Justify Cursed Equilibria via Partial Awareness

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July 4, 2007

Abstract

We show that given any finite Bayesian game, its set of cursed equilibria coincides a set of Bayesian Nash equilibria of the game augmented with players partially aware of other players' true types. Consistent with the intuition that cursedness implies scarce computational resource, partial awareness is equivalent to a reduction of the complexity of players' strategic computation.

JEL Classification: C72, D82

Keywords: incomplete information, Bayesian Nash equilibrium, cursed equilibrium, partial awareness, imperfect strategic sophistication

1 Introduction

In a game with incomplete information, if players have the right beliefs about other players' information and expect others rationally choose actions depending on their information, then the proper equilibrium concept is the standard Bayesian Nash equilibria. But if players are not sophisticated, one of the worst things they do is taking other players' actions as unrelated with their private information. In reality, the way agents make decisions often lies between the two extremes. By a χ-weighted combination of the two situations, Eyster and Rabin (2005) introduce the concept of *cursed equilibria* to explain field or experimental data and do statistical inferences.

However, the notion of Bayesian Nash equilibrium itself is legitimate even when players have wrong or inconsistent beliefs[1]. Although a large proportion of economic literature assume that prior beliefs must be common knowledge, such that there is no event that some players know could happen but others don't, sometimes we might need to relax this assumption to make the Bayesian Nash framework more flexible. In our case, we want to show that by properly choosing more general beliefs, a cursed equilibrium can be justified as a Bayesian Nash equilibrium.

The equivalence is not obvious except when χ, the weight of cursedness is positive. In the proof of the existence of cursed equilibria, Eyster and Rabin (2005) let every player have two possible payoffs at every state. With probability $1 - \chi$, it depends on others' types, and with probability χ, it does not. In this virtual game, the Bayesian Nash equilibrium is a cursed equilibrium of the original game. Such formulation shows the existence of cursed equilibria but is not a very good justification within a Bayesian Nash framework, because it is difficult to argue that players' payoff functions are as versatile as assumed.

We want to justify cursed equilibrium using a Bayesian Nash framework with

[1]See Myerson (2004) for a discussion.

beliefs which are more general but not too general to be insensible. For that matter, we propose using the idea of partial awareness.

To see the intuition of awareness, consider that a player's type is a state of mind determined by his information. In many realistic situations, the process of the information is not perfect. This information can be a multidimensional signal and the player, as a receiver of the signal, may lack the ability to either perceive, or measure, or understand the variations in certain dimensions. According to Li (2006), the dimensions of signal that one player can actually use is called his awareness. Taking a different analogy, Heifetz et al. (2006) say that all information is expressed in some language and to express complex information, a language must be rich in its expressive power. For example, a language with a restricted vocabulary in general has less expressive power. In this sense, they define awareness by the expressive power of the language used by a player.

Heifetz et al. (2007) provide a general framework formulating unawareness in games with incomplete information. In this paper, we focus on a specific situation that players may be partially aware of others' types, in the sense that the perceived types were use a coarser partition than the true partition on the underlying set of states.

As a matter of fact, it is not very surprising that partial awareness can explain some imperfect strategic behaviors. What we show is that the model with partial awareness exactly fits the definition of cursed equilibria. Thus the Bayesian Nash equilibria of a game with partial awareness is more general than cursed equilibria.

It takes two steps to show the result. First for every player's types we keep the first order belief about the exogenous parameter, then add expand the set of perceived other players' types by adding a state where all other players do not have any information. By assuming every player learns the averaged actions at this additional state, we can show a Bayesian Nash equilibrium is also a cursed equilibrium. Second we show that the augmented types implies that the information of others perceived by one player is always worse than the true information of others in the sense of Blackwell condition. We can thus use the result of Green and Stokey (1978) to show that both the perceived types and the true types can be represented by two partitions of one set of states with one probability measure, and the partition representing the true types refines the other one. This matches the meaning of partial awareness.

A closely related work is Miettinen (2007). There he first defines the original set of states as a partition of a interval of measure one. Then he defines new partition of the interval. With the new partition, at every original state, he allows Player i to be able to understand other players' type-dependent strategies with probability $1 - \chi$, and not able to do so with probability χ. Therefore at every original state, the expected payoff function is just like the one in the virtual game in Eyster and Rabin (2005). He uses the idea and the concept of analogy based expectation equilibrium to provide a learning foundation of cursed equilibrium.

Roughly speaking, the analogy based expectation equilibrium allows players to partition other players' types and assume the strategies are based the members of the partition instead of individual types. Because partial awareness can be look on as a reason for cursed players to partition the set of true types, two ideas are quite similar. However, since the new partition in Miettinen (2007) has more members than the number of original states, he concludes that when players are partially cursed, they use more complexity strategic computation. But in our framework, the partition used by partially aware players are always coarser than the true one, hence the implication is opposite.

The paper is organized as follows. We set up the framework and review Bayesian Nash Equilibrium and cursed equilibrium in Section 2. A justification with expanded type spaces is shown in Section 3. We show the main result on partial awareness in Section 4. In the end of every section, we provide a example about a lemon market. Conclusions are in Section 5. In appendix, we show a lemon market model with partial awareness without referring to cursed equilibrium.

2 Bayesian Nash Equilibria and Cursed Equilibria

The game is a static game with incomplete information denoted by $\left\langle \Theta, T_i; q; A_i; u_i \right\rangle_{i=1}^{N}$. The set of players is $\{1, 2, \ldots, N\}$. The exogenous parameter is $\theta \in \Theta$. The space of player types is $T \equiv \times_{i=1}^{N} T_i$. A common prior probability distribution q puts positive measure on every state in $\Theta \times T$. Type t_i's belief on the parameter and other players' types (θ, t_{-i}) is given by $q(\theta, t_{-i}|t_i)$.

An action of Player i is $a_i \in A_i$, and A_i being the action set. All players' action profile is a vector $a \in A \equiv \times_{i=1}^{N} A_i$. The action profile space A is assumed to be fixed for all states. Player i's payoff function is $u_i : A \times \Theta \to \mathbf{R}$. It is also assumed that this information is common knowledge.

A mixed strategy σ_i for Player i specifies a probability distribution over actions for each type, $\sigma_i : T_i \to \Delta(A_i)$. Let $\sigma_i(a_i|t_i)$ be the probability that type t_i plays action a_i. Let A_{-i} be the set of action profiles for players other than i, σ_{-i} be the set of strategies of players other than i, and $\sigma_{-i}(a_{-i}|t_{-i})$ be the probability that types $t_{-i} \in T_{-i}$ plays actions a_{-i} under strategy $\sigma_{-i}(t_{-i})$. A strategy profile is $\sigma(t) : T \to \Delta(A)$.

We will give definitions of a Bayesian Nash equilibrium (Harsanyi, 1967-1968) and a cursed equilibrium.

Definition 1. *A strategy profile σ is a Bayesian Nash equilibrium if for each Player $i = 1, \ldots, N$, each type $t_i \in T_i$, and each a_i^* such that $\sigma_i(a_i^*|t_i) > 0$,*

$$a_i^* \in \arg\max_{a_i \in A_i} \sum_{\theta \in \Theta} \sum_{t_{-i} \in T_{-i}} q(\theta, t_{-i}|t_i) \times \sum_{a_{-i} \in A_{-i}} \sigma_{-i}(a_{-i}|t_{-i}) u_i(a_i, a_{-i}; \theta). \tag{2.1}$$

In a cursed equilibrium (Eyster and Rabin, 2005), given other players' strategy σ_{-i}, Player i mistakenly believes that with a probability $\chi \in [0, 1]$ other players play mixed strategies regardless their types, and these strategies averages their true strategies over their types, which is

$$\bar{\sigma}_{-i}(a_{-i}|t_i) \equiv \sum_{t_{-i} \in T_{-i}} q(t_{-i}|t_i) \sigma_{-i}(a_{-i}|t_{-i}).$$

where $q(t_{-i}|t_i) = \sum_{\theta \in \Theta} q(\theta, t_{-i}|t_i)$.

Definition 2. *A strategy profile σ is a χ-cursed equilibrium if for each Player i, each type $t_i \in T_i$, and each a_i^* such that $\sigma_i(a_i^*|\theta_i) > 0$,*

$$a_i^* \in \arg\max_{a_i \in A_i} \sum_{\theta \in \Theta} \sum_{t_{-i} \in T_{-i}} q(\theta, t_{-i}|t_i) \times \sum_{a_{-i} \in A_{-i}} \left[\chi \bar{\sigma}_{-i}(a_{-i}|t_i) + (1 - \chi) \sigma_{-i}(a_{-i}|t_{-i}) \right] \times u_i(a_i, a_{-i}; \theta). \tag{2.2}$$

When $\chi = 0$, χ-cursed equilibrium coincides with Bayesian Nash equilibrium. When $\chi = 1$, every player assume that other players' strategy is completely unrelated with their types, namely players are *fully cursed*.

2.1 Lemon Market: Part 1

This example is taken from Eyster and Rabin (2005). There is a used car, a seller and a buyer. The exogenous parameter is the car's value $v \in \{v_h, v_l\}$. At state v_h, the value to the seller is $2,000$, the value to the buyer is $3,000$; at state v_l, the value to both is 0.

Ex ante, each state happens with probability $\frac{1}{2}$. Suppose the seller has a perfect signal $s \in \{g, b\}$ so that $Pr(v = v_h|s = g) = Pr(v = v_l|s = b) = 1$. The buyer has no information besides the prior probability distribution.

Then at a fixed price P, both sides are able to choose "deal" or "no deal". Trade happens only if both choose "deal".

Let $P = 1,000$. The seller sells only when $s = b$, and the buyer who knows this will refuse to buy. In the unique Bayesian Nash equilibrium, no trade happens.

In the cursed equilibrium, a χ-cursed buyer believes that with probability χ the seller sells with probability $\frac{1}{2}$ irrespective of the signal, the car's expected value is $3000[(1-\chi)0 + \chi\frac{1}{2}] = 1500\chi$. Hence, a buyer cursed by $\chi > \frac{2}{3}$ will buy. Also, the seller's strategy, selling whenever $s = b$, after being averaged over his types, is consistent with the buyer's belief.

3 An intermediate alternative justification

We let players to have types different from T. There will be a type $y_i \in Y_i$ corresponding to every type t_i, so that y_i shares t_i's belief on the parameter, but he believes that there is a positive probability that other players have no information at all.

To formalize the new information structure, for every Player i, we need one set of types: (Y_i, p_i) and $N-1$ sets of types: $(Y_j^i, p_j^i), \forall j \neq i, j \in \{1, \ldots, N\}$.

Here Y_i is the set of new types of Player i, with exactly the same number of elements of set T_i, or $|Y_i| = |T_i|$. Not knowing that the types of Player j are in Y_j, Player i believes that Y_j^i is the set of types of Player j, and $|Y_j^i| = |T_i| + 1$.

We define the relation among the type sets by two bijective mappings: $f_i : Y_i \to T_i$ and $f_j^i : Y_j^i \backslash y_{jx}^i \to T_j$, where y_{jx}^i denotes a special state related to the cursedness. Player i's prior belief p_i puts positive measure on every state in the sets $Y_i \times \{y_{-i} | y_j^i \in Y_j^i \backslash y_{jx}^i, j \neq i\}$ and $Y_i \times \{y_{-ix}^i\}$, where $y_{-ix}^i = (y_{jx}^i)_{j \neq i}$ such that

1) The marginal probability distributions of p_i about y_i is equal to that of q about $f_i(y_i)$.

$$\sum_{\theta \in \Theta} \sum_{y_{-i}^i \in Y_j^i} p_i(\theta, y_i, y_{-i}^i) = \sum_{\theta \in \Theta} \sum_{t_{-i} \in T_{-i}} q(\theta, f_i(y_i), t_{-i}); \tag{3.1}$$

2) The conditional probability distribution of p_i on (θ, y_{-i}^i) given y_i is

$$p_i(\theta, y_{-i}^i | y_i) = \begin{cases} (1-\chi) q(\theta, f_{-i}^i(y_{-i}^i) | f_i(y_i)) & \text{if } y_{-i}^i \in \times_{j \neq i} (Y_j^i \backslash y_{jx}^i), \\ \chi q(\theta | f_i(y_i)) & \text{if } y_{-i}^i = y_{jx}^i. \end{cases} \tag{3.2}$$

Player i also believes that j's belief p_j^i puts positive measure on every state in

$\Theta \times T_{-j} \times Y_j^i$. Actually, this is only matter in the next section, because to verify if a strategy profile is Bayesian Nash equilibria or not, only every player's beliefs about other players' types matters, unless one need to verify if the equilibria strategy profile is consistent with the belief that other players are also rational. For that matter indeed, no matter what p_j^i is, it is in general impossible for i to justify the rationality of the perceived strategy of j .

1) The marginal probability distributions of p_j^i about y_j^i is.

$$\sum_{\theta,t_{-j}} p_i(\theta, t_{-j}, y_j^i) = \begin{cases} \chi & \text{if } y_j^i = y_{jx}^i, \\ (1-\chi)\sum_{\theta,t_{-j}} q(\theta, t_{-j}, f_j^i(y_j^i)) & \text{if } y_j^i \neq y_{jx}^i. \end{cases} \tag{3.3}$$

2) The conditional probability distribution of p_i on (θ, t_{-j}) given y_{jx}^i is

$$p_j^i(\theta, t_{-j}|y_{jx}^i) = \begin{cases} \sum_{t_j\in T_j} q(\theta, t_j, t_{-i}) & \text{if } y_j^i = y_{jx}^i, \\ q(\theta, t_{-j}|f_j^i(y_{-j}^i)) & \text{if } y_j^i \in Y_j^i \backslash y_{jx}^i. \end{cases} \tag{3.4}$$

For Player i, denote the strategy of other players by a function of the perceived types of others, namely $\sigma'_{-i} : Y_{-i}^i \to \Delta(A_i)$.

Assumption 1. *Every Player i, every type y_i believes that given type profile y_{-ix}^i, every other player plays a strategy averaged over his types:*

$$\bar{\sigma}'_{-i}(a_{-i}|y_i) = \frac{1}{1 - Prob\{y_{-i}^i = y_{-ix}^i|y_i\}} \sum_{\theta\in\Theta} \sum_{y_{-i}^i\in Y_{-i}^i\backslash y_{-ix}^i} p_i(\theta, y_{-i}^i|y_i)\sigma'_{-i}(a_{-i}|y_{-i}^i).$$

Lemma 1. *With Assumption 1 hold, the augmented game's set of Bayesian Nash equilibria coincides the set of cursed equilibria of the original game.*

Proof. First by Equation 3.2,

$$Prob\{y^i_{-i} = y^i_{-ix}|y_i\} = \sum_{\theta \in \Theta} p_i(\theta, y^i_{ix}|y_i) = \sum_{\theta \in \Theta} \chi q(\theta|f_i(t_i)) = \chi.$$

Then by this and Equation 3.2, Assumption 1 implies that

$$\bar{\sigma}'_{-i}(a_{-i}|y_i) = \sum_{\theta \in \Theta} \sum_{y^i_{-i} \in Y^i_{-i} \backslash y^i_{jx}} q(\theta, f^i_j(t_{-i})|f_i(y_i))\sigma'_{-i}(a_{-i}|y^i_{-i}). \tag{3.5}$$

Second by Definition 1, in a Bayesian Nash equilibrium $\sigma'' : \times^n_{i=1} Y_i \to \Delta(A)$[2], for each Player $i = 1, \ldots, N$, each type $y_i \in Y_i$, and each a^*_i such that $\sigma''_i(a^*_i|y_i) > 0$,

$$a^*_i \in \arg\max_{a_i \in A_i} \sum_{\theta \in \Theta} p_i(\theta, y^i_{-ix}|y_i) \sum_{a_{-i} \in A_{-i}} \sigma'_{-i}(a_{-i}|y^i_{-ix})u_i(a;\theta) + \\ \sum_{\theta \in \Theta} \sum_{y^i_{-i} \in Y^i_{-i} \backslash y^i_{-ix}} p_i(\theta, y^i_{-i}|y_i) \times \sum_{a_{-i} \in A_{-i}} \sigma'_{-i}(a_{-i}|y^i_{-i})u_i(a;\theta).$$

The two components in the objective function are, first by Assumption 1 and Equation 3.2,

$$\sum_{\theta \in \Theta} p_i(\theta, y^i_{-ix}|y_i) \sum_{a_{-i} \in A_{-i}} \sigma'_{-i}(a_{-i}|y^i_{-ix})u_i(a;\theta) = \\ \sum_{a_{-i} \in A_{-i}} \bar{\sigma}'_{-i}(a_{-i}|y_i) \sum_{\theta \in \Theta} \sum_{t_{-j} \in T_{-j}} \chi q(\theta|f^i(y_i))u_i(a;\theta),$$

[2]Note that player j's true strategy $\sigma''_j(\cdot|y_j)$ is equivalent to $\sigma'_j(\cdot|y^i_j)$ when $f_j(y_j) = f^i_j(y^i_j)$.

and

$$\sum_{\theta\in\Theta}\sum_{y^i_{-i}\in Y^i_{-i}\backslash y^i_{-ix}} p_i(\theta, y^i_{-i}|y_i) \times \sum_{a_{-i}\in A_{-i}} \sigma'_{-i}(a_{-i}|y^i_{-i})u_i(a;\theta) =$$
$$\sum_{\theta\in\Theta}\sum_{y^i_{-i}\in Y^i_{-i}\backslash y^i_{-ix}} (1-\chi)q(\theta, f^i_{-i}(y^i_{-i})|f_i(y_i))\big) \times \sum_{a_{-i}\in A_{-i}} \sigma'_{-i}(a_{-i}|y^i_{-i})u_i(a;\theta).$$

Hence the objective function is equivalent to

$$\sum_{\theta\in\Theta}\sum_{y^i_{-i}\in Y^i_{-i}\backslash y^i_{-ix}} q(\theta, f^i_{-i}(y^i_{-i})|f_i(y_i)) \times \sum_{a_{-i}\in A_{-i}} \big[\chi\bar{\sigma}'_{-i}(a_{-i}|y_i) + (1-\chi)\sigma'_{-i}(a_{-i}|y^i_{-i})\big] \times u_i(a;\theta).$$

Compare it with the objective function in Definition 2.2, and in particular, compare Equation 3.5 and the definition

$$\bar{\sigma}_{-i}(a_{-i}|t_i) \equiv \sum_{t_{-i}\in T_{-i}} q(t_{-i}|t_i)\sigma_{-i}(a_{-i}|t_{-i}).$$

where $q(t_{-i}|t_i) = \sum_{\theta\in\Theta} q(\theta, t_{-i}|t_i)$. We see that the criteria for a cursed equilibrium and for a Bayesian Nash equilibrium are equivalent. The sets of two must be identical. **QED.**

3.1 Lemon Market: part 2

Let the buyer believes that there is a new signal $s' \in \{x, g, b\}$ with prior probability $Pr(s' = x) = \chi, Pr(s' = g) = \frac{1}{2}(1-\chi), Pr(s' = b) = \frac{1}{2}(1-\chi)$. The buyer believes that when the signal is g or b, it again conveys perfect information. But when it is x, the seller knows no information. This is

$$Pr(v = v_h | s' = g) = Pr(v = v_l | s' = b) = 1,$$

$$Pr(v = v_h | s' = x) = Pr(v = v_l | s' = x) = \frac{1}{2}.$$

Thus the buyer believes that the seller's expected value is 3000 at $s' = g$, 1000 at $s' = x$, and 0 at $s' = b$. The buyer expects that if the seller sells, the signal $s' \in \{x, b\}$; if the seller sells with .5 probability at $s' = x$, then the car's expected value is

$$\frac{3000Pr(v = v_h, s' = x)0.5 + 3000Pr(v = v_h, s' = b)}{0.5Pr(s' = x) + Pr(s' = b)} = \frac{3000\chi .5^2}{0.5(1-\chi) + .5\chi} = 1500\chi.$$

The buyer buys if $1500\chi > 1000$, or $\chi \geq \frac{2}{3}$, which is equivalent to the result in the first part of the example.

4 Awareness

The justification previously introduced relies on adding an artificial additional states. From the point of view of a cursed player, there could be some cognitive reason causing his belief about others types different from the true ones. We want to apply the idea of awareness on the relation between the perceived types and true types. It requires taking the sets of types as players' information structures and represent them by information partitions on a common set of states of the world with a common belief. If the relation between the partitions representing the perceived types and true types match the definition of partial awareness, we shall get our result following Lemma 1.

We first explain what we mean by representation a player's type sets by information partitions. Generally speaking, a decision maker concerns the value of parameter $\phi \in \Phi$, $\Phi = \{\phi_1, \ldots, \phi_K\}$ being finite. And let the prior probability distribution be fixed

$$r = \{r_1, \ldots, r_K\}.$$

An information structure has two alternative formalizations.

1. A set of types Y, and a prior probability distribution λ on $\Phi \times Y$. This is denoted by (Y, λ).

2. A set X, a probability measure μ on $\Phi \times X$, and a partition P on X. This is denoted by $(X, \mu, \mathcal{S})$.

Definition 3. *We say that (X, μ, P) represents (Y, q) if there is a mapping:*

$$\tau : P \to Y$$

such that

1) for each $y \in Y$, each $s \in \tau^{-1}(y)$, and all $\phi \in \Phi$,

$$\frac{\mu(\{\phi\} \times s)}{\mu(\Phi \times s)} = q(\phi|y)$$

2) for each $y \in Y$,

$$\sum_{s \in \tau^{-1}(y)} \mu(\Phi \times s) = q(\Phi, t).$$

Definition 4. *A player i is partially aware of player j's true types if for all his types, there is a set X_j, a probability measure μ_j on X_j, and two partitions of X_j: P_j and P'_j, such that*

1) The information structure (X_j, μ_j, P_j) *represents player j's true types* (T_j, q)*;*

2) The information structure (X_j, μ_j, P'_j) *represents player j's types that i perceives, or* (Y^i_j, p^i_j)*;*

3) P_j *is a refinement of* P'_j*.*

Before we show the main result, the following definition and theorem are useful. Let (Y, q) and (Y', q') be two information structures about the value of same $\phi \in \Phi$. $Y = \{y_1, \ldots, y_L\}$ and $Y' = \{y'_1, \ldots, y'_H\}$ Denote the conditional probability distribution $q(y|\phi_k)$ by a row vector π_k. Denote the conditional probability distribution $q'(y'|\phi_k)$ by a row vector π'_k.

Definition 5. *Two information structures* (Y, q) *and* (Y', q') *satisfy Blackwell's condition if and only if there exists a Markov matrix* B *such that*

$$\Pi' = \Pi B, \tag{4.1}$$

where $\Pi = (\pi_k(y_l))$*,* $\Pi' = (\pi'_k(y'_h))$*.*

Theorem 1. (Green and Stokey, 1978) *If two information structures* (Y, q) *and* (Y', q') *satisfy Blackwell's condition 4.1, then there exists* (X, μ, P, P') *such that*

- (X, μ, P) *represents* (Y, q)*;*
- (X, μ, P') *represents* (Y', q')*;*
- P *refines* P'*.*

Now we can present the main result.

Proposition 1. *Given that Player j's true types are* (T_j, q) *and Player i believes that Player j's types are* (Y^i_j, p^i_j)*, Player i is partially aware of Player j's types.*

Proof. 1. Conditional on every $\omega \in \Theta \times T_{-j}$, Player j's true type t_j's probability distribution is $q(t_j|\omega)$. Let the set T_j be indexed by $m \in \{1, \ldots, |T_j|\}$ and the set $\Theta \times T_{-j}$ be indexed by $n \in \{1, \ldots, |\Theta \times T_{-j}|\}$. We define a matrix Π_j such that an element $\pi_{nm} = q(t_{jm}|\omega_n)$.

2. Conditional on every $\omega \in \Theta \times T_{-j}$, by Conditions 3.3 and 3.4, the probability of type y^i_j given (θ, t_{-j}) is

$$p^i_j(y^i_j|\omega) = \begin{cases} \chi & \text{if } y^i_{-i} = y^i_{jx}, \\ (1-\chi)q(t_j|\omega) & \text{if } y^i_j \neq y^i_{jx} \text{ ,and } f^i_j(y^i_j) = t_j. \end{cases} \tag{4.2}$$

Let the set Y^i_j be indexed by $m' \in \{1, \ldots, |Y^i_j|\}$ we define a matrix Π^i_j such that an element $\pi'_{nm'} = q(y^i_{jm'}|\omega_n)$. Properly arrange the indexes we will have that

$$\mathbf{\Pi^i_j} = \begin{pmatrix} \chi I_{|\Theta \times T_{-j}| \times 1}, & (1-\chi)\Pi_j \end{pmatrix}.$$

Therefore there is a Markov matrix B

$$B = \begin{pmatrix} \chi & 1-\chi & 0 & \ldots & 0 \\ \chi & 0 & 1-\chi & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \chi & 0 & 0 & \ldots & 1-\chi \end{pmatrix}.$$

such that $\Pi^i_j = \Pi_j B$. Blackwell's condition 4.1 is satisfied. By Theorem 1, the result follows. **QED.**

Putting Lemma 1 and Proposition 1 together, we have the main result.

Theorem 2. *For any $\chi \in (0, 1]$, the set of cursed equilibria of a game coincides a set of Bayesian Nash equilibria of the same game augmented with types partially* *aware of other players' true types.*

Define the complexity of his strategic computation by the cardinality of the partitions of other players used by Player i. Because partition P_j refines P'_j, when Player i uses P'_j rather than P_j, the complexity decreases.

Corollary 1. *The complexity of his strategic computation decreases when the cursedness χ becomes positive.*

4.1 Construct information partitions.

The construction uses the method of Green and Stokey (1978) shown in the proof of Theorem 1. For notional brevity, we omit the subscripts of (X_j, μ_j, P_j, P'_j). The set $X = T_j \times Y_j^i$. The partitions $P = \{(t_j, y_j^i) | t_j \in T_j, y_j^i \in Y_j^i\}$ and $P' = \{T_j \times \{y_j^i\} | y_j^i \in Y_j^i\}$. We see that P refines P'.

The probability measure

$$\mu\big(\{(\omega, x)\}\big) = \mu\big(\{(\omega_n, t_{jm}, y_{jm'}^i)\}\big) = \sum_{t_j} q(\omega_n, t_j) q(t_{jm}|\omega_n) b_{mm'}.$$

where $b_{mm'}$ is the element of Markov matrix B.

Now we need to show that (X, μ, P) represents (T_j, q), namely if they satisfy Definition 3. Take $\tau((t_j, y_j^i)) = t_j$, so that $\tau^{-1}(\bar{t}_j) = \{\{(\bar{t}_j, y_j'^i)\} | y_j'^i \in Y_j^i\}$. Then

$$\frac{\mu(\omega_n, t_{jm}, y_{jm'}^i)}{\mu(\Theta \times T_{-j} \times (t_{jm}, y_{jm'}^i))} = \frac{\sum_{t_j} q(\omega_n, t_j) q(t_{jm}|\omega_n) b_{mm'}}{\sum_n \sum_{t_j} q(\omega_n, t_j) q(t_{jm}|\omega_n) b_{mm'}} = q(\omega_n | t_{jm}).$$

This verifies the first condition in the definition.

Because $\sum_{m'} b_{mm'} = 1$ for every m,

$$\begin{aligned}\sum_{x \in \tau^{-1}(t_{jm})} \mu(\Theta \times T_{-j} \times x) &= \sum_{n} \sum_{m'} \mu(\omega_n, t_{jm}, y^i_{jm'}) \\ &= \sum_{n} \sum_{m'} \sum_{t_j} q(\omega_n, t_j) q(t_{jm}|\omega_n) b_{mm'} \\ &= q(\Theta \times T_{-j}, t_{jm}).\end{aligned}$$

The second condition in the definition is also verified.

To show that (X, μ, P') represents (Y^i_j, p^i_j), define $\tau((t_j, y^i_j)) = y^i_j$ so that $\tau^{-1}(\bar{y}^i_j) = T_j \times \{\bar{y}^i_j\}$, then follow the same logic.

4.2 Lemon Market: Part 3

The probability distribution specified in part 2 implies that for the true types of the seller we have

$$\Pi = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

and $\Pi_{vs} = Pr(s|v)$, where $v \in \{v_h, v_l\}$ is the row index and $s \in \{g, b\}$ is the column index.

For the perceived types of the seller, we have

$$\Pi' = \begin{pmatrix} \chi & 1-\chi & 0 \\ \chi & 0 & 1-\chi \end{pmatrix},$$

and $\pi'_{vs'} = Pr(s'|v)$, where $v \in \{v_h, v_l\}$ is the row index and $s' \in \{x, g, b\}$ is the column index.

The Markov matrix is $B = \Pi'$.

We construct a set $X = \{g, b\} \times \{x, g, b\}$. A partition $P = \{\{gg\}, \{gx\}, \{gb\}, \{bg\}, \{bx\}, \{bb\}\}$, and a partition $P' = \{\{gg, bg\}, \{gx, bx\}, \{gb, bb\}\}$. It is easy to see that P refines P'. The measure μ is given as

$$\mu(v, s, s') = Pr(v) \times \Pi_{vs} \times \Pi'_{ss'}.$$

$$\begin{aligned}
\mu(v_h, g, g) &= \tfrac{1-\chi}{2} \\
\mu(v_h, g, x) &= \tfrac{\chi}{2} \\
\mu(v_h, g, b) &= 0 \\
\mu(v_h, b, g) &= 0 \\
\mu(v_h, b, x) &= 0 \\
\mu(v_h, b, b) &= 0 \\
\mu(v_l, g, g) &= 0 \\
\mu(v_l, g, x) &= 0 \\
\mu(v_l, g, b) &= 0 \\
\mu(v_l, b, g) &= 0 \\
\mu(v_l, b, x) &= \tfrac{\chi}{2} \\
\mu(v_l, b, b) &= \tfrac{1-\chi}{2}.
\end{aligned}$$

With a mapping $\tau(\{ss'\}) = s$, $\forall s \in \{g, b\}, \forall s' \in \{x, g, b\}$, we can check that (X, μ, P) is a representation of $s \in \{g, b\}$. A similar check for (X, μ, P') uses a mapping $\tau'(\{g, b\} \times \{s'\}) = s'$, $\forall s' \in \{x, g, b\}$.

5 Conclusions

In this paper we show the innovative cursed equilibrium concept is a special case of Bayesian Nash equilibrium with partial awareness. It justifies the first concept and suggests that the second more general concept may have potential to unify imperfect strategic sophistication. Further applications in auction and trading can be promising.

A An alternative lemon market model with partial awareness

To emphasize the idea we to present this example in a reversed order. We define set $X = \{v_h, v_l\} \times \{g, b\} \times \{g', b'\}$, partition $P = \{\{gg'\}, \{gb'\}, \{bg'\}, \{bb'\}\}$, the partition $P' = \{\{gg', bg'\}, \{gb', bb'\}\}$. Hence P refines P'. The measure μ is given as

$$\begin{aligned}
\mu(v_h, g, g') &= \tfrac{q}{2} \\
\mu(v_h, g, b') &= \tfrac{1-q}{2} \\
\mu(v_h, b, g') &= 0 \\
\mu(v_h, b, b') &= 0 \\
\mu(v_l, g, g') &= 0 \\
\mu(v_l, g, b') &= 0 \\
\mu(v_l, b, g') &= \tfrac{1-q}{2} \\
\mu(v_l, b, b') &= \tfrac{q}{2}.
\end{aligned}$$

where $q \in (\frac{1}{2}, 1]$.

Thus if the buyer believes that the seller's information is represented by partition P', he knows that the seller could essentially have two types $s' \in \{g', b'\}$. Define a mapping $\tau' : P \to \{g', b'\}$ so that $\tau'(\{gg', bg'\}) = g'$ and $\tau'(\{gb', bb'\}) = b'$. It implies that

$$Pr(s' = g') = \sum_{v \in \{v_h, v_l\}} \sum_{s \in \{g,b\}} \mu(v, s, s' = g') = \frac{1}{2};$$

$$Pr(s' = b') = \frac{1}{2};$$

$$Pr(v = v_h | s' = g') = \frac{\frac{q}{2}}{\frac{1}{2}} = q;$$

$$Pr(v = v_l | s' = b') = \frac{\frac{q}{2}}{\frac{1}{2}} = q;$$

Believing this, the buyer expects that if the seller never sells at type $s' = g'$. But if $s' = b'$, he sells because $2000(1 - q) \leq 1000$, the expected value of the car is $3000(1 - q)$ to the buyer. So if $q < \frac{2}{3}$, he wants to buy in equilibrium.

But the true types are given by partition P.

Define a mapping $\tau : P \to \{g, b\}$ so that $\tau(\{ss'\}) = s$. It implies that for every $s'' \in \tau^{-1}(s)$,

$$Pr(s'' = gg') = \sum_{v \in \{v_h, v_l\}} \mu(v, s = g, s' = g') = \frac{q}{2};$$

$$Pr(s'' = gb') = \sum_{v \in \{v_h, v_l\}} \mu(v, s = g, s' = b') = \frac{1 - q}{2};$$

$$Pr(s'' = bg') = \sum_{v \in \{v_h, v_l\}} \mu(v, s = b, s' = g') = \frac{1 - q}{2};$$

$$Pr(s'' = bb') = \sum_{v \in \{v_h, v_l\}} \mu(v, s = b, s' = b') = \frac{q}{2};$$

and

$$Pr(v = v_h | s'' = gg') = Pr(v = v_h | s'' = gb') = 1;$$

$$Pr(v = v_l | s'' = bg') = Pr(v = v_l | s'' = bb') = 1.$$

The true types implies that the seller has perfect information. He sells at $s = b$ and does not sell otherwise. So if the buyer is aware of the true types, he shall not trade in the equilibrium. But if the buyer is unaware, then the seller's averaged mixed strategy is consistent with the buyers expectation. It is just like in a cursed equilibrium.

In addition, the true types induce

$$\Pi = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

where $\pi_{11} = Pr(s = g | v = v_h)$, $\pi_{12} = Pr(s = b | v = v_h)$, $\pi_{21} = Pr(s = g | v = v_l)$, $\pi_{22} = Pr(s = b | v = v_l)$.

And the perceived types induce

$$\pi' = \begin{pmatrix} q & 1-q \\ 1-q & q \end{pmatrix},$$

where $\pi'_{11} = Pr(s' = g | v = v_h)$, $\pi'_{12} = Pr(s' = b | v = v_h)$, $\pi'_{21} = Pr(s' = g | v = v_l)$, $\pi'_{22} = Pr(s' = b | v = v_l)$.

A Markov matrix $B = \Pi'$ satisfies

$$\Pi' = \Pi B. \tag{A.1}$$

The probability mature μ is derived from Π and Π'.

B Proof

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