$.*

Proof: Sufficiency is immediate. Suppose that $f = (x, t)$ is *TBI* and transfers t are truthfully-usc. Then,

$$E_{\theta_{-i}}[V_i(x_i(\theta_i, \theta_{-i}), \theta_i)] + E_{\theta_{-i}}[t_i(\theta_i, \theta_{-i})] \geq E_{\theta_{-i}}[V_i(x_i(\hat{\theta}_i, \theta_{-i}), \theta_i)] + E_{\theta_{-i}}[t_i(\hat{\theta}_i, \theta_{-i})] \quad (6)$$

for all $\hat{\theta}_i$. For $\rho_i \in \mathbb{R}$, let

$$\delta_i(\hat{\theta}_i, \hat{\theta}_{-i}) = \sum_{j \neq i} \rho_i \hat{\theta}_i \hat{\theta}_j, \quad (7)$$

and define

$$t_i^{SM}(\hat{\theta}_i, \hat{\theta}_{-i}) = \delta_i(\hat{\theta}_i, \hat{\theta}_{-i}) + E_{\theta_{-i}}[t_i(\hat{\theta}_i, \theta_{-i})] - E_{\theta_{-i}}[\delta_i(\hat{\theta}_i, \theta_{-i})]. \quad (8)$$

Note that $E_{\theta_{-i}}[t_i^{SM}(\hat{\theta}_i, \theta_{-i})] = E_{\theta_{-i}}[t_i(\hat{\theta}_i, \theta_{-i})]$ for all $\hat{\theta}_i$. Thus (x, t^{SM}) is *TBI* by (6). Moreover, $\delta_i : \Theta \to \mathbb{R}$ is continuous and bounded. So, it follows from the Bounded Convergence Theorem that $E_\theta[\delta_i(\hat{\theta}_i(\theta_i), \hat{\theta}_{-i}(\theta_{-i})) - E_{\theta_{-i}}[\delta_i(\hat{\theta}_i(\theta_i), \theta_{-i})]]$ is continuous in $\hat{\theta}(.)$. Since transfers t are truthfully-usc, Fatou's Lemma implies that $E_\theta[t_i^{SM}(\hat{\theta}_i(\theta_i), \hat{\theta}_{-i}(\theta_{-i}))]$ is usc in $\hat{\theta}_i(.)$ for each $\hat{\theta}_{-i}(.)$. Therefore, payoffs u_i^g satisfy the continuity requirements for supermodular games. Next I show that it is possible to choose ρ_i so that u_i^g has increasing differences in $(\hat{\theta}_i(.), \hat{\theta}_{-i}(.))$. By bounded substitutes, there exists T_i such that, for all $\theta_i'' \geq \theta_i'$ and $\theta_{-i}'' \geq \theta_{-i}'$, $\Delta V_i((\theta_i'', \theta_i'), \theta_{-i}'', \theta_i) - \Delta V_i((\theta_i'', \theta_i'), \theta_{-i}', \theta_i) \geq T_i(\theta_i'' - \theta_i') \sum(\theta_j'' - \theta_j')$ for all $\theta_i \in \Theta_i$. Set $\rho_i > -T_i$. Choose any $\theta_i'' \geq_i \theta_i'$ and $\theta_{-i}'' \geq_{-i} \theta_{-i}'$. The function $u_i(x_i(\hat{\theta}_i, \hat{\theta}_{-i}), \theta_i)$ has increasing differences in $(\hat{\theta}_i, \hat{\theta}_{-i})$ for each θ_i, if the following expression is positive for all θ_i,

$$V_i(x_i(\theta_i'',\theta_{-i}''),\theta_i)+V_i(x_i(\theta_i',\theta_{-i}'),\theta_i)-V_i(x_i(\theta_i'',\theta_{-i}'),\theta_i)-V_i(x_i(\theta_i',\theta_{-i}''),\theta_i)$$
$$+\sum_{j\neq i}\rho_i\big(\theta_i''\theta_j''+\theta_i'\theta_j'-\theta_i''\theta_j'-\theta_i'\theta_j''\big). \quad (9)$$

Given $\rho_i > -T_i$, (9) is greater than

$$V_i(x_i(\theta_i'',\theta_{-i}''),\theta_i)+V_i(x_i(\theta_i',\theta_{-i}'),\theta_i)-V_i(x_i(\theta_i'',\theta_{-i}'),\theta_i)-V_i(x_i(\theta_i',\theta_{-i}''),\theta_i)$$
$$-T_i(\theta_i''-\theta_i')\sum_{j\neq i}(\theta_j''-\theta_j'). \quad (10)$$

Bounded substitutes immediately imply that (10) is positive for all θ_i, hence so is (9). By Lemma 1, the utility function u_i^g has increasing differences in $(\hat{\theta}_i(.),\hat{\theta}_{-i}(.))$. Finally, since Θ_i is a chain, Lemma 1 implies u_i^g is supermodular in $\hat{\theta}_i(.)$. **Q.E.D**

Theorem 1 shows that the class of Bayesian implementable scf that can be supermodularly implemented in Bayesian equilibrium is large, as there are only mild boundedness and continuity conditions on the utility functions and the scf. The heart of the result is (8): It is always possible to add complementarities into the transfers without affecting the incentives that appear in the expected value.

Remark. Since players receive the same expected utility in equilibrium from (x,t) and (x,t^{SM}), if (x,t) satisfies some interim participation constraints, then so does (x,t^{SM}).

Recall that, if type spaces are finite, then the assumptions of bounded substitutes and continuity are trivially satisfied for all valuation functions and scf. Furthermore, if V and $x(.)$ are twice-continuously differentiable on a compact type set, then the assumptions of bounded substitutes and continuity are satisfied. This leads to the following important corollaries which cover cases of interest.

Corollary 1 *Let type spaces Θ_i be finite subsets of $\mathbb{R}$. For any valuation functions V, if the scf $f=(x,t)$ is TBI, then there exist transfers t^{SM} such that (x,t^{SM}) is TSBI.*

Corollary 2 *Let V be C^2 and the scf $f=(x,t)$ be such that $x(.)$ is C^2. If f is TBI and $E_{\theta_{-i}}[t_i(.,\theta_{-i})]$ is usc, then there exist transfers t^{SM} such that (x,t^{SM}) is TSBI.*

The previous results state conditions that apply to *TBI* scf. In some instances it may not be obvious whether the decision rule admits truthfully-usc transfers leading to implementation. Therefore, I provide a proposition which identifies sufficient conditions for a decision rule to generate such transfers.

Standard implementation results in differentiable quasilinear environments[16] demonstrate that transfers which are part of a *TBI* scf have an explicit expected value when the other agents play truthfully. From (8), this will lead to explicit transfers that allow supermodular implementation. Letting $\Theta_i=[\underline{\theta}_i,\overline{\theta}_i]$ for $i\in N$, a necessary condition for Bayesian implementation is

$$E_{\theta_{-i}}[t_i(\hat{\theta}_i,\theta_{-i})]=-E_{\theta_{-i}}[V_i(x_i(\hat{\theta}_i,\theta_{-i}),\hat{\theta}_i)]+\int_{\underline{\theta}_i}^{\hat{\theta}_i}\frac{\partial E_{\theta_{-i}}[V_i(x_i(s,\theta_{-i}),s)]}{\partial\theta_i}\,ds+\epsilon_i(\underline{\theta}_i) \quad (11)$$

[16]See e.g Mas Colell et al. [35] for linear utility functions.

where $\epsilon_i(\underline{\theta}_i)$ is some constant. Combining Theorem 1 and Proposition 6 of Section 9.1 yields explicit sufficient conditions for supermodular Bayesian implementability as shown in Proposition 1.

Proposition 1 *If $(V, x) \in \mathcal{F}_{cont}$ has bounded substitutes such that $E_{\theta_{-i}}[V_i(x_i(\hat{\theta}_i, \theta_{-i}), \theta_i)]$ is continuous in $(\hat{\theta}_i, \theta_i)$ and $\partial E_{\theta_{-i}}[V_i(x_i(\hat{\theta}_i, \theta_{-i}), \theta_i)]/\partial\theta_i$ is increasing in $\hat{\theta}_i$, then there are transfers t^{SM} such that (x, t^{SM}) is TSBI.*

To identify the decision rules that are *TSBI* with transfers, Proposition 1 suggests to choose those decision rules $x(.)$ that lead each agent i's expected marginal valuation to be nondecreasing. Then, any such decision rule is *TSBI* with transfers t^{SM} combining (8) and (11).

The rest of this section investigates supermodular implementation under the budget balance condition. In some design problems, the planner should not realize a net gain from the mechanism. While the planner cannot sustain deficits, full efficiency requires there be no waste of numéraire. A scf is fully efficient if it maximizes the sum of the utility functions (not only the valuation functions) subject to the feasibility constraint $\sum t_i \leq 0$. The transfers then add up to zero for each vector of true types.

The next theorem provides sufficient conditions for a scf to be *TSBI* using balanced transfers. Say that a decision rule x is *allocation-efficient*, if $x(\theta) \in \text{argmax}_{x\in X} \sum_{i\in N} V_i(x_i, \theta_i)$ for all $\theta \in \Theta$. Basically, there exist transfers such that any allocation-efficient decision rule is supermodular implementable and fully efficient if substitutes are bounded.

Theorem 2 *Let $n \geq 3$. Consider an allocation-efficient decision rule $x(.)$. If $(V, x) \in \mathcal{F}_{cont}$ and has bounded substitutes, then there exist balanced transfers t^{BB} such that the scf $f = (x, t^{BB})$ is TSBI.*

The proof appears in Section 9.1 and it is constructive. Transfers t^{BB} correspond to a transformation of the transfers in the expected externality mechanism, and they rely on two observations. First, any player's transfer in the expected externality mechanism displays no complementarities or substitutes between that player's announcement and her opponents'. Second, there is a transformation of the transfers similar to that in Theorem 1 that enables to add complementarities while preserving incentives and budget balancing.

Theorem 2 can be extended to situations where, for every realization of types, enough transfers (taxes) are raised to pay the cost of x. The budget constraint takes the form $\sum_{i\in N} t_i(\theta) \geq C(x(\theta))$ for all $\theta \in \Theta$ where C is the cost function mapping X into $\mathbb{R}^+$. An additional sufficient condition to apply the theorem is that (C, x) has bounded substitutes.[17]

5.3 Optimal and Unique Supermodular Implementation

This subsection deals with the multiple equilibrium problem in supermodular implementation. Even if a mechanism has an equilibrium outcome with some desirable property, it may have other equilibrium outcomes that are undesirable. The concept of supermodular implementation relies on weak implementation, while the results in

[17]See e.g Lemma 2 in Ledyard and Palfrey [33] for transfers satisfying this budget balance condition. Note that these transfers are separable in types except (possibly) for $C(x(\theta))$, so they have are no complementarities or substitutes beyond those contained in $C(x(\theta))$.

Milgrom and Roberts [37] promise that adaptive dynamics lead to play between the greatest and the least equilibrium. This interval between the extremal equilibria is called the interval prediction. So, players may learn to play an untruthful equilibrium associated with a bad outcome. Therefore, it is important to minimize the size of the interval prediction and to take the number of equilibria into consideration. Supermodular implementation is particularly powerful when truth-revealing is the unique equilibrium.

Before presenting the results, I discuss the new concepts of this subsection. Think of the degree of complementarities as being increasing with the cross-partial derivatives, and vice-versa. *Optimal supermodular implementation* involves designing a mechanism whose induced supermodular game has the weakest complementarities in a wide class of mechanisms that supermodularly implement the scf. This mechanism turns out to produce the smallest interval prediction in this class of supermodular mechanisms. Furthermore, the interval prediction is a singleton if and only if there is a unique Bayesian equilibrium. *Unique supermodular implementation* describes that situation where truthtelling is the unique equilibrium of the supermodular induced game.

I begin with the definition of an order used in the definition of optimal supermodular implementation. As mentioned above, the cross-partial derivatives offer a way of measuring complementarities in twice-differentiable environments. It is natural to say that transfer functions $\tilde{t}$ generate larger complementarities than t, denoted $\tilde{t} \succeq_{\text{ID}} t$, if $\partial^2 \tilde{t}_i(\hat{\theta})/\partial\hat{\theta}_i\partial\hat{\theta}_j \geq \partial^2 t_i(\hat{\theta})/\partial\hat{\theta}_i\partial\hat{\theta}_j$ for all $\hat{\theta}$, j and i. For example, transfers defined by (7) and (8) generate more complementarities as ρ_i increases. The next definition formalizes the idea of the degree of complementarities and extends it to non-differentiable transfer functions.

Definition 4 *Define the ordering relation $\succeq_{ID}$ on the space of transfer functions such that $\tilde{t} \succeq_{ID} t$ if, for all $i \in N$ and for all $\theta''_i > \theta'_i$ and $\theta''_{-i} >_{-i} \theta'_{-i}$, $\tilde{t}_i(\theta''_i, \theta''_{-i}) - \tilde{t}_i(\theta''_i, \theta'_{-i}) - \tilde{t}_i(\theta'_i, \theta''_{-i}) + \tilde{t}_i(\theta'_i, \theta'_{-i}) \geq t_i(\theta''_i, \theta''_{-i}) - t_i(\theta''_i, \theta'_{-i}) - t_i(\theta'_i, \theta''_{-i}) + t_i(\theta'_i, \theta'_{-i})$.*

Transfers $\tilde{t}$ are larger than t with respect to $\succeq_{\text{ID}}$ if the double-differences are increasing from each t_i to $\tilde{t}_i$. One can verify that, for twice-differentiable transfers, it means that the cross-partial derivatives of each $\tilde{t}_i$ are larger than those of t_i.

While $\succeq_{\text{ID}}$ is transitive and reflexive on the space of transfer functions, it is not antisymmetric. Consider the set of $\succeq_{\text{ID}}$-equivalence classes of transfers, denoted $\mathcal{T}$.[18]

The next proposition shows that if a transfer function generates more complementarities than another transfer function, then the former induces a game whose interval prediction is larger than that of the game induced by the latter. This result is also of interest for the theory of supermodular games, as it relates the degree of complementarities to the size of the interval prediction.

For any $t \in \mathcal{T}$ and *TSBI* (x, t), let $\overline{\theta}^t(.)$ and $\underline{\theta}^t(.)$ be (resp.) the greatest and the smallest equilibrium in the game induced by (x, t).

Proposition 2 *Let (V, x) be such that $E_{\theta_{-i}}[V_i(x_i(\hat{\theta}_i, \theta_{-i}), \theta_i)]$ is continuous in $(\hat{\theta}_i, \theta_i)$. For any TSBI scf (x, t'') and (x, t') such that $t'', t' \in \mathcal{T}$, if $t'' \succeq_{ID} t'$, then $[\underline{\theta}^{t'}(.), \overline{\theta}^{t'}(.)] \subset [\underline{\theta}^{t''}(.), \overline{\theta}^{t''}(.)]$.*

Proof: Let (x, t'') and (x, t') be any *TSBI* scf such that $t'', t' \in \mathcal{T}$. By Proposition 6 of Section 9.1, all transfers t_i such that (x, t) is *TSBI* must have the same expected value $E_{\theta_{-i}}[t_i(\hat{\theta}_i, \theta_{-i})]$ up to a constant. Therefore, those transfers can all be written as

[18] Any quasi-order is transformed into a partially ordered set using equivalence classes.

$t_i(\hat{\theta}_i, \hat{\theta}_{-i}) = \delta_i(\hat{\theta}_i, \hat{\theta}_{-i}) - E_{\theta_{-i}}[\delta_i(\hat{\theta}_i, \theta_{-i})] + E_{\theta_{-i}}[t_i(\hat{\theta}_i, \theta_{-i})]$ for some function $\delta_i : \Theta \rightarrow \mathbb{R}$. For any *TSBI* scf, the induced game has a smallest and a greatest equilibrium along with a truthful equilibrium in between. Let $\theta_i^T(.)$ denote player i's truthful strategy, that is, $\theta_i^T(\theta_i) = \theta_i$ for all θ_i. Let $\mathcal{G}_\ell$ and $\mathcal{G}_u$ be the game $\mathcal{G}$ where the strategy spaces are restricted (resp.) from $\Sigma_i(\Theta_i)$ to $[\inf \Sigma_i(\Theta_i), \theta_i^T(.)]$, and from $\Sigma_i(\Theta_i)$ to $[\theta_i^T(.), \sup \Sigma_i(\Theta_i)]$. Since closed intervals are sublattices and $\mathcal{G}$ is supermodular, those modified games $\mathcal{G}_\ell$ and $\mathcal{G}_u$ are supermodular games. Moreover, $\mathcal{G}_\ell$ must have the same least equilibrium as game $\mathcal{G}$ and the truthful equilibrium is its largest equilibrium. Likewise, $\mathcal{G}_u$ has the same greatest equilibrium as game $\mathcal{G}$ and the truthful equilibrium is its smallest equilibrium. Let $u_i^f(\hat{\theta}(.), t) = E_\theta[V_i(x_i(\hat{\theta}(\theta)), \theta_i)] + E_\theta[t_i(\hat{\theta}(\theta))]$. I show that (i) In $\mathcal{G}_\ell$, $u_i^f(\hat{\theta}_i(.), \hat{\theta}_{-i}(.), t)$ has decreasing differences in $(\hat{\theta}_i(.), t)$ for each $\hat{\theta}_{-i}(.)$ and (ii) In $\mathcal{G}_u$, $u_i^f(\hat{\theta}_i(.), \hat{\theta}_{-i}(.), t)$ has increasing differences in $(\hat{\theta}_i(.), t)$ for each $\hat{\theta}_{-i}(.)$. In those modified games, this answers how the untruthful extremal equilibrium varies in response to changes in transfers with respect to $\succeq_{\text{ID}}$. First consider $\mathcal{G}_\ell$. Let δ'' and δ' be the δ-functions corresponding to t'' and t'. For any deception $\hat{\theta}_{-i}(.)$, note $\hat{\theta}_j(\theta_j) \leq \theta_j$ for all θ_j and $j \neq i$. Choose any $\theta_i''(.) > \theta_i'(.)$ and note $t'' \succeq_{\text{ID}} t'$ implies $\delta'' \succeq_{\text{ID}} \delta'$. Hence for all $i \in N$,

$$\begin{aligned} &E_\theta[\delta_i''(\theta_i''(\theta_i), \theta_{-i}) - \delta_i''(\theta_i''(\theta_i), \hat{\theta}_{-i}(\theta_{-i}))] - E_\theta[\delta_i''(\theta_i'(\theta_i), \theta_{-i}) - \delta_i''(\theta_i'(\theta_i), \hat{\theta}_{-i}(\theta_{-i}))] - \\ &- E_\theta[\delta_i'(\theta_i''(\theta_i), \theta_{-i}) - \delta_i'(\theta_i''(\theta_i), \hat{\theta}_{-i}(\theta_{-i}))] + E_\theta[\delta_i'(\theta_i'(\theta_i), \theta_{-i}) - \delta_i'(\theta_i'(\theta_i), \hat{\theta}_{-i}(\theta_{-i}))] \geq 0 \end{aligned} \tag{12}$$

Note (12) is equivalent to

$$u_i^f(\theta_i''(.), \hat{\theta}_{-i}(.), t'') + u_i^f(\theta_i'(.), \hat{\theta}_{-i}(.), t') - u_i^f(\theta_i''(.), \hat{\theta}_{-i}(.), t') - u_i^f(\theta_i'(.), \hat{\theta}_{-i}(.), t'') \leq 0 \tag{13}$$

for each $\hat{\theta}_{-i}(.)$. So, (13) implies that $u_i^f(\hat{\theta}_i(.), \hat{\theta}_{-i}(.), t)$ has decreasing differences in $(\hat{\theta}_i(.), t)$ for each $\hat{\theta}_{-i}(.)$. It follows from Theorem 6 in Milgrom-Roberts [37] that the smallest equilibrium in $\mathcal{G}_\ell$ is decreasing in t. The same argument applies to $\mathcal{G}_u$. There, all deceptions $\hat{\theta}_{-i}(.)$ are such that $\hat{\theta}_j(\theta_j) \geq \theta_j$ for all θ_j and $j \neq i$. As a result, the sign in (12) is reversed, which implies $u_i^f(\hat{\theta}_i(.), \hat{\theta}_{-i}(.), t)$ has increasing differences in $(\hat{\theta}_i(.), t)$ for each $\hat{\theta}_{-i}(.)$. The greatest equilibrium in $\mathcal{G}_u$ is thus increasing in t. **Q.E.D**

Before defining optimal supermodular implementation, consider the following family of transfers,

$$\begin{aligned} \mathfrak{T} \;=\; \{&t \in \mathcal{T} : t_i(\theta_i'', \theta_{-i}'') - t_i(\theta_i'', \theta_{-i}') - t_i(\theta_i', \theta_{-i}'') + t_i(\theta_i', \theta_{-i}') \geq \Delta V_i((\theta_i'', \theta_i'), \theta_{-i}', \theta_i) \\ &- \Delta V_i((\theta_i'', \theta_i'), \theta_{-i}'', \theta_i) \text{ for all } \theta_i'' > \theta_i', \theta_{-i}'' >_{-i} \theta_{-i}', \theta_i, i \in N\}. \end{aligned}$$

Transfers in $\mathfrak{T}$ make the complete information payoffs supermodular for each type $\theta \in \Theta$. If a scf is *TSBI* and its transfers offer the weakest complementarities in $\mathfrak{T}$, then it is optimally supermodular implementable in the sense defined next.[19]

Definition 5 *A scf $f = (x, t^*)$ is optimally TSBI if it is TSBI and $t \succeq_{ID} t^*$ for all transfers $t \in \mathfrak{T}$ such that (x, t) is TBI.*

[19] The ex-ante Bayesian game must be supermodular whereas transfer functions are defined at the complete information level. So it is not necessary that transfers t be in $\mathfrak{T}$ in order for a scf to be *TSBI*. For example, if the prior is mostly concentrated on some subset Θ^* of Θ, it may not be necessary to make the complete information payoffs supermodular for types in $\Theta \backslash \Theta^*$. Of course, the possibility of neglecting $\Theta \backslash \Theta^*$ depends on how unlikely that set is compared to how negative the cross-partials may be for types in that set. Therefore, I work with the condition to be in $\mathfrak{T}$.

The rationale behind optimal supermodular implementation is twofold. Games with strategic complementarities have a coordination-game "flavor" that leads to multiple equilibria (Takahashi [46]), and this relationship can be traced to how strong complementarities are. Adding complementarities improves learning and stability, but too much complementarity may yield untruthful equilibria. Optimal supermodular implementation is the best compromise. In addition, if we want to supermodularly implement the decision rule with transfers in $\mathfrak{T}$, then Proposition 2 implies that optimal transfers generate the tightest interval prediction.

Say that a scf $x : \Theta \mapsto (x_i(\theta))$ is *dimensionally reducible* if, for each $i \in N$, there are C^2 functions $h_i : \mathbb{R}^2 \to X_i$ and $r_i : \prod_{j \neq i} \Theta_j \to \mathbb{R}$ such that $r_i(.)$ is increasing and $x_i(\theta) = h_i(\theta_i, r_i(\theta_{-i}))$ for all $\theta \in \Theta$. The condition is trivially true when there are two individuals. If there are more, the announcements of each player's opponents must enter every dimension of that player's scf through a real-valued aggregate.

The next theorem says that, in twice-continuously differentiable environments, a scf is optimally supermodular implementable if it is is *TBI* and if its decision rule is dimensionally reducible.

Theorem 3 *Let V be C^2 and $f = (x, t)$ be a scf such that $x : \Theta \mapsto (x_i(\theta))$ is reducible. If f is TBI and transfers t are such that $E_{\theta_{-i}}[t_i(., \theta_{-i})]$ is usc, then there are transfers t^* such that (x, t^*) is optimally TSBI and $E_{\theta_{-i}}[t_i^*(., \theta_{-i})] = E_{\theta_{-i}}[t_i(., \theta_{-i})]$.*

Proof: Suppose $f = (x, t)$ is *TBI.* Letting

$$\delta_i(\hat{\theta}_i, \hat{\theta}_{-i}) = -\int_{\underline{\theta}_i}^{\hat{\theta}_i} \int_{r_i(\underline{\theta}_{-i})}^{r_i(\hat{\theta}_{-i})} \min_{\theta_i \in \Theta_i} \frac{\partial^2 V_i(h_i(s_i, r_i), \theta_i)}{\partial r_i \partial s_i} \, dr_i \, ds_i \tag{14}$$

for all $\hat{\theta} \in \Theta$, I show that

$$t_i^*(\hat{\theta}_i, \hat{\theta}_{-i}) = \delta_i(\hat{\theta}_i, \hat{\theta}_{-i}) - E_{\theta_{-i}}[\delta_i(\hat{\theta}_i, \theta_{-i})] + E_{\theta_{-i}}[t_i(\hat{\theta}_i, \theta_{-i})] \tag{15}$$

is well-defined and that (x, t^*) is optimally *TSBI.* Since V_i and h_i are C^2 on an open set containing compact set Θ_i, $\min_{\theta_i \in \Theta_i} \partial^2 V_i(h_i(s_i, r_i), \theta_i)/\partial r_i \partial s_i$ exists, is continuous in (r_i, s_i) by the Maximum Theorem and it is bounded. Hence $\delta_i : \Theta \to \mathbb{R}$ is continuous,[20] which implies that δ_i is Borel-measurable. Since δ_i is also bounded, $E_{\theta_{-i}}[\delta_i(., \theta_{-i})]$ is well-defined and so is $t_i^* : \Theta \to \mathbb{R}$. Next I prove that (x, t^*) is optimally *TSBI.* Note $E_{\theta_{-i}}[t_i^*(\hat{\theta}_i, \theta_{-i})] = E_{\theta_{-i}}[t_i(\hat{\theta}_i, \theta_{-i})]$ and thus (x, t^*) is *TBI.* As a continuous function on a compact set, δ_i is uniformly continuous in $\hat{\theta}$. So, $E_\theta[t^*(\hat{\theta}(\theta))]$ is continuous in $\hat{\theta}_{-i}(.)$, and upper-semicontinuity of $E_{\theta_{-i}}[t_i(\hat{\theta}_i, \theta_{-i})]$ implies $E_\theta[t_i^*(\hat{\theta}(\theta))]$ is usc in $\hat{\theta}_i(.)$. By construction, t_i^* is twice-differentiable[21] and

$$\begin{aligned} \frac{\partial^2 t_i^*(\hat{\theta}_i, \hat{\theta}_{-i})}{\partial \hat{\theta}_i \partial \hat{\theta}_j} = \frac{\partial^2 \delta_i(\hat{\theta}_i, \hat{\theta}_{-i})}{\partial \hat{\theta}_i \partial \hat{\theta}_j} &= \frac{\partial}{\partial \hat{\theta}_j} \int_{r_i(\underline{\theta}_{-i})}^{r_i(\hat{\theta}_{-i})} - \min_{\theta_i \in \Theta_i} \frac{\partial^2 V_i(h_i(\hat{\theta}_i, r_i), \theta_i)}{\partial r_i \partial s_i} \, dr_i \\ &= -\left(\min_{\theta_i \in \Theta_i} \frac{\partial^2 V_i(h_i(\hat{\theta}_i, r_i(\hat{\theta}_{-i})), \theta_i)}{\partial r_i \partial s_i} \right) \frac{\partial r_i(\hat{\theta}_{-i})}{\partial \hat{\theta}_j}. \end{aligned} \tag{16}$$

[20]See e.g Theorem 6.20 in Rudin [43].

[21]See previous footnote.

Because

$$-\min_{\theta_i\in\Theta_i}\frac{\partial^2 V_i(x_i(\hat{\theta}_i,\hat{\theta}_{-i}),\theta_i)}{\partial\hat{\theta}_i\partial\hat{\theta}_j}=-\min_{\theta_i\in\Theta_i}\left(\frac{\partial^2 V_i(h_i(\hat{\theta}_i,r_i(\hat{\theta}_{-i})),\theta_i)}{\partial r_i\partial s_i}\frac{\partial r_i(\hat{\theta}_{-i})}{\partial\hat{\theta}_j}\right) \tag{17}$$

and $r_i(.)$ is an increasing function, (16) and (17) are equal. Thus $\partial^2[V_i(x_i(\hat{\theta}),\theta_i)+t_i^*(\hat{\theta})]/\partial\hat{\theta}_i\partial\hat{\theta}_j\geq 0$ for all $\hat{\theta}$, θ_i and j,i, and so (x,t^*) is *TSBI*. Moreover, for all transfers $t\in\mathfrak{T}$ such that (x,t) is *TBI*, it must be that

$$\frac{\partial^2 t_i(\hat{\theta})}{\partial\hat{\theta}_i\partial\hat{\theta}_j}\geq-\min_{\theta_i\in\Theta_i}\frac{\partial^2 V_i(x_i(\hat{\theta}_i,\hat{\theta}_{-i})),\theta_i)}{\partial\hat{\theta}_i\partial\hat{\theta}_j}=\frac{\partial^2 t_i^*(\hat{\theta})}{\partial\hat{\theta}_i\partial\hat{\theta}_j}$$

for all $\hat{\theta}$ and j,i. This implies that (x,t^*) is optimally *TSBI*. **Q.E.D**

When the truthful equilibrium is unique, supermodular implementation is one of the most powerful forms of implementation in terms of learning and stability (Milgrom and Roberts [37]). After studying optimal supermodular implementation, it is natural to look for conditions for the interval prediction to be a singleton.

Definition 6 *A scf $f=(x,t)$ is uniquely TSBI if it is TSBI and the truthful equilibrium is the unique Bayesian equilibrium.*

The next theorem gives sufficient conditions for a scf to be uniquely supermodular implementable. Recall the definitions of Section 5.1. In particular, note that δ-increasing differences strengthen the condition of Proposition 1 that the marginal expected value is increasing in a player's announcement. Here, the marginal expected value is "sufficiently" increasing. For example, it is satisfied in environments where the valuation functions have "sufficiently" increasing differences in type and outcome and the scf is increasing enough in a player's announcement.

The main result on unique supermodular implementation is Theorem 4. If truthtelling is an equilibrium and if the mechanism induces utility functions whose complementarities between announcements are smaller than the complementarities between own announcement and type, then the truthful equilibrium is unique.[22]

Theorem 4 *Let V and scf $f=(x,t)$ be such that (V,x) has δ-increasing differences. Let V_i be C^1, x_i be differentiable in $\hat{\theta}_i$, and $V_i\circ x_i$ be ω_i-Lipschitz in $\hat{\theta}_i$. Suppose $u_i\circ f$ has complements bounded by κ_i and transfers t_i are β_i-Lipschitz in $\hat{\theta}_i$. If f is TSBI and $\kappa_i<\delta_i/(n-1)$, then it is uniquely TSBI.*

The proof appears in Section 9.1, but the intuition is as follows. On the one hand, for high values of δ_i, the complementarities between own announcement and type are so strong that players tend to announce high types regardless of their opponents' deceptions. This favors uniqueness. On the other hand, for high values of κ_i, the complementarities between players' announcements become so strong that it is source of multiplicity (See the above argument). The theorem provides a cutoff between those forces so that, for any profile greater/smaller than the truthful equilibrium, some player has a contractive best-reply between that profile and the truthful one.

[22]My results are inspired by recent theories of uniqueness in Bayesian games (Mason and Valentinyi [36]).

Optimal transfers provide the lowest bound on complements (κ), so a natural question is to ask when they actually lead to unique supermodular implementation. This is the next proposition. I defer the proof to Section 9.1.

Proposition 3 *Let V and scf $f = (x, t)$ be such that (V, x) has δ-increasing differences. Let V_i and x_i be C^2, and $V_i \circ x_i$ be ω_i-Lipschitz in $\hat{\theta}_i$. Letting*

$$\kappa_i = \max_{j \neq i} \max_{(\hat{\theta}, \theta_i) \in \Theta \times \Theta_i} \left(\frac{\partial^2 V_i(x_i(\hat{\theta}), \theta_i)}{\partial \hat{\theta}_i \partial \hat{\theta}_j} - \min_{\theta_i \in \Theta_i} \frac{\partial^2 V_i(x_i(\hat{\theta}), \theta_i)}{\partial \hat{\theta}_i \partial \hat{\theta}_j} \right),$$

if $\kappa_i < \delta_i/(n-1)$, then (x, t^*) is uniquely TSBI.

There are examples where it is straightforward to use Theorem 4 and Proposition 3. Consider the public goods example of Section 2. Recall that $N = \{1, 2\}$, $X = [0, 2]$, $\Theta_i = [0, 1]$ for $i = 1, 2$. Agents' valuation functions are $V_1(x, \theta_1) = \theta_1 x - x^2$ and $V_2(x, \theta_2) = \theta_2 x + x^2/2$. The decision rule is $x(\theta) = \theta_1 + \theta_2$. Since $\partial x_i(\theta)/\partial \theta_i = 1$ and $\partial^2 V_i(x, \theta_i)/\partial x \partial \theta_i = 1$ for $i = 1, 2$, it implies $\delta_i = 1$, $i = 1, 2$. Moreover, $\partial^2 V_i(x(\hat{\theta}), \theta_i)/\partial \hat{\theta}_1 \hat{\theta}_2 = -2$ if $i = 1$ and 1 otherwise. So, $\kappa_i = 0$ for $i = 1, 2$. By Proposition 3, (x, t^*) is uniquely supermodular implementable.

Remark: Neither unique nor optimal supermodular implementation implies the other. The truthful equilibrium may be unique, although the transfers do not generate the weakest complementarities. And the supermodular transfers could be optimal but the truthful equilibrium not unique.

The rest of this subsection deals with the multiple equilibrium problem under the budget balance condition. The next proposition shows that there are scf that are uniquely supermodular implementable with balanced transfers. It gives sufficient conditions in order that the transfers identified in Theorem 2 yield truthtelling as a unique equilibrium.

Proposition 4 *Let $n \geq 3$. Let V and decision rule $x(.)$ be C^1 such that (V, x) has δ-increasing differences and $x(.)$ is allocation-efficient. Suppose $V_i \circ x_i$ has complements bounded by τ_i and substitutes bounded by T_i. If $\tau_i - T_i < \delta_i/(n-1)$, then (x, t^{BB}) is uniquely TSBI.*

The proof is in Section 9.1. The public goods example of Section 2 again provides a nice illustration. Consider the same setting with an additional player, player 3, such that $\Theta_3 = [0, 1]$, $V_3(x, \theta_3) = \theta_3 x$ and $x(\theta) = \theta_1 + \theta_2 + \theta_3$. Then $\delta_i = 1$ for $i = 1, 2, 3$. Since $\tau_i = T_i$ for $i = 1, 2, 3$ and $T_1 = -2$, $T_2 = 1$, $T_3 = 0$, Proposition 4 says that for any $\{\rho_i\}$ such that $2 < \rho_1 < 2\frac{1}{2}$, $-1 < \rho_2 < -\frac{1}{2}$, $0 < \rho_3 < \frac{1}{2}$, (x, t^{BB}) is uniquely supermodular implementable with budget balancing.

5.4 Discussion

Optimal supermodular implementation can be viewed as an intermediary form of implementation between strictly-dominant strategy and Bayesian implementation. If the composition of any player's valuation function with the decision rule has strictly increasing differences in type and own announcement, Mookherjee and Reichelstein [41] implies that there exist transfers resulting in strictly-dominant strategy implementation.[23] The existence of strictly-dominant strategies in a game implies that each player's best-response has zero slope. But under the assumption on the composition, the slope of the best-response is given by the ratio between the cross-partials and the second-derivative in own announcement. So the game induced by these transfers will have null complementarities, that is, the cross-partials of any player's utility in equilibrium will be zero. As optimal supermodular implementation induces the supermodular game with the weakest complementarities, optimal transfers will belong to the same $\succeq_{\text{ID}}$-equivalence class as the transfers that yield strictly-dominant strategy implementation. If such transfers do not exist, then it will assign the "best Bayesian transfers."

The public goods example of Section 2 illustrates this point. The composition of 1 or 2's valuation function with the decision rule has strictly increasing differences in type and announcement. Therefore, it is not surprising that the scf be uniquely supermodular implementable. What is more surprising is that the optimal transfers transform the expected externality mechanism into dominant-strategy transfers. Note also that it is easy to find examples of scf falling into Theorem 4 or Proposition 3 that are not even dominant strategy implementable. Even if Mookherjee and Reichelstein [41] applies, my results may provide a whole range of transfers compatible with unique supermodular implementation, whereas the choice is narrow for dominant strategy implementation.[24]

Optimal implementation is based on the idea of imposing the minimal amount of complementarities (w.r.t $\succeq_{\text{ID}}$) necessary for supermodular implementation. But weak complementarities might imply a low speed of convergence of learning dynamics towards truthtelling. This is not necessarily true. Convergence is indeed fastest under strictly-dominant strategy implementation, a form of implementation with null complementarities.

Finally, there may be a conflict in supermodular implementation between budget balancing and the multiple equilibrium problem. If transfers t are balanced, then $\sum_{k\in N}\partial^2 t_k(\hat{\theta})/\partial\hat{\theta}_i\partial\hat{\theta}_j = 0$ for all $\hat{\theta}$ and any distinct i and j. But there is no reason a priori for optimal transfers to satisfy this condition. Optimal transfers indeed require some functional flexibility to minimize complementarities over all announcements, and the budget balance condition sometimes prevents it by imposing the above restriction. Beyond jargon, it seems to suggest a trade-off between learning and full efficiency. One may argue that a second-best approach could be appropriate: Choosing what is best among what players can learn.

6 Applications

6.1 Principal-Agent Problem

Consider the traditional principal-agent problem with hidden information.[25] A principal contracts with n agents. Agent i's type space is $[\underline{\theta}_i, \overline{\theta}_i]$. Types are independently distributed according to a common prior $\phi = \times\phi_i$ which admits a bounded density with full support. Let $X_i \subset \mathbb{R}$ be compact. Each agent i exerts some observable effort

[23] See Proposition 2 in Mookherjee and Reichelstein [41] and the discussion that follows.

[24] In the public goods example, Theorem 4 implies that there are infinitely many ρ_1 and ρ_2 resulting in unique supermodular implementation; but it must be that $\rho_1 = 2$ and $\rho_2 = -1$ to achieve strictly-dominant strategy implementation.

[25] See e.g Section 23.F in Mas-Colell et al. [35].

$x_i \in X_i$, and she bears a cost or disutility $c_i(x_i, \theta_i)$ of producing effort x_i when she is of type θ_i. From the vector of efforts $x = (x_1, \ldots, x_n)$, the principal receives utility $w(x)$. The principal faces the problem of designing an optimal contract subject to incentive constraints and reservation utility constraints for the agents. A contract is a function that maps each possible agents' type into effort and transfer levels. The principal's problem can be stated as

$$(x^*(\theta), t(\theta)) \in \underset{f=(x,t)}{\operatorname{argmax}}\, E_\theta[w(x^*(\theta)) - \sum_{i=1}^{n} t_i(\theta)] \tag{18}$$

subject to

$$E_{\theta_{-i}}[t_i(\theta_i, \theta_{-i}) - c_i(x_i^*(\theta_i, \theta_{-i}), \theta_i)] \;\geq\; E_{\theta_{-i}}[t_i(\theta_i', \theta_{-i}) - c_i(x_i^*(\theta_i', \theta_{-i}), \theta_i)],\ \forall\, \theta_i', \theta_i \tag{19}$$

$$E_{\theta_{-i}}[t_i(\theta_i, \theta_{-i}) - c_i(x_i^*(\theta_i, \theta_{-i}), \theta_i)] \;\geq\; \overline{u}_i,\ \forall\, \theta_i \tag{20}$$

Condition (19) requires the scf (x^*, t) to be truthfully Bayesian implementable. Condition (20) is an interim participation constraint, as agents may opt out of the mechanism if it does not meet their reservation utility.

Assume that the underlying functions w, c_i and ϕ are such that x^* is dimensionally reducible[26] and t is continuous. If c_i is twice-continuously differentiable on $X_i \times \Theta_i$, Theorem 3 applies. There are transfers t^* such that (x^*, t^*) is optimally supermodular implementable and solves (18) subject to (19) and (20).

At this level of generality, it is difficult to appreciate the strength of optimal supermodular implementation, so I present a simple application in the spirit of the team production model of McAfee and McMillan [34].

There are two agents, 1 and 2, whose types are independently uniformly distributed over $[0, 3]$. Players exert some effort to produce an observable contribution x_i. The amount of effort e_i necessary to give x_i is $e_1(x, \theta_1) = (3-\theta_1)(x_1-x_2)+x_1$ and $e_2(x, \theta_2) = (3-\theta_2)(x_2+x_1)$. Larger contributions require larger effort and higher ability levels decrease marginal effort. But agent 2 generates positive externalities on her counterpart, whereas 1 has negative externalities. Given $x = (x_1, \ldots, x_n)$, the principal only knows the density $f(y|x)$ of output y given x. The principal has utility function $u(y, x, \theta)$ and she perceives costs as $c_p(x, \theta)$. The problem is

$$x^*(\theta) \in \underset{(x_1,\ldots,x_n)}{\operatorname{argmax}}\, E_{y|x}[u(y, x, \theta)] - c_p(x, \theta). \tag{21}$$

Each agent's valuation function is $V_i(x, \theta_i) = -c(e_i(x, \theta_i))$ where $c(e_i) = e_i$. Assume u, c_p and f are such that the decision rule obtained from (21) is $(x_1^*(\theta), x_2^*(\theta)) = (\theta_2\theta_1 - \theta_1 E(\theta_2), \theta_2 - \theta_1)$.[27] Decision rule $x^*(.)$ satisfies the conditions of Proposition 6 of Section 9.1, and so there exists transfers t such that (x^*, t) is *TBI*. Constructing optimal transfers from (14) and (15) gives $t_1^*(\hat{\theta}) = -\hat{\theta}_1^2/2 - 3\hat{\theta}_1 + 4\hat{\theta}_2\hat{\theta}_1$ and $t_2^*(\hat{\theta}) = -5\hat{\theta}_2^2/4 + 3\hat{\theta}_2 + 3\hat{\theta}_2\hat{\theta}_1$. It turns out that truthtelling is the unique Bayesian equilibrium in the supermodular game $\mathcal{G}$ induced by the mechanism with optimal transfers.

[26] Recall that x^* is dimensionally-reducible if it is twice-continuously differentiable and satisfies some mild dimension condition.

[27] The one-dimensional condensation property of Mookherjee and Reichelstein [41] is violated. There exists no $h_1 : X \to \mathbb{R}$ such that $c(e_1(x, \theta_1)) = D_1(h_1(x), \theta_1)$ for some $D_1 : \mathbb{R} \times \Theta_1 \to \mathbb{R}$. Moreover, note that $V_1(x^*(\hat{\theta}), \theta_1)$ does not have increasing differences in $(\hat{\theta}_1, \theta_1)$, so $x^*(.)$ is not dominant-strategy implementable by Definition 5 in Mookherjee and Reichelstein [41].

6.2 Public Goods Problem and Approximate Supermodular Implementation

In this subsection, I apply the theory of supermodular implementation to public goods, and then I describe two binary-choice models that violate bounded substitutes (and continuity) for continuous type spaces. I circumvent this difficulty by using approximate or virtual implementation (Abreu and Matsushima [3], Duggan [16]).

Consider an economy with n consumers and two commodities, one public $x \in [0, \overline{x}]$ and one private. The consumers each have preferences for the public good and transfer t_i of the private good that can be represented by the function $u_i(x, t_i, \theta_i)$ where $\theta_i \in [\underline{\theta}_i, \overline{\theta}_i]$ is i's type. Preferences are assumed to be quasilinear in the private good: $u_i(x, t_i, \theta_i) = V_i(x, \theta_i) + t_i$. An allocation is a $(n+1)$-tuple of the form $(x, t_1, \ldots, t_n)$. In the public goods environment, the appropriate social choice function for a utilitarian planner is the mapping $(x^*(\theta), t(\theta)) \in \text{argmax}_{f=(x,t)} \sum_{i \in N} V_i(x, \theta_i) + t_i$ subject to $\sum_{i \in N} t_i(\theta) \leq 0$ for all θ. If $(V, x) \in \mathcal{F}_{\text{cont}}$ and the substitutes are bounded, then Theorem 2 implies that (x^*, t^{BB}) is supermodular implementable such that t^{BB} balances budget. In the public goods example of Section 2, (x^*, t^{BB}) is actually uniquely supermodular implementable with budget balance.

Next I describe two binary-choice models that violate bounded substitutes (and continuity) for continuous type spaces. The first is an auction model where a seller is awarding one unit of an indivisible good to the highest bidder. The second is a public goods model where agents have to choose whether to undertake a public project. These models represent a challenge for the present theory, unless type spaces are finite, in which case Theorem 1 always applies. I use approximate or virtual implementation to solve this difficulty (Abreu and Matsushima [3], Duggan [16]). A scf is approximately implementable if, in any ϵ-neighborhood of that scf, there exists an implementable scf. This requires a notion of distance that will be defined later. The main idea is that the set of twice-continuously differentiable functions is dense in the L_p-space and twice-continuously differentiable satisfy the bounded substitutes assumption on smooth domains.

There is a seller of an object who derives no value from it, and n potential buyers. Let buyer i's type space be $\Theta_i \equiv [\underline{\theta}_i, \overline{\theta}_i]$. Buyer i's utility function takes the linear form $u_i(x_i, \theta_i) = \theta_i x_i + t_i$. Consider the allocation-efficient decision rule which attributes the good to the agent with the highest type. For $i \in N$ and all θ,

$$x_i^*(\theta) = \begin{cases} 1 & \text{if } \theta_i \geq \max\{\theta_j : j \in N\} \\ 0 & \text{otherwise} \end{cases} \qquad \text{and} \quad \sum_{j \in N} x_j^*(\theta) = 1 \tag{22}$$

For $N = \{1, 2\}$, I explain why substitutes are unbounded. Note that for any $\theta_2'' > \theta_1'' > \theta_2' > \theta_1'$, $x_1(\theta_1'', \theta_2'') - x_1(\theta_1', \theta_2'') - x_1(\theta_1'', \theta_2') + x_1(\theta_1', \theta_2') = -1$. Hence, for substitutes to be bounded, there must exist T such that $-\theta_1 \geq T(\theta_1'' - \theta_1')(\theta_2'' - \theta_2')$ for all $\theta_1 \in \Theta_1$. But this is clearly impossible as we can maintain the order $\theta_2'' > \theta_1'' > \theta_2' > \theta_1'$ while $\theta_1' \uparrow \theta_2'$ and $\theta_1'' \downarrow \theta_2'$. So, Proposition 1 does not apply.

Consider now a situation in which n agents must decide whether to undertake a public project whose cost is c. The decision rule $x(.)$ takes values in $\{0, 1\}$. Let agent i's type space be $\Theta_i \equiv [\underline{\theta}_i, \overline{\theta}_i]$. Agents' utility function takes the same linear form. Consider the allocation-efficient decision rule defined as follows. For a particular $i \in N$

and all θ,

$$x^*(\theta) = \begin{cases} 1 & \text{if } \sum_{i\in N} \theta_i \geq c \\ 0 & \text{otherwise} \end{cases} \tag{23}$$

Once again, for $N = \{1, 2\}$, substitutes are unbounded. Take any $\theta''_2 > \theta'_2$ and let $\theta'_1 = -\theta'_2 + c - 1/n$ and $\theta''_1 = -\theta'_2 + c + 1/n$. Then for n large enough, $x_1(\theta''_1, \theta''_2) - x_1(\theta'_1, \theta''_2) - x_1(\theta''_1, \theta'_2) + x_1(\theta'_1, \theta'_2) = -1$. There cannot exist T such that $-\theta_1 \geq \frac{2T}{n}(\theta''_2 - \theta'_2)$ for all n. Proposition 1 does not apply.

Clearly, the problem is caused by the lack of smoothness in those decision rules. However, if one is willing to accept an ϵ-inefficiency in the process, then supermodular implementation applies.

Definition 7 *A decision rule $x(.)$ is approximately TSBI with transfers, if there exists a sequence of optimally TSBI scf $\{(x_\epsilon, t_\epsilon)\}$ such that, for $1 \leq p < \infty$, $\lim_{\epsilon\to 0}(\int_\Theta |x_{\epsilon,i} - x_i|^p)^{\frac{1}{p}} = 0$ for all i.*

The next result says that, for $1 \leq p < \infty$, L_p-decision rules are approximately supermodular implementable on C^2-domains.

Proposition 5 *Let the valuation functions V be C^2 such that $\partial V_i(x_i, \theta_i)/\partial\theta_i$ is increasing in x_i. If $x_{i,k}(.) \in L_p(\Theta, \mathbb{R})$ is increasing in $\hat{\theta}_i$ for each $k = 1, \ldots, m_i$, and $E_{\theta_{-i}}[V_i(x_i(\hat{\theta}_i, \theta_{-i}), \theta_i)]$ is continuous in $(\hat{\theta}_i, \theta_i)$, then $x(.)$ is approximately TSBI.*

Proof: Recall that V and $x(.)$ are C^2, if there exist open sets $O_i \supset \Theta_i$ and $U_i \supset X_i$, $i = 1, \ldots, n$, such that $V : U_i \times O_i \to \mathbb{R}$ and $x : \prod_{i\in N} O_i \to U_i$ are C^2. For any $\theta \in O$, let $\iota_1(\theta) = \{j \in N : \theta_j \in [\underline{\theta}_j, \overline{\theta}_j]\}$, $\iota_2(\theta) = \{j \in N : \theta_j < \underline{\theta}_j\}$ and $\iota_3(\theta) = \{j \in N : \theta_j > \overline{\theta}_j\}$. Define the extension of $x(.)$ from Θ to O, denoted x^e, such that for all $\theta \in O$, $x^e_{(i,k)}(\theta) = x_{(i,k)}(((\theta_j)_{\iota_1(\theta)}, (\underline{\theta}_j)_{\iota_2(\theta)}, (\overline{\theta}_j)_{\iota_3(\theta)}))$ for all k and $i \in N$. So, $x^e_{(i,k)}$ is an increasing function in θ_i and $x^e_{(i,k)} \in L_p(O, \mathbb{R})$. Since the space of C^2-functions on O is norm dense in $L_p(O, \mathbb{R})$, there exists a sequence $\{x_\epsilon(.)\}$ of C^2-functions from O into $\mathbb{R}$ such that $\lim_{\epsilon\to 0}(\int_O |x_{\epsilon,(i,k)} - x^e_{(i,k)}|^p)^{1/p} = 0$ for all k and i. This implies $\lim_{\epsilon\to 0}(\int_\Theta |x_{\epsilon,(i,k)} - x_{(i,k)}|^p)^{1/p} = 0$ for all k and all i. Moreover, we can take $\{x_\epsilon(.)\}$ such that $x_{\epsilon,(i,k)}(.)$ is increasing in θ_i on O_i for all k and i. Therefore, since V_i and $x_{\epsilon,i}$ are both C^2 and each Θ_i is compact, $(V, x_\epsilon) \in \mathcal{F}_{\text{cont}}$, $\partial E_{\theta_{-i}}[V_i(x_{\epsilon,(i,k)}(\hat{\theta}), \theta_i)]/\partial\theta_i = E_{\theta_{-i}}[\partial V_i(x_{\epsilon,(i,k)}(\hat{\theta}), \theta_i)/\partial\theta_i]$ is increasing in $\hat{\theta}_i$ on Θ_i, and substitutes are bounded. Thus, Proposition 1 and Theorem 3 imply that, for all $\epsilon > 0$, there exist t^{SM}_ϵ such that $f = (x_\epsilon, t^{SM}_\epsilon)$ is *TSBI*. **Q.E.D**

It follows as a corollary of Proposition 5 that, in the above auction and public goods settings, the efficient decision rules are approximately supermodular implementable.

In the environment of Proposition 5, it is not clear whether we should use approximate supermodular implementation. On the one hand, the conditions of the proposition imply dominant strategy implementability by Mookherjee and Reichelstein [41]. But in the induced game, dominant strategy implementation does not prevent adaptive dynamics from converging to an "unwanted" dominant strategy equilibrium, a "non-dominant" strategy equilibrium, a non-equilibrium profile, or simply from cycling. On the other hand, approximate supermodular implementation relies on optimal supermodular implementation. But even if optimal implementation performs well along the sequence, it remains approximate and not exact implementation; this dilemma seems to support Cabrales [7]'s argument that there is a trade-off between close implementability and stability or learning.

7 A Revelation Principle for Supermodular Bayesian Implementation

Supermodular Bayesian implementation is permissive in quasilinear environments with real type spaces. Although these assumptions are common in mechanism design and allow for a wide range of applications, it is important to consider general utility functions and type sets. One of the first questions that come to mind is that of the restrictiveness of direct mechanisms. The traditional Revelation Principle says that direct mechanisms cause no loss of generality in Bayesian (weak) implementation. How restrictive are direct mechanisms in supermodular Bayesian implementation?

Answering this question is particularly relevant, because the challenge in any supermodular design problem is to specify an ordered message space and an outcome function so that agents adopt monotone best-responding behaviors. The set of all possible message spaces and orders on those spaces is so large that it might seem intractably-complex. A Supermodular Revelation Principle gives conditions so that, if a scf is supermodular implementable, then there exists a direct-revelation mechanism that supermodularly implements this scf truthfully. So it is a technical insight that reduces the space of mechanisms to consider to the space of direct-revelation mechanisms. The question is complex because it is combinatorial in essence; it pertains to the existence of orders on type spaces that make the (induced) direct-revelation game supermodular.

The following example shows that, unfortunately, there exist supermodular implementable scf that are not truthfully supermodular implementable. Nevertheless a supermodular revelation principle exists. Although it is not as general as the traditional revelation principle, it measures the restriction imposed by direct mechanisms in supermodular implementation and gives conditions that may warrant their use.

Consider two agents, 1 and 2, with type spaces $\Theta_1 = \{\theta_1^1, \theta_1^2\}$ and $\Theta_2 = \{\theta_2^1, \theta_2^2, \theta_2^3\}$. Prior beliefs assign equal probabilities to all $\theta \in \Theta$. Let $X = \{x_1, \ldots, x_{12}\}$ be the outcome space. Agent 1's preferences are given by utility function $u_1(x_n, \theta_1)$ such that:

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}	x_{11}	x_{12}
$u_1(x_n, \theta_1^1)$	-10	0	16	-13	-2	33	-21	-2	18	-19	0	36
$u_1(x_n, \theta_1^2)$	-10	0	16	-21	-2	18	-13	-2	33	-19	0	36

For simplicity, let u_2 be a constant function. Suppose that the agents wish to supermodularly implement the scf f defined as follows

$f(.,.)$	θ_2^1	θ_2^2	θ_2^3
θ_1^1	x_4	x_5	x_6
θ_1^2	x_7	x_8	x_9

Consider the following indirect mechanism $\Gamma = ((M_1, \succeq_1), (M_2, \succeq_2), g)$. Agent 1's message space is $M_1 = \{\underline{m}_1, m_1^1, m_1^2, \overline{m}_1\}$; $\succeq_1$ is such that m_1^1 and m_1^2 are unordered, $\overline{m}_1$ is the greatest element and $\underline{m}_1$ is the smallest element. Agent 2's message space is $M_2 = \{\underline{m}_2, m_2^1, \overline{m}_2\}$; $\succeq_2$ is such that $\overline{m}_2 \succeq m_2^1 \succeq_2 \underline{m}_2$. The outcome function g is defined as follows

$g(.,.)$	$\underline{m}_2$	m_2^1	$\overline{m}_2$
$\underline{m}_1$	x_1	x_2	x_3
m_1^1	$f(\theta_1^1,\theta_2^1)$	$f(\theta_1^1,\theta_2^2)$	$f(\theta_1^1,\theta_2^3)$
m_1^2	$f(\theta_1^2,\theta_2^1)$	$f(\theta_1^2,\theta_2^2)$	$f(\theta_1^2,\theta_2^3)$
$\overline{m}_1$	x_{10}	x_{11}	x_{12}

I show that mechanism Γ supermodularly implements f in Bayesian equilibrium. Given u_2 is constant, any strategy $m_2 : \Theta_2 \to M_2$ is a best-response to any strategy of 1. So, consider strategy $m_2^*(.)$ such that $m_2^*(\theta_2^1) = \underline{m}_2$, $m_2^*(\theta_2^2) = m_2^1$ and $m_2^*(\theta_2^3) = \overline{m}_2$. Since for all m_1 we have $\sum_{m_2} u_1(g(m_1^1, m_2), \theta_1^1) > \sum_{m_2} u_1(g(m^1, m_2), \theta_1^1)$ and $\sum_{m_2} u_1(g(m_1^2, m_2), \theta_1^2) > \sum_{m_2} u_1(g(m_1, m_2), \theta_1^2)$, 1's best-response $m_1^*(.)$ to $m_2^*(.)$ must be such that $m_1^*(\theta_1^1) = m_1^1$ and $m_1^*(\theta_1^2) = m_1^2$. So, $(m_1^*(.), m_2^*(.))$ is a Bayesian equilibrium such that $g \circ m^* = f$. Moreover, for each θ_1, $u_1(g(m_1, m_2), \theta_1)$ is supermodular in m_1 and has increasing differences in (m_1, m_2). Since $\Sigma_1(\Theta_1)$ is endowed with the pointwise order, $u_1^g(m_1(.), m_2(.))$ is supermodular in $m_1(.)$ and has increasing differences in $(m_1(.), m_2(.))$. Therefore, Γ supermodularly implements f in Bayesian equilibrium, because 2's utility is constant.

Does this imply that there exists a mechanism $(\{(\Theta_i, \geq_i)\}, f)$ which truthfully implements f in supermodular game form? By means of contradiction, suppose there is such a mechanism. Then $(\Theta_1, \geq_1)$ must be totally ordered, for otherwise $\Sigma_1(\Theta_1)$ cannot be a lattice. So, assume $\theta_1^2 >_1 \theta_1^1$. Let $\theta_i^k(.)$ be the strategy where agent i always announces type θ_i^k regardless of her true type. Let $\theta_1^T(.)$ be the truthful strategy for 1 and let $\theta_1^L(.)$ be constant lying. Note $\theta_1^T(.), \theta_1^L(.) >_1 \theta_1^1(.)$. Moreover, since $\Sigma_2(\Theta_2)$ is a lattice, θ_2^1 and θ_2^2 (and thus $\theta_2^1(.)$ and $\theta_2^2(.)$) must be ordered.

Since the direct mechanism must induce a supermodular game, $u_1^f(\hat{\theta}_1(.), \hat{\theta}_2(.))$ must satisfy the single-crossing property in $(\hat{\theta}_1(.), \hat{\theta}_2(.))$.[28] Given

$$\begin{aligned} -2 = u_1^f(\theta_1^T(.), \theta_2^2(.)) \;&\geq\; u_1^f(\theta_1^1(.), \theta_2^2(.)) = -2 \\ -13 = u_1^f(\theta_1^T(.), \theta_2^1(.)) \;&>\; u_1^f(\theta_1^1(.), \theta_2^1(.)) = -17 \end{aligned}$$

u_1^f satisfies the single-crossing property in $(\hat{\theta}_1(.), \hat{\theta}_2(.))$ only if $\theta_2^1 >_2 \theta_2^2$. But

$$-2 = u_1^f(\theta_1^L(.), \theta_2^2(.)) \geq u_1^f(\theta_1^1(.), \theta_2^2(.)) = -2$$

does not imply $-21 = u_1^f(\theta_1^L(.), \theta_2^1(.)) \geq u_1^f(\theta_1^1(.), \theta_2^1(.)) = -17$. The single-crossing property is violated. Now assume $\theta_1^1 >_1 \theta_1^2$. Note $\theta_1^1(.) >_1 \theta_1^T(.), \theta_1^L(.)$. Given

$$\begin{aligned} -2 = u_1^f(\theta_1^1(.), \theta_2^2(.)) \;&\geq\; u_1^f(\theta_1^L(.), \theta_2^2(.)) = -2 \\ -17 = u_1^f(\theta_1^1(.), \theta_2^1(.)) \;&>\; u_1^f(\theta_1^L(.), \theta_2^1(.)) = -21 \end{aligned}$$

u_1^f satisfies the single-crossing property in $(\hat{\theta}_1(.), \hat{\theta}_2(.))$ only if $\theta_2^1 >_2 \theta_2^2$. But

$$-2 = u_1^f(\theta_1^1(.), \theta_2^2(.)) \geq u_1^f(\theta_1^T(.), \theta_2^2(.)) = -2$$

does not imply $-17 = u_1^f(\theta_1^1(.), \theta_2^1(.)) \geq u_1^f(\theta_1^T(.), \theta_2^1(.)) = -13$. The single-crossing property is violated. The scf f is not truthfully supermodular implementable, although it is supermodular implementable.

Even though the revelation principle fails to hold in general for supermodular implementation, a supermodular revelation principle exists, as captured by the next theorem. The proof appears in Section 9.2. This result shows that the problem which arises in the example is that the range of the equilibrium strategies is not a lattice.

[28]The single-crossing property, defined in Section 3, is implied by increasing differences.

Theorem 5 *(***The Supermodular Revelation Principle for Finite Types***) Let type space Θ_i be a finite set for $i \in N$. If there exists a mechanism $(\{(M_i, \succeq_i)\}, g)$ that supermodularly implements the scf f such that there is a Bayesian equilibrium $m^*(.)$ for which $g \circ m^* = f$ and $m_i^*(\Theta_i)$ is a lattice, then f is TSBI.*

Corollary 3 *Let type space Θ_i be a finite set for $i \in N$. If there exists a mechanism $(\{(M_i, \succeq_i)\}, g)$ that supermodularly implements the scf f such that there is a Bayesian equilibrium $m^*(.)$ for which $g \circ m^* = f$ and $(M_i, \succeq_i)$ is totally ordered for all $i \in N$, then f is TSBI.*

According to the supermodular revelation principle, limiting attention to direct mechanisms is equivalent to restricting one's scope to mechanisms where the equilibrium strategies are lattice-ranged. It is a rather strong result that supermodularity can be transmitted to the game induced by a direct mechanism. The range of the equilibrium strategies is the transmission channel. If this range is a lattice, then it is possible to construct an order such that each player's type space is order-isomorphic to the range of her equilibrium strategy. The properties of the utility functions ensue. Besides, the theorem states conditions that are verifiable a posteriori. It may be useful to know when a complex mechanism can be replaced with a simpler direct mechanism.

Corollary 3 says that if the designer is only interested in mechanisms where the message spaces are totally ordered, then she can look at direct mechanisms only without loss of generality.

The above example suggests that the conditions of Theorem 5 are somewhat minimally sufficient. Agent 1's equilibrium strategy is indeed not lattice-ranged and the scf is not truthfully supermodular implementable. Whereas this example might indicate that the pointwise-order structure causes revelation to fail, this is not the case. Theorem 6 of Section 9.2 suggests that allowing more general order structures does not weaken the conditions for a revelation principle. Those theorems only give sufficient conditions for revelation principles; but in those cases where a supermodular direct mechanism exists while the conditions are violated, the existence of an order has little or nothing to do with a revelation principle.[29]

Theorem 5 is concerned with finite type spaces. Under this assumption, if a player's type space is a (complete) lattice, then so is her set of deceptions with the pointwise order. This is no longer true for continuous types. Continuity and measurability become issues (Van Zandt [50]). In Section 9.2, I generalize the definition of supermodular implementability to incorporate orders that are not pointwise orders. This allows to prove a supermodular revelation principle for continuous types.[30]

8 Conclusion

This paper introduces a theory of implementation where the mechanisms that implement a scf must induce a supermodular game. Supermodular Bayesian implementation differs from previous literature in terms of its methodology and explicit purpose. Unlike the traditional approach, the present mechanisms derive their properties from the game

[29]In the spirit of Echenique [18], there may be conditions on the scf and the utility functions such that an order exists for which the game is supermodular. Since this existence would not follow from implementability, it is not a revelation approach.

[30]See Theorem 6 of Section 9.2.

that they induce and not from the solution concept. The paper shows that the analysis in mechanism design and implementation theory can benefit from this methodology. It may also prove useful in other contexts. The theory explicitly aims to improve learning and stability in an incentive-design framework.

The paper raises issues that have not been discussed. For the most part, type spaces are subsets of the real line. It is not straightforward to extend the theory to multidimensional type spaces. When types are real, supermodularity in a player's own type is trivial. For multidimensional types, it is not immediate that the current technique of modifying a mechanism applies. In this case, we possibly have to add complementarities between the dimension of a player's type without affecting the incentives. Similar transformations will not preserve the incentives. The condition of bounded substitutes will also have to be applied to the transfers of the original mechanism. Those transfers were trivially supermodular with real types, but they may carry substitutes in multidimensional types. For this reason and the following, indirect mechanisms seem appropriate in a supermodular implementation framework; yet the paper only considers direct mechanisms. Since the supermodular revelation principle fails in general, weak implementation calls for indirect mechanisms. So, indirect mechanisms are important to extend the frontiers of supermodular implementation in quasilinear and general environments. However weak implementation makes the issue of the interval prediction essential, so indirect mechanisms should be considered in the context of optimal or unique implementation.

The multiple equilibrium problem in supermodular implementation suggests an alternative solution, namely strong implementation. Strong implementation requires all the equilibria of the mechanisms to yield desired outcomes. Instead of relying on weak implementation, supermodular implementation could be based on strong implementation, which would also call for indirect mechanisms. Nonetheless, even under strong implementation, learning dynamics may cycle within the interval prediction or players may learn to play a non-equilibrium profile. Therefore, strong supermodular implementation cannot substitute for unique supermodular implementation. Yet it is an avenue to explore.

Like many Bayesian mechanisms, the present mechanisms are parametric in the sense that they rely on agents' prior beliefs. Thus the designer uses information other than that received from the agents (Hurwicz [25]). It may be interesting to design nonparametric supermodular mechanisms. This is yet another justification for indirect mechanisms, as nonparametric direct Bayesian mechanisms impose dominant-strategy incentive-compatibility (Ledyard [32]).

Finally, it is important to pursue testing supermodular games. Since supermodular Bayesian implementation provides a general framework, it is a good candidate for experimental tests. From a practical viewpoint, discretizing type spaces may simplify the players' task of announcing deceptions at each round. But there are also simple environments with continuous types where announcing a deception is equivalent to choosing a real number in a compact interval. For instance, in the public goods example of Section 2, announcing an optimal deception comes down to choosing an intercept in a compact set;[31] this is also the case in the team-production example of Section 6.1 where optimal deceptions are characterized by their positive slope.

[31]See Equations 1 and 2.

9 Proofs

9.1 Quasilinear Environments

The next lemma shows that if the complete information payoffs are supermodular and have increasing differences, then the ex-ante payoffs are supermodular and have increasing differences.

Lemma 1 *Assume $(M_i, \geq_i)$ is a lattice for $i \in N$. Suppose that, for each $\theta_i \in \Theta_i$, $u_i(g(m_i, m_{-i}, \theta_i))$ is supermodular in m_i for each m_{-i} and has increasing differences in (m_i, m_{-i}). Then u_i^g is supermodular in $m_i(.) \in \Sigma_i(M_i)$ for each $m_{-i}(.)$ and has increasing differences in $(m_i(.), m_{-i}(.)) \in \Sigma_i(M_i) \times \prod_{j \neq i} \Sigma_j(M_j)$.*

The proof is omitted because it is simple.

The proof of the next Proposition is also omitted, because the result is standard and its proof is similar to that of Proposition 23.D.2 in Mas-Colell et al. [35].

Proposition 6 *Consider a quasilinear family of utilities and a decision rule $x(.)$ such that $E_{\theta_{-i}}[V_i(x_i(\hat{\theta}_i, \theta_{-i}), \theta_i)]$ is continuous in $(\hat{\theta}_i, \theta_i)$.*

(i)If the scf $f = (x, t)$ is truthfully Bayesian implementable, then for all $\hat{\theta}_i$

$$E_{\theta_{-i}}[t_i(\hat{\theta}_i, \theta_{-i})] = -E_{\theta_{-i}}[V_i(x_i(\hat{\theta}_i, \theta_{-i}), \hat{\theta}_i)] + \int_{\underline{\theta}_i}^{\hat{\theta}_i} \frac{\partial E_{\theta_{-i}}[V_i(x_i(s, \theta_{-i}), s)]}{\partial \theta_i} ds + \epsilon(\underline{\theta}_i) \quad (24)$$

(ii) Let the decision rule $x(.)$ be such that $\partial E_{\theta_{-i}}[V_i(x_i(\hat{\theta}_i, \theta_{-i}), \theta_i)]/\partial \theta_i$ is increasing in $\hat{\theta}_i$ for each θ_i and $i \in N$. If transfers t satisfy (24), then the scf $f = (x, t)$ is TBI.

Proof of Theorem 4: By way of contradiction, suppose that the truthful equilibrium is not the unique Bayesian equilibrium. Since the scf is *TSBI*, there exist a greatest and a smallest equilibrium in the game induced by the mechanism. So, one of these extremal equilibria must be strictly greater/smaller than the truthful one. Suppose that the greatest equilibrium, denoted $(\overline{\theta}_i(.))_{i \in N} \in \prod \Sigma_i(\Theta_i)$, is strictly greater than the truthful equilibrium. That is, for all $i \in N$, $\overline{\theta}_i(\theta_i) \geq \theta_i$ for a.e θ_i, and there exists $N^* \neq \emptyset$ such that, for all $i \in N^*$, $\overline{\theta}_i(\theta_i) > \theta_i$ for all θ_i in some subset of types with positive measure.

I evaluate the first-order condition of agent i's maximization program at the greatest equilibrium; then, I bound it from above by an expression which cannot be positive for all players (hence the contradiction). Consider player i's interim utility for type θ_i against $\overline{\theta}_{-i}(.)$:

$$E_{\theta_{-i}}[V_i(x_i(\hat{\theta}_i, \overline{\theta}_{-i}(\theta_{-i})), \theta_i)] + E_{\theta_{-i}}[t_i(\hat{\theta}_i, \overline{\theta}_{-i}(\theta_{-i}))]. \quad (25)$$

Since $V_i \circ x_i$ and t_i are (resp.) ω_i- and β_i-Lipschitz and both differentiable in $\hat{\theta}_i$ for all $\hat{\theta}_{-i}$, we can apply the Bounded Convergence Theorem to show that for any deception $\hat{\theta}_{-i}(.)$ the first-derivative of (25) with respect to $\hat{\theta}_i$ is

$$E_{\theta_{-i}}\left[\frac{\partial V_i(x_i(\hat{\theta}_i, \overline{\theta}_{-i}(\theta_{-i})), \theta_i)}{\partial \hat{\theta}_i}\right] + E_{\theta_{-i}}\left[\frac{\partial t_i(\hat{\theta}_i, \overline{\theta}_{-i}(\theta_{-i}))}{\partial \hat{\theta}_i}\right]. \quad (26)$$

Since $u_i \circ f$ has complements bounded by κ_i, we have

$$E_{\theta_{-i}}\left[\frac{\partial V_i(x_i(\hat{\theta}_i, \overline{\theta}_{-i}(\theta_{-i})), \theta_i)}{\partial \hat{\theta}_i} + \frac{\partial t_i(\hat{\theta}_i, \overline{\theta}_{-i}(\theta_{-i}))}{\partial \hat{\theta}_i} - \frac{\partial V_i(x_i(\hat{\theta}_i, \theta_{-i}), \theta_i)}{\partial \hat{\theta}_i} - \frac{\partial t_i(\hat{\theta}_i, \theta_{-i})}{\partial \hat{\theta}_i}\right] \quad (27)$$

$$\leq \int_{\Theta_{-i}} \kappa_i \sum_{j \neq i} (\overline{\theta}_j(\theta_j) - \theta_j)\phi_{-i}(\theta_{-i}) d\theta_{-i} = \kappa_i \sum_{j \neq i} E_{\theta_j}[\overline{\theta}_j(\theta_j) - \theta_j] \quad (28)$$

By (27) and (28),

$$(26) \leq \kappa_i \sum_{j \neq i} E_{\theta_j}[\overline{\theta}_j(\theta_j) - \theta_j] + E_{\theta_{-i}}\left[\frac{\partial V_i(x_i(\hat{\theta}_i, \theta_{-i}), \theta_i)}{\partial \hat{\theta}_i}\right] + E_{\theta_{-i}}\left[\frac{\partial t_i(\hat{\theta}_i, \theta_{-i})}{\partial \hat{\theta}_i}\right]. \quad (29)$$

By part (i) of Proposition 6,

$$E_{\theta_{-i}}\left[\frac{\partial t_i(\hat{\theta}_i, \theta_{-i})}{\partial \hat{\theta}_i}\right] = -E_{\theta_{-i}}\left[\left.\frac{\partial V_i(x_i(\theta'_i, \theta_{-i}), \hat{\theta}_i)}{\partial \theta'_i}\right|_{\theta'_i = \hat{\theta}_i}\right].$$

Therefore, (29) implies

$$(26) \leq \kappa_i \sum_{j \neq i} E_{\theta_j}[\overline{\theta}_j(\theta_j) - \theta_j] + E_{\theta_{-i}}\left[\frac{\partial V_i(x_i(\hat{\theta}_i, \theta_{-i}), \theta_i)}{\partial \hat{\theta}_i} - \frac{\partial V_i(x_i(\hat{\theta}_i, \theta_{-i}), \hat{\theta}_i)}{\partial \theta'_i}\right]. \quad (30)$$

If, as claimed, it is optimal for each player i to play $\overline{\theta}_i(\theta_i)$ for a.e type θ_i, then the RHS of (30) evaluated at $\hat{\theta}_i = \overline{\theta}_i(\theta_i)$ must be positive for a.e θ_i and all $i \in N$. To see why, let $\Theta_i^* \subset \Theta_i$ be the set of types θ_i for which the RHS of (30) is strictly negative when evaluated at $\hat{\theta}_i = \overline{\theta}_i(\theta_i)$. Note that Θ_i^* is measurable by definition, because the RHS of (30) is a measurable function in θ_i when $\hat{\theta}_i = \overline{\theta}_i(\theta_i)$. By way of contradiction, suppose there is a player $i \in N$ for whom Θ_i^* has strictly positive measure. Since the RHS of (30) is greater than (26), if $\hat{\theta}_i = \overline{\theta}_i(\theta_i)$ then (26) is strictly negative for all $\theta_i \in \Theta_i^*$. But for types $\theta_i \in \Theta_i^*$, $[\underline{\theta}_i, \overline{\theta}_i(\theta_i)]$ is available to player i. Thus there exists $\varepsilon > 0$ for which the deception $\theta_i^* : \Theta_i \to \Theta_i$ defined as $\theta_i^*(\theta_i) = \overline{\theta}_i(\theta_i) - \varepsilon \mathbf{1}_{\Theta_i^*}$ for all θ_i gives i a strictly greater utility than $\overline{\theta}_i(.)$. Notice $\theta_i^*(.) \in \Sigma_i(\Theta_i)$ because $\overline{\theta}_i(.) \in \Sigma_i(\Theta_i)$, so $\theta_i^*(.)$ improves on $\overline{\theta}_i(.)$ which is a contradiction. As a result, Θ_i^* has null measure.

Since it is optimal for each player i to play $\overline{\theta}_i(\theta_i)$ for a.e type θ_i, the RHS of (30) at $\hat{\theta}_i = \overline{\theta}_i(\theta_i)$ is positive for a.e θ_i and all $i \in N$. However, this leads to the following contradiction. If the RHS of (30) is positive for a.e θ_i, then

$$\begin{aligned} 0 &\leq \kappa_i \sum_{j \neq i} E_{\theta_j}[\overline{\theta}_j(\theta_j) - \theta_j] + E_{\theta_i}\left[\frac{\partial E_{\theta_{-i}}[V_i(x_i(\overline{\theta}_i(\theta_i), \theta_{-i}), \theta_i)]}{\partial \hat{\theta}_i} - \right. \\ &\qquad \left. - \frac{\partial E_{\theta_{-i}}[V_i(x_i(\overline{\theta}_i(\theta_i), \theta_{-i}), \overline{\theta}_i(\theta_i))]}{\partial \hat{\theta}_i}\right] \\ &\leq \kappa_i \sum_{j \neq i} E_{\theta_j}[\overline{\theta}_j(\theta_j) - \theta_j] + \delta_i E_{\theta_i}[\theta_i - \overline{\theta}_i(\theta_i)] \text{ for all } i \in N, \end{aligned} \quad (31)$$

where the last inequality follows from δ_i-increasing differences. Since $\kappa_i/\delta_i < 1/(n-1)$ by hypothesis and ϕ_j has full support for all j, (31) implies

$$\sum_{j\neq i} \frac{1}{n-1} E_{\theta_j}[\overline{\theta}_j(\theta_j) - \theta_j] \geq E_{\theta_i}[\overline{\theta}_i(\theta_i) - \theta_i] \text{ for all } i \in N, \text{ and}$$

$$\sum_{j\neq i} \frac{1}{n-1} E_{\theta_j}[\overline{\theta}_j(\theta_j) - \theta_j] > E_{\theta_i}[\overline{\theta}_i(\theta_i) - \theta_i] \text{ for all } i \in \{i : \{j \neq i\} \cap N^* \neq \emptyset\}.$$

Hence

$$\sum_{i\in N}\sum_{j\neq i} \frac{1}{n-1} E_{\theta_j}[\overline{\theta}_j(\theta_j) - \theta_j] > \sum_{i\in N} E_{\theta_i}[\overline{\theta}_i(\theta_i) - \theta_i]$$

which is a contradiction because both sides are equal by definition. It is not optimal for all $i \in N$ to play $\hat{\theta}_i = \overline{\theta}_i(\theta_i)$ for a.e θ_i. Thus, there is no equilibrium that is greater than the truthful equilibrium. The same argument applies to show that there is no equilibrium that is smaller than the truthful equilibrium. Truth-revealing is the unique equilibrium. **Q.E.D**

Proof of Proposition 3: The family of valuation functions and the scf have δ-increasing differences, which implies that $\partial E_{\theta_{-i}}[V_i(x_i(\hat{\theta}_i, \theta_{-i}), \theta_i)]/\partial\theta_i$ is strictly increasing in $\hat{\theta}_i$. Given (14), let transfers be t_i^* as in (15) where t_i is taken to be (11). So, given $E_{\theta_{-i}}[V_i(x_i(\hat{\theta}_i, \theta_{-i}), \theta_i)]$ is continuous in $(\hat{\theta}_i, \theta_i)$ by assumption, Proposition 6 and Theorem 3 imply (x, t^*) is *TSBI*. Both $V_i \circ x_i$ and t_i^* are C^2, hence it follows that $u_i \circ f$ has bounded complements. The bound κ_i on complements is computed as follows,

$$\kappa_i = \max_{j\neq i} \max_{(\hat{\theta},\theta_i)\in\Theta\times\Theta_i} \left(\frac{\partial^2 V_i(x_i(\hat{\theta}), \theta_i)}{\partial\hat{\theta}_i\partial\hat{\theta}_j} - \min_{\theta_i\in\Theta_i} \frac{\partial^2 V_i(x_i(\hat{\theta}), \theta_i)}{\partial\hat{\theta}_i\partial\hat{\theta}_j} \right).$$

As C^2-functions, transfers t_i^* are β_i-Lipschitz in $\hat{\theta}_i$. Applying Theorem 4 completes the proof. **Q.E.D**

Proof of Theorem 2: Let

$$h_i(\hat{\theta}_{-i}) = -\left(\frac{1}{n-1}\right) \sum_{j\neq i} E_{\tilde{\theta}_{-j}} \left[\sum_{k\neq j} V_k(x(\hat{\theta}_j, \tilde{\theta}_{-j}), \tilde{\theta}_k) \right],$$

and for $\rho_i \in \mathbb{R}$, let

$$\delta_i(\hat{\theta}_i, \hat{\theta}_{-i}) = \sum_{j\neq i} \rho_i \hat{\theta}_i \hat{\theta}_j.$$

Define

$$t_i^{BB}(\hat{\theta}_i, \hat{\theta}_{-i}) = \delta_i(\hat{\theta}_i, \hat{\theta}_{-i}) - E_{\theta_{-i}}[\delta_i(\hat{\theta}_i, \theta_{-i})] + E_{\tilde{\theta}_{-i}} \left[\sum_{j\neq i} V_j(x(\hat{\theta}_i, \tilde{\theta}_{-i}), \tilde{\theta}_j) \right] + h_i(\hat{\theta}_{-i}) -$$

$$- \frac{1}{n-2} \sum_{j\neq i} \sum_{k\neq i,j} \rho_j \hat{\theta}_j \hat{\theta}_k + \frac{1}{n-2} \sum_{j\neq i} \sum_{k\neq i,j} \rho_j \hat{\theta}_j E(\theta_k). \quad (32)$$

First, (x, t^{BB}) is *TBI*, because $x(.)$ is allcation-efficient and

$$E_{\theta_{-i}}[t_i^{BB}(\hat{\theta}_i, \theta_{-i})] = E_{\tilde{\theta}_{-i}} \left[\sum_{j\neq i} V_j(x(\hat{\theta}_i, \tilde{\theta}_{-i}), \tilde{\theta}_j) \right] + E_{\theta_{-i}}[h_i(\theta_{-i})],$$

which is the expectation of the transfers in the expected externality mechanism (Arrow [5] and d'Aspremont and Gérard-Varet [15]). Second, note that for all θ,

$$\sum_{i\in N}\delta_i(\theta_i,\theta_{-i})-\frac{1}{n-2}\sum_{i\in N}\sum_{j\neq i}\sum_{k\neq i,j}\rho_j\theta_j\theta_k=\sum_{i\in N}\delta_i(\theta_i,\theta_{-i})-\frac{1}{n-2}\sum_{i\in N}\sum_{j\neq i}(n-2)\rho_i\theta_i\theta_j=0$$

and

$$\frac{1}{n-2}\sum_{i\in N}\sum_{j\neq i}\sum_{k\neq i,j}\rho_j\theta_jE(\theta_k)-\sum_{i\in N}E_{\theta_{-i}}[\delta_i(\theta_i,\theta_{-i})]=$$
$$=\frac{1}{n-2}\sum_{i\in N}\sum_{j\neq i}(n-2)\rho_i\theta_iE(\theta_j)-\sum_{i\in N}E_{\theta_{-i}}[\delta_i(\theta_i,\theta_{-i})]=0,$$

hence

$$\sum_{i\in N}t_i^{BB}(\theta)=\sum_{i\in N}E_{\tilde\theta_{-i}}\left[\sum_{j\neq i}V_j(x(\theta_i,\tilde\theta_{-i}),\tilde\theta_j)\right]+\sum_{i\in N}h_i(\theta_{-i})=0,$$

because transfers are balanced in the expected externality mechanism. Furthermore, t_i^{BB} is clearly continuous in $\hat\theta_{-i}$ for each $\hat\theta_i$ and usc in $\hat\theta_i$ for each $\hat\theta_{-i}$. It follows from standard arguments that $E_\theta[t_i^{SM}(\hat\theta_i(\theta_i),\hat\theta_{-i}(\theta_{-i}))]$ is continuous in $\hat\theta_{-i}(.)$ and usc in $\hat\theta_i(.)$. Next I show that it is possible to take ρ_i so that the complete information payoffs have increasing differences in $(\hat\theta_i,\hat\theta_{-i})$. By assumption, there exists a lower bound T_i on the substitutes from $V_i\circ x_i(.)$. Set $\rho_i>-T_i$. Choose any $\theta''_{-i}\geq_{-i}\theta'_{-i}$ and pick any $\theta''_i>\theta'_i$. Given (32), notice

$$t_i^{BB}(\theta''_i,\theta''_{-i})-t_i^{BB}(\theta''_i,\theta'_{-i})-t_i^{BB}(\theta'_i,\theta''_{-i})+t_i^{BB}(\theta'_i,\theta'_{-i})=$$
$$=\delta_i(\theta''_i,\theta''_{-i})-\delta_i(\theta''_i,\theta'_{-i})-\delta_i(\theta'_i,\theta''_{-i})+\delta_i(\theta'_i,\theta'_{-i}). \quad (33)$$

Therefore, $u_i(x_i(\hat\theta_i,\hat\theta_{-i}),\theta_i)$ has increasing differences in $(\hat\theta_i,\hat\theta_{-i})$ for all θ_i, if the following expression is positive for each θ_i,

$$V_i(x_i(\theta''_i,\theta''_{-i}),\theta_i)+V_i(x_i(\theta'_i,\theta'_{-i}),\theta_i)-V_i(x_i(\theta''_i,\theta'_{-i}),\theta_i)-V_i(x_i(\theta'_i,\theta''_{-i}),\theta_i)$$
$$+\sum_{j\neq i}\rho_i\big(\theta''_i\theta''_j+\theta'_i\theta'_j-\theta''_i\theta'_j-\theta'_i\theta''_j\big). \quad (34)$$

The proof then follows similarly to that of Theorem 1. **Q.E.D**

Proof of Proposition 4: Since $\tau_i-T_i<\delta_i/(n-1)$, there is $\rho_i>-T_i$ such that $\rho_i+\tau_i<\delta_i/(n-1)$. By Theorem 2, (x,t^{BB}) is supermodular implementable whenever $\rho_i>-T_i$. Because $V_i\circ x_i(.)$ has complements bounded by τ_i, the definition of t_i^{BB} implies that $u_i\circ f$ has complements bounded by $\rho_i+\tau_i$. Theorem 4 applies, which completes the proof. **Q.E.D**

9.2 Supermodular Revelation Principles

Lemma 2 *Let $(X,\geq)$ be a complete lattice. For $Y\supset X$, let $\phi: X\twoheadrightarrow Y$ be a correspondence whose range is Y and such that, for all $x\in X$, $x\in\phi(x)$ and for all $x'\neq x$, $\phi(x')\cap\phi(x)=\emptyset$. Then, there exists an extension $\geq^*$ of $\geq$ such that:*
(i) $(Y,\geq^*)$ is a complete lattice,
(ii) For all distinct $x,x'\in X$, and all $y\in\phi(x)$, $y'\in\phi(x')$, $y\geq^* y'$ iff $x\geq x'$,
(iii) For all $x\in X$, $\phi(x)$ is a complete chain.

Proof: Define $\geq^*$ on Y such that (ii) is satisfied. That is, for all distinct $x, x' \in X$, and all $y \in \phi(x)$, $y' \in \phi(x')$, $x \geq x'$ if and only if $y \geq^* y'$. By the Well Ordering Principle of set theory, for all $x \in X$, there exists $\succeq$ on $\phi(x)$ such that $(\phi(x), \succeq)$ is a chain, and any $B \subset \phi(x)$ has a least upper bound and a greatest lower bound in $\phi(x)$.[32] For each $x \in X$, define $\geq^*$ to be equal to $\succeq$ on $\phi(x)$. Therefore, for all $x \in X$, $\phi(x)$ is a complete chain and (iii) is satisfied. I show next that $(Y, \geq^*)$ is a complete lattice with the order $\geq^*$ just defined on all of Y.

First, I prove that it is a partially ordered set. For all $x \in X$, $x \in \phi(x)$ and thus $x \geq^* x$ because $(\phi(x), \geq^*)$ is a chain. This proves reflexivity. Now take $x, y, z \in Y$ such that $x \geq^* y$ and $y \geq^* z$. If $x \in \phi(x')$, $y \in \phi(y')$ and $z \in \phi(z')$ where x', y', z' are pairwise distinct in X, then $x \geq^* y$ implies $x' > y'$ and $y \geq^* z$ implies $y' > z'$. By transitivity of $\geq$, we have $x' > z'$, which implies $x \geq^* z$. Suppose that $x, y \in \phi(x')$ and $z \in \phi(z')$ for distinct $x', z' \in X$. Since $y \geq^* z$, we have $x' > z'$ which implies $x \geq^* z$. If $x, y, z \in \phi(x')$, then $x \geq^* z$ because $(\phi(x), \geq^*)$ is a chain, which completes the proof of transitivity. Now, if $x \geq^* y$ and $y \geq^* x$ for some $x \in \phi(x')$ and $y \in \phi(y')$, then $x' = y'$. Therefore, $x, y \in \phi(x')$ and so $x = y$ because $(\phi(x'), \geq^*)$ is a chain. This establishes antisymmetry of $\geq^*$.

Secondly, I prove that $(Y, \geq^*)$ is a complete lattice. For any subset $S \subset Y$, I show that $\sup_Y S$ and $\inf_Y S$ exist. Let $\mathcal{X}$ be such that $x \in \mathcal{X} \subset X$ if and only if $S \cap \phi(x) \neq \emptyset$. If $|\mathcal{X}| = 1$, then $S \subset \phi(x)$ where x is the unique element of $\mathcal{X}$. By definition of $\geq^*$, S has an infimum and a supremum in $\phi(x) \subset Y$. Now assume $|\mathcal{X}| \geq 2$ and let $S(x) = S \cap \phi(x)$ for all $x \in X$. Note $\{S(x)\}_{x \in \mathcal{X}}$ forms a partition of S. Define $\overline{s}(x) = \sup_Y S(x)$ and $\underline{s}(x) = \inf_Y S(x)$ which exist and are in $\phi(x)$ for all $x \in X$ by definition of $\geq^*$. Note that if $\sup_Y S$ and $\inf_Y S$ exist, then $\sup_Y S \equiv \sup_Y(\cup_{\mathcal{X}} \overline{s}(x))$ and $\inf_Y S \equiv \inf_Y(\cup_{\mathcal{X}} \underline{s}(x))$ by associativity. Since $(X, \geq)$ is a complete lattice, $\sup_X \mathcal{X}$ exists; call it $\overline{x}$. If $\overline{x} \in \mathcal{X}$, then $\overline{s}(\overline{x}) = \sup_Y(\cup_{\mathcal{X}} \overline{s}(x))$ and so $\sup_Y S$ exists. So suppose $\overline{x} \notin \mathcal{X}$ and define $s^* = \inf_Y \phi(\overline{x})$. Note $s^* \in \phi(\overline{x})$ is well-defined by definition of $\geq^*$. I show $s^* = \sup_Y(\cup_{\mathcal{X}} \overline{s}(x))$. Since $\overline{x} \notin \mathcal{X}$, $\overline{x} > x$ for all $x \in \mathcal{X}$. This implies $s^* \geq^* \overline{s}(x)$ for all $x \in \mathcal{X}$. Hence s^* is an upper bound for $\cup_{\mathcal{X}} \overline{s}(x)$. Take any upper bound $\overline{y} \neq s^*$ for $\cup_{\mathcal{X}} \overline{s}(x)$. Then $\overline{y} \notin \cup_{\mathcal{X}} \overline{s}(x)$, for if there were $x' \in \mathcal{X}$ such that $\overline{y} = \overline{s}(x')$ then $x' \geq x$ for all $x \in \mathcal{X}$ would imply that $\overline{x} \equiv \sup_X \mathcal{X} = x'$ is in $\mathcal{X}$, a contradiction. Therefore, $\overline{y} \in \phi(\tilde{x})$ for some $\tilde{x} \in X \backslash \mathcal{X}$ and since $\overline{y} \geq^* \overline{s}(x)$ for all $x \in \mathcal{X}$, $\tilde{x} > x$ for all $x \in \mathcal{X}$. Hence $\tilde{x} \geq \overline{x}$. If $\tilde{x} \neq \overline{x}$, then $\overline{y} >^* s^*$, and if $\tilde{x} = \overline{x}$, then $\overline{y} \in \phi(\overline{x})$ implies $\overline{y} \geq^* s^*$. As a result, $s^* = \sup_Y(\cup_{\mathcal{X}} \overline{s}(x))$. Finally, $\inf_Y S$ exists by a similar argument. Since $(X, \geq)$ is a complete lattice, $\inf_X \mathcal{X}$ exists; call it $\underline{x}$. If $\underline{x} \in \mathcal{X}$, then $\inf_Y(\cup_{\mathcal{X}} \underline{s}(x)) = \underline{s}(\underline{x})$. Otherwise $\inf_Y(\cup_{\mathcal{X}} \underline{s}(x)) = \sup_Y \phi(\underline{x})$. **Q.E.D**

Proof of Theorem 5: By the traditional revelation principle, (Θ, f) truthfully implements f in Bayesian equilibrium with any order on Θ_i. It remains to prove that there is an order $\geq_i^*$ on Θ_i such that the game induced by $(\{(\Theta, \geq_i^*)\}, f)$ is supermodular. I prove first that, for any $i \in N$, the order $\succeq_i$ on M_i induces an order $\geq_i^*$ on Θ_i such that $(\Theta_i, \geq_i^*)$ is a (complete) lattice. So, $\Sigma_i(\Theta_i)$ is a (complete) lattice with the pointwise order. Second, I establish that under $\geq_i^*$, $u_i^f(\hat{\theta}_i(.), \hat{\theta}_{-i}(.))$ is supermodular in $\hat{\theta}_i(.)$ and has increasing differences in $(\hat{\theta}_i(.), \hat{\theta}_{-i}(.))$.

[32] Take $\omega \in \phi(x)$. By the Well Ordering Principle, there is an order that well orders $\phi(x) \backslash \{\omega\}$. Extend this order to all of $\phi(x)$ by setting ω as the greatest element. Let $\succeq$ be the extension. Since $(\phi(x), \succeq)$ is also well ordered, $\inf_{\phi(x)}(S)$ exists for any $S \subset \phi(x)$. Since the set of upper bounds of S contains ω, it has a least element because $\phi(x)$ is well ordered. Hence $\sup_{\phi(x)}(S)$ exists.

Denote $M_i^* = m_i^*(\Theta_i)$ for all $i \in N$. For each $m_i \in M_i^*$, define the equivalence class $[m_i] = \{\theta_i \in \Theta_i : m_i^*(\theta_i) = m_i\}$. Let $\theta^s : M_i^* \to \Theta_i$ be a selection from the correspondence $[\,] : M_i^* \twoheadrightarrow \Theta_i$. As a mapping from M_i^* to $\theta^s(M_i^*)$, θ^s is a bijection because $m_i \neq m_i'$ necessarily implies $[m_i] \cap [m_i'] = \emptyset$, given that $m_i^*(.)$ is single-valued. Since θ^s is a bijection, we can define $\geq_i$ on a subset of Θ_i such that $\theta^s(m_i'') \geq_i \theta^s(m_i')$ if and only if $m_i'' \succeq_i m_i'$ where $m_i'', m_i' \in m_i^*(\Theta_i)$. Because θ^s is an order-isomorphism from $(M_i^*, \succeq_i)$ to $(\theta^s(M_i^*), \geq_i)$, it preserves all existing joins and meets. This implies that $(\theta^s(M_i^*), \geq_i)$ is a (complete) lattice because $(M_i^*, \succeq_i)$ is a (complete) lattice. Define the extension $\geq_i^*$ (or simply $\geq^*$) of $\geq_i$ to all of Θ_i, as follows:

1. For any $m_i, m_i' \in M_i^*$ with $m_i \neq m_i'$ and for all $\theta_i \in [m_i]$, $\theta_i' \in [m_i']$, then $\theta_i \geq^* \theta_i'$ if and only if $\theta^s(m_i) \geq_i \theta^s(m_i')$.

2. For all $m_i \in M_i^*$, $([m_i], \geq^*)$ is a chain such that any subset $B \subset [m_i]$ has a least upper bound and a greatest lower bound in $[m_i]$.

By Lemma 2, $(\Theta_i, \geq^*)$ is a (complete) lattice. Thus, $\Sigma_i(\Theta_i)$ is a (complete) lattice with the pointwise order. Endow those lattices with their order-interval topology and the Borel σ-algebra so that all functions are trivially continuous and measurable.

Proving that $m_i^*(.)$ preserves meets and joins will be useful in the last step of the proof. Take any $T \subset \Theta_i$. Since $(M_i^*, \succeq_i)$ and $(\Theta_i, \geq^*)$ are complete lattices, $\sup_{M_i^*}(m_i^*(T))$ and $\sup_{\Theta_i} T$ exist. Denote $\overline{m}_T = \sup_{M_i^*}(m_i^*(T))$. Since $\sup_{\Theta_i} T$ is an upper bound for T, $\geq^*$ implies $m_i^*(\sup_{\Theta_i} T)$ is an upper bound for $m_i^*(T)$ in M_i^*. Thus, $m_i^*(\sup_{\Theta_i} T) \succeq_i \overline{m}_T$. But $\overline{m}_T$ is an upper bound for $m_i^*(T)$, hence $\sup_{[\overline{m}_T]}([\overline{m}_T])$ is an upper bound for T. So, $\sup_{[\overline{m}_T]}([\overline{m}_T]) \geq^* \sup_{\Theta_i} T$, and therefore, $\overline{m}_T \succeq_i m_i^*(\sup_{\Theta_i} T)$. A similar argument applies to show $\inf_{M_i^*}(m_i^*(T)) = m_i^*(\inf_{\Theta_i} T)$.

Now I show that $u_i^f(\hat{\theta}_i(.), \hat{\theta}_{-i}(.))$ is supermodular in $\hat{\theta}_i(.)$ and has increasing differences in $(\hat{\theta}_i(.), \hat{\theta}_{-i}(.))$. Take any $i \in N$ and for all $j \neq i$, endow Θ_j with $\geq_j^*$ and $\Sigma_j(\Theta_j)$ with the corresponding pointwise order. Endow $\prod \Sigma_j(\Theta_j)$ with the product order. The first step is to show that $u_i^f(\hat{\theta}_i(.), \hat{\theta}_{-i}(.))$ is supermodular in $\hat{\theta}_i(.)$. For any $\theta_i''(.)$ and $\theta_i'(.)$, we know $m_i^*(\theta_i'(.)) \vee m_i^*(\theta_i''(.)) = m_i^*(\theta_i'(.) \vee \theta_i''(.))$ and similarly for $\wedge$. Since the mechanism $(\{(M_i, \succeq_i)\}, g)$ supermodularly implements f, $u_i^g(m_i(.), m_{-i}(.))$ is supermodular in $m_i(.)$ for each $m_{-i}(.)$. For any $\hat{\theta}_{-i}(.)$,

$$\begin{aligned} u_i^g(m_i^*(\theta_i'(.) \vee \theta_i''(.)), m_{-i}^*(\hat{\theta}_{-i}(.))) + u_i^g(m_i^*(\theta_i'(.) \wedge \theta_i''(.)), m_{-i}^*(\hat{\theta}_{-i}(.))) \\ \geq u_i^g(m_i^*(\theta_i'(.)), m_{-i}^*(\hat{\theta}_{-i}(.))) + u_i^g(m_i^*(\theta_i''(.)), m_{-i}^*(\hat{\theta}_{-i}(.))), \end{aligned}$$

which implies that for any $\hat{\theta}_{-i}(.)$,

$$u_i^f(\theta_i'(.) \vee \theta_i''(.), \hat{\theta}_{-i}(.)) + u_i^f(\theta_i'(.) \wedge \theta_i''(.), \hat{\theta}_{-i}(.)) \geq u_i^f(\theta_i'(.), \hat{\theta}_{-i}(.)) + u_i^f(\theta_i''(.), \hat{\theta}_{-i}(.)).$$

The second step is to show that $u_i^f(\hat{\theta}_i(.), \hat{\theta}_{-i}(.))$ has increasing differences in $(\hat{\theta}_i(.), \hat{\theta}_{-i}(.))$. For any $\theta_i''(.) \geq_i^* \theta_i'(.)$ and $\theta_{-i}''(.) \geq_{-i}^* \theta_{-i}'(.)$, we know $m_i^*(\theta_i''(.)) \succeq_i m_i^*(\theta_i'(.))$ and $m_{-i}^*(\theta_{-i}''(.)) \succeq_{-i} m_{-i}^*(\theta_{-i}'(.))$. Since the mechanism $(\{(M_i, \succeq_i)\}, g)$ supermodular implements f, $u_i^g(m_i(.), m_{-i}(.))$ has increasing differences in $(m_i(.), m_{-i}(.))$. For any θ_i,

$$\begin{aligned} u_i^g(m_i^*(\theta_i''(.)), m_{-i}^*(\theta_{-i}''(.))) - u_i^g(m_i^*(\theta_i'(.)), m_{-i}^*(\theta_{-i}''(.))) \geq \\ \geq u_i^g(m_i^*(\theta_i''(.)), m_{-i}^*(\theta_{-i}'(.))) - u_i^g(m_i^*(\theta_i'(.)), m_{-i}^*(\theta_{-i}'(.))), \end{aligned}$$

which implies that for any θ_i,

$$u_i^f(\theta_i''(.),\theta_{-i}''(.)) - u_i^f(\theta_i'(.),\theta_{-i}''(.)) \geq u_i^f(\theta_i''(.),\theta_{-i}'(.)) - u_i^f(\theta_i''(.),\theta_{-i}'(.)),$$

and completes the proof. **Q.E.D**

For the next theorem, consider a slightly different framework from that of Section 4. A mechanism is a pair $\Gamma = (M, g)$. The Bayesian game induced by mechanism Γ is $\mathcal{G} = (N, \{(\Sigma_i(M_i), \succeq_i)\}, u^g)$ where $u^g = (u_i^g)$ is the vector of ex-ante payoffs and $\succeq_i$ is some order on $\Sigma_i(M_i)$. A mechanism supermodularly implements a scf if it Bayesian implements that scf such that for all $i \in N$ there exists an order $\succeq_i$ on $\Sigma_i(M_i)$ for which $\mathcal{G}$ is supermodular. A scf is supermodular Bayesian implementable if there exists a mechanism Γ that supermodularly implements it. The only difference with Definition 2 is that $\succeq_i$ need not be a pointwise order.

Theorem 6 *(***Supermodular Revelation Principle for Continuous Types***) If there exists a mechanism (M, g) that supermodularly implements the scf f such that there is a Bayesian equilibrium $m^*(.)$ for which $g \circ m^* = f$ and $m_i^*(\Sigma_i(\Theta_i))$ is a complete lattice for all $i \in N$, then f is TSBI.*

Proof: The proof unfolds similarly to that of Theorem 5, except that continuity and measurability have to be dealt with. By the traditional revelation principle, (Θ, f) truthfully implements f in Bayesian equilibrium with any order on $\Sigma_i(\Theta_i)$. I prove first that, for any $i \in N$, the order $\succeq_i$ on $\Sigma_i(M_i)$ induces an order $\geq_i$ on $\Sigma_i(\Theta_i)$ such that $(\Sigma_i(\Theta_i), \geq_i)$ is a complete lattice. Second, I show that for all $i \in N$, $u_i^f(\hat{\theta}_i(.), \hat{\theta}_{-i}(.))$ is continuous in $\hat{\theta}_{-i}(.)$ for each $\hat{\theta}_i(.)$, and upper-semicontinuous in $\hat{\theta}_i(.)$ for each $\hat{\theta}_{-i}(.)$. Third, I establish that under $\geq_i^*$, $u_i^f(\hat{\theta}_i(.), \hat{\theta}_{-i}(.))$ is supermodular in $\hat{\theta}_i(.)$ and has increasing differences in $(\hat{\theta}_i(.), \hat{\theta}_{-i}(.))$.

Let $M_i^* = m_i^*(\Sigma_i(\Theta_i))$. For each $m_i(.) \in M_i^*$, define the equivalence class $[m_i(.)] = \{\theta_i(.) \in \Sigma_i(\Theta_i) : m_i^*(\theta_i(.)) = m_i(.)\}$. Let $\theta^s : M_i^* \to \Sigma_i(\Theta_i)$ be a selection from the correspondence $[\ ] : M_i^* \twoheadrightarrow \Sigma_i(\Theta_i)$. As a mapping from M_i^* to $\theta^s(M_i^*)$, θ^s is a bijection, so we can define $\geq_i$ such that $\theta^s(m_i''(.)) \geq_i \theta^s(m_i'(.))$ if and only if $m_i''(.) \succeq_i m_i'(.)$ where $m_i''(.), m_i'(.) \in M_i^*$. Because θ^s is an order-isomorphism from $(M_i^*, \succeq_i)$ to $(\theta^s(M_i^*), \geq_i)$, $(\theta^s(M_i^*), \geq_i)$ is a complete lattice. Define the extension $\geq_i^*$ (or simply $\geq^*$) of $\geq_i$ to all of $\Sigma_i(\Theta_i)$ as follows:

1. For any $m_i(.), m_i'(.) \in M_i^*$ with $m_i(.) \neq m_i'(.)$ and for all $\theta_i(.) \in [m_i(.)]$, $\theta_i'(.) \in [m_i'(.)]$, then $\theta_i(.) \geq^* \theta_i'(.)$ if and only if $\theta^s(m_i(.)) \geq_i \theta^s(m_i'(.))$.
2. For all $m_i(.) \in M_i^*$, $([m_i(.)], \geq^*)$ is a chain such that any subset $B \subset [m_i(.)]$ has a least upper bound and a greatest lower bound in $[m_i(.)]$.

By Lemma 2, $(\Sigma_i(\Theta_i), \geq^*)$ is a complete lattice.

A similar argument to that of Theorem 5 establishes that $m_i^* : (\Sigma_i(\Theta_i), \geq^*) \to (\Sigma_i(M_i), \succeq_i)$ preserves meets and joins.

The topological properties will follow from continuity of the equilibrium strategies. Recall $m_i^* : (\Sigma_i(\Theta_i), \geq^*, \tau_i^*) \to (\Sigma_i(M_i), \succeq_i, \tau_i)$ where τ_i^* and τ_i are order-interval topologies. Take $V \in \tau_i$. So, $V = \cup_{\lambda\in\Lambda} \cap_{i=1}^{n_\lambda} [a_{\lambda,i}, b_{\lambda,i}]^c$ and thus,

$$m_i^{*-1}(V) = \cup_{\lambda\in\Lambda} \cap_{i=1}^{n_\lambda} (m_i^{*-1}([a_{\lambda,i}, b_{\lambda,i}]))^c. \tag{35}$$

Since M_i^* is a complete lattice, $\underline{m}_{\lambda,i} \equiv \inf_{M_i^*}([a_{\lambda,i}, b_{\lambda,i}] \cap M_i^*)$ and $\overline{m}_{\lambda,i} \equiv \sup_{M_i^*}([a_{\lambda,i}, b_{\lambda,i}] \cap M_i^*)$ exist. Since $(\Sigma_i(\Theta_i), \geq^*)$ is a complete lattice, $\inf_{\Sigma_i(\Theta_i)}([\underline{m}_{\lambda,i}])$ and $\sup_{\Sigma_i(\Theta_i)}([\overline{m}_{\lambda,i}])$

exist. By definition of $\geq^*$, $m_i^{*-1}([a_{\lambda,i}, b_{\lambda,i}]) = [\inf_{\Sigma_i(\Theta_i)}([\underline{m}_{\lambda,i}]), \sup_{\Sigma_i(\Theta_i)}([\overline{m}_{\lambda,i}])]$, which is closed in τ_i^*. By (35), $m_i^{*-1}(V)$ is open.

I prove that for all $i \in N$, $u_i^f(\hat{\theta}_i(.), \hat{\theta}_{-i}(.))$ is continuous in $\hat{\theta}_{-i}(.)$ for each $\hat{\theta}_i(.)$, and upper-semicontinuous in $\hat{\theta}_i(.)$ for each $\hat{\theta}_{-i}(.)$. Take any net $\{\theta_{-i}^{\alpha}(.)\} \rightarrow \theta_{-i}^*(.)$. Since $m_{-i}^*(.)$ is continuous, $\{m_{-i}^*(\theta_{-i}^{\alpha}(.))\} \rightarrow m_{-i}^*(\theta_{-i}^*(.))$. Given $u_i^g(m_i(.), m_{-i}(.))$ is continuous in $m_{-i}(.)$ for each $m_i(.)$,

$$\begin{aligned}
\lim_{\alpha} u_i^f(\hat{\theta}_i(.), \theta_{-i}^{\alpha}(.)) &= \lim_{\alpha} u_i^g(m_i^*(\hat{\theta}_i(.)), m_{-i}^*(\theta_{-i}^{\alpha}(.))) \\
&= u_i^g(m_i^*(\hat{\theta}_i(.)), m_{-i}^*(\theta_{-i}^*(.))) \\
&= u_i^f(\hat{\theta}_i(.), \theta_{-i}^*(.)).
\end{aligned}$$

Hence, for all $i \in N$, u_i^f is continuous in $\hat{\theta}_{-i}(.)$ for each $\hat{\theta}_i(.)$. The same argument applies to establish upper-semicontinuity in $\hat{\theta}_i(.)$ for each $\hat{\theta}_{-i}(.)$.

Proving that $u_i^f(\hat{\theta}_i(.), \hat{\theta}_{-i}(.))$ is supermodular in $\hat{\theta}_i(.)$ and has increasing differences in $(\hat{\theta}_i(.), \hat{\theta}_{-i}(.))$ is analogous to Theorem 5, because $m_i^* : (\Sigma_i(\theta_i), \geq_i) \rightarrow (m_i^*(\Sigma_i(\theta_i), \succeq_i)$ preserves meets and joins. **Q.E.D**

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