The Survival of Altruistic Behavior in Public Good Games

Virginie Masson
School of Economics
University of Adelaide

Alexander Matros
Department of Economics
University of Pittsburgh

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Abstract. This paper extends the study of survival of Altruism in a public good game similar to the one by Eshel, Samuelson and Shaked (1998) to other interaction structures. We describe the short run outcomes and find sufficient conditions for the survival of Altruism in the long run.

"Egoisme. Se plaindre de celui des autres et ne pas s'apercevoir du sien."
Gustave Flaubert *(Dictionnaire des idées reçues)*

This work is still in progress. Please DO NOT Quote.

1. Introduction

Why individuals choose to be altruistic has widely been studied and is still under scrutiny. In the evolutionary literature, it is well understood that the existence and survival of altruistic behavior resides in the characteristics of the models considered: local interactions and imitation rule. See Bergstrom (2002), Bergstrom and Stark (1993), Eshel, Samuelson and Shaked (1998) and Matros (2004). In this paper, we broaden the possible range of interactions among agents, and extend the study of altruistic behavior in a public good game similar to the one by Eshel, Samuelson and Shaked (1998) to other interactions structures. We describe all possible short run outcomes and identify the necessary conditions which insure the survival of altruistic actions in the long run. We show in particular, that a condition on the number of links and the number of agents in the population is not enough to insure the survival of altruistic behavior, which differs to what Matros (2004) found. Although this paper is mainly of theoritical nature, it provides some directions on how prosocial and cooperative behavior could be promoted within an institution.

Literature Review

In their paper, Bergstrom and Stark (1993)consider a population of farmers located on a road that loops around a lake. Each farmer plays a prisoner's dilemma game with his two closest neighbors and chooses his strategy by imitating the most successful strategy within his neighborhood. It turns out that a stable outcome, one in which no one wants to revise her strategy, can be obtained only if cooperators (Altruists) are grouped in clusters of two or more.

Eshel, Samuelson and Shaked (1998) consider an evolutionary version of a public good game which benefits are only local. In their paper, they study a finite population of agents placed on a circle. An agent's neighbors are the nearest agent to her right, and the nearest agent to her left. They show that Altruism can survive within this network structure if Altruist are grouped together. Matros (2004) generalized their paper by considering neighborhoods with a larger radius. In his paper, an agent's neighbors are the k nearest agents to her right and the k nearest agents to her left. He showed that under some conditions between the number of links and the number of agents in the population, Altruism survives.

Present Work

In this paper, we consider a finite population of agents whose interactions are modeled by some undirected circulant graphs. Each period, agents need to decide whether to be an Altruist (contributing to the public good) or an Egoist. We define an Altruist as an individual who provides one unit of utility to all her neighbors and bears a certain cost for it. In a similar manner, we define an Egoist as an individual who enjoys the Altruism of her neighbors at no cost. This convention is adopted for clarity purposes and follows the terminology adopted by Eshel, Samuelson and Shaked (1998). We show that given some conditions on the number of common neighbors any two neighbors (and any two non-neighbors (strangers)) have, Altruism can survive.

The paper is organized as follows: Section 2 introduces the model. Short run outcomes are presented in Section 3. Section 4 is devoted to the long run outcomes and Section 5 concludes.

2. The Model

We first describe the graph structures we study in this paper. We then describe the public good game. Finally, an evolutionary version of the game is specified.

2.1. Graphs Structures. An *undirected* graph is a pair $\mathcal{G} = (V, E)$ of sets such that $E \subseteq [V]^2$,where V are vertices and E are edges. The vertices represent the agents, whereas the edges represent the connections among agents. Two vertices i and j of $\mathcal{G}$ are adjacent or *neighbors* if $\{ij\}$ is an edge of $\mathcal{G}$. Therefore, agent i's neighbors are all agents who possess a (double-sided) link with agent i, as in Jackson and Wolinski (1996). Let $\mathbb{G}$ be the *adjacency matrix* of the graph $\mathcal{G}$: $\mathbb{G} = (g_{ij})_n$, where $g_{ij} = g_{ji} = 1$ if agents i and j are neighbors, $g_{ii} = 0$ for all $i = 1, ..., n$. Since all link are double-sided, $g_{ij} = g_{ji}$ for all $i, j = 1, ..., n$. It means that any matrix $\mathbb{G}$ is **symmetric**,

$$\mathbb{G}^T = \mathbb{G}. \tag{1}$$

We consider neighborhoods where each agent has the same number, $2k$, of neighbors:

$$\sum_{j=1}^{n} g_{ij} = \sum_{i=1}^{n} g_{ij} = 2k, \tag{2}$$

for all $i, j = 1, ..., n$. A graph which possesses this property is called $2k-$**regular**. Moreover, any $2k-$regular graph is an **Euler** graph.[1]Furthermore, since the graphs considered are $2k-$regular, it has to be the case that n is odd.

We assume that the graph is **circulant**: all agents must have "similar type" of neighbors:

$$g_{ij} = g_{i+1,j+1[\text{mod } n]}. \tag{3}$$

A matrix which satisfies condition (3) is called **circulant**. This property means that each row is a cyclic shift of the previous row. Therefore, the i^{th} row is obtained from the first by a cyclic shift of $(i-1)$ steps,

$$g_{ij} = g_{1,j-i+1[\text{mod } n]}.$$

This means that the adjacency matrix $\mathbb{G}$ is fully determined by its first row. Thus, we denote by $\mathbf{Circ}(n, k; \mathbf{r})$ the circulant adjacency matrix $\mathbb{G}$ of size n, with $2k$ ones in each row, where $\mathbf{r} = (r_1, ..., r_{2k}) \in \mathbb{R}^{2k}$ and $r_1 = \{\min j \mid g_{1j} = 1\}$, $r_2 = \{\min j \mid j > r_1$ and $g_{1j} = 1\}$,...,$r_{2k} = \{\max j \mid g_{1j} = 1\}$.The set r is referred to as the connection set.

It follows that:

Proposition 1. *Suppose that graph $\mathbb{G}$ is symmetric and circulant. Then $g_{1j} = g_{i,((n+2)-j)[\text{mod } n]}$ for any $j = 1, 2, ..., n$.*

[1] The degree of a vertex is defined as the number of links at this vertex. Since our graph is regular, the degree of each vertex is the same and is equal to $2k$. From Diestel (2006), Theorem 1.8.1 : *A connected graph is Eulerian if and only if every vertex has even degree* (Euler 1736). It means that our graph is *Eulerian*: it admits a closed walk that traverses every edge of the graph exactly once. A closed walk is defined as a non-empty alternating sequence $v_0e_0v_1e_1...e_{k-1}v_k$ of vertices and edges such that $e_i = \{v_i, v_{i+1}\}$ for all $i < k$, and $v_0 = v_k$ (See Diestel, 2006).

Note that our assumption (3) does not follow from properties (1) and (2).Finally, we consider only connected circulant graphs. From proposition 1 in Boesch and Tindell (1984), it follows that:

Proposition 2. *The circulant graph $\mathcal{G}$ represented by the matrix $\mathbb{G}$ is connected if and only if gcd*$(r_1 - 1, ..., r_{2k} - 1, n) = 1$
Proof. *From proposition 1 in Boesch and Tindell (1984)* ■

Example 1. Consider the following matrix $\mathbb{G}$.[2]

$$\mathbb{G}=\begin{pmatrix} \mathbf{0} & \mathbf{1} & & & & \mathbf{1} & & \mathbf{1} & & \mathbf{1} \\ \mathbf{1} & \mathbf{0} & \mathbf{1} & & & & \mathbf{1} & & \mathbf{1} & \\ & \mathbf{1} & \mathbf{0} & \mathbf{1} & & & & \mathbf{1} & & \mathbf{1} \\ & & \mathbf{1} & \mathbf{0} & \mathbf{1} & \mathbf{1} & & & \mathbf{1} & \\ & & & \mathbf{1} & \mathbf{0} & \mathbf{1} & \mathbf{1} & & & \mathbf{1} \\ \mathbf{1} & & & \mathbf{1} & \mathbf{1} & \mathbf{0} & \mathbf{1} & & & \\ & \mathbf{1} & & & \mathbf{1} & \mathbf{1} & \mathbf{0} & \mathbf{1} & & \\ \mathbf{1} & & \mathbf{1} & & & & \mathbf{1} & \mathbf{0} & \mathbf{1} & \\ & \mathbf{1} & & \mathbf{1} & & & & \mathbf{1} & \mathbf{0} & \mathbf{1} \\ \mathbf{1} & & \mathbf{1} & & \mathbf{1} & & & & \mathbf{1} & \mathbf{0} \end{pmatrix} \tag{4}$$

Matrix $\mathbb{G}$ satisfies properties (1) and (2) for $k = 2$, but is not circulant.

Note that all previous papers in the literature deal with **undirected circulant Euler** graphs.

Example 2. Bergstrom and Stark (1993) and Eshel, Samuelson and Shaked (1998) consider agents on a circle where each agent has two neighbors: one agent to her right and one agent to her left. This population structure can be described by

[2]For clarity purposes, we do not write entries that are zeros and let empty spaces instead. This convention is adopted throughout the paper.

the following matrix, which represents an **undirected circulant Euler** graph

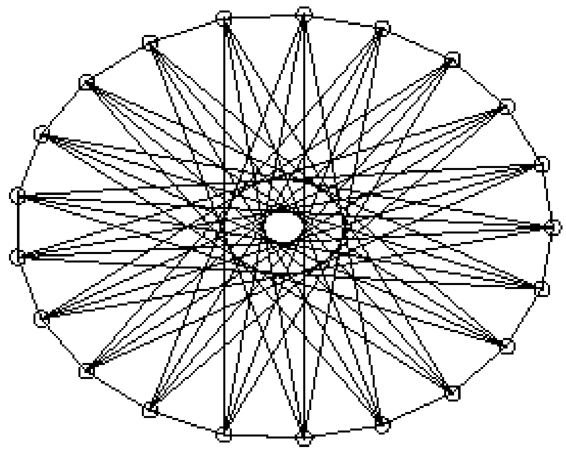
$$\begin{pmatrix} \mathbf{0} & \mathbf{1} & & & & & & & & \mathbf{1} \\ \mathbf{1} & \mathbf{0} & \mathbf{1} & & & & & & & \\ & \mathbf{1} & \mathbf{0} & \mathbf{1} & & & & & & \\ & & \mathbf{1} & \mathbf{0} & \mathbf{1} & & & & & \\ & & & \mathbf{1} & \mathbf{0} & \mathbf{1} & & & & \\ & & & & \mathbf{1} & \mathbf{0} & \mathbf{1} & & & \\ & & & & & \mathbf{1} & \mathbf{0} & \mathbf{1} & & \\ & & & & & & \mathbf{1} & \mathbf{0} & \mathbf{1} & \\ & & & & & & & \mathbf{1} & \mathbf{0} & \mathbf{1} \\ \mathbf{1} & & & & & & & & \mathbf{1} & \mathbf{0} \end{pmatrix},$$

or $\mathbf{Circ}(n, 2; \mathbf{r})$, where $\mathbf{r} = (2, n)$.

Matros(2004) also considers agents on a circle where each agent has $2k$ neighbors: k agents to her right and k agents to her left. This structure can be described by an **undirected circulant Euler** graph with adjacency matrix $\mathbf{Circ}(n, k; \mathbf{r})$, where $\mathbf{r} = (2, ..., k + 1, n - k + 1, ..., n)$.

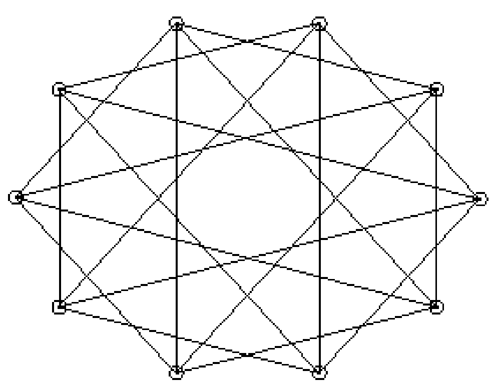
The following example illustrates that there are other graph structures which satisfy properties $(1) - (3)$.

Example 3. Let $n = 21$ and $k = 3$. Consider the matrix $\mathbf{Circ}(21, 3; \mathbf{r})$, where $\mathbf{r} = (2, 10, 11, 12, 13, 21)$. The following picture shows the corresponding **undirected circulant Euler** graph



Connected circulant Euler graph

Example 4. Let $n = 8$ and consider the matrix $\mathbf{Circ}(10, 2; \mathbf{r})$, where $\mathbf{r} = (3, 5, 7, 9)$. The following picture shows that the corresponding **undirected circulant** graph is **NOT connected**:



Disconnected circulant Euler graph

2.2. Public Good Game (PGG). We consider the following Public Good game. There are n agents. Each agent has $2k < n$, $k \geq 1$, neighbors with whom she exclusively interacts.

An agent can choose to be either an **Altruist**, or an **Egoist.** An altruist produces a public good which gives one unit of utility to all, $2k$, of her neighbors and incurs a net cost of $0 < c < 1$ to herself. An egoist produces nothing at no cost. Therefore, the payoff of an agent i is $\pi_i = N(A,i) - c$, if agent i is an Altruist, and $\pi_i = N(A,i)$, if agent i is an Egoist, where $N(A,i) \in \{0,\ldots,2k\}$ is the number of i's altruistic neighbors.

We can use our notation in order to rewrite the payoffs in the Public Good game. The payoff vector of all agents is $\pi = (\pi_1,...,\pi_n)^T = \mathbb{G}\mathbf{x} - \mathbf{c}\mathbf{x}$, where $\mathbf{x} = (x_1,...,x_n)^T$ and

$$x_i = \begin{cases} 1, & \text{if agent } i \text{ is an Altruist,} \\ 0, & \text{if agent } i \text{ is an Egoist.} \end{cases}$$

An agent chooses her strategy by examining her payoffs and her neighbors' payoffs. If her payoff is greater than the payoffs of her neighbors, she continues to play the same strategy. On the other hand, if the payoff of one or more of her neighbors is higher than hers, she adopts the strategy played by her most successful neighbor.

2.3. Evolutionary version of PGG. We consider the following evolutionary version of the Public Good game. In each discrete time period, $t = 1, 2, ...$, a population of n agents plays the Public Good game. An agent i chooses a strategy $x_i^t \in \{0,1\}$ at time t according to an imitation decision rule defined below. The play at time t is the vector $x^t = (x_1^t,\ldots,x_n^t)$.

Strategies are chosen as follows. An agent plays a strategy in period $t+1$, which gives the highest payoff among her $2k$ neighbors and herself in the previous period t.

Assume that the sampling process begins in the period $t = 1$ from some arbitrary initial play x^0. Then we obtain a finite Markov chain on the finite state space $\{0,1\}^n$ of states of the length n drawn from the strategy space $\{0,1\}$ with an arbitrary initial play x^0. Given a play x^t at time t, the process moves to a state x^{t+1} in the next period, such a state is called a successor of x^t. We call this process *unperturbed imitation dynamics* with population size n and $2k$ neighbors, $Y^{n,k,0}$.

The unperturbed imitation dynamics process describes the short-run behavior of the model when agents' behavior is mistake-free. Short-run predictions present some major interest due to fact that they arise rapidly, given the local interaction structure of the model, prevail for a long time(until a mistake is made), and depend on the initial state.

Now, suppose that agents use an imitation decision rule to choose a strategy with probability $1-\varepsilon$ and make a *mistake* and choose a strategy at random with probability $\varepsilon \geq 0$. The resulting *perturbed imitation dynamics* process $Y^{n,k,\varepsilon}$ is an ergodic Markov process on the finite state space $\{0,1\}^n$. Thus, in the long run, the initial state is irrelevant.

3. Short Run

We are able to characterize some of the absorbing states of the unperturbed imitation dynamics $Y^{n,k,0}$ - short-run outcomes with no further assumption. An **absorbing state** of the unperturbed process $Y^{n,k,0}$ is a state such that there is zero probability for the process $Y^{n,k,0}$ of moving from this state to any other state. The first result is straightforward and therefore, the proof is omitted.

Proposition 3. *For any connected graph, the followings states are absorbing:*

- *A state where all agents are Altruists,* $x = (1, ..., 1) \equiv \overline{\mathbf{1}}$,
- *A state where all agents are Egoists,* $x = (0, ..., 0) \equiv \overline{\mathbf{0}}$.

The unperturbed imitation dynamics $Y^{n,k,0}$ can have other short-run outcomes. An **absorbing set** of the unperturbed imitation dynamics $Y^{n,k,0}$ is a minimal set of states such that there is zero probability for the unperturbed process $Y^{n,k,0}$ of moving from any state in the set to any state outside, and there is a positive probability for the unperturbed process $Y^{n,k,0}$ of moving from any state in the set to any other state in the set. (A singleton absorbing set is called an **absorbing state**.) Note that any absorbing state or set different from $\overline{\mathbf{1}}$ (All agents are Altruists) and $\overline{\mathbf{0}}$ (All agents are Egoists) has to contain *both* altruistic and egoistic agents. The following proposition shows that one of the Altruists has the highest payoff in such an absorbing state.

Proposition 4. *Suppose that properties* (1)$-$(3) *hold, state* $x \notin \left\{\overline{\mathbf{1}}, \overline{\mathbf{0}}\right\}$ *is absorbing, and* $\pi(x)$ *is the corresponding payoff vector. If*

$$\pi_i(x) = \max\{\pi_1(x), ..., \pi_n(x)\},$$

then $x_i = 1$, *agent* i *is an Altruist.*

Proof. Consider an absorbing state $x \notin \left\{\overline{\mathbf{1}}, \overline{\mathbf{0}}\right\}$ and agent i such that

$$\pi_i(x) = \max\{\pi_1(x), ..., \pi_n(x)\}.$$

Since $0 < c < 1$, payoffs to an Altruist and an Egoist are always different. Therefore the highest payoff in the absorbing state can be obtained by either an Altruist or an Egoist and never both.

Suppose that agent i is an Egoist and has the highest payoff in the absorbing state. Then all her neighbors (all agents j such that $g_{ij} = 1$) have to be Egoists, $x_j = 0$, because they imitate the most successful agent - agent i. It means that $\pi_i(x) = \sum_{j=1}^{n} g_{ij}x_j = 0$, or all Egoists have the same (minimal) payoff of 0 in the absorbing state. Since we assume that $x \notin \{\overline{\mathbf{1}}, \overline{\mathbf{0}}\}$, there exist some Altruists in the absorbing state. From property (3) it follows that the graph is connected, or some Altruists have to have some Egoists as neighbors. This is not possible since the highest payoff of an Egoist is equal to zero. This is a contradiction, which means that agent i is an Altruist. **End of Proof.**

The following result describes the minimal and the maximal number of Altruists in an absorbing state different from states $\overline{\mathbf{1}}$ (All agents are Altruists) and $\overline{\mathbf{0}}$ (All agents are Egoists). Denote $\#(x, A)$ the number of Altruists in the state x.

Proposition 5. *Suppose that properties* (1) – (3) *hold and state* $x \notin \{\overline{\mathbf{1}}, \overline{\mathbf{0}}\}$ *is absorbing. Then*

$$2k + 1 \leq \#(x, A) \leq n - 2.$$

Proof. Consider an absorbing state $x \notin \{\overline{\mathbf{1}}, \overline{\mathbf{0}}\}$. It follows from Proposition 2 that there exists an Altruist (agent) i, $x_i = 1$, such that

$$\pi_i(x) = \max\{\pi_1(x), ..., \pi_n(x)\}.$$

Then all $2k$ her neighbors (all agents j such that $g_{ij} = 1$) have to be Altruists, $x_j = 1$, because they imitate the most successful agent - agent i. It means that $\#(x, A) \geq 2k + 1$ and $\pi_i(x) = \sum_{j=1}^{n} g_{ij}x_j - cx_i = 2k - c$, with $i \neq j$.

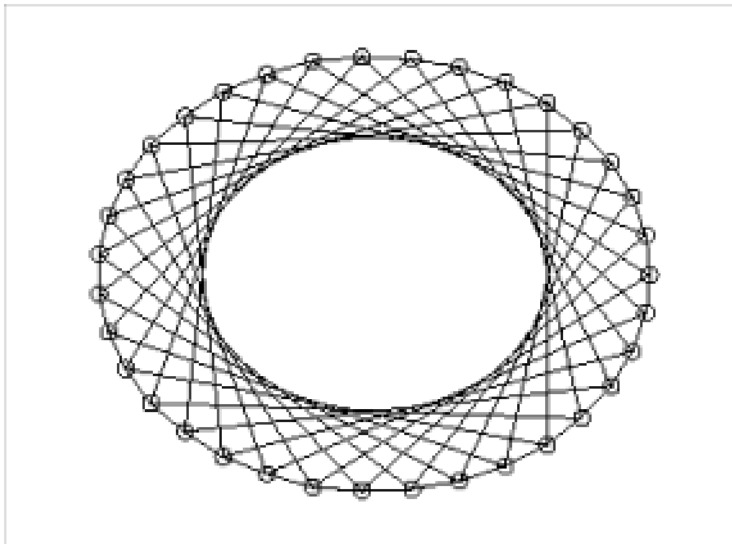
Since we assume that state $x \notin \{\overline{\mathbf{1}}, \overline{\mathbf{0}}\}$, there exists an Egoist, agent l, in the absorbing state x. This Egoist must have another Egoist in her neighborhood from Proposition 2, because otherwise

$$\pi_l(x) = 2k > 2k - c = \pi_i(x) = \max\{\pi_1(x), ..., \pi_n(x)\}.$$

Therefore, $\#(x, A) \leq n - 2$. **End of Proof.**

The following example describes an absorbing state with exactly $2k + 1$ Altruists.

Example 5. Let $n = 35$ and $k = 2$. Consider the following graph with adjacency matrix $\mathbf{Circ}(35, 2; \mathbf{r})$, where $\mathbf{r} = (2, 11, 26, 35)$. The state where agent 1 and all her neighbors, agents 2, 11, 26, and 35, are Altruists is absorbing.



Circulant graph with 35 nodes

We cannot say more in general about short-run outcomes without specifying more assumptions. Suppose that the graph structure satisfies the following *local property*: any two neighbors must have at least $k-1$ common neighbors and any two strangers can have at most k common neighbors.

$$\text{If } g_{ij} = 1, \text{ then } \sum_{l=1}^{n} g_{il} g_{jl} \geq k - 1. \tag{5}$$

$$\text{If } g_{ij} = 0, \text{ then } \sum_{l=1}^{n} g_{il} g_{jl} \leq k. \tag{6}$$

If the local property holds, we can describe all short run outcomes.

Conjecture 1. *Suppose that properties* (1) − (3) *and* (5) − (6) *hold. Then the followings are absorbing sets:*

(I) A state where all agents are Altruists,

(II) A state where all agents are Egoists,

(III) A state where two agents i and j are Egoists, $g_{ij} = 1$ such that

$$\sum_{l=1}^{n} g_{il} g_{jl} = max_h \left(\sum_{l=1}^{n} g_{il} g_{hl} \right),$$

and all other agents are Altruists,

(IV) A set of three states. In state 1, there is one agent i who is Egoist and all other agents are Altruists. In state 2, agent i and her whole neighborhood, $2k+1$ agents, are Egoists and all other agents are Altruists. In state 3, agent i and two of her neighbors, are Egoists and all other agents are Altruists.

(V) A set or state which is a combination of (III) and (IV).

The following proposition concerns some particular symmetric circulant Euler graphs.

Proposition 6. *Suppose that the graph structure is $\mathbf{Circ}(n, k; \mathbf{r})$ with $\mathbf{r} = (2, ..., k-1, j, j+1, ..., j+k, n-k, ..., n)$ and $j = \lceil \frac{n+1-k}{2} \rceil + 1$.*[3] *Then the followings are absorbing sets:*

(I) A state where all agents are Altruists,

(II) A state where all agents are Egoists,

(III) A state where two agents i and j are Egoists, $g_{ij} = 1$ such that

$$\sum_{l=1}^{n} g_{il} g_{jl} = max_h \left(\sum_{l=1}^{n} g_{il} g_{hl} \right),$$

and all other agents are Altruists,

(IV) A set of three states. In state 1, there is one agent i who is Egoist and all other agents are Altruists. In state 2, agent i and her whole neighborhood, $2k+1$ agents, are Egoists and all other agents are Altruists. In state 3, agent i and two of her neighbors, are Egoists and all other agents are Altruists.

(V) A set or state which is a combination of (III) and (IV).

Proof. It is obvious that the states where all agents are Egoists or all agents are Altruists are absorbing.

To find the remaining absorbing classes consider what happens to a cluster of Altruists. Note that any cluster of Altruists consisting of $1, 2, \ldots, k+1$ agents will immediately disappear. So, Altruism can survive in groups of the length $k+2$ or

[3] $\lceil x \rceil$ denotes the integer part of x

more. Consider what happens to a cluster of Egoists. Any cluster consisting of three or more Egoists will shrink in the next period. It will shrink until the cluster of Egoists becomes of two or one. The cluster of two Egoists will not change. However, if there is only one Egoist among her $2k$ altruistic neighbors, the whole neighborhood – all $2k+1$ agents - will become Egoists in the next period. Then this cluster of Egoists consisting of $2k+1$ agents shrinks to the cluster of three Egoists, then one. This cycle will be repeated again. ■

The above proposition is part of the conjecture regarding the short-run outcomes, and represents only some of the graphs which satisfies property 1 through 5. The decomposition of neighborhoods into more than two bocks is possible and is considered in the conjecture.

Note that the local property $(5)-(6)$ holds in Bergstrom and Stark (1993), Eshel, Samuelson and Shaked (1998), and Matros (2004) where agents are located on a circle. The following corollary describes all absorbing sets for a circle.

Corollary 1. *Suppose that the graph structure is* $\mathbf{Circ}(n, k; \mathbf{r})$ *with* $\mathbf{r} = (2, ..., k+1, n-k+1, ..., n)$. *Then the followings are absorbing sets:*

(I) A state where all agents are Altruists,

(II) A state where all agents are Egoists,

(III) A state where two agents i *and* $i+1$ *are Egoists, and all other agents are Altruists,*

(IV) A set of three states. In state 1, *there is one agent* i *who is Egoist and all other agents are Altruists. In state* 2, *agent* i *and her whole neighborhood,* $2k+1$ *agents, are Egoists and all other agents are Altruists. In state* 3, *agent* i *and two of her neighbors, are Egoists and all other agents are Altruists.*

(V) A set or state which is a combination of (III) and (IV).

Proof. Note that properties $(1)-(3)$ and $(5)-(6)$ hold for a circle $\mathbf{Circ}(n, k; \mathbf{r})$ with $\mathbf{r} = (2, ..., k+1, n-k+1, ..., n)$. The Corollary follows from Conjecture 1. **End of proof.**

4. Long Run

In order to find the states or sets of states that are stochastically stable, we need to find the resistance of the rooted trees among all states. It is easy to see that for any kind of network structure, only one mistake is needed to go from the absorbing state where all agents are Altruists to another absorbing state or set.

Conjecture 2. *Suppose that properties* $(1)-(3)$ *and* $(5)-(6)$ *hold and* $k \geq 2$. *Then*

1. If $n > 4(k+1)(k+2)$, the limiting distribution of the imitation dynamics process $Y^{n,k,\varepsilon}$ puts a positive probability on all absorbing sets except for the absorbing state where all agents are Egoists.

2. If $n < 4(k+1)(k+2)$, the limiting distribution of the imitation dynamics process $Y^{n,k,\varepsilon}$ contains only the absorbing state where all agents are Egoists.

3. If $n = 4(k+1)(k+2)$, the limiting distribution of the imitation dynamics process $Y^{n,k,\varepsilon}$ puts a positive probability on all absorbing sets.

The following corollary describes all stochastically stable sets for some particular symmetric circulant Euler graphs:

Proposition 7. *Suppose that the graph structure is* $\mathbf{Circ}(n, k; \mathbf{r})$ *with* $\mathbf{r} = (2, ..., k-1, j, j+1, ..., j+k, n-k, ..., n)$ *and* $j = \lceil \frac{n+1-k}{2} \rceil + 1$. *Then*

1. If $n > 4(k+1)(k+2)$, the limiting distribution of the imitation dynamics process $Y^{n,k,\varepsilon}$ puts a positive probability on all absorbing sets except for the absorbing state where all agents are Egoists.

2. If $n < 4(k+1)(k+2)$, the limiting distribution of the imitation dynamics process $Y^{n,k,\varepsilon}$ contains only the absorbing state where all agents are Egoists.

3. If $n = 4(k+1)(k+2)$, the limiting distribution of the imitation dynamics process $Y^{n,k,\varepsilon}$ puts a positive probability on all absorbing sets.

Proof. From Theorem 1 follows that any absorbing class can contain $n, n-2, n-3, n-4, \ldots, k+3, k+2$, or 0 Altruists. Note that it is enough to make just one mistake, ε, for moving from the absorbing class (iii) to the absorbing class (iv) from Theorem 1 and vice versa. It means that these absorbing classes have the same stochastic potential. It also means that the absorbing class (v) will have the same stochastic potential as absorbing classes (iii) and (iv).

One error, ε, is enough for moving from the absorbing state (recurrent class (ii)) to an absorbing state with $n-1$ Altruists and vice versa (absorbing class (iv)).

$(k+2)$ errors are required for moving from the absorbing state with all Egoists (from the absorbing class (i)) to the absorbing class with only 2 Egoists for n even (to absorbing class (iii)), or to sets of blinkers with $1, 2k+1$, or 3 Egoists (to the absorbing class (iv)). These $(k+2)$ errors must create a cluster consisting of $k+2$ Altruists.

What is the smallest number of mistakes which is necessary to make for moving from the absorbing class with at least $(k+2)$ Altruists to the recurrent class with all Egoists? Theorem 1 shows that the cluster of Altruists is at least of the length $(k+2)$ and the cluster of Egoists is at most of the length $(2k+1)$ in any absorbing class. There must be at least one mistake per cluster of Altruists for moving to the absorbing state where all are Egoists. After such a mistake every cluster must consist of at most $(k+1)$ Altruists in order to disappear in the next period. It is possible for a cluster of the maximal length of $(2k+3)$. That cluster must be between two clusters of Egoists and each of them consists of at most $(2k+1)$ agents. Hence, at least

$$\frac{n}{(2k+3)+(2k+1)} = \frac{n}{4\,(k+1)}$$

mistakes are necessary to move from the absorbing class (iii) or (iv) into the absorbing state with all Egoists, absorbing state (i). The statement of the theorem follows immediately. ■

The above proposition is part of the long-run conjecture, and represents only some of the graphs which satisfies property 1 through 5. The decomposition of neighborhoods into more than two bocks is possible and is considered in the conjecture.

As we already emphasized, the local property $(5)-(6)$ holds in Bergstrom and Stark (1993), Eshel, Samuelson and Shaked (1998), and Matros (2004). The following corollary describes all stochastically stable sets for a circle .

Corollary 2. *Suppose that the graph structure is* $\mathbf{Circ}(n,k;\mathbf{r})$ *with* $\mathbf{r}=(2,...,k+1,n-k+1,...,n)$ *and* $k\geq 2$. *Then*

1. If $n>4(k+1)(k+2)$, *the limiting distribution of the imitation dynamics process* $Y^{n,k,\varepsilon}$ *puts a positive probability on all absorbing sets except for the absorbing state where all agents are Egoists.*

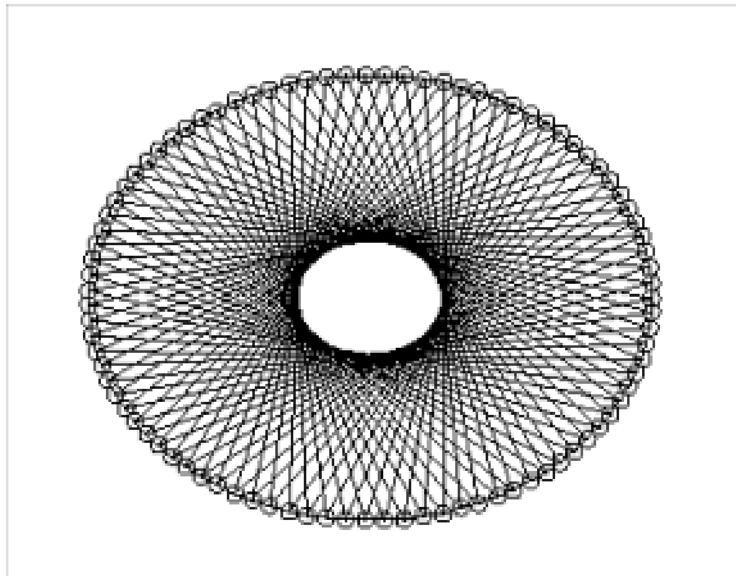
2. If $n<4(k+1)(k+2)$, *the limiting distribution of the imitation dynamics process* $Y^{n,k,\varepsilon}$ *contains only the absorbing state where all agents are Egoists.*

3. If $n=4(k+1)(k+2)$, *the limiting distribution of the imitation dynamics process* $Y^{n,k,\varepsilon}$ *puts a positive probability on all absorbing sets.*

Proof. Since properties $(1)-(3)$ and $(5)-(6)$ hold for a circle $\mathbf{Circ}(n,k;\mathbf{r})$ with $\mathbf{r}=(2,...,k+1,n-k+1,...,n)$, the Corollary follows from Conjecture 2. **End of proof.**

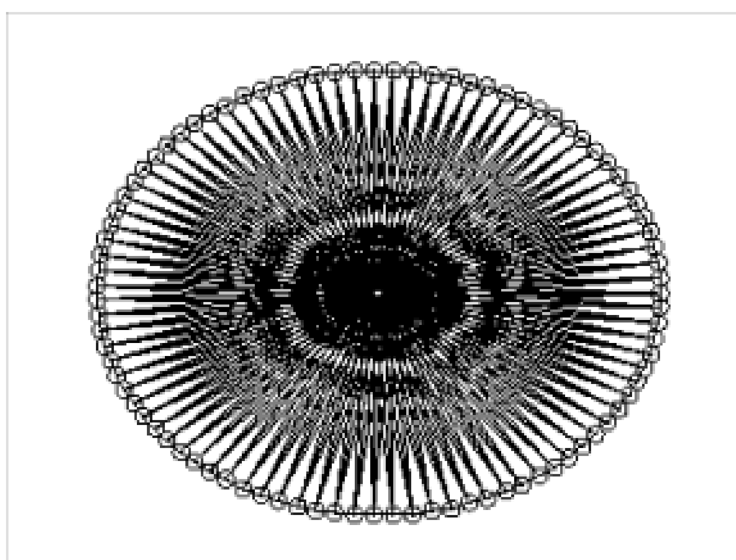
The next examples show that the local property $(5)-(6)$ must hold for the survival of Altruism.

Example 6. Let $n=91(>48)$and $k=2$. Consider the following graph with adjacency matrix $\mathbf{Circ}(\mathbf{91},2;\mathbf{r})$, where $\mathbf{r}=(2,39,54,91)$. The state where all agents are Egoist is the long-run outcome, because the local property is not satisfied.



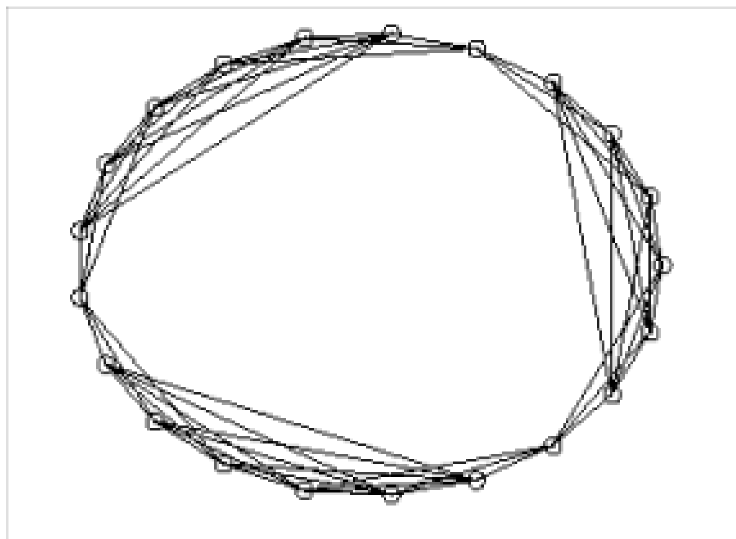
Circulant graph with 91 nodes - Local property is not satisfied

Example 7. Let $n = 91(> 80)$and $k = 3$. Consider the following graph with adjacency matrix $\mathbf{Circ}(\mathbf{91}, 3; \mathbf{r})$, where $\mathbf{r} = (2, 45, 46, 47, 48, 91)$. The state where all agents are Egoist is NOT the long-run outcome, and Altruism DOES survive in the long run.



Circulant graph with 91 nodes - Local property is satisfied

Example 8. Let $n = 21 (< 80)$ and $k = 3$. Consider the following graph with a non-circulant adjacency matrix The state where all agents are Egoist is the long-run outcome.



Non-Circulant graph with 21 nodes - Local property is satisfied

5. Conclusion and Extensions

In this paper, we focused our attention to symmetric regular circulant population structures (graphs), and studied the outcomes of a public good game which benefits are only local. We demonstrate that contrarily to what has been previously shown in the literature, a condition between the number of links and the number of agents is not enough to insure the survival of Altruism. The number of common neighbors of any two adjacent vertices (and any two non-adjacent vertices) does play a role in the outcomes selection.

Further work could be done by considering more general graphs and determine which properties may help Altruism to survive. Among the possible properties one could explore, Eulerian and Hamiltonian cycles[4] may be of particular interest.

[4] A graph is said to be Hamiltonian if it admits a Hamilton tour. This concept is similar to the

References

[1] Bergstrom, Theodore C. and O. Stark (1993) "How Altruism Can Prevail in an Evolutionary Environment," *American Economic Review* (Papers and Proceedings), 83(2), 149-55.

[2] Bergstrom, Theodore C. "Evolution of Social Behavior: Individual and Group Selection," *Journal of Economic Perspective*, 16(2), 67-88.

[3] Boesch, F. and R. Tindell (1984) "Circulants and Their Connectivities," *Journal of Graph Theory*, 8,487-499

[4] Diestel, R. *Graph Theory.* Graduate Texts in Mathematics, Springer-Verlag Berlin Heidelberg 2006.

[5] Davis, P.J. *Circulant Matrices.* Chelsea Pub Co 1994.

[6] Eshel I., L. Samuelson and A. Shaked (1998): "Altruists, Egoists, and Hooligans in a Local Interaction Model," *The American Economic Review,* 88(1), 157-179.

[7] Jackson, M., and A. Wolinsky (1996): "A Strategic Model of Social and Economic Networks," *Journal of Economic Theory*, **71**, 44-74.

[8] Matros, A. (2004): "Virtuous versus Spiteful Behavior in a Public Good Game," *Mimeo*, University of Pittsburgh.

Euler tour, but the closed walk needs to contain every vertex (not edge) exactly once. (See Diestel (2006), Chapter 10)