Group strategyproof cost sharing: the role of indifferences
(Draft)

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Summary. Every agent reports his willingness to pay for one unit of good. A mechanism allocates some goods and cost shares to some agents. A tie-breaking rule describes the behavior of an agent who is offered a price equal to his valuation. We characterize the group strategyproof (GSP) mechanisms under two alternative tie-breaking rules. With the maximalist rule (MAX) an indifferent agent is always served. With the minimalist rule (MIN) an indifferent agent does not get a unit of good.

GSP and MAX characterize the population-monotonic mechanisms. These mechanisms are appropriate for submodular cost functions. On the other hand, GSP and MIN characterize the sequential mechanisms. These mechanisms are appropriate for supermodular cost functions.

Our results are independent of an underlying cost function; they unify and strengthen earlier results for particular classes of cost functions.

Keywords: *Cost sharing, Mechanism design, Group strategyproof, Tie-breaking rule.*

1 Introduction

Units of a nontransferable, indivisible and homogeneous good (or service) are available at some non-negative cost. Agents are interested to consume at most one unit of that good and are characterized by their valuation for it (which we call their utility). We look for mechanisms that elicit these utilities from the agents, allocate some goods to some agents and charge some money only to the agents who are served.

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These mechanisms have been widely explored in the cost-sharing literature (see below). The canonical example is sharing the cost of providing service to some communities (electricity, water, Internet, etc.), where the cost function is not necessarily symmetric. Another example is auctions where the seller has multiple copies of a good.

When agents have private information about their utility, incentive compatibility, here interpreted as strategyproofness (SP), of the mechanism is an issue. The mechanisms that satisfy SP are the "auction" type mechanisms. That is, every agent is offered to buy a unit of good at a price that depends exclusively on the reports of the other agents.

A familiar strengthening of SP is group strategyproofness (GSP). This property rules out coordinated misreports of any group of agents. Unlike SP, the GSP property depends critically on the way the mechanism serves the agents who are offered a price equal to their valuation. GSP is clearly violated if such an agent can be "bossy," i.e. affect the welfare of another agent without altering his own.[1] For instance, consider the mechanism that offers to the agents in $\{1, 2\}$, following the order $1 \succ 2$, the first unit at price p and the second unit at price p', $p' > p$. If agent's 1 utility for a unit of good equals exactly p and agent's 2 utility is strictly bigger that p, then GSP requires agent 1 not to be served. Otherwise, agent 1 gets zero net utility at this profile and he can help agent 2 by reporting a utility below p. The mechanism will offer agent 2 a unit of good at price p, so he is better off.

In this paper, we consider two alternative tie-breaking rules and characterize the GSP mechanisms that satisfy these rules. With the maximalist tie-breaking rule (MAX), an agent who is indifferent between getting or not getting a unit of good will always get a unit of good. With the minimalist rule (MIN), the indifferent agents never get a unit of good.

The mechanisms that satisfy GSP and MAX are the population-monotonic mechanisms (Theorem 1). Namely, for any subset of agents S consider a vector of nonnegative payments $x^S \in [0, \infty]^N$ such that it is zero for all agents not in S. A collection of payments is cross-monotonic if the payments are weakly inclusion decreasing. Given a cross-monotonic collection of payments, we construct the mechanism as follows. For a report of utilities allocate S^* at cost x^{S^*}, where S^* is the biggest coalition of agents such that everyone in S^* is willing to pay x^{S^*} to get service –this coalition exists by cross-monotonicity of the payments.

The mechanisms that satisfy GSP and MIN are the sequential mechanisms (Theorem 2). Loosely speaking, consider any binary tree of size n such that to every node is attached exactly one agent and any path from root to end pass through all agents exactly once. At every decision node we also attach a nonnegative price. Given this tree, we construct the mechanism as follows. First we offer service to the root agent at the price attached to his node. We proceed on the right branch from the root if service is purchased and on the left branch if it is not. The key restriction on prices is that for any two nodes with

[1] In some contexts, GSP is equivalent to the combination of SP and non-bossiness: Papai[18][19], Ehlers et.al.[4], Svensson et.al.[24].

the same agent, the price on the rightist node is smaller than that on the leftist node.[2]

Surprisingly, the (welfarewise) intersection of sequential and population-monotonic mechanisms is almost empty. It contains only the fixed cost mechanisms (Corollary 1), offering the each agent a price completely independent of the reports.

The most compelling property of population-monotonic mechanisms is that they are the only GSP mechanisms that treat equal agents equally (Proposition 1). Their downplay is that they ex-ante exclude some agents when there is a limited number of units. That is, if only k units of good are available, $k < n$, then $n-k$ agents will never be served at any profile (see section 6.3). Sequential mechanisms do not ex-ante exclude any agent. In fact, when there is exactly one unit of good available, only the priority mechanisms[3] meet GSP and allocate at most one unit of good at any profile (Proposition 2).

We do not make an actual cost function part of the definition of a mechanism. That is, we place no constraint on the total cost shares collected from the agents who are served. Thus our characterization results of GSP mechanisms are entire orthogonal to budget balance and other feasibility requirements (such as bounds on the budget surplus or deficit). Naturally, one of the first questions we ask about the class of mechanisms identified in theorems 1 and 2 is when can they be chosen so as to cover exactly a given cost function. In examples 3 and 9 we answer these questions under a weak variation of symmetry. In this way, we recover most types of mechanisms identified in the earlier literature.

2 Related literature

There is some interesting literature in the design of GSP mechanisms for assignment problems of heterogeneous goods (Ehlers[3], Ehlers et.al.[4], Ergin[2], Papai[18][19] and Svensson et.al.[24]). Literature on the design of GSP mechanisms for homogeneous goods was first discussed by Moulin[10] and followed by several computer science applications (see below).

Population-monotonic mechanisms of our Theorem 1 have received the most attention in literature because, unlike the sequential mechanisms, they allow a symmetric treatment of the agents when the cost function is symmetric.

Literature of population-monotonic mechanisms starts with Moulin[10]. When the cost function is submodular (concave), these mechanisms are characterized by GSP, budget balance, voluntary participation, nonnegative transfers and strong consumer sovereignty.[4] Roughgarden et.al.[20][21], Pa'l et.al.[17] and Immorlica et.al.[6] consider population-monotonic mechanisms when the cost function is not submodular. Roughgarden et.al.[20] uses submodular population-

[2] See definition 9 for precise conditions.

[3] These are a subset of sequential mechanisms that offer to the agents, following an independent order, the unit of good at a fixed price until someone accepts the offer.

[4] Strong consumer sovereignty says that every agent has reports such that he gets (or does not get) a unit of good irrespective of other people reports.

monotonic mechanisms to approximate budget balance when the actual cost function is not submodular. Immorlica et.al.[6] shows that new population-monotonic mechanisms emerge when consumer sovereignty is relaxed.

Sequential mechanisms of our Theorem 2 are mostly discussed by Moulin[10] who imposes budget balance for a supermodular (convex) cost function. Theorem 1 there asserts wrongly that all GSP mechanisms meeting budget balance, voluntary participation, nonnegative transfers and strong consumer sovereignty charge successively marginal cost following an independent ordering of the agents. We correct this erroneous statement in example 8.

Roughgarden et.al.[22] uncovers a very clever class of weakly GSP mechanisms that are neither population-monotonic nor sequential (see also Devanur et.al.[1]). This class contains sequential, population-monotonic mechanisms and combinations of them. They apply these mechanisms to the vertex cover and Steiner tree cost sharing problems to improve the efficiency of algorithms derived by population-monotonic mechanisms. A closely related paper is the companion paper Juarez[9] developing a model where indifferences are not important. For instance, agents report an irrational number and payments are rational. It finds that the class of GSP mechanisms becomes very large. In particular, it contains mechanisms very different from the population-monotonic and sequential mechanisms (and also those discussed by Roughgarden et.al.[22]). It provides three equivalent characterizations of the GSP mechanism in this economy, two of these characterizations are generalizations of the population-monotonic and sequential mechanisms discussed in this paper.

When a cost function is specified, an important question is to evaluate the trade-offs between efficiency and budget balance. Moulin and Shenker[14] center their analysis in population-monotonic mechanisms when the underlying cost function is submodular. In particular, they find that the population-monotonic Shapley value mechanism, where the payment of a coalition equals its stand alone cost, minimizes the worst absolute surplus loss.[5] Juarez[8] analyzes similar trade-offs for the supermodular case. Contrary to the submodular case, he constructs optimal sequential mechanisms that cuts the efficiency loss by half with respect to the optimal budget balanced mechanism.

Finally a result by Goldberg et.al.[5] on fixed cost mechanisms is closely related to our Corollary 1. It characterizes these mechanisms under a very strong GSP, in particular when agents can coalitionally manipulate by misreporting, transferring goods and money between them.

3 The model

For a vector x, $x \in \mathbb{R}^M$, we denote by $x_{[S]}$ the projection of x over $S \subset M$. x_S represents the sum of the $S-$coordinates of x, $x_S = \sum_{i \in S} x_i$. When there is no confusion we denote the projection $x_{[S]}$ simply as x_S. Let 1_M the unitarian vector in $\mathbb{R}^M$, that is $1_M = (1, 1, \dots, 1)$.

[5] See also Juarez[7] and Moulin[11] for applications of this and another similar measure.

There is a finite number of agents $N = \{1, 2, \ldots, n\}$. Every agent has a utility (willingness to pay) for getting one unit of good. Let u, $u \in \mathbb{R}_+^N$, the vector of those utilities. Therefore, if agent i gets a unit paying x_i, his net utility is $u_i - x_i$. If he does not get a unit his net utility is zero.

Definition 1 *A mechanism (S, φ) allocates to every vector of utilities u a coalition of agents that gets goods $S(u) \subset N$ and the cost shares (payments) $\varphi(u) \in \mathbb{R}^N$.*

Therefore, the net utility of agent i in the mechanism, denoted by NU_i, is $NU_i(u) = \delta_i(S(u))(u_i - \varphi(u))$.[6] Let $NU(u)$ the vector of such net utilities. Notice two mechanism may be equivalent in the welfare sense, that is their net utilities at any profile are equal, but the mechanisms may be different.

We restrict our attention to mechanisms that satisfy two familiar normative properties.

- **Nonnegative Transfers (NNT):** $\varphi(u) \in \mathbb{R}_+^N$.
- **Individual Rationality (Voluntary participation (VP)):** $\varphi_i(u) \leq u_i\delta_i(S(u))$.

Nonnegative transfers requires all cost shares to be positive or zero. This is a common assumption when no transfers between agents are allowed and we do not want to subsidize any of them.

On the other hand, individual rationality implies that all agents enter the mechanism voluntarily. That is, the ex-post net utility of the agents is never smaller than their ex-ante net utility. Because we are assuming nonnegative transfer, individual rationality implies the agents with zero utility should pay nothing. However, they may get a unit for free. This is a basic equity condition protecting individual rights.

We want to characterize the mechanisms that are group strategyproof. That is, any misreport of a group of agents do not decrease their net utility and strictly increase one o them.

- **Group strategyproof (GSP):** For all $S \subset N$, and all utility profiles u and u' such that $u'_{N\setminus S} = u_{N\setminus S}$, it cannot be that $NU_i(u) \leq (u_i - \varphi_i(u'))\delta_i(S(u'))$ for all $i \in S$ and strict for at least one agent.

We define next our two systematic tie-breaking rules.

- **Maximalist tie-breaking rule (MAX):** If an agent is indifferent between getting or not getting a unit of good, then he will get it.
- **Minimalist tie-breaking rule (MIN):** If an agent is indifferent between getting or not getting a unit of good, then he will not get it.

[6] δ is the classic delta function, $\delta_i(T) = 1$ if $i \in T$, and 0 otherwise.

To properly define tie-breaking rule, consider a strategyproof mechanism (SP). Then there are functions $f_i : \mathbb{R}_+^{N\setminus i} \to [0, \infty]$ for every agent i such that this mechanism is welfare equivalent to the mechanism which offers agent i a unit of good at price $f_i(u_{-i})$. That is, if $u_i > f_i(u_{-i})$ then i is assigned a unit of good and pays $f_i(u_{-i})$. If $u_i < f_i(u_{-i})$ then i is not assigned a unit of good and pays nothing. Under MAX, if $u_i = f_i(u_{-i})$ then the agent gets a unit of good at price $f_i(u_{-i})$. On the other hand, under MIN, if $u_i = f_i(u_{-i})$ then i does not get a unit of good and pays nothing.

Remark 1 *Notice, on the space of SP mechanisms, MAX is implied by upper continuity of the mechanism. That is, we say that a rule is upper continuous if for any decreasing and convergent sequence of utility profiles $u^1 \geq u^2 \geq \cdots \to u^*$ such that $i \in S(u^k)$ for all k, then $i \in S(u^*)$. This is easy to check by taking a decreasing sequence of profiles where the utility of all agents but i is fixed and $u_i^k \to_k f_i(u_{-i})$.*

Similarly, say that a rule is lower continuous if for any increasing and convergent sequence of utility profiles $u^1 \leq u^2 \leq \cdots \to u^*$ such that $i \in S(u^k)$ for all k, then $i \in S(u^*)$. One also checks that lower continuity implies MIN.

Finally, our model is equivalent to the reduced model where agents have utility bounded above by a positive value L. A price equal to ∞, $f_i(u_{-i}) = \infty$, is reinterpreted in the new model as a price of $L + \epsilon$, $\epsilon > 0$. That is, agent i is offered a unit of good at a price above his maximum utility.

4 Population-monotonic mechanisms and MAX

Definition 2 *A cross-monotonic set of cost shares (payments) assigns to every coalition $S \subseteq N$ a vector $x^S \in [0, \infty]^N$ such that $x^S_{[N\setminus S]} = 0$ and moreover*

$$\text{If } S \subseteq T \text{ then } x^S_{[S]} \geq x^T_{[S]}.$$

We denote by χ^N a cross-monotonic set of cost shares, $\chi^N = \{x^S \mid S \subseteq N\}$.

We interpret x^S as the payment when the agents in S, and only them, are served. Therefore, by NNT and VP it should be zero for the agents outside S.

The key feature is that payments should not increase as coalition increases. This implies that for every utility profile u the set of feasible coalitions, $F(u) = \{S \in 2^N \mid x^S \leq u\}$, has a maximum element with the inclusion $\subset$. To see this, notice if $S, T \in F(u)$ then by cross-monotonicity $S \cup T \in F(u)$.

Definition 3 *Given a cross-monotonic set of cost shares χ^N, we define a population-monotonic mechanism (S, φ) as follows. For every utility profile u, $S(u)$ is the maximum feasible coalition at u and $\varphi(u) = x^{S(u)}$.*

Theorem 1 *i. A mechanism satisfies GSP and MAX if and only if it is population-monotonic.*

ii. *A mechanism is GSP and monotonic in size, that is if $u \leq \tilde{u}$ then $S(u) \subseteq S(\tilde{u})$, if and only if it is welfare equivalent to a population-monotonic mechanisms.*

Notice part i of this theorem is an exact characterization of population-monotonic mechanisms. On the other hand, part ii is weaker than part i because only holds in the welfarewise sense. An easy example can be constructed for one agent (see example 4).

Given a cross-monotonic set of cost shares χ^N, we can also implement its associated population-monotonic mechanism by playing the following demand game proposed by Moulin[10]. We offer agents in N units of good at price x^N. If all of they accept it, then everyone is served at prices x^N. If only agents in S accept, then we remove agents in $N \setminus S$ from the game and offer agents in S units of good at price x^S. Continue similarly until all of the agents in a coalition accepted or every agent in N was removed from the game.

Example 1 (Geometric description of population-monotonic mechanisms for $n = 1, 2$) *The one agent mechanisms can be described by a constant x, $x \in \mathbb{R}_+ \cup \{\infty\}$. The agent gets a unit and pays x if his utility is bigger than or equal to x. He does not get a unit and pays nothing otherwise.*

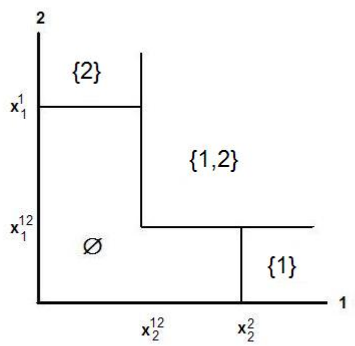

Figure 1: Generic form of 2-agent population-monotonic mechanisms.

The two agent mechanisms should be generated by a cross-monotonic set of cost shares. Thus $0 \leq x_1^{\{1,2\}} \leq x_1^1$ and $0 \leq x_2^{\{1,2\}} \leq x_2^2$ (see figure 1).

By MAX, the level set of $\{1, 2\}$ is closed. The borders between the level sets of $\{1\}$ and $\emptyset$, and $\{2\}$ and $\emptyset$, should belong to the $\{1\}$ and $\{2\}$ respectively.

As is well know from previous literature, if the actual cost of the service C is submodular with respect to coalitions, we can choose a population-monotonic mechanism to cover this cost exactly. For instance, we can choose the cross-monotonic set of cost shares χ^N where the payments of the agents in S are given by the egalitarian solution $\frac{C(S)}{|S|}$. We can alternatively choose the payments of those agents given by the Shapley value or the Dutta-Ray egalitarian solution on the stand alone cost function.

Definition 4 *We say a mechanism satisfies strong consumer sovereignty (SCS) if every agent i has utility profiles $\bar{u}_i$ and $\tilde{u}_i$ such that for any profile of the other agents u_{-i}, $i \notin S(\bar{u}_i, u_{-i})$ and $i \in S(\tilde{u}_i, u_{-i})$.*

Moulin[10] proved that, in the space of submodular cost functions, any mechanism that is budget balanced, SCS and GSP should be implemented as a population-monotonic mechanism for a set of cross-monotonic and budget-balanced cost shares. The result we propose is more general. We show that population-monotonic mechanisms are related to the cross-monotonic cost sharing function. However, as shown in example 3, this does not imply the associated budget balanced cost sharing function is submodular. Hence we capture Moulin's mechanisms and a few more.

Example 2 *Immorlica et.al.[6] proposes an example where exactly one agent pays a positive amount when a coalition of agents is served. This example relaxes the SCS condition on Moulin[10] result (see above), therefore is not captured by Moulin's mechanisms. However, it is captured by our class of population-monotonic mechanisms. For a submodular cost function, order the agent arbitrary, say $i_1 \succ, \ldots, \succ i_n$. Offer the agents, following this order, a unit of good at the cost of himself and the agents after him. The mechanism ends when someone accepts the offer or when we made an offer to all agents. That is, agent i_1 will be offer a unit at price $C(i_1, \ldots, i_n)$. If he accepts, the mechanism ends there. If he rejects, we offer agent i_2 a unit of good at price $C(i_2, \ldots, i_n)$, and so on. The cross-monotonic set of cost shares that implements this mechanism is $x^S_{i^*} = C(D_{i^*})$ and $x^S_j = 0$ for all $j \neq i^*$, where i^* is the maximal element in S and D^*_i is the set of dominated agents by i^* with $\succ$.*

Definition 5 *We say the mechanism (S, φ) meets the equal share property (ESP) if every agent in the coalition that is getting service pays the same. That is, if $\varphi_i(u) = \varphi_j(u)$ for all $i, j \in S(u)$.*

Example 3 *Consider any cost function $C : 2^N \to \mathbb{R}_+$ such that its average cost function AC, $AC(S) = \frac{C(S)}{|S|}$, is not increasing as coalition increases.*

$x^S_i = AC(S)$ if $i \in S$, $x^S_i = 0$ if $i \notin S$, defines a cross-monotonic set of cost shares that covers the cost exactly and meets the ESP.

It is easy to see that the monotonicity of AC does not imply the concavity of C. Hence, there are ESP cross-monotonic set of cost shares whose associated cost function is not concave.

Finally, notice that a ESP cross-monotonic set of cost shares covers exactly the cost of C if and only if its average cost AC is not increasing.

In general, if the cross-monotonic set of cost shares χ^N does not meet the ESP, then the cost function C such that χ^N covers exactly its cost may not be easy to describe. See Sprumont[23] and Norde et.al.[16] for characterizations of these cost functions.

5 Sequential mechanisms and MIN

Definition 6 *A sequential tree is a binary tree of length n such that:*

i. at every node there is exactly one agent in N and a price in $[0, \infty]$.

ii. Every path from the root to a terminal node contains all agents in N exactly once.

In figure 3 we show sequential trees for the agents in $N = \{1, 2, 3\}$. Every node contains a number and a letter. The number represent the agent in this node. The letter represent a prices in $[0, \infty]$.

Given this sequential tree, consider any path in the tree and a non terminal node ζ in this path. We say ζ is leftist (rightist) on this path if the edge in the path that follows ζ is a left (right) edge.

For instance, in figure 3(a), the path $[1w, 2y, 3c]$ contains one rightist node and one leftist node. $1w$ is rightist and $2y$ is leftist.

One very important path is from a node to the root of the tree. We denote by $P_0(\zeta)$ this path starting at node ζ. For instance, in figure 3(a), $P_0(3c) = [1w, 2y, 3c]$, $P_0(3d) = [1w, 2y, 3d]$ and $P_0(2x) = [1w, 2x]$.

Notice the intersection of two paths is also path. We use $\sqcap$ to denote it. For instance, $P_0(3c) \sqcap P_0(3d) = [1w, 2y]$.

Definition 7 *Let ζ and ζ' two nodes in a sequential tree. We say the node ζ is on the left of ζ' if the terminal node of $P_0(\zeta) \sqcap P_0(\zeta')$ is leftist on $P_0(\zeta)$ and rightist on $P_0(\zeta')$.*

For instance, in figure 3(a), $P_0(3c) = [1w, 2y, 3c]$, $P_0(3d) = [1w, 2y, 3d]$. Since $2y$ is leftist in $[1w, 2y, 3c]$ and rightist in $[1w, 2y, 3d]$, then $3c$ is on the left of $3d$.

Finally, if T is a path and i is an agent in this path, $i \in T$, then x_i^T is the price of i in T.

Definition 8 (Feasible tree) *Consider a sequential tree and any two nodes ζ and ζ' with a common agent k such that ζ is on the left of ζ'. Also, assume every rightist node in $P_0(\zeta)$ or $P_0(\zeta')$ has finite value. Let L the maximal path of $P_0(\zeta)$ that does nos intersect $P_0(\zeta')$, that is $L = P_0(\zeta) \setminus (P_0(\zeta) \sqcap P_0(\zeta'))$. Similarly, let $R = P_0(\zeta') \setminus (P_0(\zeta) \sqcap P_0(\zeta'))$.*

We say a sequential tree is feasible if for any two nodes ζ and ζ' as above, if the price of agent k is such that $x_k^L > x_k^R$, then there is an agent $i \in R \cap L$ such that:

(a) i is leftist in L and rightist in R and $x_i^L < x_i^R$, or

(b) i is rightist in L and leftist in R and $x_i^L \geq x_i^R$.

Notice a sufficient condition to guarantee a feasible sequential tree is that for any two nodes with the same agent, the price of leftist node is not bigger than the price of rightist node. This condition is necessary when there are three agents or less (see examples 4, 5 and 6). Example 7 shows this is not true when there are more than three agents.

Definition 9 (Sequential mechanisms) *Given a feasible sequential tree we construct a sequential mechanisms as follows:*

We offer the agent in the root of the tree a unit of good at the price of his node. If his utility is bigger than the offered price, then we allocate him a unit at this price and go right on the tree. If his utility is smaller than or equal to the offered price then we do not allocate him a unit and go left on the tree. We continue similarly with the following agent until we reach the end of the tree.

Theorem 2 *A mechanism is GSP and MIN if and only if it is sequential.*

Example 4 (Geometric description of sequential mechanisms for $n = 1, 2$) *The one agent mechanisms are easy to describe. For every $x_1 \in [0, \infty]$, agent 1 gets a unit of good at price x_1 if $u_1 > x_1$.*

A two agents mechanism such that 2 has priority over 1, is shown in figure 2. Agent 2 gets a unit of good at price x^2 if $u_2 > x^2$. If 2 gets a unit of good, then agent 1 gets a unit of good at price d^1 if $u_1 > d^1$. On the other hand, if agent 2 did not get a uit of good, then agent 1 gets a unit of good at price d^2 if $u_1 > d^2$. By feasibility of the tree $d^2 \leq d^1$.

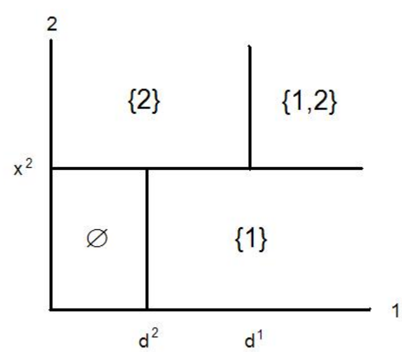

Figure 2: Generic form of 2-agent sequential mechanisms.

Example 5 *Assume there are three agents. Figure 3 shows sequential trees for three agents. Every node contains an agent from $\{1, 2, 3\}$ and a nonegative price.*

On figure 3(a), a feasible sequential tree (assuming finite values) implies: $x \leq y$, $a \leq b$ and $c \leq d$. Also, if $x < y$ then $b \leq c$.

To see this, consider nodes $2x$ and $2y$. Since they are consecutive nodes, their paths to the root of the tree only differ in $2x$ and $2y$ respectively. Then conditions (a) and (b) cannot be satisfied. Hence $x \leq y$.

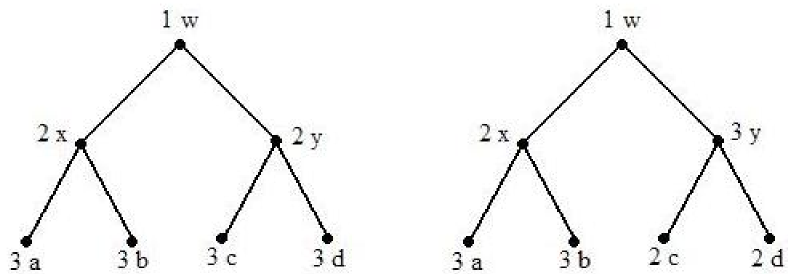

Figure 3: Possible orders for three agents. (a) Agents follow order 1,2,3. (b) Agents 2 and 3 follow different order depending on whether agent 1 is getting or not getting service.

Similarly, $a \leq b$ *and* $c \leq d$ *are satisfied by comparing nodes* $3a$ *and* $3b$*, and* $3c$ *and* $3d$ *respectively.*

Now consider the nodes $3b$ *and* $3c$*. If* $x < y$*, then condition (a) is not satisfied because* $2y$ *is not rightist. Condition (b) is clearly not satisfied. Therefore it cannot be that* $b > c$*. Hence* $x < y$ *and* $a \leq b \leq c \leq d$*.*

Finally, assume $x = y$*. Comparing nodes* $3a$ *and* $3c$*, then conditions (a) and (b) cannot be satisfied. Thus* $a \leq c$*. Similarly, comparing nodes* $3b$ *and* $3d$ *we have that conditions (a) and (b) cannot be satisfied satisfied. Thus* $b \leq d$*. Therefore,* $x = y$*,* $a \leq b \leq d$ *and* $a \leq c \leq d$*.*

If $b \leq c$ *then for every two nodes with same agent, the value on leftist node is smaller than value on rightist node.*

If $b > c$ *then because agents 1 and 2 have priority, we can exchange their order on the tree. This will look like figure 4. With this order, for every two nodes with same agent, the value on leftist node is smaller than value on rightist node.*

Example 6 *Now consider the figure 3(b). Then feasibility of the tree (assuming finite values) requires that* $a \leq b \leq y$ *and* $x \leq c \leq d$*. That is for every two nodes with same agent, the value on leftist node is smaller than value on rightist node.*

To see this, by comparing nodes $3a$ *and* $3b$*, and* $2c$ *and* $2d$*, we get (similarly as example above) that* $a \leq b$ *and* $c \leq d$ *respectively.*

Now we compare nodes $3b$ *and* $3y$*. Then there is no common agent in their path to the root, thus conditions (a) and (b) cannot be satisfied. Hence* $b \leq y$*. That is,* $a \leq b \leq y$*.*

Similarly, by comparing nodes $2x$ *and* $2c$*,* $x \leq c$*. Hence* $x \leq c \leq d$*.*

Example 7 *Consider the mechanism generated by sequential tree of figure 5 (agents are in the rectangles). For every two nodes with same agent, the value on leftist node is not bigger than value on rightist node, except for nodes (4 10)*

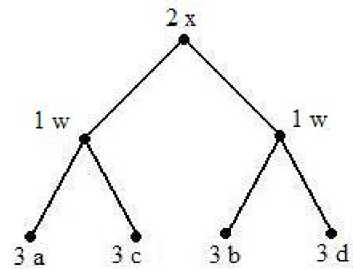

Figure 4: .

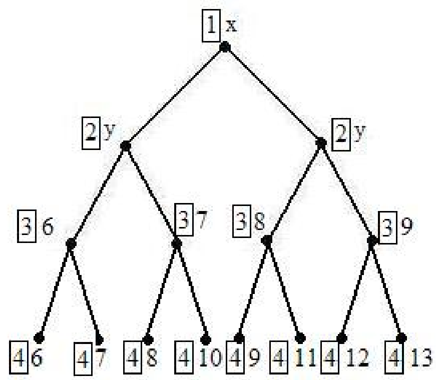

Figure 5: Four agent example such that for every two nodes with same agent, the value of rightist node may not be smaller than value of leftist node.

and (4 9). At these nodes, their paths to the root contain agent 2. This agent meets condition (b). Therefore this tree is feasible.

However, the value on leftist node (4 10) is not smaller than value on leftist node (4 9).

Since agents 1 and 2 have priority, we can also exchange their positions and leave agent agent 2 in the root. If this the case, node (3 8) is on the left of (3 7).

Consider a sequential mechanism and assume agent i^* is in the root of its feasible sequential tree. Consider the leftist (rightist) sequential mechanism for $N \setminus i^*$ agents, generated by the feasible sequential subtree where agent i^* is leftist (rightist). Then, this leftist mechanism should Pareto dominate the rightist mechanism at any profile of $N \setminus i^*$ agents. That is, for any profile of $u_{N \setminus i^*}$ agents, any agent in $N \setminus i^*$ should be better off without agent i^* than with agent i^*. To see this, assume at this profile agent $j \in N \setminus i^*$ is strictly better off with rightist mechanism. Then when the utility of agent i^* equals his offered price, $u_{i^*} = x^*_i$, by MIN we should allocate with leftist mechanism and i^* is not served. Thus agent i^* can help agent j^*. He can increase his utility profile, he will be served at a price equal to his valuation and agent j^* will be better off.

The class of sequential mechanisms resembles the incremental cost mechanism (Moulin[10]). That is, consider a supermodular (convex) cost function and a tree as above. Start with the agent i_1 in the root and offer him a unit of good at price $C(i_1)$. If he buys, continue with the agent i_2 on the right of the tree and offer him a unit of good at price $C(i_1, i_2) - C(i_1)$. If i_1 did not buy, then offer the agent on the left of the tree, k_2, a unit of good at price $C(k_2)$. Proceed similarly with the following agents until you reach the end of the tree.

Theorem 1 in Moulin[10] suggests the incremental cost mechanism capture all mechanism that are budget balanced, VP, NNT and SCS when the cost function C is supermodular. However, this is not true, as shown on next example.

Example 8 *Consider the supermodular cost function:*

$$C(i) = 1, C(1,2) = 3, C(1,3) = 5, C(2,3) = 6, C(1,2,3) = 15.$$

By choosing the ordering $1 \succ 2 \succ 3$, the cost shares are as follows:
$x^{\{1,2,3\}} = (1,2,12), x^{\{1,2\}} = (1,2,0), x^{\{1,3\}} = (1,0,4), x^{\{2,3\}} = (0,1,5), x^{\{i\}} = 1_i$.

When the utility profile is $u = (1, 1.5, 4.5)$ there are two options depending on whether 1 decides to get or not get a unit. If agent 1 gets a unit, then 2 does not get a unit and 3 gets a unit. Thus $\{1,3\}$ gets service and the cost share is $(1,0,4)$. If agent 1 does not get a unit, then 2 get a unit and 3 does not get a unit. Thus $\{2\}$ gets service and the cost share is $(0,1,0)$. Given that 1 is indifferent between getting and not getting a unit, he may help 2 or 3. Thus the mechanism cannot be GSP. The reason is clear by our analysis. The leftist mechanism without agent 1 does not Pareto dominate the rightist mechanism.

What is important from Moulin[10] is that incremental cost mechanism may not be fully GSP, but they are GSP except when agents are indifferent between getting and not getting a unit of good (see Juarez[9]). Thus the mistake is very tiny.

Whenever the supermodular cost function and the ordering of the agents give a sequential game that is GSP, it should be captured by a sequential mechanism discussed above.

On the other hand, given a sequential mechanism, the associated budget balance cost function —the cost of S defined as the sum of the payments on coalition S— may not be supermodular (see example below). So these mechanisms capture even more mechanism that those generated by the incremental cost mechanisms.

Example 9 (Sequential mechanisms that meet ESP**)** *Consider an arbitrary order of the agents, assume without loss of generality that* $1 \succ 2 \succ \cdots \succ n$*, and arbitrary values* $a_1 \geq a_2 \geq \cdots \geq a_n$*. Given this order and numbers, construct the cost function as follows:*

$$C(S) = \mid S \mid \max_{k \in S} a_k.$$

For this cost function, there is a sequential mechanism that covers its cost exactly and meets ESP. To see this, construct a sequential tree following linearly the order $\succ$ *. The price of a node* ζ *is* a_k*, where* k *is the maximal rightist agent (with* $\succ$*) in* $P_0(\zeta)$*.*

In this mechanism, the agents of every coalition that contains agent 1 *pay* a_1*. The agents of every coalition that contains agent* 2 *but not* 1 *pay* a_2*. The agents of every coalition that contains agent* 3 *but not* 1 *or* 2 *pay* a_3*, etc.*

Clearly, this mechanism meets ESP*. This tree is feasible because for every two nodes with the same agent, the price of leftist node is not smaller than the price of rightist node. Thus the mechanism is sequential and covers the cost* C *exactly.*

Similarly, it is easy to check that any sequential mechanism that meets ESP *should be of this form. Hence, the class of cost functions whose cost is covered exactly by* ESP*-sequential mechanism are those described above.*

Finally, notice this cost function may not be supermodular. We can easily find values that meet the next inequality:

$$C(1,3) + C(2,3) = 2a_1 + 2a_2 > 3a_1 + a_3 = C(1,2,3) + C(3).$$

6 Comparison between population-monotonic and sequential mechanisms

6.1 The intersection of population-monotonic and sequential mechanisms

Although the intersection of MAX and MIN is empty by definition, there is a small class of mechanisms that are welfare equivalent to both a sequential and a population-monotonic mechanism.

Definition 10 *Given* $x_1, \ldots, x_n \in [0, \infty]$*, a fixed cost mechanism is implemented by offering to agent* i *a unit of good at price* x_i*. Indifferences are broken arbitrary. That is, for the utility profile* u*, agent* i *is guaranteed a unit if* $u_i > x_i$*. Agent* i *does not get a unit if* $u_i < x_i$*. At* $u_i = x_i$ *he may or may not get a unit.*

Corollary 1 *A mechanism is welfare equivalent to a sequential and a population-monotonic mechanism if and only if it is a fixed cost mechanism.*

This result shows that the behavior of indifferences have a big impact on the class of GSP mechanism. But one can argue that indifferences are rare event, so that a better model is one where the domain of utilities and mechanisms precludes indifferences. On such domain, the class of GSP mechanisms will plow and contain (much) more than the sequential and population-monotonic mechanisms. We analyze such domain in Juarez[8] and characterize the GSP mechanisms in this domain.

6.2 Sequential trees and population-monotonic mechanisms

When there is priority of agents, we may get population-monotonic mechanisms very similar to sequential mechanisms. Indeed, consider any feasible sequential tree. Using this tree, we construct a population-monotonic mechanism as follows:

Implement the mechanism as before, but the direction to go in the tree is the opposite. That is, we offer to the agent in the root a unit of good at the price of his node. If he buys, we go left on the tree (instead of right). If he does not buy, we go right on the tree (instead of left). We continue similarly with next agents until we reach the end of the tree.

Since the payments in a feasible sequential tree are decreasing as coalition increases, then the payments are cross-monotonic when we follow the opposite direction. Therefore this mechanism is population-monotonic.

Thus, we have an injective relation between sequential and population-monotonic mechanisms. Hence a social planner has much more freedom when he chooses a population-monotonic mechanism rather than a sequential mechanism.

6.3 Treatment of equal agents

Definition 11 *We say a mechanism satisfies equal treatment of equals (ETE) if for any u such that $u_i = u_j$, $i \in S(u)$ then $j \in S(u)$ and $\varphi_i(u) = \varphi_j(u)$.*

Proposition 1 *A mechanism meets GSP and ETE if and only if it is welfare equivalent to a population-monotonic mechanism that meets ESP.*

This proposition not only talks in favor of population-monotonic mechanisms as GSP mechanisms meeting this basic equity requirement. It also shows the incompatibility of GSP and fairness for any other mechanism that is not welfare equivalent to a population-monotonic. In particular, it rules out sequential mechanisms and also those GSP mechanisms discussed by Juarez[9] and Roughgarden[22].

6.4 Limited number of goods

When a social planner or seller has (can produce) less than n units of good, it is impossible to meet simultaneously ETE and GSP.[7] This is easy to check by looking at the utility profiles of the form $(x, \ldots, x)$, $x > 0$. By ETE,

[7] Except by the trivial mechanism that does not serve anyone at any profile.

$S(x,\ldots,x)=\emptyset$ for all x. Hence, by proposition 1 above and taking into account that the smallest cost share in a population-monotonic mechanism is achieved when serving N, the mechanism should not allocate any unit at all.

Instead of ETE, another equity requirement is to not exclude ex-ante any agent from the mechanisms. As we see, this cannot be satisfied by population-monotonic mechanisms. That is, if only k units of good are available, $k<n$, then any population-monotonic mechanism is such that $n-k$ agents are not served at any profile. To see this, notice coalition N never gets service, therefore the cost shares of N should have at least one coordinate equal to ∞. Thus the agent i with such coordinate never participates in the game because his smallest payment is achieved when serving N. We remove this agent from the game and proceed similarly with the remaining coalition $N\setminus i$, until we have removed at least $n-k$ agents.

On the other hand, there are many sequential mechanisms that do not ex-ante exclude any agent. If $k\geq 2$, some easy combination of sequential and population-monotonic mechanisms can be constructed.

Definition 12 *Given an arbitrary an arbitrary order of the agents $i_1,\ldots,i_n$ and arbitrary prices (some of them may be infinity) $x^1,x^2,\ldots,x^n$, we define a priority mechanism as follows: Start with agent i_1 and offer him a unit of good at price x^1. If he buys the mechanism stops there. If he does not buy, then continue with agent i_2 and offer him a unit of good at price x^2. Continue similarly until some agent buy or we offered a unit to all agents.*

Notice priority mechanisms are sequential mechanisms for the feasible sequential tree such that agents are ordered linearly following the order $i_1,\ldots,i_n$; only the most leftist branch of the tree has prices equal to $(x^1,x^2,\ldots,x^n)$ and any other node has a price equal to ∞.

Proposition 2 *Suppose a mechanism is GSP and allocates at most one unit of good at any profile, then the mechanism is welfare equivalent to a priority mechanism.*

Notice this proposition is independent of the tie-breaking rule. In particular, it shows that when there is only one unit of good, sequential mechanisms are the only GSP mechanisms that do not exclude ex-ante any agent.

Proofs

Proof of Theorem 1.

Population-monotonic mechanisms meet MAX and GSP.

Population-monotonic mechanisms clearly meet MAX.

We prove by contradiction that these mechanisms meet GSP. Assume coalition $\tilde{S}$ misreports $\tilde{u}_{\tilde{S}}$ when the true profile is u.

Let $\bar{u}_{\tilde{S}} = \max(u_{\tilde{S}}, \tilde{u}_{\tilde{S}})$, where max is taken element by element. By cross-monotonicity, coalition $\tilde{S}$ also misreports $\bar{u}_{\tilde{S}}$ when the true profile is u.

By cross-monotonicity and since there is one agent who strictly increases his net utility, $S(\bar{u}_{\tilde{S}}, u_{-\tilde{S}}) \supset S(u)$. Finally, since $S(\bar{u}_{\tilde{S}}, u_{-\tilde{S}})$ is not feasible at u, $\tilde{u}_i \geq \varphi_i(\bar{u}_{\tilde{S}}, u_{-\tilde{S}}) > u_i$ for some $i \in \tilde{S}$. This is contradiction because agent i is worse off by misreporting.

Any mechanism that is MAX and GSP is population-monotonic.

Let (S, φ) a mechanism that meets these properties. We denote by $f_i(u_{-i})$ the price agent i should pay to get a unit of good when the utilities of the remaining agents is u_{-i}.

Step 0.[Monotonicity] $f_j(\tilde{u}_i, u_{-ij}) \leq f_j(u_i, u_{-ij})$ for all $\tilde{u}_i > u_i$.

Proof.

We prove this by contradiction. Suppose $f_j(\tilde{u}_i, u_{-ij}) > f_j(u_i, u_{-ij})$. Let $\bar{u}_j$ such that $f_j(\tilde{u}_i, u_{-ij}) > \bar{u}_j > f_j(u_i, u_{-ij})$.

Case 1. $f_i(\bar{u}_j, u_{-ij}) > \tilde{u}_i$ or $f_i(\bar{u}_j, u_{-ij}) \leq u_i$

By SP and MAX, at the profiles $(\tilde{u}_i, \bar{u}_j, u_{-ij})$ and $(u_i, \bar{u}_j, u_{-ij})$ agent i is simultaneously served or not served at price $f_i(\bar{u}_j, u_{-ij})$. Hence when the true utility is $(\tilde{u}_i, \bar{u}_j, u_{-ij})$, agent i can help j by misreporting u_i. This contradicts GSP.

Case 2. $u_i < f_i(\bar{u}_j, u_{-ij}) \leq \tilde{u}_i$.

Let $\hat{u}_i = f_i(\bar{u}_j, u_{-ij})$. By SP and MAX, agent i is being served at price $\hat{u}_i$ at the profiles $(\tilde{u}_i, \bar{u}_j, u_{-ij})$ and $(\hat{u}_i, \bar{u}_j, u_{-ij})$. Thus, by GSP, $f_j(\hat{u}_i, u_{-ij}) \geq \bar{u}_j$. To see this, assume $f_j(\hat{u}_i, u_{-ij}) < \bar{u}_j$. Then, when the true profile is $(\tilde{u}_i, \bar{u}_j, u_{-ij})$, agent i helps j by misreporting $\hat{u}_i$. This contradicts GSP.

Hence, at the true profile $(\hat{u}_i, \bar{u}_j, u_{-ij})$, agents i and j get zero net utility because $f_j(\hat{u}_i, u_{-ij}) \geq \bar{u}_j$ and $\hat{u}_i = f_i(\bar{u}_j, u_{-ij})$. Thus agent i helps j by reporting u_i. This contradicts GSP.

Step 1. If $S(u) = S^*$ and $\varphi(u) = \varphi^*$ then for all $\tilde{u}$ such that $\tilde{u}_{[S^*]} \geq \varphi_{[S^*]}$ and $\tilde{u}_{[N \setminus S^*]} \leq u_{[N \setminus S^*]}$, $S(\tilde{u}) = S^*$ and $\varphi(\tilde{u}) = \varphi^*$.

Proof.

Let $i \in S^*$. Since $\tilde{u}_i \geq f_i(u_{-i}) = \varphi_i^*$, then by SP and MAX, $i \in S(\tilde{u}_i, u_{-i})$ and $\varphi_i(\tilde{u}_i, u_{-i}) = \varphi_i^*$.

Let j, $j \neq i$. Then $NU_j(\tilde{u}_i, u_{-i}) = NU_j(u)$. To see this, if $NU_j(\tilde{u}_i, u_{-i}) > NU_j(u)$, then when the true profile is u, agent i helps j by reporting $\tilde{u}_i$. This contradicts *GSP*. The case $NU_j(\tilde{u}_i, u_{-i}) < NU_j(u)$ is analogous. Thus $NU_j(\tilde{u}_i, u_{-i}) = NU_j(u)$ for all $j \neq i$.

Therefore, if $j \in S^*$ and $NU_j(\tilde{u}_i, u_{-i}) = NU_j(u) > 0$ then $j \in S(\tilde{u}_i, u_{-i})$ and $\varphi_j(\tilde{u}_i, u_{-i}) = \varphi_j^*$.

Assume $j \in S^*$ and $NU_j(\tilde{u}_i, u_{-i}) = NU_j(u) = 0$. Moreover, to get a contradiction, assume $j \notin S(\tilde{u}_i, u_{-i})$. Thus, $f_j(\tilde{u}_i, u_{-ij}) > u_j = f_j(u_{-j})$. By step 0, $u_i > \tilde{u}_i \geq \varphi_i^*$. Let $\bar{u}_j$ such that $\bar{u}_j > u_j$. By step 0,

$$f_i(\bar{u}_j, u_{-ij}) \leq f_i(u_j, u_{-ij}) \leq \tilde{u}_i < u_i.$$

Hence, when the true profile is $(\bar{u}_j, \tilde{u}_i, u_{-ij})$, agent i can help j by misreporting u_i : Agent i is served in both profiles at price $f_i(\bar{u}_j, u_{-ij})$, however agent j is offered a unit at a cheaper price $f_j(u_{-j})$. This contradicts GSP. Hence $j \in S(\tilde{u}_i, u_{-i})$ and $\varphi_j(\tilde{u}_i, u_{-i}) = \varphi_j^*$.

Similarly as above, assume to get a contradiction that $j \notin S^*$ but $j \in S(\tilde{u}_i, u_{-i})$. Thus, $f_j(\tilde{u}_i, u_{-ij}) = u_j > f_j(u_{-j})$. So, we are in exactly in the previous case but changing the role of $\tilde{u}_i$ and u_i. Therefore this cannot occur, thus $j \notin S(\tilde{u}_i, u_{-i})$

Hence $S(\tilde{u}_i, u_{-i}) = S^*$ and $\varphi(\tilde{u}_i, u_{-i}) = \varphi^*$. By changing one agent at a time, we have that $S(\tilde{u}_{S^*}, u_{-S^*}) = S^*$ and $\varphi(\tilde{u}_{S^*}, u_{-S^*}) = \varphi^*$.

Let $j \notin S^*$. By SP and MAX, $\tilde{u}_j \leq u_j < f_j(\tilde{u}_{S^*}, u_{-S^* \cup j})$, hence $j \notin S(\tilde{u}_j, \tilde{u}_{S^*}, u_{-S^* \cup j})$. Similarly as above, by GSP $NU_k(\tilde{u}_j, \tilde{u}_{S^*}, u_{-(S^* \cup j)}) = NU_k(\tilde{u}_{S^*}, u_{-S^*})$ for all $k \neq j$.

By step 0, $f_k(\tilde{u}_j, \tilde{u}_{S^* \setminus k}, u_{-S^* \cup j}) \geq f_k(u_j, \tilde{u}_{S^* \setminus k}, u_{-S^* \cup j})$ for all $k \neq j$. Clearly, if $NU_k(\tilde{u}_{S^*}, u_{-S^*}) > 0$, then $f_k(\tilde{u}_j, \tilde{u}_{S^* \setminus k}, u_{-(S^* \cup j)}) = f_k(u_j, \tilde{u}_{S^* \setminus k}, u_{-(S^* \cup j)})$.

Assume, $k \in S^*$ and $NU_k(\tilde{u}_j, \tilde{u}_{S^*}, u_{-S^* \cup j}) = NU_k(\tilde{u}_{S^*}, u_{-S^*}) = 0$. Also, assume to get a contradiction that $k \notin S(\tilde{u}_j, \tilde{u}_{S^*}, u_{-S^* \cup j})$. Then, $f_k(\tilde{u}_j, \tilde{u}_{S^* \setminus k}, u_{-S^* \cup j}) > u_k = f_k(u_j, \tilde{u}_{S^* \setminus k}, u_{-S^* \cup j})$.

Let $\bar{u}_k$ such that $\bar{u}_k > u_k$. By monotonicity

$$f_j(\bar{u}_k, \tilde{u}_{S^* \setminus k}, u_{-S^* \cup j}) \leq f_j(u_k, \tilde{u}_{S^* \setminus k}, u_{-S^* \cup j}).$$

First assume $f_j(\bar{u}_k, \tilde{u}_{S^* \setminus k}, u_{-(S^* \cup j)}) > u_j > \tilde{u}_j$. Then when the true profile is $(\bar{u}_k, \tilde{u}_j, \tilde{u}_{S^* \setminus k}, u_{-(S^* \cup j)})$, agent j can help agent k by misreporting u_j : Agent j does not get a unit in both profiles, however agent k gets a unit at a cheaper price $f_k(u_j, \tilde{u}_{S^* \setminus k}, u_{-S^* \cup j})$. This contradicts *GSP*.

Second, assume $f_j(\bar{u}_k, \tilde{u}_{S^* \setminus k}, u_{-(S^* \cup j)}) \leq u_j$. Let $\bar{u}_j$ such that $\bar{u}_j > u_j$. By step 1,

$$f_k(\bar{u}_j, \tilde{u}_{S^* \setminus k}, u_{-S^* \cup j}) \leq f_k(u_j, \tilde{u}_{S^* \setminus k}, u_{-S^* \cup j}) = u_k < \bar{u}_k.$$

Thus when true profile is $(u_k, \bar{u}_j, \tilde{u}_{S^* \setminus k}, u_{-S^* \cup j})$, agent k helps j by misreporting $\bar{u}_k$: In both profiles agent k is served at a price $f_k(\bar{u}_j, \tilde{u}_{S^* \setminus k}, u_{-S^* \cup j})$. On the other hand, agent j is served at a cheaper price $f_j(\bar{u}_k, \tilde{u}_{S^* \setminus k}, u_{-S^* \cup j})$. This contradicts GSP.

Hence, if $k \in S^*$ then $k \in S(\tilde{u}_{S^* \cup j}, u_{-S^* \cup j})$ and $\varphi_k(\tilde{u}_{S^* \cup j}, u_{-(S^* \cup j)}) = \varphi_k(\tilde{u}_{S^*}, u_{-S^*})$.

Finally, notice the case $k \notin S^*$, $NU_k(\tilde{u}_j, \tilde{u}_{S^*}, u_{-(S^* \cup j)}) = NU_k(\tilde{u}_{S^*}, u_{-S^*}) = 0$, but $k \in S(\tilde{u}_{S^* \cup j}, u_{-(S^* \cup j)})$ cannot occur. This is analogous to previous case by exchanging the role of $\tilde{u}_j$ and u_j.

By changing one agent at a time, $S(\tilde{u}) = S^*$ and $\varphi(\tilde{u}) = \varphi^*$.

Step 2. If $S(u) = S(\tilde{u})$ then $\varphi(u) = \varphi(\tilde{u})$.

Proof.

Let $S^* = S(u) = S(\tilde{u})$, $\bar{u}_{[S]} = \max(\tilde{u}_{[S]}, u_{[S]})$ and $\bar{u}_{[N \setminus S]} = \min(\tilde{u}_{[N \setminus S]}, u_{[N \setminus S]})$ (where max and min are taken element by element).

By step 1, comparing $\bar{u}$ and u, $S(\bar{u}) = S^*$ and $\varphi(\bar{u}) = \varphi(u)$. Similarly, comparing $\bar{u}$ and $\tilde{u}$, $\varphi(\bar{u}) = \varphi(\tilde{u})$.

Step 3.

In this final step we prove the theorem by induction on the number of agents. The base of induction is the case $n = 1$. The mechanisms are easy to construct. Given $x \in [0, \infty]$, if $u_1 \geq x$ then $(S, \varphi)(u_1) = (1, x)$. On the other hand, if $u_1 < x$ then $(S, \varphi)(u_1) = (\emptyset, 0)$. These mechanisms are clearly population-monotonic.

For the induction hypothesis, assume that any GSP and MAX mechanism for k agents, $k < n$, is population-monotonic. We prove this for a mechanism (S, φ) defined for the agents $N = \{1, \ldots, n\}$.

First, assume there is a utility profile u such that $S(u) = N$. Let $x^N = \varphi(u)$. Then, by step 1, for all $\tilde{u} \geq x^N$ $S(\tilde{u}) = N$ and $\varphi(\tilde{u}) = x^N$.

For every agent $j \in N$, consider the set of utility profiles such that $u_j = 0$, that is

$$U^j = \{u \in \mathbb{R}^N_+ \mid u_j = 0\}.$$

By induction, there is a population-monotonic mechanism for $N \setminus j$ agents defined on U^j. Let ρ^i the cross-monotonic set of cost shares in U^j (where we define $x^{N_1} = x^N$ in case $x_i^N = 0$).

Notice that if S^* is such that $i, j \notin S^*$, then for $x^{S^*} \in U^j$ and $y^{S^*} \in U^j$ it should be that $x^{S^*} = y^{S^*}$. To see this, assume $S(u) = S^*$. By step 1, $S(u_{-ij}, 0, 0) = S^*$. Since $(u_{-ij}, 0, 0) \in U^j \cap U^i$ then by step 2 $x^{S^*} = y^{S^*}$.

Finally, notice that the set of cost shares defined this way is cross-monotonic. To see this, by induction we just need to check that $x_i^N \leq x_i^{S^*}$ for all i and S^* such that $i \in S^*$. Assume this is not true, so $i \in S(u) = S^*$ and $\varphi_i(u) < x_i^N$ for some u. By step 1, $S(\varphi_i(u), u_{-i}) = S^*$. By step 0, $i \in S(\varphi_i(u), \tilde{u}_{-i})$ where $\tilde{u}_{-i} = \max(x_{-i}^N, u_{-i})$. This is a contradiction because $\varphi_i(u) < x_i^N = f_i(\tilde{u}_{-i})$.

From above, we have the mechanism (S, φ) satisfies:

- If $u \geq x^N$ then $S(u) = N$.
- If $u_i < x_i^N$ then $i \notin S(u)$. Thus $S(u) = S(0, u_{-i})$.

These two properties, the induction hypothesis and cross-monotonicity of the cost-shares clearly implies the mechanism is population-monotonic.

Now assume there is no u such that $S(u) = N$. Then there is $j \in N$ such that $j \notin S(\tilde{u})$ for all $\tilde{u}$. We prove this by contradiction. Assume for any j there is u^j such that $j \in S(u^j)$. Let $\bar{u} = max(u^1, \ldots, u^n)$. By step 0, at $\bar{u}$ every agent j is offered a unit of good at price not bigger than u^j_j, thus $j \in S(\bar{u})$ for all $j \in N$. This is a contradiction.

Since there is an agent who is not serviced at any profile, say agent j^*, then $S(u) = S(u_{-j^*}, 0)$ for all u. Hence by induction the mechanism is population-monotonic. Notice in this case the cross-monotonic set of cost shares are such that:

Proof of Theorem 2.

Sequential mechanisms meet MIN and GSP.

Sequential mechanisms trivially meet MIN.

We prove by contradiction these mechanisms meet GSP. Assume coalition $\tilde{S}$ misreports $\tilde{u}_{\tilde{S}}$ at the true profile u. Let $k \in \tilde{S}$ the agent that strictly increases his net utility by misreporting. Let ζ and ζ' in the path that generate $S(u)$ and $S(\tilde{u}_{\tilde{S}}, u_{-\tilde{S}})$ respectively such that they contain agent k.

First notice ζ is on the left of ζ'. To see this, let i^* the terminal node in $P_0(\zeta) \cap P_0(\zeta')$. Then, in order to move from $P_0(\zeta)$ to $P_0(\zeta')$, agent i^* misreports. If i^* is rightist in $P_0(\zeta)$ then by MIN his net utility is positive, so he will never agree to move to $P_0(\zeta')$ because he is not served there.

Let L and R as in definition 9. Since agent k strictly increases his net utility, then $x^L_k > x^R_k$. Assume condition (a) of feasibility is satisfied. That is, there is $i \in L \cap R$ such that i is leftist in L and rightist in R and $x^L_i < x^R_i$. Since i is leftist in L then $u_i \leq x^L_i < x^R_i$. Then $i \notin S^*$ and $P_0(\zeta')$ does not realize with $(\tilde{u}_{\tilde{S}}, u_{-\tilde{S}})$ because i is rightist at $P_0(\zeta')$ and $u_i < x^R_i$. This is a contradiction.

On the other hand, assume condition (b) of feasibility is satisfied. That is, there is $i \in L \cap R$ such that i is rightist in L and leftist in R and $x^L_i \geq x^R_i$. Since i is rightist in L then $u_i > x^L_i \geq x^R_i$. Then agent i cannot be leftist in R. This is a contradiction.

Any GSP and MIN mechanism is sequential.

Step 1. If $S(u) = S^*$ and $\varphi(u) = \varphi^*$ then for all $\tilde{u}$ such that $\tilde{u}_{[S^*]} >> \varphi_{[S^*]}$ and $\tilde{u}_{[N \setminus S^*]} \leq u_{[N \setminus S^*]}$, $S(\tilde{u}) = S^*$ and $\varphi(\tilde{u}) = \varphi^*$.

Proof.

First notice that by MIN, an agent gets positive net utility if and only if he is served.

Let $i \in S^*$. Then $S(\tilde{u}_i, u_{-i}) = S^*$ and $\varphi(\tilde{u}_i, u_{-i}) = \varphi^*$. To see this, if $i \notin S(\tilde{u}_i, u_{-i})$ or $\varphi(\tilde{u}_i, u_{-i}) > \varphi^*$, then agent i misreports u_i when the true profile is $(\tilde{u}_i, u_{-i})$, which contradicts SP. On the other hand, if $i \in S(\tilde{u}_i, u_{-i})$ and $\varphi(\tilde{u}_i, u_{-i}) < \varphi^*$, then agent i misreports $\tilde{u}_i$ when the true profile is u, which also contradicts SP. Therefore, $i \in S(\tilde{u}_i, u_{-i})$ and $\varphi_i(\tilde{u}_i, u_{-i}) = \varphi^*_i$.

Let j, $j \neq i$. If $NU_j(\tilde{u}_i, u_{-i}) > NU_j(u)$, then agent i helps j by reporting $\tilde{u}_i$ when the true profile is u. This contradicts GSP. The case $NU_j(\tilde{u}_i, u_{-i}) < NU_j(u)$ is analogous. Thus $NU_j(\tilde{u}_i, u_{-i}) = NU_j(u)$ for all $j \neq i$. Therefore, by MIN, $S(\tilde{u}_i, u_{-i}) = S^*$ and $\varphi(\tilde{u}_i, u_{-i}) = \varphi^*$.

By changing one agent at a time, we have that $S(\tilde{u}_{S^*}, u_{-S^*}) = S^*$ and $\varphi(\tilde{u}_{S^*}, u_{-S^*}) = \varphi^*$.

Let $j \notin S^*$. Then $S(\tilde{u}_{S^* \cup j}, u_{-S^* \cup j}) = S^*$ and $\varphi(\tilde{u}_{S^* \cup j}, u_{-S^* \cup j}) = \varphi^*$. First notice that $j \notin S(\tilde{u}_{S^* \cup j}, u_{-S^* \cup j})$, otherwise by voluntary participation

$$\varphi_j(\tilde{u}_{S^* \cup j}, u_{-S^* \cup j}) < \tilde{u}_j \leq u_j.$$

Thus agent j misreports $\tilde{u}_j$ when true profile is $(\tilde{u}_{S^*}, u_{-S^*})$. This contradicts SP.

On the other hand, if $NU_k(\tilde{u}_{S^* \cup j}, u_{-(S^* \cup j)}) < NU_k(\tilde{u}_{S^*}, u_{-S^*})$ for some $k \neq j$, then agent j helps k by reporting $\tilde{u}_j$ when true profile is $(\tilde{u}_{S^*}, u_{-S^*})$, which contradicts GSP. Similarly by GSP $NU_k(\tilde{u}_{S^* \cup j}, u_{-(S^* \cup j)}) > NU_k(\tilde{u}_{S^*}, u_{-S^*})$ cannot occur. Thus $NU_k(\tilde{u}_{S^* \cup j}, u_{-(S^* \cup j)}) = NU_k(\tilde{u}_{S^*}, u_{-S^*})$ for all $k \neq j$. Hence, by MIN, $S(\tilde{u}_{S^* \cup j}, u_{-(S^* \cup j)}) = S^*$ and $\varphi(\tilde{u}_{S^* \cup j}, u_{-S^* \cup j}) = \varphi^*$.

By changing one agent at a time, $S(\tilde{u}) = S^*$ and $\varphi(\tilde{u}) = \varphi^*$.

Step 2. If $S(u) = S(\tilde{u})$ then $\varphi(u) = \varphi(\tilde{u})$.

Proof.

Let $S^* = S(u) = S(\tilde{u})$, $\bar{u}_{[S]} = \max(\tilde{u}_{[S]}, u_{[S]})$ and $\bar{u}_{[N \setminus S]} = \min(\tilde{u}_{[N \setminus S]}, u_{[N \setminus S]})$ (where max and min are taken element by element).

By step 1, comparing $\bar{u}$ and u, $S(\bar{u}) = S^*$ and $\varphi(\bar{u}) = \varphi(u)$. Similarly, comparing $\bar{u}$ and $\tilde{u}$, $\varphi(\bar{u}) = \varphi(\tilde{u})$.

By step 2, there exist at most one vector of payments for every coalitions. Let x^{S^*} the payment of coalition S^* when S^* is served at some profile.

Step 3. Let u such that $S(u) = S^*$ and $\varphi(u) = \varphi^*$. Then for every $i \in S^*$ and $u_i^* \leq \varphi_i^*$, $S^* \backslash i \subseteq S(u_i^*, u_{-i})$ and $\varphi_{S^* \backslash i}(u_i^*, u_{-i}) \leq \varphi^*_{S^* \backslash i}$.

Let $j \in S^*$, then $j \in S(\varphi_i^*, u_{-i})$ and $\varphi_j(\varphi_i^*, u_{-i}) \leq \varphi_j^*$. Indeed, by MIN the net utility of agent j with u is positive. If $j \notin S(\varphi_i^*, u_{-i})$ or $\varphi_j(\varphi_i^*, u_{-i}) > \varphi^*$ then agent i can help j when true profile is (φ_i^*, u_{-i}). He is indifferent between misreporting u_i and geting a unit at price φ_i^*, or truly reporting φ_i^* and not getting a unit.

Finally, by MIN $i \not\in S(\varphi_i^*, u_{-i})$. Thus, by step 1, $S(u_i^*, u_{-i}) = S(\varphi_i^*, u_{-i})$ and $\varphi(u_i^*, u_{-i}) = \varphi(\varphi_i^*, u_{-i})$ for all $u_i^* \leq \varphi_i^*$.

In particular, notice step 3 implies that if $S(u) = S^*$, then for any $T, T \subset S^*$, there exist $\tilde{u}$ such that $S(\tilde{u}) = T$ and $x^T_{[T]} \leq x^{S^*}_{[T]}$.

To see this, let $u = x^{S^*} + \epsilon 1_{[S^*]}$, $\epsilon > 0$. By step 1, $S(u) = S^*$. Let $i \in S^*$. By MIN $i \notin S(x_i^{S^*}, u_{-i})$. By step 3, $S^* \setminus i \subset S(x_i^{S^*}, u_{-i})$. Since utilities of agents outside S^* are zero, then by MIN $S^* \setminus i = S(x_i^{S^*}, u_{-i})$. Thus by step 3, $x^{S \backslash i^*}_{[S \backslash i^*]} \leq x^S_{[S \backslash i^*]}$.

Step 4. Assume there is u such that $S(u) = N$. Then there is an agent who has priority.

We prove this by induction in the size of N.

If $N = \{1\}$ then the mechanism are the fixed cost mechanisms. That is, there is a fixed value x, $x \in [0, \infty]$ such that If $u_1 > x$ then 1 is served at price x. If $u_1 \leq x$ then he is not served.

Assume there is an agent who has priority for every mechanism of at most $n-1$ agents. Let (S, φ) a mechanism for the agents in $N = \{1, \ldots, n\}$.

For every j, consider the restriction of the mechanism to the profiles

$$U^j = \{u \mid u_j = 0\}.$$

By MIN, agent j is not getting a unit of good at any profile of U^j. Thus, this defines a GSP mechanism for the agents in $N \setminus j$. Let $\rho^j = \{x^S \mid j \notin S\}$ the set of payments in this mechanism. Notice because N is being served, then by step 3 every coalition $S \subset N$ is being served. In particular ρ^j contains a payment for every coalition that does not contain agent j.

By induction hypothesis, on ρ^1 there is an agent i_1 who has priority. That is, $x_{i_1}^{N \setminus 1} = x_{i_1}^{i_1}$. Similarly, for agent i_1 there is an agent who has priority on ρ^{i_1}. Call this agent i_2, thus $x_{i_2}^{N \setminus i_1} = x_{i_2}^{i_2}$. We continue this procedure until we reach a cycle. Without loss of generality, we assume the cycle is $i_1, i_2, \ldots, i_k$. This means i_{j+1} has priority on ρ^j for $j = 1, \ldots, k-1$, and i_k has priority on ρ^1.

Case 1. The cycle has size less than n, that is $k < n$.

Let $\bar{u}_{[N \setminus \{i_1, i_2, \ldots, i_k\}]}$ such that $\bar{u}_{[N \setminus \{i_1, i_2, \ldots, i_k\}]} >> x^N_{[N \setminus \{i_1, i_2, \ldots, i_k\}]}$.
Consider the profiles

$$U = \{u \mid u_{[N \setminus \{i_1, i_2, \ldots, i_k\}]} = \bar{u}_{[N \setminus \{i_1, i_2, \ldots, i_k\}]}\}.$$

Notice that for every $u \in U$, $N \setminus \{i_1, i_2, \ldots, i_k\} \subset S(u)$. Indeed, consider $(\tilde{u}_{\{i_1, i_2, \ldots, i_k\}}, u_{-\{i_1, i_2, \ldots, i_k\}})$, where $\tilde{u}_{\{i_1, i_2, \ldots, i_k\}} \geq x^N_{[i_1, i_2, \ldots, i_k]}$. By step 1, $S(\tilde{u}_{\{i_1, i_2, \ldots, i_k\}}, u_{-\{i_1, i_2, \ldots, i_k\}}) = N$. By steps 1 and 3, $N \setminus i_1 \subset S(u_{i_1}, \tilde{u}_{\{i_2, \ldots, i_k\}}, u_{-\{i_1, i_2, \ldots, i_k\}})$. Similarly, $N \setminus i_1, i_2 \subset S(u_{i_1, i_2}, \tilde{u}_{\{i_3, \ldots, i_k\}}, u_{-\{i_1, i_2, \ldots, i_k\}})$. Continuing this way, $N \setminus i_1, i_2, \ldots, i_k \subset S(u_{\{i_1, i_2, \ldots, i_k\}}, u_{-\{i_1, i_2, \ldots, i_k\}})$.

By step 3, for every coalition T such that $N \setminus \{i_1, i_2, \ldots, i_k\} \subset T$, there is $\tilde{u} \in U$ such that $S(\tilde{u}) = T$. This is clear because coalition N is being served at some profile of U, so we can reduce (one step at a time) the utility of the agents not in T to zero.

Clearly, the mechanism restricted to U defines a GSP mechanism for the agents in $\{i_1, i_2, \ldots, i_k\}$. By the induction hypothesis, there is an element who has priority, say i_1. Thus, $x_{i_1}^{N \setminus \{i_2, \ldots, i_k\}} = x_{i_1}^N$. Therefore, $x_{i_1}^N = x_{i_1}^{i_1}$. Hence agent i_1 has priority over the other agents in N.

Case 2. The cycle has size n, that is $k = n$.

Without loss of generality, assume agent 2 has priority over $N \setminus 1$, agent 3 has priority over $N \setminus 2, \ldots,$ etc. Thus,

$$x_2^2 = x_2^{N\setminus 1}, \ldots x_3^3 = x_3^{N\setminus 2}, \ldots x_1^1 = x_1^{N\setminus n}. \tag{1}$$

Also, assume to get a contradiction that there is no agent who has priority. That is,

$$x_2^{N\setminus 1} < x_2^N, x_3^{N\setminus 2} < x_3^N, \ldots, x_1^{N\setminus n} < x_1^N.$$

Let u such that $S(u) = N$. By MIN, $u >> x^N$.

By step 3, $2 \in S(x_1^N, u_{-1})$ and 2 pays x_2^2, $x_2^2 < x_N^2$. Also by step 3, $2 \in S(x_{1,3}^N, u_{-1,3})$ and 2 pays not more than x_2^2. Continuing similarly, $2 \in S(x_{-2}^N, u_2)$ and 2 pays not more than x_2^2. By step 1, $2 \in S(x^N)$ because $u_2 > x_2^N > x_2^2$.

Finally, since everything is symmetric, $S(x^N) = N$. This is a contradiction to MIN.

Step 5. Assume there is no u such that $S(u) = N$. If the mechanism is not trivial ($S(u) \neq \emptyset$ for some u), there is an agent who has finite priority. That is, there is an agent i^* and a payment x^*, $0 \leq x^* < \infty$ such that $i^* \in S(u)$ for all u, $u_{i^*} > x^*$.

First notice there is a group of agents S^* who has priority. That is, for all $\tilde{u}$ such that $\tilde{u}_{[S^*]} \geq x_{[S^*]}^{S^*}$, $S(\tilde{u}) = S^*$. To see this, consider $\tilde{u}$ such that $\tilde{u} >> x^T$ for all T. Let S^* such that $S(\tilde{u}) = S^*$. First notice that, for any i, $i \notin S^*$, $S(\tilde{u}_{-i}, \bar{u}_i) = S^*$ for all $\bar{u}_i$. To see this, if $\bar{u}_i \leq \tilde{u}_i$ then by step 1 $i \notin S(\tilde{u}_{-i}, \bar{u}_i)$. On the other hand, if $\bar{u}_i > \tilde{u}_i$, then $i \notin S(\tilde{u}_{-i}, \bar{u}_i)$. This is easy to see by contradiction, assume $i \in S(\tilde{u}_{-i}, \bar{u}_i)$, then by the choice of $\tilde{u}$, $\varphi_i(\tilde{u}_{-i}, \bar{u}_i) < \tilde{u}_i < \bar{u}_i$. Therefore, by step 1, $i \in S(\tilde{u})$, which is a contradiction.

By step 1, $S(\tilde{u}_{-i}, \bar{u}_i) = S^*$ for all $\bar{u}_i$. Thus, by changing the utilities of the agents in $N \setminus S^*$ one at a time, $S(\tilde{u}_{S^*}, u_{-S^*}) = S^*$ for all $\tilde{u}_{S^*} \geq x_{[S^*]}^{S^*}$.

We now prove step 5 by induction. For $n = 1$, if $S(u) \neq 1$ for all u then clearly the mechanism is trivial ($S(u) = \emptyset$ for all u). So the claim is true.

For the induction hypothesis, assume the claim is true for any mechanism of $n - 1$ agents. Furthermore, assume the mechanism is not trivial.

Let S^* defined as above and $j \notin S^*$. Consider the restriction of the mechanism to $U^j = \{u \mid u_j = 0\}$. Then this restriction is GSP for the agents in $N \setminus j$. By induction and step 4, there is an agent i^* who has (finite) priority for the agents $N \setminus j$. Clearly $i^* \in S^*$, otherwise his payment is dependent on other agents.

We now prove by contradiction that for any profile u, i^* has priority. Assume there is u such that $f_{i^*}(u_{-i^*}) \neq x_{i^*}^{S^*}$. Let $u_{i^*} > x_{i^*}^{S^*}$, then $j \in S(u)$. Otherwise, by step 1 $S(u) = S(0, u_{-j})$. Thus i^* is served at price $x_{i^*}^{S^*}$, which is a contradiction to GSP. Hence $j \in S(u)$. By step 3, $f_{i^*}(u_{-i^*}) > x_{i^*}^{S^*}$.

Let $k \in S^*$ and $\tilde{u}_k > max(u_k, x_k^{S^*})$. If $k \in S(u)$, then $S(\tilde{u}_k, u_{-k}) = S(u)$, thus $f_{i^*}(\tilde{u}_k, u_{-k,i^*}) = f_{i^*}(u_k, u_{-k,i^*}) > x_{i^*}^{S^*}$. If $k \notin S(u)$ and $k \notin S(\tilde{u}_k, u_{-k})$, then by GSP $S(\tilde{u}_k, u_{-k}) = S(u)$, thus $f_{i^*}(\tilde{u}_k, u_{-k,i^*}) = f_{i^*}(u_k, u_{-k,i^*}) > x_{i^*}^{S^*}$.

Finally, if $k \notin S(u)$ and $k \in S(\tilde{u}_k, u_{-k})$, then by step 3 $f_{i^*}(\tilde{u}_k, u_{-i^*,k}) \geq f_{i^*}(u_k, u_{-i^*,k}) > x_{i^*}^{S^*}$.

By increasing step by step every agent in S^* we conclude that $f_{i^*}(u_{-S^*}, \tilde{u}_{S^*}) > x_{i^*}^{S^*}$ for some $\tilde{u}_{S^*} \geq x_{[S^*]}^{S^*}$. This contradicts the priority of coalition S^*.

Step 6. The mechanism is implemented by a feasible sequential tree.

Proof.

By steps 4 and 5 and MIN, the mechanism is implemented by a sequential tree as in definition 10. We just have to prove the sequential tree is feasible.

Assume it is not feasible, let ζ and ζ' two achievable nodes that contain the same agent k as in definition 9. Furthermore, to get a contradiction assume $x_k^L > x_k^R$ and for every common agent $i \in R \cap L$ one of the next conditions hold:

1. i is leftist in L and rightist in R and $x_i^L \geq x_i^R$.
2. i is rightist in L and leftist in R and $x_i^L < x_i^R$.
3. i is leftist in L and R.
4. i is rightist in L and R.

Let i^* the agent in the terminal node of $P_0(\zeta) \cap P_0(\zeta')$. We choose u such that:

- u_{i^*} equal the value of his node.
- u_k such that $x_k^L > u_k > x_k^R$
- $u_i = x_i^R$ if condition 1 holds.
- $u_i = \frac{x_i^L + x_i^R}{2}$.
- $u_i = 0$ if condition 3 holds.
- u_i such that $u_i > \max(x_i^L, x_i^R)$.
- If j is rightist node in $(P_0(\zeta) \cup P_0(\zeta')) \setminus (L \cap R)$ then u_j is bigger than the price of its node.
- If j is leftist node in $(P_0(\zeta) \cup P_0(\zeta')) \setminus (L \cap R)$ then $u_j = 0$.
- Any other agent has zero utility.

First notice the profile u realizes the path $P_0(\zeta)$. Indeed, an agent $j \in (P_0(\zeta) \cup P_0(\zeta')) \setminus (L \cap R)$ is served if rightist and not served of leftist. Agent i^* is not served because his valuation equals the offered price. If $i \in L \cap R$ is leftist in L, then he is not being served because $x_i^L \geq x_i^R = u_i$. On the other hand, if $i \in L \cap R$ is rightist in R, then he is being served because $u_i = \frac{x_i^L + x_i^R}{2} > x_i^L$.

Let T the common agents of condition 1 and $S = T \cup \{i^*, k\}$. We now check that when the true profile is u, coalition S can misreport. First notice all agents in S are not being served at u, so they get zero net utility.

Let $\tilde{u}_S$ such that:

- $\tilde{u}_i > u_i$ if $i \in T \cup \{i^*\}$.
- $\tilde{u}_k = u_k$

Then at the profile $(\tilde{u}_S, u_{-S})$ the path $P_0(\zeta')$ realizes. Indeed, an agent $j \in (P_0(\zeta) \cup P_0(\zeta')) \setminus (L \cap R)$ is served if rightist and not served of leftist. If $i \in L \cap R$ is leftist in R, then by condition 2 $u_i = \frac{x_i^R + x_i^L}{2} < x_i^R$, so he is not served. On the other hand, i^* is served at a price equal to his true valuation u_i, thus his net utility is zero. If $i \in T$, that is $i \in L \cap R$ is rightist in R, then he is being served at a price equal to his valuation because $\tilde{u}_i > u_i = x_i^R$, thus his net utility is zero. Finally, agent k is being served at a price x_k^R, $u_k > x_k^R$. Hence his net utility is positive.

Proof of Corollary 1.

If the mechanism meets GSP and MIN (MAX), then for every agent i his payment does not decrease (increase) when coalition increases.

Therefore, in order to have a common point at every coalition, it must be that $x_i^N = x_i^i$ for all i. Hence, the cost share of agent i is fixed.

Proof of Proposition 1.

By ETE, $S(x \cdot 1_N)$ serve N or $\emptyset$.

First notice that $S(x \cdot 1_N) = \emptyset$ for all $x > 0$ implies the mechanism is welfare equivalent to the trivial mechanism where no agent is served at any profile. To see this, if $NU_k(u) > 0$ for some k at some profile u, then when the true profile is $u_k \cdot 1_N$ agents in N misreport u. This contradicts GSP.

On the other hand, assume $S(x \cdot 1_N) = N$ for some $x > 0$ and $\varphi_i(x \cdot 1_N) = \varphi^*$ for all i. Notice we can assume w.l.g. that $x > \varphi^*$. Indeed, assume $x = \varphi^*$. Consider $\tilde{x}$ such that $\tilde{x} > x$. By GSP and ESP, $S(\tilde{x} \cdot 1_N) = N$ and $\varphi_i(\tilde{x} \cdot 1_N) = \varphi^*$. Otherwise if $\varphi_i(\tilde{x} \cdot 1_N) < \varphi^*$ then agents in N misreport $\tilde{x} \cdot 1_N$ when the true profile is $x \cdot 1_N$. Similarly, if $\varphi_i(\tilde{x} \cdot 1_N) > \varphi^*$ then agents in N misreport $x \cdot 1_N$ when the true profile is $\tilde{x} \cdot 1_N$.

By GSP, for all $u >> \varphi^* \cdot 1_N$, $S(u) = N$ and $\varphi_i(u) = \varphi^*$ for all i. To see this, let $v = x \cdot 1_N$. By SP, $1 \in S(v_{-1}, u_1)$ and $\varphi_1(v_{-1}, u_1) = \varphi^*$. Thus, by GSP, $S(v_{-1}, u_1) = N$ and $\varphi_i(v_{-1}, u_1) = \varphi^*$ for all i. Changing the profiles one agent at a time $S(u) = N$ and $\varphi_i(u) = \varphi^*$ for all i.

We now prove the proposition by induction in the number of agents. This is obvious when there is only one agent. Assume this is true for any number of agents less than n. We prove it for n agents.

Consider U^j the set of utility profiles where agent j has utility zero. By induction, the restriction of the mechanism to U^j is welfare equivalent to a ESP population-monotonic mechanism of $N \setminus j$ agents. Let x^S the payment of coalition S on U^j and $x^N = \varphi^* \cdot 1_N$. First notice $x_i^S \geq \varphi^*$ for all $S \subseteq N \setminus j$.

To see this, by population-monotonicity we just need to check that $x_i^{N\setminus j} \geq \varphi^*$. Assume $x_i^{N\setminus j} < \varphi^*$.

Let $\epsilon > 0$ such that $\varphi^* - \epsilon > x_i^{N\setminus j}$ and $u = (x^N + \epsilon 1_N)$. Then $S(u) = N$ and $\varphi(u) = x^N$. By SP, $i \notin S(\varphi^* - \epsilon, u_{-j})$. Thus by GSP $S(\varphi^* - \epsilon, u_{-j}) = N \setminus j$ and $\varphi(\varphi^* - \epsilon, u_{-j}) = x^{N\setminus j}$.

Since $u_i > x_i^{N\setminus j}$ for all $i \in N \setminus j$, then by GSP $S((\varphi^* - \epsilon) \cdot 1_N) = N \setminus j$ and $\varphi((\varphi^* - \epsilon) \cdot 1_N) = x^{N\setminus j}$. This contradicts ETE. Hence $x_i^S \geq \varphi^*$ for all $j \in N$, $S \subseteq N \setminus j$.

Thus the mechanism is clear. If $u \geq x^N$ then the mechanism is welfare equivalent to $S(u) = N$ and $\varphi(u) = x^N$. If $u_i < x^N$ then $i \notin S(u)$. Hence by GSP the mechanism is welfare equivalent to $S(u) = (0, u_{-i})$ and $\varphi(u) = \varphi(0, u_{-i})$. Since the restriction to U^i is welfare equivalent to a population-monotonic mechanism with cost-shares not smaller than x^N, then $S(u)$ is the biggest feasible coalition.

Proof of Proposition 2.

First notice if agent i is not served at any profile, then by GSP $NU_k(u) = NU_k(\tilde{u}_i, u_{-i})$ for all u, $k \neq i$, and $\tilde{u}_i$. Thus we can remove this agent from the game without any welfare consequence.

Then, assume without loss of generality that every agent in N is served at least in one profile and that there is no agent who has priority. Then for every agent i there exist profiles u^i and $\tilde{u}^i$ such that $i \in S(u^i)$, $i \notin S(\tilde{u}^i)$, $u_i^i, \tilde{u}_i^i > \bar{x}_i$ where $\bar{x}_i = \varphi_i(u^i)$.

Let $\bar{u} >> \max_{k\in N}(u^k, \tilde{u}^k)$ where max is taken element by element over all utility profiles u^k, $\tilde{u}^k$.

By GSP, $S(\bar{u}) \neq \emptyset$, otherwise coalition N misreport u^1 when true profile is $\bar{u}$. Assume $S(\bar{u}) = i^*$. By GSP, $\varphi_{i^*}(\bar{u}) = \bar{x}_i$, otherwise coalition N misreport u^1 when true profile is $\bar{u}$ or viceversa.

By SP, $k \notin S(\tilde{u}_k^{i^*}, \bar{u}_{-k})$ for all $k \neq i^*$. Thus by GSP, $S(\tilde{u}_k^{i^*}, \bar{u}_{-k}) = i^*$ and $\varphi_{i^*}(\tilde{u}_k^{i^*}, \bar{u}_{-k}) = \bar{x}_{i^*}$. Changing the profiles one agent at a time, $S(\tilde{u}_{-i^*}^{i^*}, \bar{u}_{i^*}) = i^*$ and $\varphi_{i^*}(\tilde{u}_{-i^*}^{i^*}, \bar{u}_{i^*}) = \bar{x}_{i^*}$.

Since $\tilde{u}_{i^*} > \bar{x}_{i^*}$ then by strategyproof $S(\tilde{u}^{i^*}) = i^*$. This is a contradiction.

References

[1] Devanur N., Mihail M., Vazirani.: Strategyproof cost-sharing mechanisms for set cover and facility location games. Decision Support Systems 11-22 (2005)

[2] Ergin H.:Efficient Resource Allocation on the Basis of Priorities. Econometrica **70(6)**, 24892497 (2002)

[3] Ehlers L.: Coalitional Strategy-Proof House Allocation. Journal of Economic Theory **105(2)**, 298-317 (2002)

[4] Ehlers L., Klaus B.: Coalitional strategy-proof and resource-monotonic solutions for multiple assignment problems. Social Choice and Welfare **21**, 265-280 (2003)

[5] Goldberg A., Hartline J.: Collusion-Resistant Mechanisms for Single-Parameter Agents. Mimeo Microsoft (2004)

[6] Immorlica N., Mahdian M., Mirrokni V.: Limitations of cross-monotonic cost sharing schemes. Mimeo, MIT (2005)

[7] Juarez R.: The worst absolute surplus loss in the problem of commons: Random Priority vs. Average Cost. Forthcoming Economic Theory (2006)

[8] Juarez, R.: Group strategyproof cost sharing: budget balance vs. efficiency. Mimeo Rice University (2007).

[9] Juarez, R.: Group strategyproof cost sharing: Mechanisms without indifferences. Mimeo Rice University (2007).

[10] Moulin H.: Incremental Cost Sharing: characterization by coalitional strategy-proofness. Social Choice and Welfare **16**, 279-320 (1999)

[11] Moulin H.: The price of anarchy of serial cost sharing and other methods. Mimeo, Rice University (2005)

[12] Moulin H, Shenker S: Serial Cost Sharing. Econometrica **50(5)**: 1009-1039 (1992)

[13] Moulin H, Shenker S: Average Cost Pricing Versus Serial Cost Sharing: an axiomatic comparison. Journal of Economic Theory **64(1)**, 178-201 (1994)

[14] Moulin H., Shenker S.: Strategyproof sharing of submodular costs: budget balance versus efficiency. Economic Theory **18**, 511-533 (2001)

[15] Myerson R.: Optimal Auction Design. Mathematics of Operation Research **6** 58-73 (1981)

[16] Norde H., Reijnierse H.: A dual description of the class of games with a population monotonic scheme. Games and Economic Behavior **41**, 322-343 (2002)

[17] P'al M., Tardos E.: Group strategyproof mechanisms via primal-dual algorithms. In Proceedings of 44th Annual IEEE Symposium on Foundations of Computer Science (FOCS), pages 584593, 2003.

[18] Papai S.: Strategyproof Assignment by Hierarchical Exchange. Econometrica **68**, 1403-1433 (2000)

[19] Papai S.: Strategy-proof and nonbossy assignments. Journal of Public Economic Theory **3**, 257-271 (2001)

[20] Roughgarden T.: Sundararajan M. Approximately efficient cost-sharing mechanisms. Mimeo Stanford (2006)

[21] Roughgarden T.: Sundararajan M. New TradeOffs in Cost Sharing Mechanisms. STOC (2006)

[22] Roughgarden T., Mehta A., Sundararajan M.: Beyond Moulin Mechanisms. ACM Conference on Electronic Commerce (2007)

[23] Sprumont Y.: Population Monotonic Allocation Schemes for Cooperative Games with Transferable Utility. Games and Economic Behavior **2**, 378-394 (1990)

[24] Svensson L., Larsson B. Strategy-proof and nonbossy allocation of indivisible goods and money. Economic Theory **20(3)**, 483-502 (2002)