Auction Design with Opportunity Cost*

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Summary. This paper considers the revenue-maximizing auctions design in an independent private value setting where potential bidders have the same known positive opportunity cost of bidding. Firstly, we show that there is no loss of generality in deriving the revenue-maximizing auctions within the class of threshold-participation mechanisms, under which a bidder participates in the auction if and only if his value exceeds a threshold. Secondly, for any given set of participation thresholds whose corresponding virtual values exceed the seller's value, a modified second-price sealed-bid auction with properly set reserve prices and participation subsidy for participants is revenue-maximizing. Thirdly, we establish a variety of revenue-maximizing auctions within the symmetric threshold-participation class. Two of them involve no entry subsidy (fee). Fourthly, we identify sufficient conditions under which it is in the seller's interest to limit the number of potential bidders even if the revenue-maximizing symmetric threshold-participation auction is adopted. Lastly, we illuminate that the revenue-maximizing auction must be discriminatory in many cases, in the sense of implementing asymmetric participation thresholds across symmetric bidders.

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1 Introduction

Opportunity costs are ubiquitous in an economic world. For example, it is costly and time-consuming for potential bidders to travel to the auction site and stay for the entire duration. The bidders could have invested their resources in other opportunities and enjoyed a positive payoff. In this paper, we study the revenue-maximizing auctions when any bidder who does not bid earns a positive payoff from an outside option. We consider an independent private value (IPV) setting where the potential bidders learn their private values before their entry decisions as in Samuelson [10] and Stegeman [11].[1] Samuelson [10] studies the ex ante efficient auction and revenue-maximizing auction within the first-price sealed-bid auction class allowing no entry fee (subsidy). Stegeman [11] considers ex ante efficient auction without restricting the auction format. The revenue-maximizing auction with positive opportunity cost for bidders is yet to be derived without restrictions.[2]

The existence of opportunity cost leads to an auction design problem, in which every bidder has a positive instead of a zero reservation utility adopted by Myerson [9]. To induce participation, an auction must provide the participants with at least this positive reservation utility (the opportunity cost of bidding). Due to this positive reservation utility of the bidders, the revenue-maximizing mechanism potentially involves nonparticipation of the bidders with lower private values. Since participation can be uncertain, the procedure of deriving the revenue-maximizing auctions in Myerson [9] needs to be modified to accommodate for the nonparticipation of the bidders. We use "shutdown" to refer to the outcomes where some types of bidders prefer not to participate. To catch the essence of the shutdown policy of the seller and the endogenous entry of the potential

[1]Since Samuelson [10] and Stegeman [11] assume positive real participation cost and zero opportunity cost for bidders, their setting is essentially equivalent to ours if their participation cost equals our opportunity cost.

[2]Menezes and Monteiro [8] attempted a "restricted" revenue-maximizing auction design problem with participation cost by assuming symmetric participation across bidders.

bidders, based on a semirevelation principle in Stegeman [11, Lemma 1], our mechanism requires only the participating bidders to reveal their types, whereas those who do not participate are not required to submit signals. Our allocation rule thus depends on both the participation decisions of all potential bidders and the signals of all participants.

We first establish that there is no loss of generality in deriving the revenue-maximizing auctions within the class of threshold-participation mechanisms, under which a bidder participates in the auction if and only if his value exceeds (weakly) a threshold. Following this result, we then look for the revenue-maximizing auction within the mentioned class while allowing asymmetric participation thresholds across bidders. Myerson [9] introduced the concept of bidders' "virtual value", defined by $J(v) = v - 1/\rho(v)$, where $\rho(v)$ is the hazard rate of the distribution of the bidders' values v. The virtual value $J(v)$ measures the maximal surplus the seller can extract from a winning bidder whose private value is v. Following the literature, we assume that $J(\cdot)$ is an increasing function.[3] For a given set of participation thresholds whose corresponding virtual values exceed the seller's value, the revenue-maximizing auction takes the form of a modified second-price sealed-bid auction with appropriately set reserve prices and participation subsidy for participants whose bids are higher than the reserve prices for them.[4] The entry subsidy equals their opportunity cost, and the reserve price for each bidder equals his participation threshold. The reserve prices and the discriminating entry subsidy turn away those bidders whose values are below the participation thresholds. More importantly, the participation subsidy functions to eliminate the impact of entry cost on the entry and bidding behavior of the higher types whose values exceed the entry thresholds.

We further restrict the participation thresholds to be symmetric across bidders and study the revenue-maximizing symmetric threshold-participation auctions. The condition

[3] An increasing hazard rate function $\rho(\cdot)$ guarantees an increasing $J(\cdot)$ function. Please also refer to pages 69, 72 and 73 in Krishna [3] for further interpretation of the virtual value.

[4] As will be shown later in Corollary 2, a threshold whose corresponding virtual value is less than the seller's value is never revenue-maximizing.

characterizing the revenue-maximizing symmetric participation-threshold is provided. We find that the revenue-maximizing symmetric threshold-participation auctions can take a variety of first-price or second-price auctions with or without entry subsidy (fee), including the revenue-maximizing auction identified by Samuelson [10] within the first-price sealed-bid auction class. We further provide sufficient conditions under which it is in the seller's interest to limit the number of potential bidders even if the revenue-maximizing symmetric threshold-participation auction is adopted. This implies that the revenue-maximizing auction must be discriminatory in many cases, in the sense that asymmetric threshold participation is implemented across symmetric bidders. Thus asymmetry among the bidders is not a necessary condition for obtaining a discriminatory revenue-maximizing auction. The following example shows that at optimum, the seller wants to shut down the potential bidders asymmetrically, even if his expected revenue increases with the number of potential bidders when the revenue-maximizing symmetric-participation auctions are adopted. There are two potential bidders with opportunity cost 0.2, and the potential bidders' values follow a cumulative distribution function of $F(t) = \frac{t-0.6}{0.4}$ on $[0.6, 1.0]$. The seller's value of the object is 0. The revenue-maximizing symmetric participation threshold is 0.76, which gives the seller an expected revenue of 0.427 when the corresponding revenue-maximizing auction is taken. When the seller completely shuts down one potential bidder, he gets the expected revenue of 0.40 by optimally setting a participation threshold for the remaining bidder at 0.60. However, participation thresholds of 0.66 and 0.86 respectively for bidder 1 and bidder 2 provide the seller with the best expected revenue of 0.431. The optimality of the discriminatory auction from the view of the seller is parallel to that in Stegeman [11] in terms of efficiency. One policy implication of the above result is that the seller does not necessarily engage in limiting the number of potential bidders. What the seller needs to do is to discriminate the ex ante symmetric bidders by shutting down the potential bidders asymmetrically, which can be implemented through setting different reserve prices across bidders.

In Section 2, we first set up the model for symmetric bidders with known positive opportunity costs. We show that any mechanism that implements other participation patterns renders less expected revenue to the seller than a threshold-participation mechanism. In Section 3, we present the revenue-maximizing auction that implements a given set of participation thresholds. In Section 4, the paper derives the revenue-maximizing symmetric participation-threshold and shows that the revenue-maximizing symmetric threshold-participation auctions can take a variety of first-price or second-price auctions with or without entry subsidy (fee). In Section 5, we investigate how the seller's expected revenue is affected by the number of potential bidders, as well as the asymmetry in the participation thresholds across bidders. Finally, the conclusion will be presented in Section 6.

2 The Optimality of Threshold-Participation Mechanisms

The existence of opportunity costs means that bidders have a positive reservation utility in the auction design problem. Due to the positive reservation utility, the revenue-maximizing mechanism potentially involves nonparticipation of a portion of bidders with low private values. In this section, we show that there is no loss of generality in deriving the revenue-maximizing auctions within the class of threshold-participation mechanisms, under which a bidder participates in the auction if and only if his value exceeds (weakly) a threshold.

2.1 The Model

There is one seller who wants to sell one indivisible object to $N(\geq 2)$ potential bidders through an auction, N is assumed to be public information. We use $\mathcal{N} = \{1, 2, .., N\}$ to denote the set of all potential bidders. The ith bidder's private value of the object is t_i, which is his private information. These values $t_i,\ i \in \mathcal{N}$ are $i.i.d.$ distributed on interval $[\underline{t}, \overline{t}]$ following cumulative distribution function $F(\cdot)$ and density function $f(\cdot)(>0)$. The density $f(\cdot)$ is assumed to be public information. We adopt the regularity condition that the virtual value function $J(t) = t - \frac{1-F(t)}{f(t)}$ increases with t on interval $[\underline{t}, \overline{t}]$. The concept of "virtual value" measures the maximal surplus the seller can extract from a winning bidder whose private value is t. Krishna [3] interprets the virtual value as the seller's marginal revenue from the corresponding bidder. The regularity condition of increasing virtual value is well-adopted in the auction design literature, since it was first introduced by Myerson [9]. The seller's private value for the object is $t_0 \in [0, \overline{t})$, which is public information. Any bidder who does not bid earns a positive payoff U_0 from an outside option. U_0 is also public information. Every bidder observes his private value before he makes his participation decision.[5] The seller and bidders are assumed to be risk neutral.

Following Stegeman [11], we introduce a null message $\emptyset$ to denote the signal of a nonparticipant. Without loss of generality, we consider message space $\mathcal{M} = [\underline{t}, \overline{t}] \cup \{\emptyset\}$ for every bidder. The outcome functions of a general **mechanism** announced by the seller accommodate for all participation possibilities in the following form: payment function $x_i(\mathbf{s})$ and winning probability function $p_i(\mathbf{s})$ of bidder i, $\forall i \in \mathcal{N}$, where $\mathbf{s} = (s_1, s_2, ..., s_N)$ is the message vector and $s_i \in \mathcal{M}$ is the message from bidder i. A positive $x_i(\mathbf{s})$ means a payment to the seller from bidder i, and $p_i(\mathbf{s})$ refers to the probability that bidder i receives the object. We denote the above mechanism by $(\mathbf{p}, \mathbf{x})$, where $\mathbf{p} = (p_1(\mathbf{s}), p_2(\mathbf{s}), ..., p_N(\mathbf{s}))$ and $\mathbf{x} = (x_1(\mathbf{s}), x_2(\mathbf{s}), ..., x_N(\mathbf{s}))$. We assume that $(\mathbf{p}, \mathbf{x})$ satisfy the following conditions:

(c1) $p_i(\mathbf{s}) \geq 0,\ \forall i \in \mathcal{N},\ \forall \mathbf{s} \in \mathcal{M}^N$, and $\sum_{i=1}^{N} p_i(\mathbf{s}) \leq 1,\ \forall \mathbf{s} \in \mathcal{M}^N$, and

(c2) $p_i(\mathbf{s}) = x_i(\mathbf{s}) = 0\ if\ s_i = \emptyset,\ \forall i \in \mathcal{N}$.

[5] This setting differs from the other strand of literature on endogenous entry and participation cost, which includes McAfee and McMillan [6], Engelbrecht-Wiggans [1, 2], Tan [12], and Levin and Smith [4]. They study the case where bidders learn their values after paying the entry cost.

While (c1) says that the sum of all bidders' winning probabilities should not be higher than 1, (c2) means that the nonparticipating bidders have no chance to win the object and their payments to the seller are zero. Restriction (c2) reflects the fundamental relationship between the seller and a nonparticipant: if a bidder does not participate, then the bidder and seller are not bonded by any agreement. Thus it is reasonable to assume that both the seller and the nonparticipant maintain their status quo. This implies that the nonparticipants get their outside option U_0. In addition, (c2) is consistent with the **no passive reassignment** (NPR) assumption adopted by Stegeman [11].[6] We use S_0 to denote the set of all the mechanisms $(\mathbf{p}, \mathbf{x})$ specified above.

Before we proceed, we first introduce the timing of the auction.

Time 0: The number of potential bidders N, their gain U_0 from the outside option, the seller's private value t_0 and the distributions of the bidders' values are revealed by Nature as public information. Every bidder i observes his private value t_i, $i \in \mathcal{N}$.

Time 1: The seller announces an allocation rule $(\mathbf{p}, \mathbf{x})$ defined on the message space $\mathcal{M}^N$. The seller's strategy space is S_0.

Time 2: The bidders simultaneously and confidentially make their participation probability decisions and announce their messages $s_i \in [\underline{t}, \overline{t}]$ if they participate.

[6]The approach in Menezes and Monteiro [8] is flawed and is not adopted in this paper. To begin, the first 2 paragraphs on page 84 in Menezes and Monteiro [8] assume that $q^i(y)$ (the winning probability of bidder i **conditional** on his participation) does not depend on the participation of the other bidders, although other bidders may not participate with positive probabilities. This restriction is unreasonable because the entry decisions of the other potential bidders do affect the winning probability of a participant. Second, there is the constraint $\sum_{i=1}^{n} q^i(y) \leq 1$ at the bottom of page 84, which requires the **conditional** winning probabilities to sum to less than one for any state y. However, this restriction makes little sense for the states where bidders do not participate with positive probabilities. It seems that at various points, the reasoning in Menezes and Monteiro [8] is based on incompatible interpretations of $q^i(y)$. Third, the mechanism in Menezes and Monteiro [8] requires all potential bidders to reveal their true types irrelevant of their participation decisions, and the allocation rules are based on the signals from all potential bidders. Readers can refer to the first 2 paragraphs on page 84 in Menezes and Monteiro [8] for details.

Time 3: The payoffs of the seller and the participating bidders are determined according to the rule announced at time 1. The nonparticipants simply get the outside option U_0, as implied by restriction (c2).

Bidder i's **strategy** is denoted by $\pi_i(t_i) = (q_i^e(t_i), m_i(t_i, 1))$, where $q_i^e(t_i) \in [0, 1]$ is the participation probability of bidder i if his type is t_i, $m_i(t_i, 1) \in [\underline{t}, \overline{t}]$ is the signal of bidder i with value t_i if he participates, where "1" means he participates. The strategy space S_i of bidder i is the set of all these π_i. For convenience, we use $m_i(t_i, 0) \equiv \emptyset$ to denote bidder i's null message if he does not participate, where "0" means he does not participate. We denote $(\pi_1(t_1), \pi_2(t_2), ..., \pi_N(t_N))$ by π. A **strategy profile** π is **semidirect** if $m_i(t_i, 1) = t_i,\ \forall t_i \in [\underline{t}, \overline{t}],\ \forall i \in \mathcal{N}$.

Following Stegeman [11], we define triple $\theta = (\pi, \mathbf{p}, \mathbf{x})$ a **procedure**. The payoff of bidder i is U_0 if he does not participate; his payoff is $-x_i$ if he participates but does not win the object; his payoff is $t_i - x_i$ if he participates and wins the object, where x_i is his payment to the seller when he participates. Bidder i's objective is to maximize his expected payoff conditional on his value t_i. We use $U_i(t_i, \pi_i'; \theta)$ to denote the expected payoff of bidder i with value of t_i, if he takes strategy $\pi_i' = (q_i'^e, t_i')$ where $q_i'^e$ is his participation probability and $t_i' \in [\underline{t}, \overline{t}]$ is his signal when participating. Then we have

$$\begin{aligned} U_i(t_i, \pi_i'; \theta) = (1 - q_i'^e)U_0 + q_i'^e E_{\mathbf{t_{-i}}}\Big\{ \sum_{\mathbf{d_{-i}}} \Big[\Big(\prod_{j \neq i} q_j^e(t_j)^{d_j}(1 - q_j^e(t_j))^{1-d_j}\Big) \\ \cdot \Big(t_i p_i(t_i', \mathbf{m_{-i}}(\mathbf{t_{-i}}, \mathbf{d_{-i}})) - x_i(t_i', \mathbf{m_{-i}}(\mathbf{t_{-i}}, \mathbf{d_{-i}}))\Big)\Big]\Big\}, \end{aligned} \tag{1}$$

where $\mathbf{t_{-i}} = (t_1, ..., t_{i-1}, t_{i+1}, ..., t_N)$, $\mathbf{d_{-i}} = (d_1, ..., d_{i-1}, d_{i+1}, ..., d_N)$ and $\mathbf{m_{-i}}(\mathbf{t_{-i}}, \mathbf{d_{-i}}) = (m_1(t_1, d_1), ..., m_{i-1}(t_{i-1}, d_{i-1}), m_{i+1}(t_{i+1}, d_{i+1}), ..., m_N(t_N, d_N))$. Binary variable $d_i,\ i \in \mathcal{N}$ denotes the entry status of bidder i, it takes value of 0 or 1. If $d_i = 1$, bidder i enters; if $d_i = 0$, bidder i does not enter. The support of $\mathbf{t_{-i}}$ is $\mathcal{T}_{-i} = [\underline{t}, \overline{t}]^{N-1}$. The support of $\mathbf{d_{-i}}$ is $\{0, 1\}^{N-1}$.

A procedure is **incentive compatible** if the strategy profile π is a Bayesian Nash equilibrium of the mechanism $(\mathbf{p}, \mathbf{x})$. In other words, the following conditions hold:

$$U_i(t_i, \pi_i(t_i); \theta) \geq U_i(t_i, \pi_i'; \theta), \ \forall t_i, \ \forall \pi_i', \ \forall i \in \mathcal{N}. \tag{2}$$

We denote entry pattern $(q_1^e(\cdot), q_2^e(\cdot), ..., q_N^e(\cdot))$ by $\mathbf{q^e}$. If procedure $(\pi, \mathbf{p}, \mathbf{x})$ is incentive compatible, we say π is implemented by mechanism $(\mathbf{p}, \mathbf{x})$.

The seller's revenue equals the total payments of the bidders, plus t_0 if he retains the item. Thus, for an incentive compatible procedure $\theta = (\pi, \mathbf{p}, \mathbf{x})$, the seller's expected revenue is:

$$R(\theta) = E_{\mathbf{t}}\{\sum_{\mathbf{d}} (\prod_i q_i^e(t_i)^{d_i}(1 - q_i^e(t_i))^{1-d_i})[t_0(1 - \sum_i p_i(\mathbf{m}(\mathbf{t}, \mathbf{d}))) + \sum_i x_i(\mathbf{m}(\mathbf{t}, \mathbf{d}))]\}, \tag{3}$$

where $\mathbf{t} = (t_1, ..., t_N)$, $\mathbf{d} = (d_1, ..., d_N)$ and $\mathbf{m}(\mathbf{t}, \mathbf{d}) = (m_1(t_1, d_1), ..., m_N(t_N, d_N))$. The support of $\mathbf{t}$ is $\mathcal{T} = [\underline{t}, \overline{t}]^N$. The support of $\mathbf{d}$ is $\{0, 1\}^N$.

A procedure $(\pi, \mathbf{p}, \mathbf{x})$ is **feasible** if it is incentive compatible and if restrictions (c1) and (c2) hold for $(\mathbf{p}, \mathbf{x})$. A procedure $(\pi, \mathbf{p}, \mathbf{x})$ is **semidirect** if π is semidirect. If procedure $(\pi, \mathbf{p}, \mathbf{x})$ is feasible and semidirect, we say mechanism $(\mathbf{p}, \mathbf{x})$ is a feasible and semidirect mechanism which implements participation $\mathbf{q^e}$. Stegeman [11, Lemma 1] provides a "semirevelation" principle for the case with participation costs, which justifies that we only need to consider the feasible semidirect procedures/mechanisms for the revenue-maximizing auction.

2.2 The Optimality of Threshold-Participation Mechanisms

For a threshold-participation procedure, a bidder participates with probability 1 if his value exceeds (weakly) a threshold, and he participates with probability 0 if his value is lower than the threshold. If $(\pi, \mathbf{p}, \mathbf{x})$ is a feasible semidirect threshold-participation procedure, we say $(\mathbf{p}, \mathbf{x})$ is a feasible semidirect threshold-participation mechanism. The following Lemma justifies why we only need to consider the feasible semidirect threshold-participation procedures/mechanisms for the revenue-maximizing auction.

Lemma 1: *Among all feasible semidirect procedures, a threshold-participation procedure maximizes the seller's expected revenue.*

Proof: see Appendix.

The intuition behind this result can be explained here: For any given feasible procedure θ, bidder i's expected payoff $U_i(t_i, \pi_i(t_i); \theta)$ must be nondecreasing with t_i. Thus, there must exist a critical value $t_i^* \in [\underline{t}, \overline{t}]$ satisfying the following properties. If $t_i > t_i^*$, $U_i(t_i, \pi_i(t_i); \theta) > U_0$; if $t_i \leq t_i^*$, $U_i(t_i, \pi_i(t_i); \theta) = U_0$. If a new feasible procedure shuts down bidder i if and only if $t_i < t_i^*$ while ensuring that every participating type of bidder wins with the same expected probability and makes the same expected payment as in the original procedure, then every type of bidder enjoys the same expected surplus. While the bidders' expected surplus does not change, the seller's expected revenue increases. The reason is that the total surplus of the seller and bidders increases by the amount of savings in the entry costs of the types $t_i < t_i^*$ who may participate in the original procedure with positive probabilities. In the Appendix, we prove Lemma 1 by constructing a new feasible semidirect procedure along the above lines.

According to Lemma 1, there is no loss of generality to consider only the feasible semidirect procedures/mechanisms implementing threshold-participation for the revenue-maximizing auction, as long as the participation thresholds are not restricted.

3 Revenue-Maximizing Auction Implementing Given Threshold-Participation

For convenience, we define $r_0 = J^{-1}(t_0)$ if $t_0 \in [J(\underline{t}), J(\overline{t})]$; $r_0 = \underline{t}$ if $t_0 < J(\underline{t})$; and $r_0 = \overline{t}$ if $t_0 > J(\overline{t}) = \overline{t}$. It is well known that r_0 is the revenue-maximizing reserve price established by Myerson [9]. In this section, we establish the revenue-maximizing auction implementing any given participation thresholds $\mathbf{t^c} = (t_1^c, t_2^c, ..., t_N^c)$ where $t_i^c \geq r_0$, $\forall i \in \mathcal{N}$. Meanwhile, we will show that any threshold which is lower than r_0 cannot be revenue-maximizing.

Consider any given participation-threshold vector $\mathbf{t^c}$ where $t_i^c \in [\underline{t}, \overline{t}], \forall i \in \mathcal{N}$. Define $\mathcal{M}_i^* = [t_i^c, \overline{t}] \cup \{\emptyset\}$ and $\mathcal{M}^* = \prod_{i=1}^N \mathcal{M}_i^*$. Since we focus on mechanisms implementing given participation-threshold vector $\mathbf{t^c}$ in this section, according to the semirevelation principle, we can further restrict the message space of the bidders to be $\mathcal{M}^*$ without loss of generality.

Based on Lemma 1, we only need to consider feasible semidirect procedures that implements participation thresholds $\mathbf{t^c}$. This means that we can fix the bidders' strategy profile as $\pi_{\mathbf{t^c}} = (\pi_1(t_1), ..., \pi_N(t_N))$ where $\pi_i(t_i) = (q_i^e(t_i), t_i)$. Here, $q_i^e(t_i) = 1$ if $t_i \in [t_i^c, \overline{t}]$ and $q_i^e(t_i) = 0$ if $t_i \in [\underline{t}, t_i^c)$.

For convenience, we define $m_i(x) = x$ if $x \in [t_i^c, \overline{t}]$ and $m_i(x) = \emptyset$ if $x \in [\underline{t}, t_i^c)$. For any feasible semidirect procedure $\theta = (\pi_{\mathbf{t^c}}, \mathbf{p}, \mathbf{x})$, (3) gives the seller's expected revenue as:

$$R(\theta) = E_{\mathbf{t}}\{t_0(1 - \sum_{i=1}^N p_i(m_1(t_1), ..., m_N(t_N))) + \sum_{i=1}^N x_i(m_1(t_1), ..., m_N(t_N))\}. \tag{4}$$

For procedure θ, we use $U_i(t_i, (1, t_i'); \theta)$ to denote the interim expected payoff of bidder i with private value t_i, if he adopts strategy $(1, t_i')$, which means that he participates with probability 1 and submits signal $t_i' \in [t_i^c, \overline{t}]$. (1) gives the following:

$$U_i(t_i, (1, t_i'); \theta) = E_{\mathbf{t_{-i}}}\{t_i \; p_i(m_1(t_1), ..., t_i', ..., m_N(t_N)) - x_i(m_1(t_1), ..., t_i', ..., m_N(t_N))\}. \tag{5}$$

The seller's optimization problem of finding the revenue-maximizing feasible semidirect mechanism that implements participation thresholds $\mathbf{t^c}$ is to maximize (4) subject to the following restrictions (6) $\sim$ (9). Note that in this optimization problem, the choice variable of the seller is the mechanism $(\mathbf{p}, \mathbf{x})$.

$$(i)\ U_i(t_i, (1, t_i); \theta) \geq U_0;\ \forall i \in \mathcal{N},\ \forall t_i \in [t_i^c, \overline{t}], \tag{6}$$

$$(ii)\ U_i(t_i, (1, t_i); \theta) \geq U_i(t_i, (1, t_i'); \theta);\ \forall i \in \mathcal{N},\ \forall t_i \in [t_i^c,\ \overline{t}],\ t_i' \in [t_i^c,\ \overline{t}], \tag{7}$$

$$(iii)\ U_i(t_i, (1, t_i'); \theta) \leq U_0;\ \forall i \in \mathcal{N},\ \forall t_i < t_i^c,\ \forall t_i' \in [t_i^c,\ \overline{t}], \tag{8}$$

$$(iv)\ p_i(\mathbf{s}) = x_i(\mathbf{s}) = 0\ if\ s_i = \emptyset,\ p_i(\mathbf{s}) \geq 0,\ \forall i \in \mathcal{N}, \sum_{i=1}^N p_i(\mathbf{s}) \leq 1,\ \forall \mathbf{s} \in \mathcal{M}^*. \tag{9}$$

Conditions $(i) \sim (iii)$ come from the incentive compatible conditions (2). Conditions (i) and (ii) require that bidder i with private values equal to or higher than t_i^c participates and reveals truthfully his value. Condition (iii) requires that bidder i with private values lower than t_i^c does not participate, i.e., if he participates, he gets at most his reservation utility U_0. Thus these types of bidders submit the null signal. Conditions (iv) are the restrictions (c1) and (c2), which are listed in Section 2.1. Compared to Myerson [9], (8) is an additional constraint guaranteeing that low-type bidders do not participate; (7) guarantees that high-type bidders reveal truthfully their values when participating while (6) guarantees that higher-type bidders do participate.

We define:

$$Q_i(t_i;\theta) = E_{\mathbf{t_{-i}}}\ p_i(m_1(t_1), ..., m_i(t_i), ..., m_N(t_N)). \tag{10}$$

Note that if $t_i < t_i^c$, then $Q_i(t_i;\theta) = 0$. The following Lemma that is parallel to Myerson [9, Lemma 2], gives the necessary and sufficient conditions for a semidirect procedure $\theta = (\pi_{\mathbf{t^c}}, \mathbf{p}, \mathbf{x})$ to be feasible.

Lemma 2: *A semidirect procedure $\theta = (\pi_{\mathbf{t^c}}, \mathbf{p}, \mathbf{x})$ is feasible, if and only if the following conditions and (8) and (9) hold:*

$$Q_i(r_i;\theta) \leq Q_i(t_i;\theta),\ \forall t_i^c \leq r_i \leq t_i \leq \overline{t},\ \forall i \in \mathcal{N}, \tag{11}$$

$$U_i(t_i,(1,t_i);\theta) = U_i(t_i^c,(1,t_i^c);\theta) + \int_{t_i^c}^{t_i} Q_i(r_i;\theta)dr_i,\ \forall t_i \in [t_i^c, \overline{t}],\ \forall i \in \mathcal{N}, \tag{12}$$

$$U_i(t_i^c,(1,t_i^c);\theta) \geq U_0,\ \forall i \in \mathcal{N}. \tag{13}$$

Proof: see Appendix.

Based on Lemma 2, we can replace (6) and (7) by (11) $\sim$ (13) in the seller's optimization problem. The following Lemma then rewrites the seller's expected revenue from a feasible semidirect procedure $\theta = (\pi_{\mathbf{t^c}}, \mathbf{p}, \mathbf{x})$.

Lemma 3: *For a feasible semidirect procedure $\theta = (\pi_{\mathbf{t^c}}, \mathbf{p}, \mathbf{x})$, the seller's expected revenue* *can be written as:*

$$R(\theta) = t_0 - \sum_{i=1}^{N}(1 - F(t_i^c))U_i(t_i^c, (1, t_i^c); \theta) + E_{\mathbf{t}}[\sum_{i=1}^{N} p_i(m_1(t_1), ..., m_N(t_N))(J(t_i) - t_0)]. \quad (14)$$

Proof: see Appendix.

From Lemmas 2 and 3, we immediately have the following revenue equivalence theorem with endogenous entry in the next Corollary. This result will be applied in Section 4.

Corollary 1: *The seller and bidders' expected payoffs from a feasible auction mechanism that implements participation thresholds* $\mathbf{t^c}$ *are completely determined by the participation thresholds* $\mathbf{t^c}$*, the expected payoffs of the lowest participating types of* t_i^c, $i \in \mathcal{N}$ *and the bidders' winning probabilities for all* $\mathbf{t} \in \mathcal{T}$.

Proof: According to the semirevelation principle, for any feasible procedure that implements $\mathbf{t^c}$, there must exist an equivalent feasible semidirect procedure which delivers the same participation and allocation for any $\mathbf{t} \in \mathcal{T}$, including the winning probabilities and payments for every bidder. The result in this Corollary immediately comes from applying Lemmas 2 and 3 to this equivalent feasible semidirect procedure. $\Box$

We now turn to the revenue-maximizing auction that implements given participation thresholds $\mathbf{t^c}$. We first consider any given participation thresholds $\mathbf{t^c}$ where $t_i^c \geq r_0, \ \forall i \in \mathcal{N}$. Then we will show that any threshold which is lower than r_0 cannot be revenue-maximizing.

We define the following mechanism $(\mathbf{p}^*, \mathbf{x}^*)$. The winning probability functions are defined as following: $\forall i \in \mathcal{N}, \ \forall \mathbf{s} = (s_1, s_2, ..., s_N) \in \mathcal{M}^N$,[7]

$$p_i^*(\mathbf{s}) = \begin{cases} 0 & \text{if} \quad s_i = \emptyset \ \textit{or} \ s_i < z_i(\mathbf{s}_{-i}), \\ 1 & \text{if} \quad s_i > z_i(\mathbf{s}_{-i}), \end{cases} \quad (15)$$

[7]Clearly, we can also define the following mechanism on message space $\mathcal{M}^* \subset \mathcal{M}^N$.

and the payment functions are defined as:

$$x_i^*(\mathbf{s}) = \begin{cases} 0 & \text{if} \quad s_i = \emptyset \ or \ s_i < t_i^c, \\ -U_0 & \text{if} \quad s_i \in [t_i^c, z_i(\mathbf{s}_{-i})), \\ z_i(\mathbf{s}_{-i}) - U_0 & \text{if} \quad s_i > z_i(\mathbf{s}_{-i}), \end{cases} \tag{16}$$

where $\mathbf{s}_{-i} = (s_1, ..., s_{i-1}, s_{i+1}, ..., s_N)$ and $z_i(\mathbf{s}_{-i}) = \max\{t_i^c, \ \max_{\{j \neq i, s_j \neq \emptyset\}}\{s_j\}\}, \ i \in \mathcal{N}$.

To complete the definition of the winning probability and payment functions, we still need to consider the situation when $s_i = z_i(\mathbf{s}_{-i})$. When this happens, all participants submitting the highest signal share equally the winning probability of 1. The winner pays the highest signal while the losers do not pay anything. All participants bidding higher than their entry thresholds are subsidized by U_0. Clearly, the above defined mechanism satisfies restriction (9).

Mechanism $(\mathbf{p}^*, \mathbf{x}^*)$ constitutes exactly the rule for a modified second-price sealed-bid auction with a reserve price of t_i^c for bidder i, and an entry subsidy of U_0 for the participants whose bids exceed their corresponding reserve prices. The following proposition shows that $\theta^* = (\pi_{\mathbf{t^c}}, \mathbf{p}^*, \mathbf{x}^*)$ is a feasible semidirect procedure and it delivers higher expected revenue than any feasible semidirect procedure $(\pi_{\mathbf{t^c}}, \mathbf{p}, \mathbf{x})$ for given $\mathbf{t^c}$.

Proposition 1: *A modified second-price sealed-bid auction with a reserve price of t_i^c for bidder i and an entry subsidy of U_0 for the participants whose bids exceed their corresponding reserve prices, is the revenue-maximizing auction that implements any given entry thresholds $\mathbf{t^c}$ where $t_i^c \geq r_0, \ \forall i \in \mathcal{N}$. The highest bidder among the participants wins the object. If there is a tie in the highest bid, all highest bidders win in equal probability. The winner pays the second highest bid or the reserve price, whichever is higher. Bidder i participates and bids his true value if and only if his value exceeds his entry threshold t_i^c.*

Proof: see Appendix.

The Proposition 1 auction selects the bidders with values higher than their participation thresholds to participate through the combination of the discriminating entry subsidy and reserve prices. While the entry subsidy functions to encourage participation by compensating exactly the entry cost, the reserve prices function to implement the participation thresholds that equal the reserve prices. More importantly, the participation subsidy of the amount U_0 eliminates exactly the impact of the entry cost on the participation and bidding behavior of the higher types whose values exceed the participation thresholds. These higher types of bidders would participate and bid as if they are in the environment of Myerson [9]. This explains why the modified second-price auction in Proposition 1 is revenue-maximizing.

Define $\tilde{\mathbf{t}}^{\mathbf{c}} = (\max\{t_1^c, r_0\}, \max\{t_2^c, r_0\}, ..., \max\{t_N^c, r_0\})$, for any given $\mathbf{t}^{\mathbf{c}}$ where $t_i^c \in [\underline{t}, \overline{t}]$, $\forall i \in \mathcal{N}$. For any feasible semidirect procedure $\theta = (\pi_{\mathbf{t}^{\mathbf{c}}}, \mathbf{p}, \mathbf{x})$, the seller's expected revenue is $R(\theta)$ as in (4) or (14). Define $\tilde{\theta}^* = (\pi_{\tilde{\mathbf{t}}^{\mathbf{c}}}, \tilde{\mathbf{p}}^*, \tilde{\mathbf{x}}^*)$ where $(\tilde{\mathbf{p}}^*, \tilde{\mathbf{x}}^*)$ is the Proposition 1 revenue-maximizing feasible semidirect mechanism that implements $\tilde{\mathbf{t}}^{\mathbf{c}}$. The corresponding seller's expected revenue is then $R(\tilde{\theta}^*)$. We have the following result regarding the revenue-maximizing participation thresholds.

Corollary 2: *(i) $R(\theta) \leq R(\tilde{\theta}^*)$; (ii) If there exists $i_0 \in \mathcal{N}$ such that $t_{i_0}^c < r_0$, then $R(\theta) < R(\tilde{\theta}^*)$. Thus any threshold which is lower than r_0 cannot be revenue-maximizing.*

Proof: see Appendix.

The intuition for Corollary 2 is as follows: If $t_i^c < r_0$, then $J(t_i^c) < t_0$. Note that virtual value $J(t_i^c)$ is the maximal surplus the seller can extract if bidder i with value of t_i^c wins the object. Since $J(\cdot)$ is an increasing function and $J(t_i^c) < t_0$, the seller is better off if he does not allocate the object to bidder i with $t_i \in [t_i^c, r_0)$ even if he participates. On top of that, in order to ensure those types $t_i \in [t_i^c, r_0)$ to participate, the seller has to provide them with a subsidy of at least U_0. Therefore, a revenue-maximizing seller prefers to strictly shut down those types $t_i < r_0$.

For any participation thresholds $\mathbf{t}^{\mathbf{c}} \in \mathcal{T}_0 = [r_0, \overline{t}]^N$, define $R^*(\mathbf{t}^{\mathbf{c}}) = R(\theta^*)$ where θ^* is the revenue-maximizing feasible semidirect procedure that implements $\mathbf{t}^{\mathbf{c}}$. Define $\mathbf{t}^{*\mathbf{c}} = Argmax_{\mathbf{t}^{\mathbf{c}} \in \mathcal{T}_0}\{R^*(\mathbf{t}^{\mathbf{c}})\}$. Then the Proposition 1 revenue-maximizing auction that implements $\mathbf{t}^{*\mathbf{c}}$ is the revenue-maximizing auction in the setting with opportunity costs, and $\mathbf{t}^{*\mathbf{c}}$ gives the revenue-maximizing participation thresholds.

As $R^*(\mathbf{t}^{\mathbf{c}})$ is a highly nonlinear function of $\mathbf{t}^{\mathbf{c}}$, we are unable to give an analytical solution for $\mathbf{t}^{*\mathbf{c}}$. On the other hand, we are usually interested in the auctions treating bidders symmetrically. In the next section, we will restrict the participation thresholds to be symmetric across bidders and solve for the revenue-maximizing auctions and the corresponding participation thresholds. The obtained results will help us to investigate further issues, including the relationship between the seller's expected revenue and the number of potential bidders, and conditions under which the revenue-maximizing participation thresholds are asymmetric across bidders.

4 The Revenue-Maximizing Symmetric Threshold-Participation Auctions

In this section we solve for the revenue-maximizing symmetric-participation auctions where potential bidders have a strictly positive reservation utility. The condition characterizing the revenue-maximizing symmetric participation-threshold is provided, and we find that the revenue-maximizing symmetric threshold-participation auctions can take a variety of modified first-price or second-price auctions with or without entry subsidy (fee).

4.1 The Revenue-Maximizing Symmetric Threshold-Participation Auctions

As shown in Corollary 2, the revenue-maximizing symmetric participation threshold t^c must not be less than r_0. When $t^c \in [r_0, \overline{t}]$, according to Proposition 1, the revenue-maximizing auction that implements uniform participation threshold t^c across bidders can take the form of a modified second-price sealed-bid auction with a reserve price of t^c and a discriminating entry subsidy of U_0 for participants who bid higher than the reserve price. We denote the seller's expected revenue from this auction that implements symmetric participation threshold t^c by $R^*(t^c)$. We have:

$$R^*(t^c) = t_0 F(t^c)^N - NU_0(1 - F(t^c)) + Nt^c(1 - F(t^c))F(t^c)^{N-1} + \int_{t^c}^{\bar{t}} tf^{(2)}(t)dt, \quad (17)$$

where $f^{(2)}(t) = N(N-1)(1-F(t))F(t)^{N-2}f(t)$ is the density of the second order statistics of $(t_1, t_2, ..., t_N)$.[8] Thus, we have:

$$\frac{dR^*(t^c)}{dt^c} = Nf(t^c)F(t^c)^{N-1}(t_0 + \frac{U_0}{F(t^c)^{N-1}} - J(t^c)). \quad (18)$$

Proposition 2: *i) If $U_0 \in (0, \bar{t}-t_0)$,[9] then the revenue-maximizing symmetric participation-threshold t^{c*} is given by the unique solution of:*

$$t_0 + \frac{U_0}{F(t^c)^{N-1}} = J(t^c) = t^c - \frac{1 - F(t^c)}{f(t^c)}. \quad (19)$$

We have $t^{c*} \in (\underline{t}, \bar{t})$ and $t^{c*} > r_0$. In addition, t^{c*} increases with N and U_0, and approaches $\bar{t}$ as N goes to ∞.

ii) If $U_0 \geq \bar{t} - t_0$ then $t^{c*} = \bar{t}$.

iii) If $U_0 = 0$, then $t^{c*} = r_0$, the revenue-maximizing reserve price in Myerson [9].

Proof: see Appendix.

Based on the revenue equivalence theorem in Corollary 1 and Propositions 1 and 2, the following proposition presents the revenue-maximizing auctions implementing symmetric participation.

[8] (17) can also be obtained directly through (14) under the allocation rule defined in (15) and (16) for the uniform threshold t^c. From (14) $\sim$ (16), we have $R^*(t^c) = t_0 - NU_0(1 - F(t^c)) + \int_{t^c}^{\bar{t}}(t - \frac{1-F(t)}{f(t)} - t_0)f^{(1)}(t)dt$, here $f^{(1)}(t) = NF(t)^{N-1}f(t)$ is the density function of the first order statistics of $(t_1, t_2, ..., t_N)$. Thus, we have $R^*(t^c) = t_0 F(t^c)^N - NU_0(1 - F(t^c)) + \int_{t^c}^{\bar{t}} NtF(t)^{N-1}f(t)dt - \int_{t^c}^{\bar{t}} N(1 - F(t))F(t)^{N-1}dt$. Since $\int_{t^c}^{\bar{t}} N(1 - F(t))F(t)^{N-1}dt = Nt(1 - F(t))F(t)^{N-1}|_{t^c}^{\bar{t}} - \int_{t^c}^{\bar{t}} Nt[(N - 1)(1 - F(t))F(t)^{N-2} - F(t)^{N-1}]f(t)dt$, we have (17).

[9] Note that if $U_0 + t_0 \geq \bar{t}$, then selling the object always creates a loss of total surplus.

Proposition 3: *Assume $U_0 \in (0, \bar{t} - t_0)$. The following auctions are revenue-maximizing auctions implementing symmetric participation: (i) a first-price or second-price sealed-bid auction with a reserve price of t^{c*} and a participation subsidy of U_0 to buyers whose bids exceed t^{c*}, where t^{c*} is defined in (19); (ii) a first-price or second-price sealed-bid auction with an entry subsidy $E \in [0, U_0)$ and a reserve price $r = t_0 + \frac{1-F(t^{c*})}{f(t^{c*})} + \frac{E}{F^{N-1}(t^{c*})} < t^{c*}$. In all these auctions, the bidders participate if and only if their values exceed t^{c*}.*

Proof: A second-price sealed-bid auction with a reserve price of t^{c*} and a participation subsidy of U_0 to buyers whose bids exceed t^{c*} is revenue-maximizing as immediately implied by Proposition 1. This auction implements the revenue-maximizing participation threshold t^{c*}, which is defined in Proposition 2, and the lowest participating type gains an expected payoff of U_0. In addition, the bidder with the highest value among all participants wins.

The optimality of the other formats of auctions comes from the revenue equivalence theorem in Corollary 1.

First, we show that all these auctions implement participation threshold t^{c*}.

For the first-price sealed-bid auction with a reserve price t^{c*} and a participation subsidy U_0 to buyers whose bids exceed t^{c*}, the types of bidders whose values are lower than t^{c*} strictly prefer not to participate. The reason is this: If those bidders bid lower than t^{c*}, they just incur their entry costs. If they bid higher than t^{c*}, they have a positive probability of winning. As they win, they have to pay at least the reserve price which is higher than their values.

We now show that a first-price or second-price sealed-bid auction with an entry subsidy $E \in [0, U_0)$ and a reserve price $r = t_0 + \frac{1-F(t^{c*})}{f(t^{c*})} + \frac{E}{F^{N-1}(t^{c*})}$ implements symmetric participation threshold t^{c*}. From (19), we have $r = t^{c*} - \frac{U_0 - E}{F(t^{c*})^{N-1}}$. To implement threshold $\tilde{t}$, we should have that r satisfies $(\tilde{t} - r)F^{N-1}(\tilde{t}) + E = U_0$. Thus to implement threshold t^{c*}, we need $r = t^{c*} - \frac{U_0 - E}{F(t^{c*})^{N-1}}$. Under $r = t^{c*} - \frac{U_0 - E}{F(t^{c*})^{N-1}}$, clearly, it is a Nash equilibrium that the bidders participate if and only if their values exceed t^{c*}.

Second, the lowest participating type gains an expected payoff of U_0 in all these auctions from the construct of the reserve prices and entry subsidies.

Third, the participant with the highest value wins as the equilibrium bidding functions for all these auctions are strictly increasing. □

From Proposition 3(ii), we see that a positive entry subsidy (fee) is not a necessary feature for the revenue-maximizing symmetric-participation auctions. Samuelson [10] focuses on the first-price sealed-bid auctions with no entry fee (subsidy) in a procurement setting with bidding costs. He characterizes the revenue-maximizing reserve price. An immediate implication of Proposition 3(ii) is that the Samuelson [10] auction actually is among the revenue-maximizing auctions that implement symmetric-participation.

From Proposition 3(i), the revenue-maximizing reserve price is t^{c*} when entry subsidy is U_0, while the revenue-maximizing subsidy-free reserve price is $r_f = t_0 + \frac{1-F(t^{c*})}{f(t^{c*})}$. Clearly, we have $r_f = t^{c*}$ when $U_0 = 0$. From Proposition 2(i), t^{c*} increases with U_0. Thus r_f decreases with U_0 if $\frac{1-F(\cdot)}{f(\cdot)}$ is a decreasing function.[10] The intuition behind the result that an increase in the opportunity cost of bidders may lead to lower revenue-maximizing subsidy-free reserve price is as follows: An increase in U_0 leads to an increase in t^{c*}. A higher participation threshold t^{c*} requires a higher subsidy-free reserve price to implement the threshold, while a higher U_0 requires a lower subsidy-free reserve price to subsidize entry. The marginal change in the revenue-maximizing subsidy-free reserve price is thus determined by these two conflicting needs.

The following arguments provide the intuition for obtaining (19) that determines the revenue-maximizing symmetric participation threshold t^{c*}. A modified Vickrey auction with a reserve price of t^c and a discriminating participation subsidy of U_0 to those who bid higher than the reserve price, is the revenue-maximizing auction that implements participation threshold $t^c(\geq r_0)$ according to Proposition 1. Determining the revenue-maximizing symmetric participation threshold can be achieved through ascer-

[10] $J(t)$ increases with t and $\frac{1-F(t)}{f(t)}$ decreases with t for $F(t) = t^k$ on $[0, 1]$ where $k \geq 1$.

taining the revenue-maximizing reserve price t^c in a second-price auction with discriminating entry subsidy of U_0. Consider an infinitesimal increase in reserve price t^c, say $\triangle t^c > 0$. The seller's savings in participation subsidy from this change in t^c are $N(F(t^c + \triangle t^c) - F(t^c))U_0$; the seller's gain from this higher reserve price is approximately $N(1 - F(t^c + \triangle t^c))F^{N-1}(t^c)\triangle t^c$ and the seller's loss due to less participation is $(F^N(t^c + \triangle t^c) - F^N(t^c))(t^c - t_0)$. For t^c to be revenue-maximizing, the marginal gain should be equal to the marginal loss. This condition corresponds to (19).

From the above analysis, the t^{c*} satisfying (19) is the revenue-maximizing reserve price in a second-price auction with discriminating entry subsidy of U_0. We next show that U_0 is the revenue-maximizing entry subsidy in a second-price auction with a reserve price equal to t^{c*}. Consider an infinitesimal decrease of magnitude $\triangle E$ from U_0 in the entry subsidy. We want to show that the marginal impact of $\triangle E$ on seller's expected revenue is zero. As the entry subsidy decreases, the new participation threshold $\tilde{t}^c$ must be higher than t^{c*}, as $\tilde{t}^c$ must satisfy $(\tilde{t}^c - t^{c*})F^{N-1}(\tilde{t}^c) = \triangle E$. Let $\triangle t^c = \tilde{t}^c - t^{c*}$. We have that $\triangle t^c$ approximately equals $\frac{\triangle E}{F^{N-1}(t^{c*})}$. The marginal impact of $\triangle E$ on the seller's expected revenue can be divided into three components: the seller's savings in the entry subsidy due to the lower entry subsidy; the seller's savings in the entry subsidy due to less participation; and the seller's loss in his expected revenue due to less participation. The seller's savings in the entry subsidy due to the lower entry subsidy are approximately $N(1 - F(t^{c*}))\triangle E$; the seller's savings in the entry subsidy due to less participation are approximately $N(F(t^{c*} + \triangle t^c) - F(t^{c*}))U_0 \approx NU_0 f(t^{c*})\frac{\triangle E}{F^{N-1}(t^{c*})}$; the seller's loss in his expected revenue due to less participation is approximately $(t^{c*} - t_0)(F^N(t^{c*} + \triangle t^c) - F^N(t^{c*})) \approx (t^{c*} - t_0)Nf(t^{c*})\triangle E$. Since t^{c*} satisfies (19), we have the sum of these three components is $N\triangle E f(t^{c*})\{t_0 + \frac{U_0}{F^{N-1}(t^{c*})} - J(t^{c*})\} = 0$. In other words, the aggregate marginal impact of $\triangle E$ on the seller's expected revenue is zero. This means U_0 is the revenue-maximizing entry subsidy.

Based on Proposition 2, we can easily verify that changing the values of every player (including the seller and bidders) leads to a corresponding and equivalent change in the revenue-maximizing entry threshold. However, changing every buyer's value and participation cost by the same amount does not change the revenue-maximizing entry threshold by the same amount. Direct computations using (19) show the change in the entry threshold is greater. The intuition behind this result is as follows: Suppose the revenue-maximizing second-price auction in Proposition 3(i) is adopted. Assume initially the opportunity cost is zero. Consider a simultaneous increase of δ in both the value and opportunity cost for every bidder. If the new threshold increases by the same amount δ, the increase in seller's participation subsidy is δ multiplied by the number of participants. Thus the increase in the expected total entry subsidy is strictly higher than the increase δ in the winning bidder's payment to the seller. For this reason, the increase in the revenue-maximizing entry threshold has to be higher than δ to decrease the participation.

4.2 Seller's Revenue and Participation in the Limit

The following Proposition 4 gives the seller's expected revenue and the expected participation in the limit when the number of potential bidders goes to ∞. Both the seller's expected revenue and the expected participation converge and the limit values only depend on the maximal private value $\bar{t}$ and the opportunity cost U_0.

Proposition 4: $\forall U_0 \in (0, \bar{t} - t_0)$, $R^*(t^{c*})$ *approaches* $(\bar{t} - U_0) + U_0 \log U_0 - U_0 \log(\bar{t} - t_0)$ *and the expected number of participants approaches* $\log \frac{\bar{t}-t_0}{U_0}$ *as* N *approaches* ∞. *Note that the limit values only depend on* $\bar{t}$ *and* U_0.

Proof: see Appendix.

From Proposition 4, we see that the expected participation in the limit decrease with the opportunity cost U_0 and the seller's value t_0. As $\frac{d[(\bar{t}-U_0)+U_0 \log U_0 - U_0 \log(\bar{t}-t_0)]}{dU_0} = \log \frac{U_0}{\bar{t}-t_0} < 0$, the seller's expected revenue in the limit also decreases with the opportunity cost U_0 of bidders.

5 Number of Bidders and Asymmetry in Participation Thresholds

Samuelson [10] studies the first-price sealed-bid auctions with no entry fee (subsidy) in a procurement setting with bidding costs. He claims that the seller's expected revenue may vary in almost any way with the number of potential bidders, if the revenue-maximizing reserve price is adopted. Since Proposition 3(ii) has shown that the Samuelson [10] auction is one of the revenue-maximizing symmetric-participation auctions, we immediately see that his claim applies to all the revenue-maximizing symmetric participation auctions of Proposition 3. In Proposition 5, we further provide some sufficient conditions for the seller's expected revenue to increase or decrease with the number of potential bidders when $U_0 > 0$.

Proposition 5: *Assume $F(t) = t^k$ on support [0, 1], where $k \geq 1$. If $U_0 \in (0, 1 - t_0)$, $R^*(t^{c*})$ increases with the number of the potential bidders N if $k \in [1, 2]$. However, if $k > \frac{2}{1-t_0}(1 - \frac{\log(U_0/(1-t_0))}{U_0/(1-t_0)})$, we have $\frac{dR^*(t^{c*})}{dN} < 0$ as N approaches ∞.*

Proof: see Appendix. Note that $J'(t) > 0$ if $k \geq 1$, so Propositions 2 and 3 apply. In addition, $\frac{2}{1-t_0}(1 - \frac{\log(U_0/(1-t_0))}{U_0/(1-t_0)}) \geq \frac{2}{1-t_0} \geq 2$. $\square$

Proposition 5 tells us that if the revenue-maximizing symmetric-participation auctions are adopted, the seller's expected revenue can decrease as the number of potential bidders increases if the bidders' opportunity cost is positive. Therefore in this case, it is in the seller's benefit to limit the number of potential bidders even if the revenue-maximizing symmetric-participation auctions are adopted. The following example illustrates this point. When $k = 4$, $U_0 = 0.3$ and $t_0 = 0$, $R^*(t^{c*})$ reaches a maximum of 0.3483 when $N = 3$. When $N = 2$, $R^*(t^{c*}) = 0.3459$, when $N = 4$, $R^*(t^{c*}) = 0.3475$. In addition, it approaches 0.3388 when N approaches ∞.

Furthermore, this example suggests that searching the revenue-maximizing auction within the "symmetric-participation" class is definitely restrictive when there are more than 3 potential bidders in the above example. In other words, at the optimum, the seller should discriminate the ex ante symmetric bidders by shutting them down asymmetrically. Proposition 5 suggests that this kind of situation is very common. This discrimination can be implemented through adopting different reserve prices across bidders as pointed out in Proposition 1. In particular, setting the reserve price for a bidder at his maximal private value is equivalent to shutting down this bidder completely.

Why may the seller prefer to reduce the number of bidders by shutting them down asymmetrically rather than symmetrically if entry cost exists? For given participation thresholds, the probability of sale is pinned down. The expected selling price conditional on sale is also pinned down by Proposition 1. If the item is unsold, the seller gets t_0; if the item is sold, his expected income is the expected selling price conditional on sale. However, because of the bidders' entry cost, the seller has to pay entry subsidy to the participants. The net revenue of the seller is the difference between his expected income and the entry subsidy to the bidders. The revenue-maximizing participation has to balance between these two terms. An asymmetric participation will optimally balance the seller's expected income and the entry subsidy to the bidders, when $k > \frac{2}{1-t_0}(1 - \frac{\log(U_0/(1-t_0))}{U_0/(1-t_0)})$ and N is big, where k is the parameter in Proposition 5. This can be seen when we compare the symmetric threshold participation where $t_c^{(i)} = t_N^{c*},\ i \in \mathcal{N}$ with a particular asymmetric threshold participation where $t_c^{(i)} = t_{N-1}^{c*},\ i \leq N-1$ and $t_c^{(N)} = \bar{t}$. Here, t_N^{c*} and t_{N-1}^{c*} respectively are the thresholds given in Proposition 2 for the cases of N and $N-1$ potential bidders. Based on the calculations found in the proof of Proposition 5, the difference in the seller's revenues from the two participation patterns is approximately $(1 - F(t_N^{c*}))^2[\frac{U_0}{2} + \frac{1}{k}\log\frac{U_0}{1-t_0} - \frac{1}{k}\frac{U_0}{1-t_0}]$ which is positive when $k > \frac{2}{1-t_0}(1 - \frac{\log(U_0/(1-t_0))}{U_0/(1-t_0)})$.

We now understand that generally it is necessary to allow asymmetric participation thresholds across bidders in order to solve for the revenue-maximizing auction. However, as pointed out in Section 3, we did not provide an analytical solution for the revenue-maximizing participation thresholds as the seller's highest expected revenue $R^*(\mathbf{t^c})$ is a highly nonlinear function of the participation thresholds $\mathbf{t^c}$. We thus solved the following 2-bidder example numerically. Consider a setting in which $N = 2$, $t_0 = 0$, $U_0 = 0.2$ and $F(t) = \frac{t-0.6}{0.4}$ on $[0.6, 1.0]$. Setting the reserve prices at 0.66 and 0.86 respectively for bidder 1 and bidder 2 provides the seller with the highest expected revenue of 0.431 in a second-price auction with a participation subsidy of $U_0 = 0.2$. This auction is the revenue-maximizing auction in Proposition 1 which shuts down bidder 1 and bidder 2 respectively at 0.66 and 0.86.

For the above example with two potential bidders, the seller gets the expected revenue of 0.427 from the revenue-maximizing symmetric-participation auctions while he gets the expected revenue of 0.40 if he excludes one bidder completely and sets optimally the participation threshold for the other bidder. These results show that even if the seller's expected revenue from the revenue-maximizing symmetric-participation auctions increases with the number of potential bidders, at the optimum, the seller will still want to implement asymmetric participation thresholds across bidders.

Moreover, direct calculations show that it is impossible for a second-price auction with no entry fee and a uniform reserve price to implement the revenue-maximizing asymmetric participation in the above example.[11] This result is in contrast with the Stegeman [11] result that the efficient participation maximizing the total surplus of the seller and bidders is always implementable through a second-price auction with no entry fee and a reserve price equal to the seller's value.

McAfee and McMillan [7] show that the revenue-maximizing procurement is in general discriminatory if the bidders' cost distributions are different. Proposition 5 suggests that if the bidders' opportunity cost is positive, discriminatory revenue-maximizing auction

[11] We show this result by contradiction. Suppose there is a second-price auction with no entry fee and a uniform reserve price r which implements participation thresholds $t_1^c = 0.66$, $t_2^c = 0.86$. Then r must satisfy $(t_1^c - r)F(t_2^c) = U_0$ and $(t_2^c - r)F(t_1^c) + \int_{t_1^c}^{t_2^c} tf(t)dt = U_0$. However, the r satisfies $(t_1^c - r)F(t_2^c) = U_0$ does not satisfy $(t_2^c - r)F(t_1^c) + \int_{t_1^c}^{t_2^c} tf(t)dt = U_0$.

can arise even with symmetric bidders.

6 Conclusion

In this paper we relax the assumption of zero reservation utility for bidders postulated in Myerson [9], and study the revenue-maximizing auctions. The revenue-maximizing auction characterized implements a threshold participation and takes the form of a modified second-price sealed-bid auction with appropriately set reserve prices and discriminating participation subsidy for participants. The participation subsidy equals the opportunity cost of bidders. While the discriminating participation subsidy and reserve prices turn away bidders whose values are below the participation thresholds, the participation subsidy functions to eliminate the impact of entry cost on the participation and bidding behavior of the bidders whose values exceed the participation thresholds. This explains why the modified second-price auction is revenue-maximizing.

We analytically characterized the revenue-maximizing symmetric participation threshold. Two revenue-maximizing symmetric-participation auctions were discovered, which provide no entry subsidy but set lower reserve price. When the revenue-maximizing symmetric-participation auctions are adopted, the seller's expected revenue can decrease as the number of the potential bidders increases, if the bidders' private values are distributed near the highest value. This result has two implications. On one hand, if the seller cannot discriminate bidders, he wants to limit the number of potential bidders in many situations. On the other hand, the revenue-maximizing auction must be discriminatory in many cases, in the sense of implementing asymmetric participation across symmetric bidders. These implications have appeal in general when considering contracting problems with symmetric agents who have positive opportunity cost.

The revenue-maximizing mechanism when opportunity cost is private information of bidders should be considered. Following this direction, Lu [5] considers auction design when bidders have two-dimensional private signals, namely their private values and their opportunity costs.

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Appendix

Proof of Lemma 1: It is sufficient to show that for any feasible semidirect procedure, we can construct a threshold-participation feasible semidirect procedure, which leads to the seller higher expected revenue.

Consider a feasible semidirect procedure $\theta = (\pi, \mathbf{p}, \mathbf{x})$. The equilibrium participation is described by $\mathbf{q^e} = (q_1^e(t_1), q_2^e(t_2), ..., q_N^e(t_N))$ where $q_i^e(t_i) \in [0,1]$ is the participation probability of bidder i if his type is t_i. Since the procedure θ is semidirect, we use $m_i(t_i, 1) = t_i$ to denote bidder i's truthful revelation of his type if he participates. Clearly $(\mathbf{p}, \mathbf{x})$ has to satisfy the conditions (c1) and (c2) in Section 2.1.

$U_i(t_i, \pi_i(t_i); \theta)$ is the expected payoff of bidder i with value t_i under procedure θ. From (1), we have

$$U_i(t_i, \pi_i(t_i); \theta) = (1 - q_i^e(t_i))U_0 + q_i^e(t_i)U_i(t_i, (1, t_i); \theta), \tag{A.1}$$

where $(1, t_i)$ denotes bidder i's strategy of participating with probability 1 and revealing truthfully his value when participating.

Since θ is a feasible semidirect procedure, it satisfies the following properties. (i) $U_i(t_i, \pi_i(t_i); \theta) \geq U_0, \ \forall i \in \mathcal{N}$ as participating with probability of zero is an available choice to bidder i; (ii) $U_i(t_i, \pi_i(t_i); \theta)$ is nondecreasing in t_i because a higher type can always mimic a lower type; (iii) since bidders truthfully reveal their values when participating, we have $U_i(t_i, (1, t_i); \theta) \geq U_i(t_i, (1, t_i'); \theta), \ \forall t_i, \ t_i' \in [\underline{t}, \overline{t}], \ \forall i \in \mathcal{N}$.

Without loss of generality, we consider the case that $U_i(\underline{t}, \pi_i(\underline{t}); \theta) = U_0, \ \forall i \in \mathcal{N}$.[12] If $U_i(\overline{t}, \pi_i(\overline{t}); \theta) > U_0$, we define $t_i^* = \inf_{\{U_i(t_i, \pi_i(t_i); \theta) > U_0\}}\{t_i\}$; if $U_i(\overline{t}, \pi_i(\overline{t}); \theta) = U_0$, we define $t_i^* = \overline{t}$. We have $t_i^* \in [\underline{t}, \overline{t}], \ \forall i \in \mathcal{N}$. We'll construct a new feasible semidirect procedure $\tilde{\theta}$ which implements participation threshold t_i^* for bidder i, and at the same time provides the seller with an expected revenue higher than that from the original θ.

Hereafter, owing to space constraint, we focus on the most intricate case in which $t_i^* \in (\underline{t}, \overline{t}), \ \forall i \in \mathcal{N}$. Other cases can be similarly dealt with. Before we construct $\tilde{\theta}$, we first show

[12] If $U_i(\underline{t}, \pi_i(\underline{t}); \theta) > U_0$, then we must have $q_i^e(t_i) = 1, \ \forall t_i \in [\underline{t}, \overline{t}]$. The seller's expected revenue can be increased if bidder i is asked to pay an additional fixed amount of $U_i(\underline{t}, \pi_i(\underline{t}); \theta) - U_0$ when he participates.

the following two Lemmas that characterize the implementable participation patterns and other properties of the feasible semidirect procedure θ.

Lemma A.1: *$\forall i \in \mathcal{N}, \forall t_i > t_i^*$, we have $U_i(t_i, \pi_i(t_i); \theta) > U_0$, $U_i(t_i, (1, t_i); \theta) > U_0$ and $q_i^e(t_i) = 1$; $\forall t_i < t_i^*$, we have $U_i(t_i, \pi_i(t_i); \theta) = U_0$, $U_i(t_i, (1, t_i); \theta) \leq U_0$. Moreover, we have $U_i(t_i^*, \pi_i(t_i^*); \theta) = U_0$ and $U_i(t_i^*, (1, t_i^*); \theta) = U_0$.*

Proof of Lemma A.1: When $t_i > t_i^*$, we have $U_i(t_i, \pi_i(t_i); \theta) > U_0$ by definition of t_i^*. From (A.1), this implies $U_i(t_i, (1, t_i); \theta) > U_0$ and $q_i^e(t_i) = 1$. When $t_i < t_i^*$, from the definition of t_i^*, we must have $U_i(t_i, (q_i^e(t_i), t_i); \theta) = U_0$. This leads to $U_i(t_i, (1, t_i); \theta) \leq U_0$.

We show $U_i(t_i^*, (1, t_i^*); \theta) = U_0$ in two steps. First, as $t_i - t_i^* \to 0^+$, we have $U_i(t_i, (1, t_i); \theta) - U_i(t_i^*, (1, t_i); \theta) \to 0$, which implies that $U_i(t_i^*, (1, t_i^*); \theta) \geq U_0$. Suppose this is not true, then $U_i(t_i^*, (1, t_i); \theta) \leq U_i(t_i^*, (1, t_i^*); \theta) < U_0$. As $U_i(t_i, (1, t_i); \theta) > U_0$, then $U_i(t_i, (1, t_i); \theta) - U_i(t_i^*, (1, t_i); \theta) \to 0$ cannot hold as $t_i - t_i^* \to 0^+$. Second, as $t_i - t_i^* \to 0^-$, we have $U_i(t_i, (1, t_i^*); \theta) - U_i(t_i^*, (1, t_i^*); \theta) \to 0$, which implies that $U_i(t_i^*, (1, t_i^*); \theta) \leq U_0$. Suppose this is not true, then $U_i(t_i^*, (1, t_i^*); \theta) > U_0$. As $U_i(t_i, (1, t_i^*); \theta) \leq U_i(t_i, (1, t_i); \theta) \leq U_0$, $U_i(t_i, (1, t_i^*); \theta) - U_i(t_i^*, (1, t_i^*); \theta) \to 0$ cannot hold as $t_i - t_i^* \to 0^-$. Aggregating the above results leads to $U_i(t_i^*, (1, t_i^*); \theta) = U_0$, which implies $U_i(t_i^*, \pi_i(t_i^*); \theta) = U_0$. $\square$

Lemma A.2: *If $t_i < t_i^*$ and $q_i^e(t_i) > 0$, we must have $U_i(t_i, (1, t_i); \theta) = U_0$. Moreover, when bidder i with such t_i participates, his expected winning probability is zero, and the expected subsidy to him from the seller is U_0.*

Proof of Lemma A.2: First, for any type $t_i < t_i^*$ who participates with probability $q_i^e(t_i) > 0$, we must have $U_i(t_i, (1, t_i); \theta) = U_0$, because $U_i(t_i, \pi_i(t_i); \theta) = U_0$ according to Lemma A.1.

The intuition for the zero expected winning probability of bidder i with such t_i is the following. If this is not true, we will have that for $\tilde{t}_i \in (t_i, t_i^*)$, $U_i(\tilde{t}_i, (1, t_i); \theta) > U_i(t_i, (1, t_i); \theta) = U_0$. This conflicts with the $U_i(\tilde{t}_i; \pi_i(\tilde{t}_i); \theta) = U_0$ result of Lemma A.1. The detailed proof is as follows.

Consider any $t_i < t_i^*$ with $q_i^e(t_i) > 0, \forall i \in \mathcal{N}$. When other bidders participate with the equilibrium entry probabilities $q_j^e(t_j)$ and reveal their true types when participating, (1) and

Lemma A.1 give

$$U_i(t_i,(1,t_i);\theta) = E_{\mathbf{t_{-i}}}\Big\{\sum_{\mathbf{d_{-i}}}\Big[\Big(\prod_{j\neq i} q_j^e(t_j)^{d_j}(1-q_j^e(t_j))^{1-d_j}\Big) \cdot\Big(t_i p_i(t_i,\mathbf{m_{-i}}(\mathbf{t_{-i}},\mathbf{d_{-i}})) - x_i(t_i,\mathbf{m_{-i}}(\mathbf{t_{-i}},\mathbf{d_{-i}}))\Big)\Big]\Big\} = U_0. \quad (A.2)$$

Bidder i's expected winning probability when he participates with signal t_i is

$$G_i(t_i;\theta) = E_{\mathbf{t_{-i}}}\{\sum_{\mathbf{d_{-i}}}[(\prod_{j\neq i} q_j^e(t_j)^{d_j}(1-q_j^e(t_j))^{1-d_j})\ p_i(t_i,\mathbf{m_{-i}}(\mathbf{t_{-i}},\mathbf{d_{-i}}))],\ t_i \in [\underline{t},\overline{t}]. \quad (A.3)$$

For $\forall\ \tilde{t_i} \in (t_i, t_i^*)$, we have

$$U_i(\tilde{t}_i,(1,t_i);\theta) = U_i(t_i,(1,t_i);\theta) + (\tilde{t}_i - t_i)G_i(t_i;\theta). \quad (A.4)$$

Since we have $U_0 = U_i(t_i,(1,t_i);\theta) \le U_i(\tilde{t}_i,(1,t_i);\theta) \le U_i(\tilde{t}_i,(1,\tilde{t}_i);\theta) \le U_0$, we must have that the expected winning probability of bidder i with $t_i < t_i^*$ when participating is

$$G_i(t_i;\theta) = 0, \qquad \forall t_i < t_i^*,\ q_i^e(t_i) > 0, \forall i \in \mathcal{N}. \quad (A.5)$$

$(A.2)$ and $(A.5)$ lead to the expected payment of bidder i with $t_i < t_i^*$ when participating

$$\begin{aligned} X_i(t_i;\theta) &= E_{\mathbf{t_{-i}}}\{\sum_{\mathbf{d_{-i}}}[(\prod_{j\neq i} q_j^e(t_j)^{d_j}(1-q_j^e(t_j))^{1-d_j})\ x_i(t_i,\mathbf{m_{-i}}(\mathbf{t_{-i}},\mathbf{d_{-i}}))] \\ &= -U_0, \qquad \forall t_i < t_i^*,\ q_i^e(t_i) > 0, \forall i \in \mathcal{N}. \end{aligned} \quad (A.6)$$

(A.5) indicates that at the equilibrium if bidder i with $t_i < t_i^*$ participates with a positive probability, then he has no chance of winning when he participates. (A.6) indicates that in order to get him to participate with a positive probability, the seller's expected subsidy to him is U_0 when he participates. □

Based on Lemma A.2, if a feasible semidirect procedure $\tilde{\theta}$ implements participation threshold t_i^* for bidder i while maintaining the same expected winning probability and the same expected payment as in θ for every participating type, then the seller saves the subsidy U_0 and has nothing to lose. Thus, the seller's expected revenue will be increased. The reason why this result holds can also be seen from the following argument. By doing so, the total surplus of the seller and bidders is increased by the savings in the participating costs of the types $t_i < t_i^*$ who never win, while the bidders' expected surplus does not change. This is the basic intuition why a threshold-participation feasible semidirect procedure maximizes seller's expected revenue.

The procedure $\tilde{\theta} = (\tilde{\pi}, \tilde{\mathbf{p}}, \tilde{\mathbf{x}})$ is defined as follows. The strategy profile $\tilde{\pi}$ is $(\tilde{\pi}_1(t_1), ..., \tilde{\pi}_N(t_N))$ where $\tilde{\pi}_i(t_i) = (\tilde{q}_i^e(t_i), t_i)$. Here $\tilde{q}_i^e(t_i) = 1$ if $t_i \geq t_i^*$, and $\tilde{q}_i^e(t_i) = 0$ if $t_i < t_i^*$. Note that $\tilde{\pi}$ is semidirect. We now turn to the construction of the mechanism $(\tilde{\mathbf{p}}, \tilde{\mathbf{x}})$.

For any message vector $\mathbf{s} = (s_1, s_2, ..., s_N) \in \mathcal{M}^N$, denote set $\{i | s_i = \emptyset\}$ by $\mathbf{I}(\mathbf{s})$, vector $(t_i)_{i \in \mathbf{I}(\mathbf{s})}$ by $\mathbf{t}_{\mathbf{I}(\mathbf{s})}$, set $\prod_{i \in \mathbf{I}(\mathbf{s})} [\underline{t}, t_i^*)$ by $\mathcal{T}_{\mathbf{I}(\mathbf{s})}$ and set $\{0,1\}^{N(\mathbf{I}(\mathbf{s}))}$ by $\mathbf{D}(\mathbf{I}(\mathbf{s}))$, where $N(\mathbf{I}(\mathbf{s}))$ is the cardinality of $\mathbf{I}(\mathbf{s})$. Denote a vector in $\mathbf{D}(\mathbf{I}(\mathbf{s}))$ by $\mathbf{d}_{\mathbf{I}(\mathbf{s})} = (d_i)_{i \in \mathbf{I}(\mathbf{s})}$. Denote vector $(s_i)_{i \notin \mathbf{I}(\mathbf{s})}$ by $\mathbf{s}_{\mathcal{N} \setminus \mathbf{I}(\mathbf{s})}$, and vector $(m_i(t_i, d_i))_{i \in \mathbf{I}(\mathbf{s})}$ by $\mathbf{m}(\mathbf{t}_{\mathbf{I}(\mathbf{s})}, \mathbf{d}_{\mathbf{I}(\mathbf{s})})$.

The idea of constructing $(\tilde{\mathbf{p}}, \tilde{\mathbf{x}})$ is to let the participating types $t_i \geq t_i^*,\ i \in \mathcal{N}$ in procedure $\tilde{\theta}$ win in the same expected probability and pay the same expected payment as in the procedure θ, conditioning on the types of other participants in procedure $\tilde{\theta}$. The set of outcome functions $\tilde{\mathbf{p}}$ and $\tilde{\mathbf{x}}$ are defined as follows.

For any signal vector $\mathbf{s} = (s_1, s_2, ..., s_N) \in \mathcal{M}^N$, define

$$\tilde{p}_i(\mathbf{s}) = \begin{cases} 0 & \text{if} \quad s_i = \emptyset \textit{ or } s_i < t_i^*, \\ A_i(\mathbf{s}) & \text{if} \quad s_i \geq t_i^*, \end{cases}$$

where $A_i(\mathbf{s}) = E_{\mathbf{t}_{\mathbf{I}(\mathbf{s})} \in \mathcal{T}_{\mathbf{I}(\mathbf{s})}} [\sum_{\forall \mathbf{d}_{\mathbf{I}(\mathbf{s})}} (\prod_{\forall j \in \mathbf{I}(\mathbf{s})} q_j^e(t_j)^{d_j} (1 - q_j^e(t_j))^{1-d_j}) p_i(\mathbf{s}_{\mathcal{N} \setminus \mathbf{I}(\mathbf{s})}, \mathbf{m}(\mathbf{t}_{\mathbf{I}(\mathbf{s})}, \mathbf{d}_{\mathbf{I}(\mathbf{s})}))]$, and

$$\tilde{x}_i(\mathbf{s}) = \begin{cases} 0 & \text{if} \quad s_i = \emptyset, \textit{ or } s_i < t_i^* \\ B_i(\mathbf{s}) & \text{if} \quad s_i \geq t_i^*, \end{cases}$$

where $B_i(\mathbf{s}) = E_{\mathbf{t}_{\mathbf{I}(\mathbf{s})} \in \mathcal{T}_{\mathbf{I}(\mathbf{s})}} [\sum_{\forall \mathbf{d}_{\mathbf{I}(\mathbf{s})}} (\prod_{\forall j \in \mathbf{I}(\mathbf{s})} q_j^e(t_j)^{d_j} (1 - q_j^e(t_j))^{1-d_j}) x_i(\mathbf{s}_{\mathcal{N} \setminus \mathbf{I}(\mathbf{s})}, \mathbf{m}(\mathbf{t}_{\mathbf{I}(\mathbf{s})}, \mathbf{d}_{\mathbf{I}(\mathbf{s})}))]$.

First, it is clear that $(\tilde{\mathbf{p}}, \tilde{\mathbf{x}})$ satisfies (c1) and (c2) in Section 2.1. Second, by the construct of $(\tilde{\mathbf{p}}, \tilde{\mathbf{x}})$, the expected winning probability and expected payment of bidder i signaling $s_i \geq t_i^*$ remain the same as those from $(\mathbf{p}, \mathbf{x})$, given that the other bidders participate with probabilities $\tilde{q}_j^e(t_j)$ and reveal their true types when participating. Third, by the construct of $(\tilde{\mathbf{p}}, \tilde{\mathbf{x}})$, both the expected winning probability and expected payment of bidder i are 0 if $s_i = \emptyset$ or $s_i < t_i^*$.

We now verify that it is a Bayesian Nash equilibrium of the mechanism $(\tilde{\mathbf{p}}, \tilde{\mathbf{x}})$ that the bidders participate with probabilities $\tilde{q}_i^e(t_i)$ and reveal their true types when participating. Suppose other bidders participate with probabilities $\tilde{q}_j^e(t_j)$ and reveal their true types when participating, we show that the best strategy for bidder i is to participate with probability $\tilde{q}_i^e(t_i)$ and reveal his true type when participating.

Use $U(t_i, (q_i'^e, t_i'); \tilde{\theta})$ to denote the expected payoff of bidder i with value t_i under procedure $\tilde{\theta}$ if he participates with probability of $q_i'^e$ and signals t_i' when participating, given that the others participate with probabilities $\tilde{q}_j^e(t_j)$ and reveal their true types when participating.

For $t_i < t_i^*$, if bidder i participates with probability 1 and submits $t_i' \geq t_i^*$ when participating, then we have $U_i(t_i, (1, t_i'); \tilde{\theta}) = U_i(t_i, (1, t_i'); \theta)$ by the construct of $\tilde{\theta}$. Since we have truthful telling when participating with procedure θ, $U_i(t_i, (1, t_i'); \theta) \leq U_i(t_i, (1, t_i); \theta)$. From Lemma A.1, we have $U_i(t_i, (1, t_i); \theta) \leq U_0$. Thus $U_i(t_i, (1, t_i'); \tilde{\theta}) = U_i(t_i, (1, t_i'); \theta) \leq U_i(t_i, (1, t_i); \theta) \leq U_0$. If bidder i participates with probability 1 and submits $t_i' < t_i^*$ when participating, then $U_i(t_i, (1, t_i'); \tilde{\theta}) = 0$. Therefore bidder i has no incentive to deviate from no participation when $t_i < t_i^*$.

When $t_i \geq t_i^*$, if bidder i participates with probability 1 and submits $t_i' < t_i^*$ when participating, then $U_i(t_i, (1, t_i'); \tilde{\theta}) = 0$. If bidder i participates with probability 1 and submits $t_i' \geq t_i^*$ when participating, then $U_i(t_i, (1, t_i'); \tilde{\theta}) = U_i(t_i, (1, t_i'); \theta) \leq U_i(t_i, (1, t_i); \theta) = U_i(t_i, (1, t_i); \tilde{\theta})$. Thus bidder i participates with probability of 1 and reveals his true type when participating if $t_i \geq t_i^*$, since $U_i(t_i, (1, t_i); \theta) \geq U_0$ from Lemma A.1. Therefore, it is a Bayesian Nash equilibrium of the game $(\tilde{\mathbf{p}}, \tilde{\mathbf{x}})$ that the bidders participate with probabilities of $\tilde{q}_i^e(t_i)$ and reveal their true types when participating. In other words, $\tilde{\theta}$ is a feasible semidirect procedure.

Furthermore, the expected winning probability and expected payment of the participants in procedure $\tilde{\theta}$ are the same as those from θ. Recall that Lemma A.2 says that the types $t_i < t_i^*$ never win in procedure θ when they participate. As a result, $\tilde{\theta}$ increases the total surplus of the seller and bidders by the amount of the savings in the entry cost of the nonparticipating types $t_i < t_i^*$ in $\tilde{\theta}$. Meanwhile, every type of bidder still enjoys the same expected surplus. Consequently, the seller's expected revenue increases by the amount of savings in the entry costs of the types $t_i < t_i^*, \ i \in \mathcal{N}$.

Alternatively, we next show the above result through directly calculating the seller's expected revenue. Define $\hat{m}_i(t_i) = t_i$ if $t_i \geq t_i^*$, otherwise $\hat{m}_i(t_i) = \emptyset$. For $\forall \mathbf{t} \in \mathcal{T}$, denote set $\{i | \hat{m}_i(t_i) = \emptyset\}$

by $\mathbf{I}(\mathbf{t})$, vector $(t_i)_{i\in\mathbf{I}(\mathbf{t})}$ by $\mathbf{t}_{\mathbf{I}(\mathbf{t})}$, and set $\{0,1\}^{N(\mathbf{I}(\mathbf{t}))}$ by $\mathbf{D}(\mathbf{I}(\mathbf{t}))$, where $N(\mathbf{I}(\mathbf{t}))$ is the cardinality of $\mathbf{I}(\mathbf{t})$. Denote a vector $(d_i)_{i\in\mathbf{I}(\mathbf{t})}$ in $\mathbf{D}(\mathbf{I}(\mathbf{t}))$ by $\mathbf{d}_{\mathbf{I}(\mathbf{t})}$. Denote vector $(t_i)_{i\notin\mathbf{I}(\mathbf{t})}$ by $\mathbf{t}_{\mathcal{N}\setminus\mathbf{I}(\mathbf{t})}$ and vector $(m_i(t_i,d_i))_{i\in\mathbf{I}(\mathbf{t})}$ by $\mathbf{m}(\mathbf{t}_{\mathbf{I}(\mathbf{t})},\mathbf{d}_{\mathbf{I}(\mathbf{t})})$.

From (3), the seller's expected revenue from the procedure $\tilde{\theta}$ is:

$$\begin{aligned}
R(\tilde{\theta}) &= E_{\mathbf{t}}\Big\{t_0(1-\sum_{i=1}^{N}\tilde{p}_i(\hat{m}_1(t_1),...,\hat{m}_N(t_N))) + \sum_{i=1}^{N}\tilde{x}_i(\hat{m}_1(t_1),...,\hat{m}_N(t_N))\Big\} \\
&= E_{\mathbf{t}}\Big\{t_0(1-\sum_{\forall \mathbf{d}_{\mathbf{I}(\mathbf{t})}}(\prod_{\forall j\in\mathbf{I}(\mathbf{t})} q_j^e(t_j)^{d_j}(1-q_j^e(t_j))^{1-d_j})\sum_{\forall i\in\mathcal{N}\setminus\mathbf{I}(\mathbf{t})} p_i(\mathbf{t}_{\mathcal{N}\setminus\mathbf{I}(\mathbf{t})},\mathbf{m}(\mathbf{t}_{\mathbf{I}(\mathbf{t})},\mathbf{d}_{\mathbf{I}(\mathbf{t})}))) \\
&\qquad + \sum_{\forall \mathbf{d}_{\mathbf{I}(\mathbf{t})}}(\prod_{\forall j\in\mathbf{I}(\mathbf{t})} q_j^e(t_j)^{d_j}(1-q_j^e(t_j))^{1-d_j})\sum_{\forall i\in\mathcal{N}\setminus\mathbf{I}(\mathbf{t})} x_i(\mathbf{t}_{\mathcal{N}\setminus\mathbf{I}(\mathbf{t})},\mathbf{m}(\mathbf{t}_{\mathbf{I}(\mathbf{t})},\mathbf{d}_{\mathbf{I}(\mathbf{t})})))\Big\} \\
&= R(\theta) + E_{\mathbf{t}}\Big\{t_0\sum_{\forall \mathbf{d}_{\mathbf{I}(\mathbf{t})}}(\prod_{\forall j\in\mathbf{I}(\mathbf{t})} q_j^e(t_j)^{d_j}(1-q_j^e(t_j))^{1-d_j})\sum_{\forall i\in\mathbf{I}(\mathbf{t})} p_i(\mathbf{t}_{\mathcal{N}\setminus\mathbf{I}(\mathbf{t})},\mathbf{m}(\mathbf{t}_{\mathbf{I}(\mathbf{t})},\mathbf{d}_{\mathbf{I}(\mathbf{t})})) \\
&\qquad - \sum_{\forall \mathbf{d}_{\mathbf{I}(\mathbf{t})}}(\prod_{\forall j\in\mathbf{I}(\mathbf{t})} q_j^e(t_j)^{d_j}(1-q_j^e(t_j))^{1-d_j})\sum_{\forall i\in\mathbf{I}(\mathbf{t})} x_i(\mathbf{t}_{\mathcal{N}\setminus\mathbf{I}(\mathbf{t})},\mathbf{m}(\mathbf{t}_{\mathbf{I}(\mathbf{t})},\mathbf{d}_{\mathbf{I}(\mathbf{t})})))\Big\} \\
&= R(\theta) + t_0\sum_{\forall i}\int_{\underline{t}}^{t_i^*} q_i^e(t_i)G_i(t_i;\theta)f(t_i)dt_i - \sum_{\forall i}\int_{\underline{t}}^{t_i^*} q_i^e(t_i)X_i(t_i;\theta)f(t_i)dt_i \\
&= R(\theta) + \sum_{\forall i}\int_{\underline{t}}^{t_i^*} q_i^e(t_i)U_0 f(t_i)dt_i \\
&\geq R(\theta),
\end{aligned}$$

where $R(\theta)$ is the seller's expected revenue from θ, i.e,

$$\begin{aligned}
R(\theta) &= E_{\mathbf{t}}\Big\{t_0(1-\sum_{\forall \mathbf{d}_{\mathbf{I}(\mathbf{t})}}(\prod_{\forall j\in\mathbf{I}(\mathbf{t})} q_j^e(t_j)^{d_j}(1-q_j^e(t_j))^{1-d_j})\sum_{\forall i\in\mathcal{N}} p_i(\mathbf{t}_{\mathcal{N}\setminus\mathbf{I}(\mathbf{t})},\mathbf{m}(\mathbf{t}_{\mathbf{I}(\mathbf{t})},\mathbf{d}_{\mathbf{I}(\mathbf{t})}))) \\
&\qquad + \sum_{\forall \mathbf{d}_{\mathbf{I}(\mathbf{t})}}(\prod_{\forall j\in\mathbf{I}(\mathbf{t})} q_j^e(t_j)^{d_j}(1-q_j^e(t_j))^{1-d_j})\sum_{\forall i\in\mathcal{N}} x_i(\mathbf{t}_{\mathcal{N}\setminus\mathbf{I}(\mathbf{t})},\mathbf{m}(\mathbf{t}_{\mathbf{I}(\mathbf{t})},\mathbf{d}_{\mathbf{I}(\mathbf{t})})))\Big\}.
\end{aligned}$$

□

Proof of Lemma 2: From (5) and (10), we have

$$U_i(t_i,(1,t_i');\theta) = U_i(t_i',(1,t_i');\theta) + (t_i - t_i')Q_i(t_i';\theta),\ \forall t_i,\ t_i' \in [t_i^c,\ \bar{t}],\ \forall i\in\mathcal{N}. \tag{A.7}$$

From (7) and (A.7), we have

$$U_i(t_i,(1,t_i);\theta) \geq U_i(t_i',(1,t_i');\theta) + (t_i - t_i')Q_i(t_i';\theta),\ \forall t_i,\ t_i' \in [t_i^c,\ \bar{t}],\ \forall i\in\mathcal{N}. \tag{A.8}$$

Note (7) is equivalent to (A.8).

Using (A.8) twice, we have for $\forall\ t_i,\ t_i' \in [t_i^c,\ \bar{t}],\ \forall\ i \in \mathcal{N}$,

$$(t_i - t_i')Q_i(t_i';\theta) \leq U_i(t_i,(1,t_i);\theta) - U_i(t_i',(1,t_i');\theta) \leq (t_i - t_i')Q_i(t_i;\theta). \tag{A.9}$$

(A.9) implies (11). From (A.9), we have for $\forall\ s_i,\ s_i + \delta \in [t_i^c,\ \bar{t}],\ \forall\ i \in \mathcal{N}$,

$$Q_i(s_i;\theta)\delta \leq U_i(s_i+\delta,(1,s_i+\delta);\theta) - U_i(s_i,(1,s_i);\theta) \leq Q_i(s_i+\delta;\theta)\delta. \tag{A.10}$$

Since $Q_i(s_i;\theta)$ increases with s_i, (A.10) implies

$$\frac{dU_i(s_i,(1,s_i);\theta)}{ds_i} = Q_i(s_i;\theta),\ \forall i \in \mathcal{N},\ \forall\ s_i \in [t_i^c,\bar{t}],$$

where $Q_i(s_i;\theta)$ is Riemann integrable, so

$$\int_{t_i^c}^{t_i} Q_i(s_i;\theta)ds_i = U_i(t_i,(1,t_i);\theta) - U_i(t_i^c,(1,t_i^c);\theta),\ \forall\ t_i \in [t_i^c,\bar{t}]. \tag{A.11}$$

(A.11) is (12), and (13) is directly from (6). Thus (11) $\sim$ (13) are derived from (6) and (7). Now we show (6) and (7) from (9), (11) $\sim$ (13).

$\forall t_i^c \leq r_i \leq t_i \leq \bar{t},\ \forall i \in \mathcal{N}$, (11) and (12) imply

$$\begin{aligned} U_i(t_i,(1,t_i);\theta) &= U_i(r_i,(1,r_i);\theta) + \int_{r_i}^{t_i} Q_i(x;\theta)dx \\ &\geq U_i(r_i,(1,r_i);\theta) + \int_{r_i}^{t_i} Q_i(r_i;\theta)dx \\ &= U_i(r_i,(1,r_i);\theta) + (t_i - r_i)Q_i(r_i;\theta). \end{aligned}$$

Similarly, $\forall t_i^c \leq t_i \leq r_i \leq \bar{t},\ \forall i \in \mathcal{N}$, (11) and (12) imply

$$\begin{aligned} U_i(t_i,(1,t_i);\theta) &= U_i(r_i,(1,r_i);\theta) + \int_{r_i}^{t_i} Q_i(x;\theta)dx \\ &\geq U_i(r_i,(1,r_i);\theta) - \int_{t_i}^{r_i} Q_i(r_i;\theta)dx \\ &= U_i(r_i,(1,r_i);\theta) + (t_i - r_i)Q_i(r_i;\theta). \end{aligned}$$

Thus we have (A.8), i.e., (7) is shown. (6) is from (9), (12) and (13). $\square$

Proof of Lemma 3: Define

$$V_i(t_i;\theta) = E_{\mathbf{t_{-i}}}\{t_i\ p_i(m_1(t_1),...,m_i(t_i),...,m_N(t_N)) - x_i(m_1(t_1),...,m_i(t_i),...,m_N(t_N))\}. \tag{A.12}$$

Note that $V_i(t_i;\theta) = U_i(t_i,(1,t_i);\theta)$ if $t_i \geq t_i^c$, and $V_i(t_i;\theta) = 0$ if $t_i < t_i^c$. From (A.12),

$$\begin{aligned}
&E_{t_i} V_i(t_i;\theta) \\
&= E_{t_i} E_{\mathbf{t_{-i}}} \{t_i \ p_i(m_1(t_1),...,m_N(t_N)) - x_i(m_1(t_1),...,m_N(t_N))\} \\
&= E_{\mathbf{t}} \{t_i \ p_i(m_1(t_1),...,m_N(t_N)) - x_i(m_1(t_1),...,m_N(t_N))\}. \qquad \text{(A.13)}
\end{aligned}$$

From (A.13), we have

$$\sum_{i=1}^{N} E_{t_i} V_i(t_i;\theta) = E_{\mathbf{t}} \{\sum_{i=1}^{N} [t_i \ p_i(m_1(t_1),...,m_N(t_N)) - x_i(m_1(t_1),,...,m_N(t_N))]\}. \qquad \text{(A.14)}$$

From (4) and (A.14),

$$R(\theta) = t_0 - \sum_{i=1}^{N} E_{t_i} V_i(t_i;\theta) + E_{\mathbf{t}} \{\sum_{i=1}^{N} p_i(m_1(t_1),...,m_N(t_N))(t_i - t_0)\}. \qquad \text{(A.15)}$$

From (12), we have

$$\begin{aligned}
&E_{t_i} V_i(t_i;\theta) = \int_{t_i^c}^{\bar{t}} V_i(t_i;\theta) f(t_i) dt_i = \int_{t_i^c}^{\bar{t}} U_i(t_i,(1,t_i);\theta) f(t_i) dt_i \\
&= \int_{t_i^c}^{\bar{t}} [U_i(t_i^c,(1,t_i^c);\theta) + \int_{t_i^c}^{t_i} Q_i(s_i;\theta) ds_i] f(t_i) dt_i \\
&= (1 - F(t_i^c)) U_i(t_i^c,(1,t_i^c);\theta) + \int_{t_i^c}^{\bar{t}} [\int_{t_i^c}^{t_i} Q_i(s_i;\theta) ds_i] f(t_i) dt_i \\
&= (1 - F(t_i^c)) U_i(t_i^c,(1,t_i^c);\theta) + \int_{t_i^c}^{\bar{t}} [\int_{s_i}^{\bar{t}} f(t_i) dt_i] Q_i(s_i;\theta) ds_i \\
&= (1 - F(t_i^c)) U_i(t_i^c,(1,t_i^c);\theta) + \int_{t_i^c}^{\bar{t}} [1 - F(s_i)] Q_i(s_i;\theta) ds_i \\
&= (1 - F(t_i^c)) U_i(t_i^c,(1,t_i^c);\theta) + \int_{\underline{t}}^{\bar{t}} [1 - F(s_i)] Q_i(s_i;\theta) ds_i. \qquad \text{(A.16)}
\end{aligned}$$

From (10), we have

$$\begin{aligned}
&\int_{\underline{t}}^{\bar{t}} [1 - F(s_i)] Q_i(s_i;\theta) ds_i \\
&= \int_{\underline{t}}^{\bar{t}} [1 - F(s_i)] E_{\mathbf{t_{-i}}} p_i(m_1(t_1),...,m_i(s_i),...,m_N(t_N)) ds_i \\
&= E_{\mathbf{t}} p_i(m_1(t_1),...,m_i(t_i),...,m_N(t_N)) \frac{1 - F(t_i)}{f(t_i)}. \qquad \text{(A.17)}
\end{aligned}$$

From (A.16) and (A.17), we have

$$\sum_{i=1}^{N} E_{t_i} V_i(t_i;\theta)$$
$$= \sum_{i=1}^{N}(1-F(t_i^c))U_i(t_i^c,(1,t_i^c);\theta) + E_{\mathbf{t}}\{\sum_{i=1}^{N} p_i(m_1(t_1),...,m_i(t_i),...,m_N(t_N))\frac{1-F(t_i)}{f(t_i)}\}. \quad (A.18)$$

From (A.15) and (A.18), we have the desired result. □

Proof of Proposition 1: We first show that $\theta^* = (\pi_{\mathbf{t^c}}, \mathbf{p}^*, \mathbf{x}^*)$ is a feasible semidirect procedure. Mechanism $(\mathbf{p}^*, \mathbf{x}^*)$ constitutes exactly the rule for a modified second-price sealed-bid auction with a reserve price of t_i^c for bidder i, and an entry subsidy of U_0 for all participants whose bids exceed their corresponding reserve prices. Therefore, what we need to show is that the given auction implements participation thresholds $\mathbf{t^c}$, and the types $t_i \in [t_i^c, \bar{t}]$ bid their true values in such an auction. First of all, clearly bidder i with types of $t_i \in [t_i^c, \bar{t}]$ would participate, and bidding his true value is his weakly dominant strategy in this second-price auction. Second, those lower types with $t_i \in [\underline{t}, t_i^c)$ cannot get the entry subsidy and they cannot win if they bid lower than t_i^c, they just incur an entry cost. Moreover, if they win they have to pay at least a reserve price t_i^c which is higher than their values. Therefore, they would bid t_i^c if they would participate, in order to minimize their winning probability. Without loss of generality, we assume that $t_1^c \geq t_2^c \geq ... \geq t_N^c$. Suppose $t_1^c > \underline{t}$. If bidder 1 with $t_1 \in [\underline{t}, t_1^c)$ participates and bids t_1^c, he will win in a positive probability which is no less than $\frac{F(t_1^c)^{N-1}}{N}$. As he wins, he has to pay at least a reserve price t_1^c which is higher than his value. Thus, his expected gain from participating is strictly less than his reservation utility. Therefore, those lower types of bidder 1 with $t_1 \in [\underline{t}, t_1^c)$ prefers not to participate. Suppose $t_2^c > \underline{t}$. We can similarly show that those lower types of bidder 2 with $t_2 \in [\underline{t}, t_2^c)$ will not participate. This process continues until the last participation threshold which is higher than $\underline{t}$.

Second, $\mathbf{p}^*$ maximizes component $E_{\mathbf{t}}\{\sum_{i=1}^{N} p_i(m_1(t_1),...,m_N(t_N))(t_i - t_0 - \frac{1-F(t_i)}{f(t_i)})\}$ in (14) among all $\mathbf{p}$ satisfying (9). Based on the above discussion, we see that the given mechanism allocates the object to the participant with the highest value. For the given $\mathbf{t^c}$ and the corresponding truthful semirevelation functions $m_i(\cdot), \forall i \in \mathcal{N}$, the above defined winning probability functions $\mathbf{p}^*$ maximize $\sum_{i=1}^{N} p_i(m_1(t_1),...,m_N(t_N)(t_i - t_0 - \frac{1-F(t_i)}{f(t_i)}), \forall \mathbf{t} \in \mathcal{T}$ among all the possibilities for $\mathbf{p}$ satisfying (9), because $J(t) = t - \frac{1-F(t)}{f(t)}$ increases with t and winner's value ($\geq r_0$) is the highest among all participants. Therefore, the winning probability functions $\mathbf{p}^*$ maximize $E_{\mathbf{t}}\{\sum_{i=1}^N p_i(m_1(t_1), ..., m_N(t_N))(t_i - t_0 - \frac{1-F(t_i)}{f(t_i)})\}$.

Third, we show $U_i(t_i^c, (1, t_i^c); \theta) = U_0, \ \forall i \in \mathcal{N}$. For the given mechanism, this is true because the threshold types of bidders receive the entry subsidy of U_0 when they participate, and they do not gain further from winning because of the reserve prices which equal their values.

Based on (13) and the above results, (14) is maximized under the mechanism $(\mathbf{p}^*, \mathbf{x}^*)$ or the corresponding modified second-price auction. Thus the modified second-price auction is the revenue-maximizing auction implementing given participation thresholds $\mathbf{t^c}$. □

Proof of Corollary 2: Denote the truthful semirevelation functions corresponding to $\tilde{\mathbf{t}}^{\mathbf{c}}$ by $\tilde{m}_i(t_i)$, $\forall i \in \mathcal{N}$, where $\tilde{m}_i(t_i) = t_i$ if $t_i \geq \tilde{t}_i^c$, and $\tilde{m}_i(t_i) = \emptyset$ if $t_i < \tilde{t}_i^c$. First, note that $\sum_{i=1}^N p_i(m_1(t_1), ..., m_N(t_N))(t_i - t_0 - \frac{1-F(t_i)}{f(t_i)}) \leq \sum_{i=1}^N \tilde{p}_i^*(\tilde{m}_1(t_1), ..., \tilde{m}_N(t_N))(t_i - t_0 - \frac{1-F(t_i)}{f(t_i)})$, $\forall \mathbf{t} \in \mathcal{T}$, where $\tilde{p}_i^*(\mathbf{s})$ is bidder i's winning probability function in $(\tilde{\mathbf{p}}^*, \tilde{\mathbf{x}}^*)$ and $p_i(\mathbf{s})$ is bidder i's winning probability function in $(\mathbf{p}, \mathbf{x})$. Second, we have $(1 - F(t_i^c))U_i(t_i^c, (1, t_i^c); \theta) \geq (1 - F(\tilde{t}_i^c))U_0 = (1 - F(\tilde{t}_i^c))U_i(\tilde{t}_i^c, (1, \tilde{t}_i^c); \tilde{\theta}^*)$, since $\tilde{t}_i^c \geq t_i^c$ and we have $U_i(t_i^c, (1, t_i^c); \theta) \geq U_0$ from (13). Note that $U_i(\tilde{t}_i^c, (1, \tilde{t}_i^c); \tilde{\theta}^*) = U_0$. Therefore, based on Lemma 3, we have $R(\theta) \leq R(\tilde{\theta}^*)$.

When there exists $i_0 \in \mathcal{N}$ such that $t_{i_0}^c < r_0$, we have $(1 - F(t_{i_0}^c))U_{i_0}(t_{i_0}^c, (1, t_{i_0}^c); \theta) > (1 - F(\tilde{t}_{i_0}^c))U_0 = (1 - F(\tilde{t}_{i_0}^c))U_{i_0}(\tilde{t}_{i_0}^c, (1, \tilde{t}_{i_0}^c); \tilde{\theta}^*)$. Thus $R(\theta) < R(\tilde{\theta}^*)$. It follows that any $t_i^c < r_0$ cannot be a revenue-maximizing participation threshold. □

Proof of Proposition 2: Clearly, we have $t^{c*} > r_0$. Note that $\frac{U_0}{F(t^c)^{N-1}}$ decreases with t^c and $J(t^c)$ increases with t^c. Thus $t_0 + \frac{U_0}{F(t^c)^{N-1}} - J(t^c)$ decreases with t^c. If $U_0 \in (0, \bar{t} - t_0)$, $t_0 + \frac{U_0}{F(t^c)^{N-1}} - J(t^c)$ goes to $+\infty$ when t^c goes to $\underline{t}$; while $t_0 + \frac{U_0}{F(t^c)^{N-1}} - J(t^c)$ goes to $t_0 + U_0 - \bar{t}$ (< 0) when t^c goes to $\bar{t}$. Therefore, the revenue-maximizing symmetric participation threshold is determined by the unique solution of $t_0 + \frac{U_0}{F(t^c)^{N-1}} - J(t^c) = 0$, since a unique solution exists in $(\underline{t}, \bar{t})$. We can use contradiction method to show that the revenue-maximizing t^{c*} increases with the number of the potential bidders N. Suppose t^{c*} does not increase when the number of the potential bidders increases from N_1 to N_2. Then we have that the left-hand-side of (19) strictly increases when the number of the potential bidders increases from N_1 to N_2, while the right-hand-side does not. Thus (19) cannot hold simultaneously for both the revenue-maximizing participation thresholds for N_1 and N_2. Similarly we can show the monotonicity of t^{c*} with regard to U_0. We next show that t^{c*} approaches $\bar{t}$ as N approaches ∞. Suppose there is an upper bound for t^{c*} which is smaller than $\bar{t}$, then the left-hand-side of (19) approaches to ∞ as N approaches ∞, while the right-hand-side is lower than $\bar{t}$. This means that (19) cannot hold for the revenue-maximizing participation thresholds as N approaches ∞.

If $U_0 \geq \bar{t} - t_0$, clearly the revenue-maximizing threshold is a boundary solution, i.e., $t^{c*} = \bar{t}$.

If $U_0 = 0$, then clearly $t^{c*} = r_0$ according to (18). □

Proof of Proposition 4: From (17), we have $R^*(t^{c*})$

$$= t_0 F(t^{c*})^N - NU_0(1 - F(t^{c*})) + \bar{t} - t^{c*}F^{(2)}(t^{c*}) + Nt^{c*}(1 - F(t^{c*}))F(t^{c*})^{N-1} - \int_{t^{c*}}^{\bar{t}} F^{(2)}(t)dt,$$

where $F^{(2)}(t) = F(t)^N + NF(t)^{N-1}(1 - F(t))$. Thus

$$R^*(t^{c*}) = t_0 F(t^{c*})^N - NU_0(1 - F(t^{c*})) + \bar{t} - t^{c*}F^N(t^{c*}) - \int_{t^{c*}}^{\bar{t}} F^{(2)}(t)dt. \tag{A.19}$$

We first show that $N(1 - F(t^{c*})) = -\log\Big(\frac{U_0}{\bar{t}-t_0}\Big) + o(1)$ as $N \to \infty$ (i.e., $F(t^{c*}) \to 1$ from Proposition 2(i)). (19) gives $F(t^{c*})^{N-1} = \frac{U_0}{t^{c*} - t_0 - \frac{1-F(t^{c*})}{f(t^{c*})}}$, we thus have

$$(N-1)\log F(t^{c*}) = \log U_0 - \log\Big(t^{c*} - t_0 - \frac{1 - F(t^{c*})}{f(t^{c*})}\Big). \tag{A.20}$$

Thus

$$(N-1)\log F(t^{c*}) = \log\frac{U_0}{\bar{t} - t_0} + o(1). \tag{A.21}$$

(A.21) implies that

$$1 - F(t^{c*}) = O\Big(\frac{1}{N-1}\Big). \tag{A.22}$$

From (A.20), we have

$$\begin{aligned}
N(1 - F(t^{c*})) &= [N(1 - F(t^{c*})) + (N-1)\log F(t^{c*})] - \Big[\ \log U_0 - \log\Big(t^{c*} - t_0 - \frac{1 - F(t^{c*})}{f(t^{c*})}\Big)\Big] \\
&= \Big[N(1 - F(t^{c*})) - (N-1)(1 - F(t^{c*})) - \frac{N-1}{2}(1 - \tilde{F})^2\Big] - \log\frac{U_0}{\bar{t} - t_0} + o(1) \\
&= -\log\frac{U_0}{\bar{t} - t_0} - \frac{N-1}{2}(1 - \tilde{F})^2 + o(1),
\end{aligned} \tag{A.23}$$

where $\tilde{F} \in [F(t^{c*}), 1]$.

(A.22) and (A.23) lead to that

$$N(1 - F(t^{c*})) = -\log \frac{U_0}{\bar{t} - t_0} + o(1). \tag{A.24}$$

From (A.19), $R^*(t^{c*}) = (t_0 - t^{c*})F(t^{c*})^N - NU_0(1 - F(t^{c*})) + \bar{t} - \int_{t^{c*}}^{\bar{t}} F^{(2)}(t)dt$

$$= -(\frac{1 - F(t^{c*})}{f(t^{c*})} + \frac{U_0}{F(t^{c*})^{N-1}})F(t^{c*})^N - NU_0(1 - F(t^{c*})) + \bar{t} - \int_{t^{c*}}^{\bar{t}} F^{(2)}(t)dt.$$

Thus from (A.24), we have $R^*(t^{c*}) \to \ (\bar{t} - U_0) + U_0 \log U_0 - U_0 \log(\bar{t} - t_0)$ as $N \to \infty$.

Note that the left hand side of (A.24) is the expected participations when there are N potential bidders, thus we have that the expected participations in the limit are $\log \frac{\bar{t}-t_0}{U_0}$. □

Proof of Proposition 5: Applying the Envelope Theorem to (A.19), we have $\frac{dR^*(t^{c*})}{dN}$

$$\begin{aligned}
&= t_0 F(t^{c*})^N \log(F(t^{c*})) - U_0(1 - F(t^{c*})) - t^{c*}F(t^{c*})^N \log(F(t^{c*})) \\
&- \int_{t^{c*}}^{\bar{t}} F(t)^{N-1}\{[\ \log F(t) + (1 - F(t))\ ] + (N-1)(1 - F(t))\log F(t)\}dt \\
&= t_0 F(t^{c*})^N \log(F(t^{c*})) - U_0(1 - F(t^{c*})) + t^{c*}F(t^{c*})^{N-1}(1 - F(t^{c*})) \\
&\qquad -t^{c*}F(t^{c*})^{N-1}[F(t^{c*})\log(F(t^{c*}) + (1 - F(t^{c*}))] \\
&- \int_{t^{c*}}^{\bar{t}} F(t)^{N-1}\{[\ \log F(t) + (1 - F(t))] + (N-1)(1 - F(t))\log F(t)\}dt \\
&= t_0 F(t^{c*})^N \log(F(t^{c*})) + t_0 F(t^{c*})^{N-1}(1 - F(t^{c*})) + (1 - F(t^{c*}))[-U_0 + (t^{c*} - t_0)F(t^{c*})^{N-1}] \\
&-t^{c*}F(t^{c*})^{N-1}[F(t^{c*})\log(F(t^{c*}) + (1 - F(t^{c*}))] \\
&- \int_{t^{c*}}^{\bar{t}} F(t)^{N-1}\{[\ \log F(t) + (1 - F(t))] + [(N-1)(1 - F(t))\log F(t)]\}dt \\
&= t_0 F(t^{c*})^{N-1}[F(t^{c*})\log(F(t^{c*})) + (1 - F(t^{c*}))] + (1 - F(t^{c*}))\frac{1 - F(t^{c*})}{f(t^{c*})}F(t^{c*})^{N-1} \\
&-t^{c*}F(t^{c*})^{N-1}[F(t^{c*})\log(F(t^{c*}) + (1 - F(t^{c*}))] \\
&- \int_{t^{c*}}^{\bar{t}} F(t)^{N-1}\{[\ \log F(t) + (1 - F(t))] + [(N-1)(1 - F(t))\log F(t)]\}dt. \qquad (A.25)
\end{aligned}$$

Since $[F(t^{c*})\log(F(t^{c*})) + (1 - F(t^{c*}))] > 0$, we have $t_0 F(t^{c*})^{N-1}[F(t^{c*})\log(F(t^{c*})) + (1 - F(t^{c*}))] > 0$. Since both $[\ \log F(t) + (1 - F(t))]$ and $(N-1)(1 - F(t))\log F(t)$ are negative, we have $-\int_{t^{c*}}^{\bar{t}} F(t)^{N-1}\{[\ \log F(t) + (1 - F(t))] + [(N-1)(1 - F(t))\log F(t)]\}dt > 0$. For

$k \in [1,2]$, we have $(1-F(t^{c*}))^2 - f(t^{c*})t^{c*}[F(t^{c*})\log(F(t^{c*})) + (1-F(t^{c*}))] = (1-F(t^{c*}))^2 - kF(t^{c*})[F(t^{c*})\log(F(t^{c*})) + (1-F(t^{c*}))] > 0$. Thus the sum of the other two components in (A.25) is positive. Therefore, $\frac{dR^*(t^{c*})}{dN} > 0$ holds for $k \in [1,2]$.

From (A.25), $\frac{dR^*(t^{c*})}{dN}$

$$
\begin{aligned}
&= (t_0 - t^{c*})F(t^{c*})^{N-1}[F(t^{c*})\log(F(t^{c*})) + (1-F(t^{c*}))] + (1-F(t^{c*}))\frac{1-F(t^{c*})}{f(t^{c*})}F(t^{c*})^{N-1} \\
&\quad - \int_{t^{c*}}^{\bar{t}} F(t)^{N-1}\{[\,\log F(t) + (1-F(t))] + [(N-1)(1-F(t))\log F(t)]\}dt \\
&= -(\frac{1-F(t^{c*})}{f(t^{c*})} + \frac{U_0}{F(t^{c*})^{N-1}})F(t^{c*})^{N-1}[F(t^{c*})\log(F(t^{c*})) + (1-F(t^{c*}))] \\
&\quad +(1-F(t^{c*}))\frac{1-F(t^{c*})}{f(t^{c*})}F(t^{c*})^{N-1} \\
&\quad - \int_{t^{c*}}^{\bar{t}} F(t)^{N-1}\{[\,\log F(t) + (1-F(t))] + [(N-1)(1-F(t))\log F(t)]\}dt \\
&= -\frac{1-F(t^{c*})}{f(t^{c*})}F(t^{c*})^N \log(F(t^{c*})) - U_0[F(t^{c*})\log(F(t^{c*})) + (1-F(t^{c*}))] \\
&\quad - \int_{t^{c*}}^{\bar{t}} F(t)^{N-1}\{[\,\log F(t) + (1-F(t))] + [(N-1)(1-F(t))\log F(t)]\}dt. \qquad \text{(A.26)}
\end{aligned}
$$

Let $h = 1 - F(t^{c*})$, using $t^{c*} = (1-h)^{1/k}$, $f(t^{c*}) = k(1-h)^{\frac{k-1}{k}}$ and $F(t^{c*})^{N-1} = \frac{-U_0}{t_0 - t^{c*} + \frac{1-F(t^{c*})}{f(t^{c*})}}$, we have

$$
\begin{aligned}
&-\frac{1-F(t^{c*})}{f(t^{c*})}F(t^{c*})^N \log(F(t^{c*})) \\
&= -\frac{h}{k}(1-h)^{1/k}U_0 \log(1-h)\frac{1}{(1-h)^{1/k} - t_0 - \frac{h(1-h)^{1/k}}{k(1-h)}} \\
&= \frac{U_0}{(1-t_0)k}h^2 + o(h^2),\ as\ N \to \infty\ (\ i.e.,\quad F(t^{c*}) \to 1), \qquad \text{(A.27)}
\end{aligned}
$$

and

$$
\begin{aligned}
&-U_0(F(t^{c*})\log(F(t^{c*})) + (1-F(t^{c*}))) \\
&= -\frac{U_0}{2}h^2 + o(h^2),\ as\ N \to \infty\ (\ i.e.,\ F(t^{c*}) \to\ 1). \qquad \text{(A.28)}
\end{aligned}
$$

Using (A.24), we have that

$$
-\int_{t^{c*}}^{\bar{t}} F(t)^{N-1}\{[\,\log F(t) + (1-F(t))] + [(N-1)(1-F(t))\log F(t)]\}dt
$$

$$
\begin{aligned}
&\leq -\{[\log(F(t^{c*})) + (1 - F(t^{c*}))] + N(1 - F(t^{c*}))\log(F(t^{c*}))\}(1 - F(t^{c*})^{1/k}) \\
&= -\{[\ \log(1-h) + h] + (-\log(\frac{U_0}{1-t_0}) + o(1))\log(1-h)\}(1 - (1-h)^{1/k}) \\
&= \frac{-1}{k}\log(\frac{U_0}{1-t_0})h^2 + o(h^2),\ as\ N \to \infty\ (\ i.e.,\ F(t^{c*}) \to 1). \qquad \text{(A.29)}
\end{aligned}
$$

Form (A.26), (A.27), (A.28) and (A.29), when $k > \frac{2}{1-t_0}(1 - \frac{\log(U_0/(1-t_0))}{(U_0/(1-t_0))})$, $\frac{dR^*(t^{c*})}{dN} < 0$ holds as $N \to\ \infty$. $\square$

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