The Combinatorial Seller's Bid Double Auction: An Asymptotically Efficient Market Mechanism*

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We consider the problem of efficient mechanism design for multilateral trading of multiple goods with independent private types for players and incomplete information among them. The problem is partly motivated by an efficient resource allocation problem in communication networks where there are both buyers and sellers. In such a setting, ex post budget balance and individual rationality are key requirements, while efficiency and incentive compatibility are desirable goals. Such mechanisms are difficult if not impossible to design [36]. We propose a combinatorial market mechanism which in the complete information case is efficient, budget-balanced, ex post individual rational and "almost" dominant strategy incentive compatible. In the incomplete information case, it is budget-balanced, ex post individual rational and asymptotically efficient and Bayesian incentive compatible. Thus, we are able to achieve efficiency, budget-balance and individual rationality by compromising on incentive compatibility, achieving only a weak version of it.

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1. Introduction

We study a multilateral trading problem with multiple indivisible goods and independent private types in which ex post budget-balance is required. The problem is partly motivated by the need to design mechanisms for efficient resource allocation exchange between strategic internet service providers such as AOL and Comcast who lease transmission capacity (or bandwidth) to form desired routes and networks and carriers such as Qwest and MCI who own capacity on individual links. Bandwidth is traded in discrete amounts, say multiples of 100 Mbps, and hence is an indivisible good. Thus, the buyers want bandwidth on combinations of several links available in multiples of some indivisible unit. This makes the problem *combinatorial*. We consider the interaction in several settings. (Similar problems also occur in other settings such as electricity markets [40] and spectrum auctions [34])

We propose a 'combinatorial sellers' bid double auction' (c-SeBiDA) mechanism for such settings that achieves a socially desirable interaction among strategic agents. The mechanism is *combinatorial* since buyers make bids on combinations of goods, such as several links that form a route. However, each seller offers to sell only a single type of item (e.g., bandwidth on a single link). The mechanism mimics a competitive market: it takes all buy and sell bids, solves a mixed-integer program that matches bids to maximize the social surplus, and announces prices at which the matched (i.e., accepted) bids are settled. The settlement price for an item is the highest price asked by a matched seller (hence 'sellers' bid' auction). As a result there is a *uniform* price for each item.

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It is shown that in the c-SeBiDA auction game with complete information, a Nash equilibrium exists; it is not generally a competitive equilibrium, nor is it unique. Nevertheless, there is an allocatively efficient Nash equilibrium wherein it is a weakly dominant strategy for all buyers and for all sellers except the matched seller with the highest-ask price to be truthful. Moreover, every Nash equilibrium in weakly rationalizable strategies is efficient in the single good case. In the combiantorial case, every Nash equilibrium with a non-zero trade for each good is efficient. Since in an auction, players usually have incomplete information, following Harsanyi [15], we then consider the Bayesian-Nash equilibrium of the auction game. We show that if the players use only *ex post individual rational (IR)* strategies [32], the *semi-symmetric* Bayesian-Nash equilibrium strategies (wherein all sellers selling the same item use the same strategy) converge to truth-telling as the number of players becomes very large.

Previous Work and Our Contribution.

The k-double auction was introduced by Chatterjee and Samuelson [8] as a model of bilateral bargaining. It was shown by Myerson and Satterthwaite [36] that when there is incomplete information, there exists no bilateral mechanism which is Bayesian incentive compatible, individual rational, budget-balanced and efficient. Thus, the notion of constrained incentive efficiency was considered by Wilson [50]. The k-double auction mechanism was further generalized to the single-item multilateral case by Satterthwaite and Williams [45, 46]. In this paper, we consider a multilateral trading mechanism for multiple objects. The mechanism may be considered to be a generalization and modification of the k-double auction mechanism (please see remark 1 and example 2 in section 2 for similarities and differences).

A survey of the vast auction theory literature is provided in [26, 49]. Many are extensions of Vickrey's ideas [48]. Recently, [27] introduced a generalization of the VCG mechanism with participation costs for multi-dimensional types and multiple objects. Also, [9] extends the VCG mechanism to the case of common values, and shows it is constrained efficient. Some multi-round ascending bid auctions [5, 39] achieve the same outcome as VCG. However, these are single-sided auction mechanisms. A Vickrey double auction mechanism for single goods is proposed in [52] but it is neither (ex post) budget-balanced nor individual rational. It appears very difficult to achieve ex post budget balance (along with efficiency and individual rationality) in double-sided auction mechanisms [38].

Our interest is in a double-sided auction mechanism for multiple goods with independent private types (and quasi-linear utility functions). We propose a *combinatorial double auction* mechanism which is individual rational and budget-balanced by design, makes a small compromise on incentive-compatibility and yet is efficient. It is a non-VCG-type double-sided auction mechanism for multiple goods. Like the proposal in [6], our mechanism is also NP-hard. But the mechanism's mixed-integer linear program structure makes the computation manageable for many practical applications [23].

The interplay between economic, game-theoretic and computational issues has sparked interest in algorithmic mechanism design [42, 49]. The generalized Vickrey auction mechanisms for multiple heterogeneous goods are not computationally tractable [37, 38]. Thus, mechanisms that rely on approximation of the integer program [37, 44] or linear programming [7] have been proposed. The results here also relate to the recent efforts in the network pricing literature [29]. There is an ongoing effort to propose mechanisms for divisible resource allocation in networks through auctions [25] and to understand the worst case Nash equilibrium efficiency loss of such mechanisms when users act strategically [22]. Optimal mechanisms for single divisible goods that minimize this efficiency loss have been proposed [51, 30] though not extended to the incomplete information case nor for multiple goods. Most of this literature regards the good (in this case, bandwidth) as divisible, with complete information for all players. The case of combinatorial bids on multiple indivisible goods or incomplete information case is harder.

The results in this paper are significant from several perspectives. It is well known that the only known positive result in the mechanism design theory is the VCG class of mechanisms [32]. The generalized Vickrey auction (GVA) (with complete information) is ex post individual rational, dominant strategy incentive compatible and efficient. It is however not budget-balanced. With incomplete information, the expected version of GVA (dAGVA) [2, 4] is Bayesian incentive compatible, efficient and budget-balanced. It is, however, not ex post individual rational. Indeed, in the complete information setting there can be no mechanism that is efficient, budget-balanced, ex post individual rational and dominant strategy incentive compatible (Hurwicz impossibility theorem) [16]. In the incomplete information setting there is no mechanism which is efficient, budget-balanced, ex post individual rational and Bayesian incentive compatible (Myerson-Satterthwaite impossibility theorem) [36].

In this paper, we provide a non-VCG combinatorial (market) mechanism which in the complete information case is efficient, budget-balanced, ex post individual rational and "almost" dominant strategy incentive compatible. In the incomplete information case, it is budget-balanced, ex post individual rational and asymptotically efficient and Bayesian incentive compatible. Thus, we are able to achieve efficiency, budget-balance and individual rationality by compromising on incentive compatibility, achieving only a weak version of it. Moreover, we show that a Nash equilibrium allocation (say of a network resource allocation game) is efficient (zero efficiency loss) and *any* (semi-symmetric) Bayesian-Nash equilibrium allocation is asymptotically efficient.

This work can also be seen as a contribution to the bargaining games literature. The proposed multilateral trading mechanism for multiple indivisible goods yields an asymptotically efficient allocation even in the case of incomplete information. To our knowledge, this seems to be the only known generalization of the Myerson-Satterthwaite [36] trading environment for multiple heterogeneous goods. Moreover, we provide a positive result: While it is impossible to achieve Bayesian incentive compatibility and efficiency along with ex post budget balance and individual rationality, it is possible to achieve these properties asymptotically even in a multilateral, multiple good trading environment.

The rest of this paper is organized as follows. In Section 2 we present the combinatorial seller's bid double auction (c-SeBiDA) mechanism. In Section 3 we consider Nash equilibrium of the complete information auction game. In Section 4 we consider the Bayesian-Nash equilibrium of the incomplete information auction game for multiple goods.

2. The Combinatorial Sellers' Bid Double Auction

A buyer places buy bids for a bundle of items. A buyer's bid is *combinatorial*: he must receive all items in his bundle or nothing. A buy-bid consists of a buy-price per unit of the bundle and maximum demand, the maximum number of units of the bundle that the buyer needs. On the other hand, each seller makes *non-combinatorial* bids. A sell-bid consists of an ask-price and maximum supply, the maximum number of units the seller offers for sale.

The mechanism collects all announced bids, matches a subset of these to maximize the 'surplus' (equation (1), below) and declares a settlement price for each item at which the matched buy and ask bids—which we call the winning bids—are transacted. This constitutes the payment rule. As will be seen, each matched buyer's buy bid is larger, and each matched seller's ask bid is smaller than the settlement price, so the outcome respects individual rationality.

There is an asymmetry: buyers make multi-item combinatorial bids, but sellers only offer one type of item. This yields uniform settlement prices for each item.

Players' bids may not be truthful. They know how the mechanism works and formulate their bids to maximize their individual returns.

In the *combinatorial sellers' bid double auction* (c-SeBiDA), each player places only one bid. c-SeBiDA is a 'double' auction because both buyers and sellers bid; it is a 'sellers' bid' auction because the settlement price depends only on the matched sellers' bids.

Formal mechanism.

There are L items $l_1, \cdots, l_L$, m buyers and n sellers. Buyer i has (true) reservation value v_i per unit for a bundle of items $R_i \subseteq \{l_1, \cdots, l_L\}$, and submits a buy bid of b_i per unit and demands up to δ_i units of the bundle R_i. Thus, the buyers have quasi-linear utility functions of the form $u_i^b(x; \omega, R_i) = \bar{v}_i(x) + \omega$ where ω is money and

$$\bar{v}_i(x) = \begin{cases} x \cdot v_i, & \text{for } x \leq \delta_i, \\ \delta_i \cdot v_i, & \text{for } x > \delta_i. \end{cases}$$

Seller j has (true) per unit cost c_j and offers to sell up to σ_j units of l_j at a unit price of a_j. Note that there may be many sellers j, j', etc., selling the same good $l_j = l_{j'} = l$, etc. Denote $L_j = \{l_j\}$. The sellers also have quasi-linear utility functions of the form $u_j^s(x; \omega, L_j) = -\bar{c}_j(x) + \omega$ where ω is money and

$$\bar{c}_j(x) = \begin{cases} x \cdot c_j, & \text{for } x \leq \sigma_j, \\ \infty, & \text{for } x > \sigma_j. \end{cases}$$

The mechanism receives all these bids, and matches some buy and sell bids. The possible matches are described by integers x_i, y_j: $0 \leq x_i \leq \delta_i$ is the number of units of bundle R_i allocated to buyer i and $0 \leq y_j \leq \sigma_j$ is the number of units of item l_j sold by seller j.

The mechanism determines the allocation (x^*, y^*) as the solution of the surplus maximization problem **MIP**:

$$\begin{aligned} \max_{x,y} \quad & \textstyle\sum_i b_i x_i - \sum_j a_j y_j \\ s.t. \quad & \textstyle\sum_j y_j \mathbf{1}(l \in L_j) - \sum_i x_i \mathbf{1}(l \in R_i) \geq 0, \forall l \in [1:L], \\ & x_i \in \{0, 1, \cdots, \delta_i\}, \forall i, \quad y_j \in [0, \sigma_j], \forall j. \end{aligned} \tag{1}$$

MIP is a mixed integer program: Buyer i's bid is matched up to his maximum demand δ_i; Seller j's bid will also be matched up to his maximum supply σ_j. x_i^* is constrained to be integral; y_j^* will be integral due to the demand less than equal to supply constraint.

The settlement price is the highest ask-price among matched sellers,

$$\hat{p}_l = \max\{a_j : y_j^* > 0, l \in L_j\}. \tag{2}$$

The payments are determined by these prices. If no seller of item l is matched, i.e., item l is not traded, the price of $\hat{p}_l$ is unspecified. Matched buyers pay the sum of the prices of items in their bundle; matched sellers receive a payment equal to the number of units sold times the price for the item. Unmatched buyers and sellers do not get any allocation and do not make or receive any payments. This completes the mechanism description.

If i is a matched buyer ($x_i^* > 0$), it must be that his bid $b_i \geq \sum_{l \in R_i} \hat{p}_l$; for otherwise, the surplus (1) can be increased by eliminating the corresponding matched bid. Similarly, if j is a matched seller ($y_j^* > 0$), and $l \in L_j$, his bid $a_j \leq \hat{p}_l$, for otherwise the surplus can be increased by eliminating his bid. Thus the outcome of the auction respects individual rationality.

It is easy to understand how the mechanism picks matched sellers. For each item j, a seller with lower ask bid will be matched before one with a higher bid. So sellers with bid $a_j < \hat{p}_l$ sell all their supply ($y_j^* = \sigma_j$). At most one seller with ask bid $a_j = \hat{p}_l$ sells only a part of his total supply ($y_j^* < \sigma_j$). On the other hand, because their bids are combinatorial, the matched buyers are selected only after solving the MIP.

Example 1. Consider one item, three buyers each of whom wants one unit and three sellers each of whom has one unit to offer. Suppose buyers bid $b_1 = 3.1, b_2 = 2.1, b_3 = 1.1$ and sellers bid $a_1 = 1, a_2 = 2, a_3 = 3$. Then, the *revealed social surplus* in MIP (1) is maximized when buyers 1 and 2, and sellers 1 and 2 are matched. The price then is $\hat{p} = 2$. Note that if bids of buyer 3 and seller 3 are also accepted, this will result in a lower revealed social surplus.

Remarks. 1. We designed c-SeBiDA so that its outcome mimics a competitive equilibrium with a particular interest in the combinatorial case. The single item version SeBiDA resembles the k-double auction (a special case being called the buyer's bid double auction [46, 45, 47]).

The k-DA is defined as follows: Sellers submit offers $a_j, j = 1, \cdots, n$ and buyers bids $b_i, i = 1, \cdots, n$. To determine who trades, list these offers/bids as $s_{(1)} \leq s_{(2)} \leq \cdots \leq s_{(2n)}$ where $s_{(l)}$ denotes the lth order-statistic. Thus, $s_{(n)}$ could either be a buy-bid or a sell-offer. Then, for given $k \in [0,1]$, pick price to be $p(k) = (1-k)s_{(n)} + ks_{(n+1)}$. Sell-offers below p and buy-bids above p are accepted. Others are not. For the special case of $k = 1$, the k-DA mechanism is the same as the buyer's bid double auction (BBDA) mechanism [45]. The "sell-side version" would take $k = 0$ with $p = s_{(n)}$. But note that despite similar nomenclature and spirit, BBDA and c-SeBiDA determine prices differently. We illustrate the difference through an example.

EXAMPLE 2. Consider one item, three buyers each of whom wants one unit and three sellers each of whom has one unit to offer. Suppose buyers bid $b_1 = 6.1, b_2 = 3.1, b_3 = 1.1$ and sellers bid $a_1 = 2, a_2 = 4, a_3 = 5$. (i) *BBDA:* Then, $s_{(3)} = 3.1$ and $s_{(4)} = 4$, and the price determined by BBDA is $p = 4$ with one trade between buyer 1 and seller 1. The "sell-side" version of BBDA would determine a price $p = 3.1$ with a single trade. k-DA determines a price $p \in [3.1, 4]$. (ii) *c-SeBiDA:* The mechanism proposed in this paper, on the other hand, determines one trade between buyer 1 and seller 1 with price $p = 2$.

Thus, the mechanism proposed in this paper is distinct from BBDA [46]. It is also not clear what a generalization of the k-double auction or BBDA would be to the combinatorial case.

2. The issue of computational complexity for such mechanisms becomes very important when there are a large number of players. Similar concerns arise in [6] as well. However, the computational problem here involves solving a mixed linear program, for which computationally efficient approximation algorithms have been devised. Developing an approximation algorithm for the particular MIP here will be undertaken in the future.
3. The ties between players will be broken by randomly picking the winners. This has no effect on the auction's outcome, or its properties unlike other mechanisms.

3. Complete Information Nash Equilibrium Analysis: c-SeBiDA is Efficient

We first focus on how strategic behavior of players affects price when they have complete information. We will assume that players don't strategize over the bundles R_i and the quantities (namely, δ_i, σ_j), which will be considered fixed in the players' bids. A strategy for buyer i is a buy bid b_i, a strategy for seller j is an ask bid a_j. Let θ denote a collective strategy. Given θ, the mechanism determines the allocation (x^*, y^*) and the prices $\{\hat{p}_l\}$. So the payoff to buyer i and seller j is, respectively,

$$u_i^b(\theta) = \bar{v}_i(x_i^*) - x_i^* \cdot \sum_{l \in R_i} \hat{p}_l, \tag{3}$$

$$u_j^s(\theta) = y_j^* \cdot \sum_{l \in L_j} \hat{p}_l - \bar{c}_j(y_j^*). \tag{4}$$

The bids b_i, a_j may be different from the true valuations v_i, c_j, which however figure in the payoffs.

A collective strategy θ^* is a *Nash equilibrium* if no player can increase his payoff by unilaterally changing his strategy [11]. Define *social welfare* function for the auction game as

$$SW(x,y) = \sum_i v_i x_i - \sum_j c_j y_j.$$

where (x, y) satisfy the feasibility conditions of MIP (1). An auction mechanism is said to be *(allocatively) efficient* if every Nash equilibrium allocation maximizes *social welfare*.

We say that a strategy $\tilde{b}_i$ is *weakly dominated* for player i if there exists a strategy b_i of player i such that

$$u_i(b_i, b_{-i}) \geq u_i(\tilde{b}_i, b_{-i}), \forall b_{-i}$$

with strict inequality for at least one b_{-i} where b_{-i} are the strategies of the other players. Such strategies are considered unlikely to be played. Strategies which are not weakly dominated will be called *undominated*.

Strategies which remain undominated after iterated elimination of weakly dominated strategies will be called *weakly rationalizable strategies* [11]. They are so called because it is considered rational for players to play only such strategies when it is *common knowledge* that all players are rational.

We now construct a Nash equilibrium and show it yields an efficient allocation (Theorem 1). We then show that when players only play weakly rationalizable strategies, all resulting Nash equilibria are efficient in the single good case.

For simplicity, we assume that each buyer bids for at most one unit, and each seller sells at most one unit of the item (so δ_i, σ_j equal 1 in (3), (4)). Suppose there are m buyers and n sellers, whose true valuations and costs lie in $[0, 1]$. To avoid trivial cases of non-uniqueness, assume all buyers have different valuations and all sellers have different costs.

THEOREM 1. *(i) A Nash equilibrium (b^*, a^*) exists in the c-SeBiDA game. (ii) There is a Nash equilibrium wherein except for the matched seller with the highest bid on each item, each player bids truthfully. (iii) Furthermore, in case of a single good, any Nash equilibrium in weakly rationalizable strategies has an efficient allocation. (iv) In the combinatorial case, if there is a trade for each good, then every Nash equilibrium is efficient.*

Proof: Suppose buyer i demands the bundle R_i with reservation value v_i and the seller (l, j) (the j-th seller offering item l) has reservation cost $c_{l,j}$. Assume without loss of generality that $c_{l,1} \leq \cdots \leq c_{l,n_l}$, in which n_l is the number of sellers offering item l.

We will iteratively construct a set of strategies to consider as Nash equilibrium.

Consider the surplus maximization problem (1) with true valuations and costs. Let I be the set of matched buyers and k_l the number of matched sellers offering item l determined by the **MIP**. Set $b_i^* = v_i$ for all i; $a_{l,j}^0 = c_{l,j}$; $\gamma_i^t = b_i^* - \sum_{l \in R_i} a_{l,k_l}^t$, the revealed surplus of a matched buyer i at stage $t \geq 0$, and

$$\hat{l} \in \arg\min_{l} \{ \min_{i \in I : l \in R_i} \gamma_i^t : \gamma_i^t > 0 \}, \tag{5}$$

the item with the smallest surplus among the matched buyers at stage t, with each l being picked only once. Denote the corresponding surplus by $\gamma_{\hat{l}}^t$. We will denote the corresponding minima by $\gamma_{\hat{l}}^t$. Now, define

$$a_{\hat{l},k_{\hat{l}}}^{t+1} := \min\{ a_{\hat{l},k_{\hat{l}}+1}^t, a_{\hat{l},k_{\hat{l}}}^t + \gamma_{\hat{l}}^t \}, \tag{6}$$

which is the strategy of seller $(\hat{l}, k_{\hat{l}})$ at the t-th stage: His ask bid is increased to decrease the surplus of the matched buyer with the smallest surplus up to the ask bid of the unmatched seller with the lowest bid. For all other $(l, j) \neq (\hat{l}, k_{\hat{l}})$, the ask bid remains the same, $a_{l,j}^{t+1} = a_{l,j}^t$. This procedure is repeated until the strategies converge such that each l is picked only once. In fact, it is repeated at most L times. Observe that at each stage, the matches and the allocations from the MIP using the current bids (b^*, a^t) do not change. Let a^* denote the seller ask bids when the procedure converges.

We prove that (b^*, a^*) is a Nash equilibrium, by showing that no player has an incentive to deviate.

First, an unmatched seller offering item l has no incentive to bid lower than a_{l,k_l}^*: Because his reservation cost is higher than that, by bidding lower than his reservation cost, it may get matched but his payoff will be negative. Next, consider a matched seller $(l,j) \neq (l,k_l)$ offering item l. By bidding higher or lower he cannot change the price of the item but may end up getting unmatched. Thus, it is the dominant strategy of all sellers except the 'marginal' seller (l,k_l) to bid truthfully.

Now, consider this marginal matched seller (l,k_l). If he bids lower then a^*_{l,k_l}, his payoff will decrease. He could bid higher but because of (6), either there is an unmatched seller of the item with the same ask bid, or there is a marginal buyer whose surplus has been made zero by (6). So if he bids higher than a^*_{l,k_l}, either he will become unmatched and the first unmatched seller of the item will become matched, or the 'marginal' buyer with zero surplus will become unmatched causing this marginal seller to be unmatched as well. Thus, a^*_{l,k_l} is a Nash strategy of the marginal seller given that all other players (except the marginal sellers of the other items) bid truthfully.

Now, let us consider the buyers. First, observe that due to our construction of the strategies above, the payoff (and the surplus γ_i) of the 'marginal' buyer for each item is zero. Now, consider an unmatched buyer i. Clearly, he has no incentive to bid lower than b^*_i since he wouldn't match anyway. Further, it must be that $b^*_i(=v_i) < \sum_{l\in R_i} a^*_{l,k_l}$. For if not, then it would be possible to increase the auction surplus (1) by accepting bid of i, and not accepting bids of 'marginal' buyers with accepted bids on each of the links $l \in R_i$ (as also some other 'marginal' sellers with accepted bids). But this would contradict the fact that the earlier allocation did not in fact maximize the auction surplus (1). Now, if buyer i bids high enough, he will become matched but he will have to pay prices a^*_{l,k_l} on the links and since $v_i < \sum_{l\in R_i} a^*_{l,k_l}$, his payoff will become negative. Next, a matched buyer with a positive payoff has no incentive to bid lower since by bidding lower he can lower the prices but only when he becomes unmatched. Also, he certainly has no incentive to bid higher since by so doing he will not be able to lower the price. Lastly, consider the 'marginal' matched buyers with zero payoff: Clearly, if they bid higher, their payoff will not increase; and if they bid lower, they will become unmatched. Thus, it is the best response of all buyers to bid truthfully.

The Nash equilibrium allocation (x^*, y^*) as determined above is efficient since it maximizes (1) with true valuations.

We now show that in case of a single good any Nash equilibrium allocation in weakly rationalizable strategies is efficient. (We will drop the subscript l for sellers).

First, observe that a seller's bid below his cost is weakly dominated by his bid at cost: Thus, $a_j \geq c_j, \forall j$. Further, since this elimination of strategy space of the sellers is common knowledge, no buyer will bid below $c_{min} = \min_j c_j$.

Let $\tilde{B}_{matched}$ and $\tilde{S}_{matched}$ denote the set of buyers and sellers that are matched at a Nash equilibrium $(\tilde{b}, \tilde{a})$. It is worth noting that at an equilibrium, the transaction price $\tilde{p} = \min\{b_i : i \in \tilde{B}_{matched}\} = \max\{a_j : j \in \tilde{S}_{matched}\}$.

Now suppose $\tilde{z} := (\tilde{x}, \tilde{y})$ is an allocation, corresponding to the Nash equilibrium $(\tilde{b}, \tilde{a})$, which is not efficient. There are two main cases:

(1) No Trade is Efficient Case: Suppose that the efficient allocation $(z^* := (x^*, y^*))$ involves no trade, but the allocation $\tilde{z}$ does. This implies that $v_i < c_j,\ \forall i, j$ but there exists some buyer $\hat{i}$ and seller $\hat{j}$ such that $b_{\hat{i}} \geq c_{\hat{j}}$. Then, either $b_{\hat{i}} > v_{\hat{i}}$ or $a_{\hat{j}} < c_{\hat{j}}$. In both cases, one of the buyer $\hat{i}$ or the seller $\hat{j}$ has an incentive to deviate.

(2) Non-zero Trade is Efficient Case: (a) First, suppose that the efficient allocation z^* involves a trade but the allocation $\tilde{z}$ involves no trade. Let i^* denote the buyer with highest value v_i and j^* denote the seller with the least cost c_j $(c^*_j = c_{min})$. Then, $v_{i^*} \geq c_{j^*}$ but $c_{min} \leq b_{i^*} < a_{j^*}$. But then this cannot be a Nash equilibrium since either the buyer or the seller will have an incentive to deviate.

(b) Now, suppose that the efficient allocation z^* involves a trade and the allocation $\tilde{z}$ involves a trade but is not efficient. Then, the two allocations must differ in one of the following ways as we go from z^* to $\tilde{z}$:

(i) z^* and $\tilde{z}$ differ only among sellers: A (non-empty) set of sellers S_{out} matched in z^*, is no longer matched in $\tilde{z}$ and a (non-empty) set of sellers S_{in} are now matched;

(ii) z^* and $\tilde{z}$ differ only among buyers: A (non-empty) set of buyers B_{out} matched in z^*, is no longer matched in $\tilde{z}$ and a (non-empty) set of buyers B_{in} are now matched;

(iii) All buyers and sellers matched in z^* remain matched in $\tilde{z}$, and some new buyers B_{in} and some new sellers S_{in} now get matched;

(iv) No new buyers and sellers are matched in $\tilde{z}$ and some old buyers B_{out} and some old sellers S_{out} are now not matched;

(v) (*General Case*) A set of buyers B_{out} and a set of sellers S_{out} are no longer matched and a set of buyers B_{in} and a set of sellers S_{in} are now matched in $\tilde{z}$.

Case (i) Suppose $j_1 \in S_{in}$ and $j_2 \in S_{out}$. Then, it must be that $c_{j_1} > c_{j_2}$ but $\tilde{a}_{j_1} < \tilde{a}_{j_2}$. But then either j_1's payoff is negative or j_2 can also bid just below j_1's bid. In either case $\tilde{z}$ cannot be a Nash equilibrium allocation.

Case (ii) Suppose $i_1 \in B_{in}$ and $i_2 \in B_{out}$. Then it must be that $v_{i_1} < v_{i_2}$ and $\tilde{b}_{i_1} > \tilde{b}_{i_2}$. But then either i_1's payoff is negative or i_2 can also bid just above i_1's bid. In either case $\tilde{z}$ cannot be a Nash equilibrium allocation.

Case (iii) Denote $\check{i} := \arg\max_{i \in B_{in}} \tilde{b}_i$ and $\check{j} := \arg\min_{j \in S_{in}} \tilde{a}_j$. Then, $v_{\check{i}} < c_{\check{j}}$ and $\tilde{b}_{\check{i}} \geq \tilde{a}_{\check{j}}$. But then at least one of the two has a negative payoff at $(\tilde{b}, \tilde{a})$, and so will deviate, in which case it cannot be a Nash equilibrium outcome.

Case (iv) Denote $\check{i} := \arg\max_{i \in B_{out}} v_i$ and $\check{j} := \arg\min_{j \in S_{out}} c_j$. And denote the transaction price with bids $(\tilde{b}, \tilde{a})$ by $\tilde{p}$. Then, $v_{\check{i}} \geq c_{\check{j}}$ and $\tilde{b}_{\check{i}} < \tilde{a}_{\check{j}}$. Now, if $\tilde{a}_{\check{j}} < v_{\check{i}}$, then clearly, buyer $\check{i}$ has an incentive to bid just above $\tilde{a}_{\check{j}}$ and match. Similarly, if $\tilde{b}_{\check{i}} > c_{\check{j}}$, then seller $\check{j}$ has an incentive to bid just below $\tilde{b}_{\check{i}}$ and match. In either of these cases, the bids under consideration cannot be a Nash equilibrium.

Now, let us consider the case $\tilde{b}_{\check{i}} \leq c_{\check{j}} \leq v_{\check{i}} \leq \tilde{a}_{\check{j}}$. There are three sub-cases: if $\tilde{p} \in (\tilde{b}_{\check{i}}, c_{\check{j}}]$, then buyer $\check{i}$ can raise his bid and match; if $\tilde{p} \in [v_{\check{i}}, \tilde{a}_{\check{j}})$, then seller $\check{j}$ can lower his bid and match; and if $\tilde{p} \in (c_{\check{j}}, \tilde{a}_{\check{i}})$, then both the buyer $\check{i}$ and the seller $\check{j}$ have an incentive to deviate from their current bids and match. Thus, in none of the above sub-cases can the bids under consideration be a Nash equilibrium.

Case (v) Denote $\hat{i} := \arg\min_{i \in B_{in}} \tilde{b}_i$ and $\hat{j} := \arg\max_{j \in S_{in}} \tilde{a}_j$, and $\check{i} := \arg\max_{i \in B_{out}} \tilde{b}_i$ and $\check{j} := \arg\min_{j \in S_{out}} \tilde{a}_j$. And denote the transaction price with bids $(\tilde{b}, \tilde{a})$ by $\tilde{p}$. Then, $\tilde{b}_{\hat{i}} \geq \tilde{p} \geq \tilde{a}_{\hat{j}}$ and $\tilde{b}_{\check{i}} \leq \tilde{p} \leq \tilde{a}_{\check{j}}$.

Now, observe that $v_{\check{i}} > v_{\hat{i}}$ and $c_{\check{j}} < c_{\hat{j}}$ since players $\check{i}$ and $\check{j}$ are matched in z^*, the efficient allocation but players $\hat{i}$ and $\hat{j}$ are not. Further, $b_{\hat{i}} \geq \tilde{p}$. So, either $v_{\hat{i}} \geq \tilde{p}$, in which case $v_{\check{i}} \geq \tilde{p}$ as well and so buyer $\check{i}$ can increase his bid to match; or $v_{\hat{i}} < \tilde{p}$, in which case buyer $\hat{i}$ has negative payoff and so it will decrease his bid. Thus, in either case, the buyer has an incentive to deviate, and hence the allocation $\tilde{z}$ cannot correspond to a Nash equilibrium. A similar argument can also be given for sellers.

Thus, for every case above, the corresponding bids cannot be a Nash equilibrium. This proves claim (iii) of the theorem.

The proof of part (iv) is similar in some details to that for part (iii) and can be found in the appendix. ■

Remarks. 1. It is obvious that if the minimum in step (5) is not unique, the efficient Nash equilibrium will not be unique.
2. Parts (i) and (ii) of the above result still hold when buyers make multiple unit combinatorial bids and sellers make single unit non-combinatorial bids.
3. Note that there are other Nash equilibria where buyers may not bid their true valuation. Consider the setting of example 1.

EXAMPLE 3. Consider the bids to be $a_1 = 2.05$, $a_2 = 2.05$, $a_3 = 3$, $b_1 = 2.05$, $b_2 = 2.05$ and $b_3 = 1.1$. It is easy to check this is a Nash equilibrium with efficient allocation. But note that buyer 2 does not bid true valuation. Thus, in c-SeBiDA it is not a dominant-strategy for buyers or sellers to be truthful.

4. We have considered Nash equilibrium in weakly rationalizable strategies since it is not rational for players to play weakly dominated strategies. However, if we do consider all strategies, there are no-trade Nash equilibria which may not be efficient as the following examples show.

EXAMPLE 4. Consider a buyer with $v = 0.7$ and a seller with $c = 0.3$. Clearly a trade is possible and in fact any $b^* = a^* \in [0.3, 0.7]$ is a Nash equilibrium with an efficient outcome. However, consider the bids $b = 0$ and $a = 1$. Clearly, this is a Nash equilibrium with no trade, which is inefficient. But, these strategies are strictly dominated by other strategies, e.g., the buyer can bid anything above 0.3 and the seller anything below 0.7.

EXAMPLE 5. Consider a two goods (A and B) case. There is one buyer with $v = 0.7$ for one unit of both goods, and zero otherwise. There is one seller who offers good A and has $c_1 = 0.2$ and another seller who offers good B and has $c_2 = 0.3$. Clearly, the efficient allocation involves an exchange between these players. Now, consider $b = 0.6$, $a_1 = 0.4$ and $a_2 = 0.5$. It is a no-trade Nash equilibrium. In fact, it is easy to check that there does not exist an efficient Nash equilibrium even in weakly rationalizable strategies.

It is interesting to note that

THEOREM 2. *With multiple unit buy-bids and single unit sell-bids, i.e., $\sigma_j = 1, \forall j$, the Nash equilibrium allocation and prices $((x^*, y^*), \hat{p})$ is a competitive equilibrium.*

Proof: Consider a matched seller. He supplies exactly one unit at prices $\hat{p}$ while an unmatched, non-marginal seller (l, j) for $j > k_l + 1$, supplies zero units. The unmatched marginal seller (l, k_l) will supply zero units since $\hat{p}_l \geq a_{l,k_l+1}$. Now, consider a matched buyer i. At prices $\hat{p}$, he demands up to δ_i units of its bundle. If it is the "marginal" matched buyer, its surplus is zero and it may receive anything up to δ_i. If it is a "non-marginal" matched buyer, it receives δ_i units. An unmatched buyer, on the other hand, has zero demand at prices $\hat{p}$. Thus, total demand equals total supply, and the market clears. ■

4. Asymptotic Bayesian Incentive Compatibility of c-SeBiDA

We now consider the incomplete information case for the combinatorial-SeBiDA. Analysis for the simpler non-combinatorial setting can be found in [20]. We analyze the c-SeBiDA market mechanism in the limit of a large number of players. Suppose there are n_l sellers of good l, $l = 1, \cdots, L$ and m buyers with m_l buyers who want good l, i.e., have l in their bundle.

We will consider a Bayesian game to model incomplete information. Let $c_{l,j}$ and $a_{l,j}$ denote the cost and ask-bid of the jth seller of good l respectively, and v_i and b_i denote the valuation and buy-bid of the ith buyer with bundle R_i respectively. Suppose nature draws $c_{l,1},\cdots,c_{L,n_L}$ independently from the probability distribution $U[0,1]$ and draws $v_1,\cdots,v_m$ independently from probability distributions, $v_i \sim U[0,|R_i|]$. Each player is then revealed his own valuation or cost. It is common information that the seller (l,j)'s costs are drawn from $U[0,1]$ and a buyer i's valuations are drawn from $U[0,|R_i|]$, his R_i being known to all. Let $\alpha_{l,j}:[0,1]\to[0,1]$ denote the strategy of the seller (l,j) and $\beta_i:[0,|R_i|]\to[0,|R_i|]$ denote the strategy of the buyer i. Then, the payoff received by the buyers and sellers is as defined by equations (3) and (4). Let $\theta=(\alpha_{1,1},\cdots,\alpha_{L,n_L},\beta_1,\cdots,\beta_m)$ denote the collective strategy of the buyers and the sellers. A buyer i chooses strategy β_i to maximize $\mathbb{E}[u_i^b(\theta);\beta_i]$, the conditional expectation of the payoff given its strategy β_i. The seller (l,j) chooses strategy $\alpha_{l,j}$ to maximize $\mathbb{E}[u_{l,j}^s(\theta);\alpha_{l,j}]$, the conditional expectation of the payoff given its strategy $\alpha_{l,j}$. The *Bayesian-Nash equilibrium* of the game is then the Nash equilibrium of the Bayesian game defined above [11].

We consider *semi-symmetric* Bayesian-Nash equilibria, i.e., equilibria where all the sellers of the same good use the same strategy α_l while the buyers may use different strategies β_i, since they may demand bundles of different sizes. Let $\tilde{\alpha}_l(c):=c$ and $\tilde{\beta}_i(v):=v$ denote the truth-telling strategies. Under the strategy profile $(\alpha_1,\cdots,\alpha_L,\beta_1,\cdots,\beta_m)$, we denote the distribution of ask-bids $a_{l,\cdot}$ and buy-bids b_i as F_l and G_i respectively. We denote $[1-F(x)]$ by $\bar{F}(x)$. Under $\tilde{\alpha}_l$ and $\tilde{\beta}_i$, $F_l=U[0,1]$ and $G_i=U[0,|R_i|]$. We will assume that players are *risk-averse* and consider only those bid strategies which satisfy the *ex post individual rationality* constraint, i.e., $\alpha_l(c)\geq c$ and $\beta_i(v)\leq v$. Denote $\mathcal{X}_l=\{\alpha_l:\alpha_l(c)\geq c\}$, $\mathcal{X}=\mathcal{X}_1\times\cdots\times\mathcal{X}_L$, $\boldsymbol{\alpha^n}=(\alpha_1^n,\cdots,\alpha_L^n)$ and $\tilde{\boldsymbol{\alpha}}=(\tilde{\alpha}_1,\cdots,\tilde{\alpha}_L)$ when there are n sellers of each good. Also denote $\mathcal{Y}_i=\{\beta_i:\beta_i(v)\leq v\}$, $\mathcal{Y}=\mathcal{Y}_1\times\cdots\times\mathcal{Y}_m$ and $\boldsymbol{\beta^n}=(\beta_1^n,\cdots,\beta_m^n)$ and $\tilde{\boldsymbol{\beta}}=(\tilde{\beta}_1,\cdots,\tilde{\beta}_m)$ when there are m buyers and n sellers for each good. Let m_l denote the number of buyers who want good l. We will assume that $m_l=O(n)$.

We consider single unit bids and assume that a semi-symmetric Bayesian-Nash equilibrium exists. And following Wilson [50, 45, 46, 47, 43], we make the following assumption:

Assumption 1. *There exist symmetric Bayesian-Nash equilibria which have seller's strategies such that $\alpha_n'(c)$ is uniformly bounded in n and c.*

Theorem 3. *Consider the c-SeBiDA auction game with $(\boldsymbol{\alpha},\boldsymbol{\beta})\in\mathcal{X}\times\mathcal{Y}$, i.e., both buyers and sellers have ex post individual rationality constraint. Let $(\boldsymbol{\alpha^n},\boldsymbol{\beta^n})$ be a semi-symmetric Bayesian Nash equilibrium with m buyers and n sellers of each good. Then, (i) $\beta_i^n(v)=\tilde{\beta}_i(v)=v$ for $i=1,\cdots,m$ and $\forall n\geq 2$, and (ii) $(\boldsymbol{\alpha^n},\boldsymbol{\beta^n})\to(\tilde{\boldsymbol{\alpha}},\tilde{\boldsymbol{\beta}})$ in the sup norm as $n\to\infty$, i.e., c-SeBiDA is* asymptotically Bayesian incentive compatible.

We proceed in three steps and first prove two lemmas.

Lemma 1. *Consider the c-SeBiDA auction game with m buyers and n_l sellers for item l. Suppose the sellers use a bid strategy profile $\boldsymbol{\alpha}=(\alpha_1,\cdots,\alpha_L)$ with $f_l(a)$, the pdf of its ask-bid under strategy α_l. Then, the best-response strategy profile of the buyers $\boldsymbol{\beta^n}$ satisfies $\beta_i^n(v)\geq v$ for $i=1,\cdots,m$ and $\forall n\geq 2$.*

Remarks. 1. As we noted in the single good case as well, a buyer's strategy is to bid more than his true value. This at first glance seems surprising. However, intuitively it makes sense for this mechanism since the prices are determined by the sellers' bids alone, and by bidding higher, a buyer only increases his probability of being matched. Of course, if he bids too high, he may end up with a negative payoff. The result implies that under the ex post individual rationality constraint, the buyers always use the strategy profile $\beta^n=\tilde{\beta}$.
2. It is also worth noting that the result can be easily extended to the case when all the sellers may use different strategies.

The next step is to look at the best response strategy of the sellers when the buyers bid truthfully.

LEMMA 2. *Consider the c-SeBiDA auction game with $n_l = n$ sellers of good l and m_l buyers who want the good in their bundle, and suppose the buyers bid truthfully, i.e., $\beta_i^n = \tilde{\beta}_i$, and let $\boldsymbol{\alpha^n}$ be the sellers' best-response strategy. Then, $(\boldsymbol{\alpha^n}, \tilde{\boldsymbol{\beta}}) \to (\tilde{\boldsymbol{\alpha}}, \tilde{\boldsymbol{\beta}})$ in the sup norm as $n \to \infty$.*

The conclusion of this lemma is what we would expect intuitively. If all buyers bid truthfully, then as the number of sellers increases, increased competition forces them to bid closer and closer to their true costs.

We can use the above two lemmas to prove the main result of this section.

Proof: (Theorem 3) By Lemma 1, when the sellers use the strategy profile $\boldsymbol{\alpha} = \boldsymbol{\alpha^n}$, the buyers under the ex post individual rationality constraint use the strategy profile $\tilde{\boldsymbol{\beta}}$. By Lemma 2, when the buyers bid truthfully, sellers' best-response is $\boldsymbol{\alpha^n}$. Thus, $(\boldsymbol{\alpha^n}, \tilde{\boldsymbol{\beta}})$ is a Bayesian-Nash equilibrium with n sellers on each good. Further, Lemma 2 shows that $(\boldsymbol{\alpha^n}, \boldsymbol{\beta^n}) = (\boldsymbol{\alpha^n}, \tilde{\boldsymbol{\beta}}) \to (\tilde{\boldsymbol{\alpha}}, \tilde{\boldsymbol{\beta}})$ as $n \to \infty$, which is the conclusion we wanted to establish. ■

Thus, under the ex post individual rationality constraint, c-SeBiDA is ex ante budget balanced (of course, ex post budget balanced as well), asymptotically Bayesian incentive compatible and efficient. Unlike in the complete information case when the mechanism is not incentive compatible, yet the outcome is efficient, in the incomplete information case, the mechanism is asymptotically efficient.

Remarks. 1. The above result holds for any arbitrary m, the number of buyers, and in particular, for the case where m increases with n to infinity. When m is finite, W_* and W^* in proposition 1 both converge to zero in probability.

2. The result above depends on assuming Wilson's hypothesis. Such an assumption has also been made in [50, 47, 43]. In fact, we have been able to show that the seller strategies we consider above are strictly increasing and uniformly continuous on [0,1] for every n. Assuming that the strategies are monotonically decreasing in n (it might be possible to argue this using results from monotone comparative statics [35]), we can conclude using Dini's theorem [1] that the strategies converge uniformly. This yields equicontinuity of the strategies and it might be possible then to conclude existence of strategies that satisfy Wilson's hypothesis. However, we have not been able to completely resolve this open problem as of now.

3. Existence of semi-symmetric pure strategy Bayesian-Nash equilibria has been considered in the literature. A very general result is obtained using fixed point theory on perturbed games [10] (see also [39, 19]). They establish existence of monotone pure strategy equilibria in large enough uniform-price double-sided auctions. In [24], existence of monotone pure strategy bayesian-nash equilibrium in market mechanisms with general values has been studied using Lattice-theoretic methods [35, 33]. While [47, 28] show existence for particular auctions by showing the existence of solutions to the differential equations that describe the equilibria.

4. The mechanism proposed in this paper is related to the buyer's bid double auction (BBDA) mechanism [45, 47] and its generalization for single items, the k-double auction mechanism. For the special case of $k = 1$, the k-DA mechanism is the same as BBDA. But note that despite similar nomenclature and spirit, BBDA and SeBiDA determine prices differently. While the spirit of the two mechanisms is the same (maximizing the efficiency of trading), the prices and the payments are different. Please see remark 1 and example 2 in section 2 for more details on differences and similarities. An example illustrating that in c-SeBiDA, neither the buyers nor the sellers have a dominant strategy to be truthful was given in example 3 of section 3. This is also the case for BBDA as proved by the following counterexample.

EXAMPLE 6. Consider one item type with two buyers who have valuations $v_1 = 3.1, v_2 = 2.1$ and two sellers who have costs $c_1 = 1, c_2 = 2$. Consider the bids $b_1 = 2.05, b_2 = 2.05$ and $a_1 = 2.05, a_2 = 2.05$. BBDA then determines a price of $p = 2.05$ with two trades. Moreover, this is a full information Nash equilibrium. But note that neither the buyers nor the sellers are truthful.

5. Finally, the ex post individual rationality constraint seems restrictive at first glance. However, in two human subject experiments we have conducted using this mechanism [23], it was observed that all subjects acted *risk-averse* and in fact always used strategies that were ex post individual rational. Thus, the predictive power of the result does not seem diminished in real settings despite the assumption made.

5. Conclusions

We have introduced a combinatorial, sellers' bid, double auction (c-SeBiDA). The first result concerned the Nash equilibria for c-SeBiDA with full information. In c-SeBiDA, settlement prices are determined by sellers' bids. We showed that the allocation of c-SeBiDA is efficient. Moreover, there is a Nash equilibrium in undominated strategies wherein truth-telling is a dominant strategy for all players except the highest matched seller for each item.

The second result concerned the Bayesian-Nash equilibrium of the mechanism under incomplete information. We showed that under the ex post individual rationality constraint, the semi-symmetric Bayesian-Nash equilibrium strategies converge to truth-telling. Thus, the mechanism is asymptotically Bayesian incentive compatible, and hence asymptotically efficient.

Thus, we have proposed an exchange mechanism for the multilateral Myerson-Satterthwaite [36] trading environment with multiple goods. In such an environment it is impossible to achieve all the four desirable properties of an auction mechanism. Nevertheless, we have shown that it is still possible to achieve ex post budget balance and individual rationality, and *asymptotic* Bayesian incentive compatibility and efficiency.

In [21], we considered a more general setting and showed that a competitive equilibrium exists in a continuum model of an exchange economy with indivisible items and money (a divisible item). There, using results from optimal control, we also showed that within the continuum model, c-SeBiDA outcome is a competitive equilibrium. This again suggests that in the finite setting, the auction outcome is close to efficient.

We have tested the proposed mechanism c-SeBiDA through human-subject experiments. Those results can be found elsewhere [23].

Finally, while our work was primarily motivated by a market mechanism design problem, it can also be considered as an indirect contribution to the strategic foundations of competitive markets [12]. This body of literature relates Nash and Bayesian-Nash equilibrium with competitive equilibrium. The basic idea is that as the economy gets large (in our context the number of buyers and sellers and quantities of items all go to infinity), Nash equilibrium strategies should converge to competitive equilibrium strategies, because the 'market power' diminishes.

The relationship is first investigated in [41]. In a later paper [14], it is shown that under certain regularity conditions, a sufficiently replicated economy has an allocation which is incentive-compatible, individually-rational and ex-post ϵ-efficient. Similarly [17] shows that the demand functions that an agent might consider based on strategic considerations converge to the competitive demand functions. Further, [18] shows that under certain conditions on beliefs of individual agents, not only do the strategic behaviors of individual agents converge to the competitive behavior but the Nash equilibrium allocations also converge to the competitive equilibrium allocation. The formulation in [50] is a buyer's bid double auction with a single type of item that maximizes surplus. It is shown that with Bayesian-Nash strategies, the mechanism is asymptotically "incentive efficient," the notion of incentive efficiency being different from that of incentive compatibility and efficiency that we use here. Along a different line of investigation, [13, 46, 43] investigate the rate of convergence of the Nash equilibria to the competitive equilibria for buyer's bid double auction. Finally, implementation and mechanism design in a setting with a continuum of players is discussed in [31]. We have provided a market mechanism that asymptotically achieves competitive behavior in multilateral, multiple good trading environment with incomplete information.

Appendix A: Proof of Lemma 1

Proof: Set $a_{l,0}=c_{l,0}=0$ and $b_0=v_0=L$. Fix a buyer i with valuation v and bundle R_i. Suppose the sellers use a fixed bidding strategy α and denote the buyers' best-response strategy profile by β^n. Let θ_{-i} denote the strategy of all the other players. Then, there is a level U^*, a function of θ_{-i} such that the bid b of i is accepted if $b>U^*$. It is easy to see that the allocation $z(b)=(x(b),y(b))$ is some $z^*=(x^*,y^*)$ for all $b>U^*$. Suppose not: Let z_1 be the allocation for $U^*<U_1<b<U_2$ and z_2 be the allocation for $b>U_2$. But clearly, the auction surplus, $b-U_1>b-U_2$ for bids $b>U_2$ as well. Thus, the allocation z_1 will yield higher auction surplus than z_2 for $b>U_2$ as well. Thus, $z_2=z_1$ and the corresponding price Y^* is the same for all $b>U^*$. Note that $Y^*\le U^*$.

Thus, buyer i's payoff when he bids b is

$$\pi_i'(b)=\begin{cases} v-Y^*, & \text{if } b>U^* \\ 0, & \text{otherwise.}\end{cases} \tag{7}$$

The expected payoff denoted by $\bar{\pi}_i'$ then is given by

$$\bar{\pi}_i'(b)=\int_0^b\int_0^u (v-y)f_{Y^*,U^*}(y,u)\ dydu \tag{8}$$

and the buyer i's best response satisfies the differential equation

$$\frac{d\bar{\pi}_i'}{db}=\int_0^b (v-y)f_{Y^*,U^*}(y,b)\ dy\ \ =0 \tag{9}$$

The boundary condition for the differential equation is $\bar{\pi}_i'(0)=0$. Since the left-hand side of the equation above is always non-negative (and in fact positive) for all $b\le v$, it is clear that the best response $b=\beta_i^n(v)\ge v,\ \forall n\ge 2$. ■

Appendix B: Proof of Lemma 2

Proof: Fix a good l (say =1). Set $a_{l,0}=c_{l,0}=0$, and $b_0=v_0=L$. Fix a seller (l,j) with cost c (in the rest of the proof we will refer to this seller as seller j). Consider the auction game, denoted $\mathcal{G}_{-(l,j)}$, in which seller j bids very high and his bid is not accepted, and all buyers bid truthfully. Let $\underline{z}=(\underline{x},\underline{y})$ denote the corresponding allocation. Denote the number of matched buyers and sellers on good l by K_l, $X=a_{l(K_l)}$, the bid of the highest matched seller, $Y=a_{l(K_l+1)}$, the bid of the lowest unmatched seller, and $Z=a_{l(K_l-1)}$, the bid of the next highest matched seller. Suppose seller j bids a and let $\tilde{z}_t=(\tilde{x}_t,\tilde{y}_t)$ be the corresponding allocation. Let the allocation $\tilde{z}_t$ differ from $\underline{z}$ in the following way: There is a set of buyers $\underline{B}_t$ and a set of sellers $\underline{S}_t$ whose bids are accepted in $\underline{z}$ but not in $\tilde{z}_t$. And there is a set of buyers $\tilde{B}_t$ and a set of sellers $\tilde{S}_t$ (excluding j) whose bids are accepted in $\tilde{z}_t$ but not in $\underline{z}$. Then, the seller j's bid a is accepted if the auction surplus now is greater, i.e., if

$$v(\tilde{B}_t)-a(\tilde{S}_t)-a>v(\underline{B}_t)-a(\underline{S}_t), \tag{10}$$

Thus, if $a<W_t:=(v(\tilde{B}_t)-a(\tilde{S}_t))-(v(\underline{B}_t)-a(\underline{S}_t))$, the bids corresponding to allocation $\tilde{z}_t$ result in higher auction surplus than the bids corresponding to the allocation $\underline{z}$.

Now, for various levels of bid a, there may be many allocations $\tilde{z}_t,t=0,\cdots,T$ with corresponding levels $W_t,t=0,\cdots,T$. Observe that one possible allocation is $\tilde{B}_t=\underline{B}_t=\varnothing$, $\tilde{S}_t=\varnothing,\underline{S}_t=\{(l,(K_l))\}$ with (say) $W_0=X$. This is the case when the only change is that the seller j displaces the highest matched seller $(l,(K_l))$ on the good. Denote $W:=\max_{t\ge 1}W_t$. Note that out of the various levels W_t, only the maximum matters since the bid a is accepted as long as $a<\max_{t\ge 0}W_t$. Further, when that is true, the resulting allocation will be the one corresponding to $t^*=\arg\max_{t\ge 0}W_t$.

Thus, the payoff of the j-th seller when he bids $a = \alpha(c)$ is given by

$$\pi_j(a) = \begin{cases} x - c, & \text{if } a < Z < X < W, \text{ or} \\ & Z < a < X < W; \\ a - c, & \text{if } Z < X < a < W, \text{ or} \\ & Z < a < W < X, \text{ or} \\ & Z < W < a < X, \text{ or} \\ & W < Z < a < X; \\ z - c, & \text{if } a < Z < W < X, \text{ or} \\ & a < W < Z < X, \text{ or} \\ & W < a < Z < X. \end{cases} \tag{11}$$

The payoff of the seller as his bid a varies is shown graphically in figure 2. The reader can convince himself that the only relevant quantities for payoff calculation are X, Z and W. Thus, there are three cases: (i) $Z < X < W$, (ii) $Z < W < X$ and (iii) $W < Z < X$.

It is easy to verify that the expected payoff of seller j, denoted by $\bar{\pi}_j$ satisfies the differential equation

$$d\bar{\pi}_j(a) = [P^n(A_a) + P^n(B_a) + P^n(C_a)]da - (a - c)[dP^n(D_a) + dP^n(E_a)], \tag{12}$$

with the boundary condition $\bar{\pi}_j(1) = 0$, where A_a denotes the event $\{X < a < W\}$. As a is increased by da, the payoff to the seller increases by da since seller j is the price-determining seller. Similarly, B_a denotes the event $\{Z < a < W < X\}$ and seller j is the price-determining seller. In the same way, C_a denotes the event $\{\max(Z, W) < a < X\}$ and seller j is the price-determining seller. D_a denotes the event $\{X < a \text{ and } W \in [a, a + da]\}$, so that the seller j becomes unmatched as it increases its bid from a to $a + da$. Similarly, E_a is the event $\{W < a \text{ and } X \in [a, a + da]\}$. And so, as he increases his bid, he becomes unmatched.

Figure 2 shows these events graphically. Events A_a, B_a and C_a correspond to various cases when the change in the bid from a to $a + da$, causes a change in payoff of da. Events D_a and E_a correspond to cases when the change in the bid a from $a + da$, causes a change in payoff of $-(a - c)$.

Given the strategy profile α used by the sellers, the strategy profile $\tilde{\beta}$ used by the buyers, let the probability distribution of ask-bid of a seller on good l be F (with pdf f). Note that α and F depends on n.

We first obtain asymptotic upper and lower bounds on W (here called W_n to stress its dependence on n).

Proposition 1. *Define* $W_* := X_{1(K_1)}$ *and* $W^* := X_{1(K_1+1)}$. *Then, (i)* $W_* \le W_n \le W^*$ *in probability, i.e.,* $P(W_n \le W^*), P(W_* \le W_n) \to 1$ *as* $n \to \infty$. *(ii) For any* $\epsilon > 0$ *and large enough* n, $P(W_n > \epsilon) \le P(W^* > \epsilon)$ *and* $P(W_n \le \epsilon) \le P(W_* \le \epsilon)$.

Proof: (i) Let $\mathcal{B}_1$ denote the set of buyers who want good $l = 1$, and whose bids are not accepted when seller "a" is not "present". Consider any buyer $t \in \mathcal{B}_1$. Then,

$$\begin{aligned} W_t = [v_t - a(S(L_{1t}) \cup \overline{S}(L_{2t}))] \; &+ \; [v(\overline{B}_t) - a(S(L_{3t}) \cup \overline{S}(L_{4t}))] \\ &- \; [v(\underline{B}_t) - a(S(L_{1t}) \cup S(L_{3t}) \cup S(L_{5t}))], \end{aligned} \tag{13}$$

where $S(L)$ denotes the highest matched sellers on the set of goods L, $\overline{S}(L)$ denotes the lowest unmatched sellers on goods L, $a(S)$ denotes the sum of bids of the sellers S, $\overline{B}_t$ is the set of buyers (excluding t) whose bids can get accepted at seller bid "a", $\underline{B}_t$ is the set of buyers which become unmatched at new seller bid "a". Above, L_{1t} is the set of goods also demanded by buyer t and on which highest matched sellers remain matched; L_{2t} is the set of goods also demanded by buyer t where formerly unmatched sellers become matched; L_{3t} is the set of goods demanded by buyers $\overline{B}_t$ where highest matched sellers remain matched; L_{4t} is the set of goods demanded by buyers $\overline{B}_t$ where formerly unmatched sellers become matched; and L_{5t} is the set of goods demanded by $\underline{B}_t$ which now become unmatched. The first term in square brackets in equation (13) represents the contribution to the auction surplus when buyer t is matched; the third term represents the contribution to the auction surplus by buyers $\underline{B}_t$ which is being lost when seller "a" is introduced; the second term is the contribution to the auction surplus by buyers $\overline{B}_t$ whose acceptance becomes possible since buyers $\overline{B}_t$ are now unmatched. Thus, the sets $L_{1t}, \cdots, L_{5t}$ are disjoint and do not include $l = 1$. Thus, bid "a" can be accepted if $W_t > a$ for some $t \in \mathcal{B}_1$, i.e., if $W := \max_{t \in \mathcal{B}_1} W_t > a$.

Clearly, the third term in the square brackets of equation (13) is greater than the second term in the square brackets, otherwise the bids of $\overline{B}_t, S(L_{3t}), \overline{S}(L_{4t})$ would have been accepted before instead of bids of players $\underline{B}_t, S(L_{1t} \cup L_{3t} \cup L_{5t})$. Thus,

$$W_t \leq v_t - a(S(L_{1t}) \cup \overline{S}(L_{2t})) \leq v_t - a(S(L_{1t}) \cup S(L_{2t})),$$

where the second inequality is obvious.

Suppose a buyer t wants only good $l = 1$. Then, $W_t \leq v_t \leq X_{1(K_1+1)}$, the bid of the lowest unmatched seller of good 1, where K_1 is the number of matches for good $l = 1$. Next consider a buyer t who wants goods $l = 1, 2$. Then, $W_t \leq v_t - X_{2(K_2)}$ where K_2 is the number of matches on good 2. Further note that v_t must be smaller than $X_{1(K_1+1)} + X_{2(K_2+1)}$, otherwise buyer t could have matched with the lowest unmatched sellers on the two goods. Thus, we have

$$W_t \leq X_{1(K_1+1)} + (X_{2(K_2+1)} - X_{2(K_2)}).$$

Defining $\Delta_l(k) = (X_{l(k+1)} - X_{l(k)})$, we see that in general for a buyer t who wants goods R_t (including $l = 1$),

$$W_n := \max_{t \in \mathcal{B}_1} W_t \leq X_{1(K_1+1)} + \sum_{l \neq 1} \Delta_l(K_l) =: W_n^* \quad \text{a.s.} \tag{14}$$

Now, as $n \to \infty$, $\Delta_l(K_l) \xrightarrow{P} 0$ (convergence in probability) for every l. This implies that

$$W_n^* \xrightarrow{P} W^* := X_{1(K_1+1)}.$$

Thus, for $n \to \infty$

$$P(W_n \leq W^*) \ \to \ 1.$$

Let us now consider equation (13) to obtain a lower bound.

$$W_t \geq [v_t - a(\overline{S}(L_{1t} \cup L_{2t}))] - [a(\overline{S}(L_{3t})) - a(S(L_{3t}))]$$

since $v(\underline{B}_t) < a(\overline{S}(L_{1t} \cup L_{2t}) \cup S(L_{5t}))$ (otherwise the set of buyers $\underline{B}_t$ could still match). Also, note that the second term in the square brackets is $\sum_{l \in L_{3t}} \Delta_l(K_l) \xrightarrow{P} 0$ as $n \to \infty$.

Now, if buyer t wants only one good $l = 1$, then $L_{1t}, L_{2t} = \varnothing$ and $W_t \geq v_t \geq X_{1(K_1)}$ otherwise it cannot match. If buyer wants two goods (say 1 and 2), then $v_t > X_{1(K_1)} + X_{2(K_2)}$ otherwise it cannot match. Thus,

$$W_t \geq X_{1(K_1)} - \Delta_2(K_2) - \sum_{l \in L_{3t}} \Delta_l(K_l).$$

And, in general, we have

$$W_n := \max_{t \in \mathcal{B}_1} W_t \geq X_{1(K_1)} - \sum_{l \neq 1} \Delta_l(k) - \sum_{l \in L_{3t}} \Delta_l(K_l) =: W_*^n. \tag{15}$$

Since $\Delta_l(K_l) \xrightarrow{P} 0$ as $n \to \infty$ and for all l, we have

$$W_*^n \xrightarrow{P} W_* := X_{1(K_1)},$$

which implies for $n \to \infty$,

$$P(W_n \geq W_*) \ \to \ 1.$$

(ii) We will prove only the first part. We know that $W_n \leq W_n^*$ a.s. and $W_n^* \to W^*$ i.p. Thus, for some n and $0 < \delta < \epsilon$, we have

$$\begin{aligned} P(W_n > \epsilon) &= P(W_n > \epsilon, W_n^* \geq W^* + \delta) + P(W_n > \epsilon, W_n^* < W^* + \delta) \\ &\leq P(W_n^* \geq W^* + \delta) + P(W^* > \epsilon - \delta) \end{aligned}$$

and we get that

$$\limsup_n P(W_n > \epsilon) \leq P(W^* > \epsilon - \delta)$$

since $\limsup_n P(W_n^* \geq W^* + \delta) = 0$. Since, the inequality above is valid for any $0 < \delta < \epsilon$, we have that for large enough n, $P(W_n > \epsilon) \leq P(W^* > \epsilon)$. ∎

W_t can be interpreted as the "effective bid" of an unmatched buyer t (who wants good 1) on good 1. W is the highest such "effective bid". As long as a is smaller than W, bid a can be accepted. The proposition above shows that W in fact lies between $X = X_{1(K_1)}$ and $Y = X_{1(K_1+1)}$ when n becomes large (we will drop the subscript 1 for good $l = 1$ below). For a single good case, $W = b_{K+1}$, the highest unmatched buy-bid on the good, which is smaller than Y, and can only be accepted upon introducing another seller with bid "a" if it is bigger than X.

Now, observe that

$$\begin{aligned} P^n(A_a) &= \sum_k P(X < a | W > a, K = k) P(W > a, K = k) \\ &\leq \sum_k P(X_{(k)} < a < X_{(k+1)}) P(W^* > a, K = k) \\ &\lessapprox \sum_k P(X_{(k)} < a < X_{(k+1)}) P(X_{(k+1)} > a) \\ &= \sum_{k=0}^{n-1} \binom{n-1}{k} F^k(a) \overline{F}^{n-1-k}(a) \sum_{i=0}^{k} \binom{n-1}{i} F^i(a) \overline{F}^{n-1-i}(a) \end{aligned} \tag{16}$$

The first equality follows from conditioning and Bayes' rule and uses proposition 1. The second inequality holds asymptotically (for large n). The last equality is obtained using order statistics arguments.

In the same way, we can obtain the following:

$$\begin{aligned} P^n(B_a) &\lessapprox \sum_k P(X_{(k-1)} < a < X_{(k)}) P(X_{(k+1)} > a) \\ &= \sum_k \binom{n-1}{k-1} F^{k-1}(a) \bar{F}^{n-k}(a) \sum_{i=0}^{k} \binom{n-1}{i} F^i(a) \overline{F}^{n-1-i}(a), \end{aligned} \tag{17}$$

$$\begin{aligned} P^n(C_a) &\lessapprox \sum_k P(X_{(k-1)} < a < X_{(k)}) P(X_{(k-1)} < a < X_{(k)}) \\ &= \sum_k \binom{n-1}{k-1} F^{k-1}(a) \bar{F}^{n-k}(a) \binom{n-1}{k} F^k(a) \overline{F}^{n-1-k}, \end{aligned} \tag{18}$$

$$\begin{aligned} dP^n(D_a) &\gtrapprox \sum_k P(X_{(k)} \in [a, a + da)) P(X_{(k)} < a < X_{(k+1)}) \\ &= (n-1) f(a) \sum_k \binom{n-2}{k-1} F^{k-1}(a) \bar{F}^{n-1-k}(a) \binom{n-1}{k} F^k(a) \overline{F}^{n-k-1}(a) da. \end{aligned} \tag{19}$$

Let $a = \alpha_n(c)$ be the best-response strategy of the sellers on good $l = 1$. Further, $f(\alpha_n(c)) = 1/\alpha_n'(c)$ when the costs are uniformly distributed over [0,1]. Then, $\frac{d\bar{\pi}_i}{da} = 0$ at $a = \alpha^n(c)$. Now, for any $a \leq c$, $\frac{d\bar{\pi}_i}{da} > 0$ from (12). Thus,

$$a = \alpha^n(c) \geq c, \ \forall n \geq 2. \tag{20}$$

If $a > c$, from (12) after some rearrangement, we get

$$\alpha_n(c) - c \leq \frac{\sum_k [P^n(A_a) + P^n(B_a) + P^n(C_a)]}{dP^n(D_a)/da}$$

and using equations (16), (25), (26 and (27), we obtain that

$$\begin{aligned}[\alpha_n(c) - c] \leq \sup_{0<x<1} \alpha_n'(x) \cdot \sup_{0<z<1} [& \left(\frac{\sum_{k=0}^{n-1} \binom{n-1}{k} z^k (1-z)^{n-1-k} \sum_{i=0}^{k} \binom{n-1}{i} z^i (1-z)^{n-1-i}}{(n-1) \sum_{k=1}^{n-1} \binom{n-2}{k-1} z^{k-1} (1-z)^{n-k-1} \binom{n-1}{k} z^k (1-z)^{n-k-1}} \right) \\ & + \left(\frac{\sum_{k=1}^{n-1} \binom{n-1}{k-1} z^{k-1} (1-z)^{n-k} \sum_{i=0}^{k} \binom{n-1}{i} z^i (1-z)^{n-1-i}}{(n-1) \sum_{k=1}^{n-1} \binom{n-2}{k-1} z^{k-1} (1-z)^{n-k-1} \binom{n-1}{k} z^k (1-z)^{n-k-1}} \right) \\ & + \left(\frac{\sum_{k=1}^{n-1} \binom{n-1}{k-1} z^{k-1} (1-z)^{n-k} \binom{n-1}{k} z^k (1-z)^{n-1-k}}{(n-1) \sum_{k=1}^{n-1} \binom{n-2}{k-1} z^{k-1} (1-z)^{n-k-1} \binom{n-1}{k} z^k (1-z)^{n-k-1}} \right)]\end{aligned}$$

It can be checked that each of the terms in the square brackets converges to zero for all $0 < z < 1$ as $n \to \infty$. Thus, from assumption 1 and proposition 1, we get that $(\boldsymbol{\alpha^n}, \tilde{\boldsymbol{\beta}}) \to (\tilde{\boldsymbol{\alpha}}, \tilde{\boldsymbol{\beta}})$. ■

Appendix C: Proof of Theorem 1(iv)

We now show that in case of a multiple goods any Nash equilibrium allocation with non-zero trade for each good is efficient.

Let $\tilde{B}_{matched}$ and $\tilde{S}_{matched}$ denote the set of buyers and sellers that are matched at a Nash equilibrium $(\tilde{b}, \tilde{a})$.

Now suppose $\tilde{z} := (\tilde{x}, \tilde{y})$ is an allocation, corresponding to the Nash equilibrium $(\tilde{b}, \tilde{a})$, which is not efficient. Denote the efficient allocation that involves a trade for all goods by z^* and the allocation $\tilde{z}$ that also involves a trade for all goods but is not efficient. Then, the two allocations must differ in one of the following ways as we go from z^* to $\tilde{z}$:

(i) z^* and $\tilde{z}$ differ only among sellers: A (non-empty) set of sellers S_{out} matched in z^*, is no longer matched in $\tilde{z}$ and a (non-empty) set of sellers S_{in} are now matched;

(ii) z^* and $\tilde{z}$ differ only among buyers: A (non-empty) set of buyers B_{out} matched in z^*, is no longer matched in $\tilde{z}$ and a (non-empty) set of buyers B_{in} are now matched;

(iii) All buyers and sellers matched in z^* remain matched in $\tilde{z}$, and some new buyers B_{in} and some new sellers S_{in} now get matched;

(iv) No new buyers and sellers are matched in $\tilde{z}$ and some old buyers B_{out} and some old sellers S_{out} are now not matched;

(v) (*General Case*) A set of buyers B_{out} and a set of sellers S_{out} are no longer matched and a set of buyers B_{in} and a set of sellers S_{in} are now matched in $\tilde{z}$.

Case (i) Suppose $(l,j_1)\in S_{in}$ and $(l,j_2)\in S_{out}$. Then, it must be that $c_{l,j_1}>c_{l,j_2}$ but $\tilde{a}_{l,j_1}<\tilde{a}_{l,j_2}$. But then either (l,j_1)'s payoff is negative or (l,j_2) can also bid just below (l,j_1)'s bid. In either case $\tilde{z}$ cannot be a Nash equilibrium allocation.

Case (ii) Now, given the sets of buyers B_{in} and B_{out}, let $\tilde{p}_l$ denote the set of prices on the links at the allocation $\tilde{z}$. Then, it must be that

$$\sum_{i_1\in B_{out}} v_{i_1} \geq \sum_{i_2\in B_{in}} v_{i_2} \geq \sum_{i\in B_{in}}\sum_{l\in R_i}\tilde{p}_l.$$

The first inequality follows because the set of Buyers B_{out} match ahead of the buyers B_{in} at the efficient allocation z^*. The second inequality follows because the buyers B_{out} are matched with allocation $\tilde{z}$ and pay prices $\tilde{p}_l$. Furthermore,

$$\sum_{i_2\in B_{in}} \tilde{b}_{i_2} \geq \sum_{i\in B_{in}}\sum_{l\in R_i}\tilde{p}_l.$$

Now, clearly there exists an $i_1\in B_{in}$ can bid just above $\sum_{i_2\in B_{in}}\tilde{b}_{i_2}$ and match, and still have a positive payoff since the price it will at most have to pay is $\sum_{l\in R_{i_1}}\tilde{p}_l<v_{i_1}$. Thus, $(\tilde{b},\tilde{a})$ cannot be a Nash equilibrium in this case.

Case (iii) For a fixed l, denote $(l,\check{j}):=\arg\min_{(l,j)\in S_{in}}\tilde{a}_{l,j}$ and let $\check{i}\in B_{in}$ be any such buyer. Then, $v_{\check{i}}<\sum_{l\in R_{\check{i}}}c_{l,\check{j}}$ since these bidders are not matched at z^* and $\tilde{b}_{\check{i}}\geq\sum_{l\in R_{\check{i}}}\tilde{a}_{l,\check{j}}$ since they are matched at $\tilde{z}$. But then either buyer $\check{i}$ or one of the sellers $(l,\check{j})$ with $l\in R_{\check{i}}$ has a negative payoff at $(\tilde{b},\tilde{a})$, and so will deviate, in which case it cannot be a Nash equilibrium outcome.

Case (iv) Denote $\check{j}(l):=\arg\min_{(l,j)\in S_{out}}c_{l,j}$ and $\check{i}\in B_{out}$, any such buyer. And denote the prices with bids $(\tilde{b},\tilde{a})$ by $\tilde{p}$. Then, $\tilde{p}_l\leq c_{l,\check{j}(l)}$ otherwise any seller $(l,\check{j}(l))$ can outbid the highest matched seller on l in the allocation $\tilde{z}$.

Furthermore, $v_{\check{i}}\geq\sum_{l\in R_{\check{i}}}c_{l,\check{j}(l)}$ and $\tilde{b}_{\check{i}}<\sum_{l\in R_{\check{i}}}\tilde{a}_{l,\check{j}(l)}$. Now, if $\sum_{l\in R_{\check{i}}}\tilde{a}_{l,\check{j}(l)}\leq v_{\check{i}}$, then clearly, buyer $\check{i}$ has an incentive to bid just above $\sum_{l\in R_{\check{i}}}\tilde{a}_{l,\check{j}(l)}$ and match. And if $\sum_{l\in R_{\check{i}}}\tilde{a}_{l,\check{j}(l)}>v_{\check{i}}\geq\sum_{l\in R_{\check{i}}}c_{l,\check{j}(l)}\geq\sum_{l\in R_{\check{i}}}\tilde{p}_l$, then again the buyer $\check{i}$ can bid high enough to match and pay $\sum_{l\in R_{\check{i}}}\tilde{p}_l$. Thus, in either of these cases, the bids under consideration cannot be a Nash equilibrium.

Case (v) Denote $\hat{j}(l):=\arg\max_{(l,j)\in S_{in}}\tilde{a}_{l,j}$ and $\check{j}(l):=\arg\min_{(l,j)\in S_{out}}\tilde{a}_{l,j}$, and let $\hat{i}\in B_{in}$ and $\check{i}\in B_{out}$ be any buyers. Let $R_{\check{i}\hat{i}}:=R_{\check{i}}\cap R_{\hat{i}}$ be the set of common goods in the bundles of the two buyers. Denote $R_{\check{i}\check{i}}:=R_{\check{i}}\setminus R_{\check{i}\hat{i}}$ and $R_{\hat{i}\hat{i}}:=R_{\hat{i}}\setminus R_{\check{i}\hat{i}}$. We shall use the shorthand $\tilde{p}(R)$ to mean the sum $\sum_{l\in R}\tilde{p}_l$.

Now, suppose that there is a common good between sellers S_{in} and S_{out}. Then, clearly by case (i), the bids under consideration cannot be a Nash equilibrium.

Thus, suppose that the sellers S_{in} and S_{out} do not have any good in common. Now, suppose that no two buyers $\hat{i}\in B_{in}$ and $\check{i}\in B_{out}$ have a good in common, i.e., $R_{\check{i}\hat{i}}=\emptyset$. Then, again this will reduce to cases (iii) and (iv) above, and we can conclude that the bids under consideration cannot be a Nash equilibrium.

Now, we are left with the case where sellers S_{in} and S_{out} do not have a good in common and some buyers $\hat{i}$ and $\check{i}$ do, i.e., $R_{\check{i}\hat{i}}\neq\emptyset$. Further, we are given that there is at least one trade for each good. Thus, $\tilde{p}_l\leq c_{l,\check{j}(l)}\leq p^*_l$ and $\tilde{p}_l\geq c_{l,\hat{j}(l)}\geq p^*_l$ for all relevant l otherwise some sellers would have an incentive to deviate. Further note that

$$v_{\check{i}}-\sum_{l\in R_{\check{i}}}p^*_l > v_{\hat{i}}-\sum_{l\in R_{\hat{i}}}p^*_l.$$

This implies that either

$$v_{\check{i}} - \tilde{p}(R_{\check{i}\check{i}}) \geq \min\{v_{\hat{i}}, b_{\hat{i}}\} - \tilde{p}(R_{\hat{i}\hat{i}}) \geq \tilde{p}(R_{\check{i}\hat{i}})$$

and so buyer $\check{i}$ can outbid the buyer $\hat{i}$, or

$$v_{\hat{i}} - \tilde{p}(R_{\hat{i}\hat{i}}) < \tilde{p}(R_{\check{i}\hat{i}})$$

and buyer $\hat{i}$ has a negative payoff.

In either case, the bids under consideration cannot be a Nash equilibrium.

Thus, for every case above, the corresponding bids cannot be a Nash equilibrium. This proves claim (iv) of the theorem. ∎

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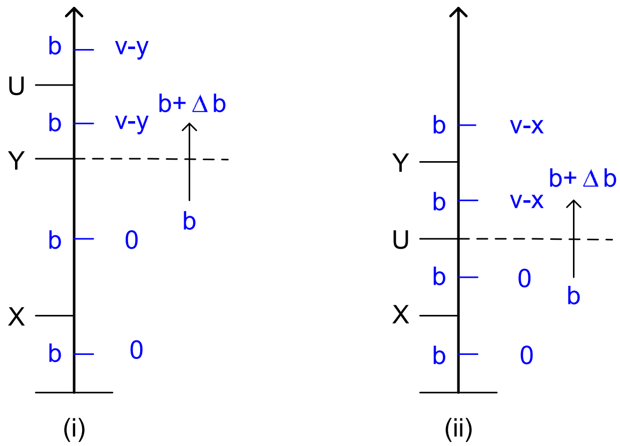

Figure 1 The payoff of the buyer as a function of its bid b for various cases.

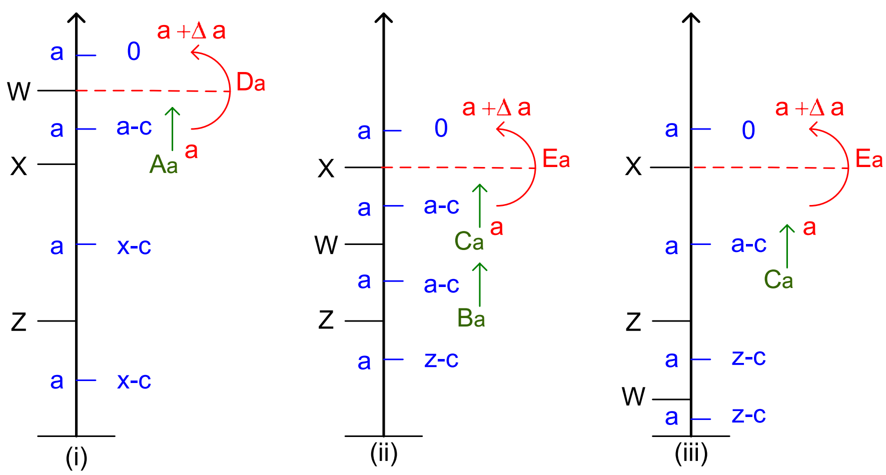

Figure 2 The payoff of the seller as a function of its bid a for various cases.