The Stability of the Roommate Problem Revisited*

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Abstract

The purpose of this paper is to propose the solution of absorbing sets, that extends the notion of core stability, for solving the *entire* class of the roommate problems with strict preferences. We characterize the solution, that alwasys exists, and determine the number of absorbing sets. In case of multiplicity we show that all of them share a similar structure. Moreover, we find that the absorbing sets solution satisfies a property of outer stability.

Keywords: roommate problems, stability and absorbing sets.

Very preliminary draft

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1 Introduction

The analysis of the stability in matching literature can be traced back to Gale and Shapley (1962). These authors showed that stable matchings (in the sense of core) always exist in simple two-sided markets, however, this property is not true neither in problems with more complicated agents' preferences nor in one or three-sided markets.

In this paper we consider the special case of the roommate problem, an instance of one-sided market, introduced by Gale and Shapley (1962) as a generalization of the classical marriage problem. A roommate problem consists of a set of agents and for each agent a preference list which is a total order over the set of agents. They showed by means of an example that, in contrast to the marriage problem, the roommate problem is one of those matching problems where stable matchings do not always exist. Empirical results in Pittel and Irving (1994) suggest that as the number of agents increases, the probability that a roommate problem with n agents admits a stable matching decreases fairly steeply. Hence it is clear the necessity of the study of the stability for the entire class of the roommate problem.

While (core) stability for matching problems has been investigated extensively, Gale and Shapley (1962) Irving (1985) Roth and Sotomayor (1990) Tan (1991) Abeledo and Isaak (1991) Chung (2000) and Diamantoudi, Miyagawa and Xue (2004), to the best of our knowledge, with the exception of few papers (see for instance Tan (1990) and Abraham, Biro and Manlove (2005) there are not works applying alternative notions of stability to unsolvable roommate problems, that is roommate problems that do not admit core stable matchings.

In this paper we model the roommate problem as a matching system[1] where the vertices are the matchings that can be formed and the edges depict the existence of a myopic blocking pair which allows to go from one matching to another by satisfying it. To solve the matching system we propose the absorbing set solution[2], which extends the notion of (core) stability and that solves the *entire* class of the roommate problems with strict preferences. This solution applied to a matching system can be illustrate as follows:

[1] An abstract system is a pair (X, R), where X is a set of elements and R is a binary relation defined on X.

[2] An absorbing set is a minimal closed subset of the *admissible set*, Kalai, Pazner and Schmeidler, 1976, and Kalai and Schmeidler [1977] and it coincides with the *elementary dynamic solution* of Shenoy [].

> Consider a point for each matching and imagine that a point of light is switched on each of these matching points. Then the following process follows indefinitely: each time each matching-point turns off and passes the light to all matchings-points that can result from it by some blocking couple (in this way a matching can keep its light only if receives the light from another). After a finite number of periods only the matchings-points belonging to some "stable constellation" will keep permanently on light. Moreover, a particular stable constellation can be individualized as such among all the stars by the fact that it is "energetically closed", that is, more precisely, it is a minimal set of self-lightening matching-points. In this terminology a stable matching is then just a stable constellation which consists of a single star.

In this paper we explore the solution of absorbing sets for the roommate problem with strict preferences.

With respect to the existence of the solution proposed we find that every roommate problem have at least one absorbing set. We also prove that if the roommate problem is solvable then the absorbing sets are singletons consisting of a single stable matching. Consequently this solution concept extends the notion of core stability.

We characterize the absorbing sets in terms of the *stable partitions*[3]. This characterization allows to determine the matchings belonging to the absorbing sets. Moreover, we find exactly the stable partitions which determine the absorbing sets.

In case of multiplicity we first determine the number of absorbing sets. Next, we show that all the absorbing sets share the same structure in the following sense: Given an arbitrary absorbing set, the set of agents can be partitioned into the set of agents which form part of some blocking pair of some matching belonging to the absorbing sets and the set of non blocking agents. Then we find for any two absorbing sets blocking and non blocking agents coincide. Moreover

[3]Tan (1991) introduces the notion of stable partition and gives a necessary and sufficient condition for the existance of a stable matching in roommate problems with stric preferences.

the matchings restricted to the blocking agents coincide in the two sets, whereas those restricted to non blocking agents are singletons consisting of a stable matching. Informally, it can be said that the constellations are quasi-replicas of one another.

Finally, we show that the absorbing sets has the property of outer stability in the following sense: Every matching not in any absorbing set can be dominated by a matching belonging to some absorbing set. Following the metaphor used above it can be said that after a finite number of periods only the matching-points belonging to some constellations will keep illuminated.

The paper is organized as follows. Section 2 contains the preliminaries of the paper. In Section 3 we recall the notion of stable partition and define some specific matchings related to them. Section 4 contains the analysis of the absorbing sets.

2 Preliminaries

A *roommate problem* is a pair $(N, (\succcurlyeq_x)_{x\in N})$ where N is a finite set of agents and for each agent $x \in N$, $\succcurlyeq_x$ is a complete, transitive preference relation defined over N. Let $\succ_x$ be the strict preference associated with $\succcurlyeq_x$. In this paper we only consider roommate problems with strict preferences, which we denote by $(N, (\succ_x)_{x\in N})$.

A *matching* μ is a one to one mapping from N onto itself such that for all x, $y \in N$, if $\mu(x) = y$, then $\mu(y) = x$. Let $\mu(x)$ denote the partner of agent x under the matching μ. If $\mu(x) = x$, then agent x is single under μ.

Let $\mathcal{M}$ be the set of all matchings. Let $S \subseteq N$. For any $\mu \in \mathcal{M}$ such that $\mu(S) = S$, $\mu\mid_S$ denotes the restriction of μ to S. Moreover, if $\mathcal{A} \subseteq \mathcal{M}$ and, for any $\mu \in \mathcal{M}$, $\mu(S) = S$, then $\mathcal{A}\mid_S = \{\mu\mid_S : \mu \in \mathcal{A}\}$.

A pair of agents $\{x, y\} \subseteq N$ (without ruling out $x = y$) blocks the matching μ if

$$y \succ_x \mu(x) \text{ and } x \succ_y \mu(y). \qquad [1]$$

That is, x and y prefer each other to their current partners at μ. If $x = y$, [1] means that agent x prefers being alone to being matched with $\mu(x)$. When [1] holds, we call $\{x, y\}$ a *blocking pair* of μ.

A matching satisfies *individual rationality* if it is not blocked by any pair $\{x, y\}$ such that $x = y$. A matching is called *stable* if it is not blocked by any pair.

Let $\{x, y\}$ be a blocking pair of μ. A matching μ' is obtained from μ by satisfying $\{x, y\}$ if $\mu'(x) = y$ and for all $z \in N \backslash \{x, y\}$,

$$\mu'(z) = \begin{cases} z & if\ \mu(z) \in \{x, y\} \\ \mu(z) & \text{otherwise.} \end{cases}$$

That is, once $\{x, y\}$ is formed, their partners (if any) at μ are alone in μ' while the remaining agents are matched as in μ.

The *abstract system associated with a roommate problem* $(N, (\succ_x)_{x \in N})$ is the pair $(\mathcal{M}, R)$ where R is the binary relation defined over $\mathcal{M}$ as follows: Given μ, $\mu' \in \mathcal{M}$, $\mu' R \mu$ if and only if μ' is obtained from μ by satisfying a blocking pair of μ. Let R^T denote the transitive closure of R. Then $\mu' R^T \mu$ if and only if there exists a finite sequence of matchings $(\mu = \mu_0, \mu_1, ..., \mu_m = \mu')$ such that for all $i \in \{1, ..., m\}$, $\mu_i R \mu_{i-1}$.

R^T is called a *domination relation*. Then, if $\mu' R^T \mu$ we say that μ' *dominates* to μ and that μ' *directly dominates* to μ when $\mu' R \mu$.

3 P-stable matchings

In [] we define the P-stable matching concept associated with the notion of a stable partition introduced by Tan (1991) as follows.

Let $A = \{a_1, ..., a_k\} \subseteq N$ be an ordered set of agents. The set A is a *ring* if $k \geq 3$ and for all $i \in \{1, ..., k\}$, $a_{i+1} \succ_{a_i} a_{i-1} \succ_{a_i} a_i$ (subscript modulo k). The set A is a pair of mutually acceptable agents if $k = 2$ and for all $i \in \{1, 2\}$, $a_{i-1} \succ_{a_i} a_i$ (subscript modulo 2)[4]. The set A is a singleton if $k = 1$.

A *stable partition* is a partition P of N such that:
(i) For all $A \in P$, the set A is a ring, a mutually acceptable pair of agents or a singleton, and
(ii) For any sets $A = \{a_1, ..., a_k\}$ and $B = \{b_1, ..., b_l\}$ of P (possibly $A = B$), the following condition holds:

$$\text{if } \ b_j \succ_{a_i} a_{i-1} \text{ then } b_{j-1} \succ_{b_j} a_i,$$

[4] Hereafter we omit subscript modulo k.

for all $i \in \{1, ..., k\}$ and $j \in \{1, ..., l\}$ such that $b_j \neq a_{i+1}$.

Condition (ii) may be interpreted as a notion of stability over partitions.

Remark 1 *The following assertions are proved by Tan (1991):*
(i) For any roommate problem $(N, (\succ_x)_{x \in N})$, there exists at least one stable partition. Furthermore, any two stable partitions have exactly the same odd sets[5].
(ii) Each even ring of a stable partition can be broken into pairs of mutually acceptable agents preserving stability.
(iii) A roommate problem $(N, (\succ_x)_{x \in N})$ has no stable matchings if and only if there exists a stable partition with some odd ring.

Definition 1 *Let P be a stable partition. A P-stable matching is a matching μ such that for each $A = \{a_1, ..., a_k\} \in P$, $\mu(a_i) \in \{a_{i+1}, a_{i-1}\}$ for all $i \in \{1, ..., k\}$ except for a unique j where $\mu(a_j) = a_j$ if A is odd.*

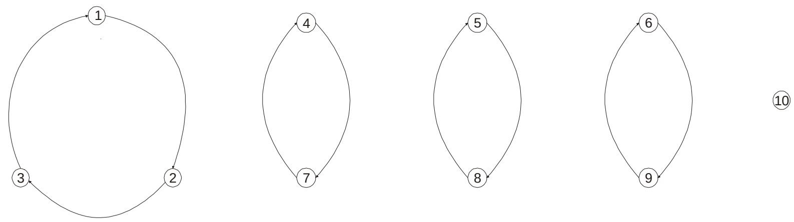
To illustrate the notion of P-stable matching let us consider the following example.

Example 1 *Consider the following 10-agent roommate problem:*

$$\begin{array}{l}
2 \succ_1 3 \succ_1 4 \succ_1 5 \succ_1 6 \succ_1 7 \succ_1 8 \succ_1 9 \succ_1 1 \succ_1 10 \\
3 \succ_2 1 \succ_2 4 \succ_2 5 \succ_2 6 \succ_2 7 \succ_2 8 \succ_2 9 \succ_2 2 \succ_2 10 \\
1 \succ_3 2 \succ_3 4 \succ_3 5 \succ_3 6 \succ_3 7 \succ_3 8 \succ_3 9 \succ_3 3 \succ_3 10 \\
7 \succ_4 8 \succ_4 9 \succ_4 5 \succ_4 6 \succ_4 1 \succ_4 2 \succ_4 3 \succ_4 4 \succ_4 10 \\
8 \succ_5 9 \succ_5 7 \succ_5 4 \succ_5 6 \succ_5 5 \succ_5 1 \succ_5 2 \succ_5 3 \succ_5 10 \\
9 \succ_6 7 \succ_6 8 \succ_6 4 \succ_6 5 \succ_6 6 \succ_6 1 \succ_6 2 \succ_6 3 \succ_6 10 \\
5 \succ_7 6 \succ_7 1 \succ_7 4 \succ_7 9 \succ_7 8 \succ_7 7 \succ_7 2 \succ_7 3 \succ_7 10 \\
6 \succ_8 4 \succ_8 5 \succ_8 7 \succ_8 9 \succ_8 8 \succ_8 1 \succ_8 2 \succ_8 3 \succ_8 10 \\
4 \succ_9 5 \succ_9 6 \succ_9 7 \succ_9 8 \succ_9 9 \succ_9 1 \succ_9 2 \succ_9 3 \succ_9 10 \\
10 \succ_{10} 1 \succ_{10} ...
\end{array}$$

It is easy to verify that $P = \{\{1, 2, 3\}, \{4, 7\}, \{5, 8\}, \{6, 9\}, \{10\}\}$ is a stable partition where $A_1 = \{1, 2, 3\}$ is an odd ring, $A_2 = \{4, 7\}$, $A_3 = \{5, 8\}$, $A_4 = \{6, 9\}$ are pairs of mutually acceptable agents and $A_5 = \{10\}$ is a singleton. Partition P can be represented graphically as follows:

[5] We say that A is an odd (even) set of P if the cardinal of A is odd (even).



The P-stable matchings associated with the stable partition P are:

$$\begin{aligned}
\mu_1 &= [\{1\},\{2,3\},\{4,7\},\{5,8\},\{6,9\},\{10\}] \\
\mu_2 &= [\{2\},\{1,3\},\{4,7\},\{5,8\},\{6,9\},\{10\}] \\
\mu_3 &= [\{3\},\{1,2\},\{4,7\},\{5,8\},\{6,9\},\{10\}].
\end{aligned}$$

Remark 2 *If μ is a P-stable matching, then the matching that results if the single agents from odd rings are excluded from μ is stable.*[6]

Remark 3 *For a solvable roommate problem $(N, (\succ_x)_{x \in N})$ the set of P-stable matchings for all stable partitions coincides with the set of stable matchings*[7]*.*

For solvable roommate problems with strict preferences Diamantoudi et al.(2004) prove that for any unstable matching, there exists a finite sequence of successive myopic blockings leading to a stable matching. In Theorem 1 of [] we generalize the previous result by proving for any roommate problem with strict preferences, that *"for any matching μ, there exists a finite sequence of matchings $(\mu = \mu_0, \mu_1, ..., \mu_m = \overline{\mu})$ such that for all $i \in \{1, ..., m\}$, μ_i is obtained from μ_{i-1} by satisfying a blocking pair of μ_{i-1} and $\overline{\mu}$ is a P-stable matching for some stable partition P"*.

4 Absorbing sets

In this section we introduce our solution concept for a roommate problem with strict preferences.

[6] See Tan (1990).
[7] See [].

Definition 2 *A set $\mathcal{A} \subseteq \mathcal{M}$ is an absorbing set if*
i) for any two distinct $\mu, \mu' \in \mathcal{A}$, $\mu R^T \mu'$, and
ii) for any $\mu \in \mathcal{A}$ there is no a $\mu' \notin \mathcal{A}$ such that $\mu' R \mu$.

Notice that condition *i)* means that the matchings of $\mathcal{A}$ are symmetrically connected by the relation R^T and condition *ii)* that the set $\mathcal{A}$ is R closed.

We know some such solution exists from Theorem 1 of [] [8]. Our purpose is to characterize this solution concept in terms of stable partitions.

In what follows whenever we write stable partition we refer to a stable partition which no contains any even ring.[9]
Given a stable partition P denoted by $\mathcal{A}_P$ to the set of all P-stable matchings and those that dominate to them.

We first establish that an absorbing set is one of these sets $\mathcal{A}_P$.

Lemma 1 *Let $(N, (\succ_x)_{x \in N})$ be a roommate problem. If $\mathcal{A}$ is an absorbing set then $\mathcal{A} = \mathcal{A}_P$ for some stable partition P.*

Proof. First we prove that there exists a P-stable matching $\overline{\mu}$ such that $\overline{\mu} \in \mathcal{A}$. Let $\mu \in \mathcal{A}$. If μ is a P-stable matching for some stable partition P, $\overline{\mu} = \mu$ and we are done. Otherwise, by Theorem 1 of [ ], there exists a P-stable matching $\overline{\mu}$ such that $\overline{\mu} R^T \mu$. But then, by condition *ii)* of Definition 2, $\overline{\mu} \in \mathcal{A}$.

Now we prove that $\mathcal{A} = \mathcal{A}_P$. By Lemma 7 of the Appendix, we have $\mathcal{A}_P = \{\overline{\mu}\} \cup \left\{\mu \in \mathcal{M} : \mu R^T \overline{\mu}\right\}$.
($\subseteq$): Let $\mu \in \mathcal{A}$. We must show that $\mu \in \mathcal{A}_P$. If $\mu = \overline{\mu}$ as $\overline{\mu} \in \mathcal{A}_P$ we are done. Suppose that $\mu \neq \overline{\mu}$. As $\overline{\mu} \in \mathcal{A}$, by condition *i)* of Definition 2, we have $\mu R^T \overline{\mu}$. Hence $\mu \in \mathcal{A}_P$.
($\supseteq$): Let $\mu \in \mathcal{A}_P$. We must show that $\mu \in \mathcal{A}$. If $\mu = \overline{\mu}$ since $\overline{\mu} \in \mathcal{A}$ we are done. If $\mu \neq \overline{\mu}$ then $\mu R^T \overline{\mu}$. But as $\overline{\mu} \in \mathcal{A}$, by condition *ii)* of Definition 2, it follows that $\mu \in \mathcal{A}$. ■

Then as an easy corollary of Lemma 1 and Remark 3 we have:

[8] The *admissible set* of $(\mathcal{M}, R)$ (Kalai and Schmeidler[]) is the set

$$\mathcal{A}^* = \{\mu \in \mathcal{M} : \mu' R^T \mu \Leftrightarrow \mu R^T \mu'\}$$

This set is the union disjoint of the minimally R-closed subsets of $\mathcal{M}$, which are the absorbing sets.

[9] By Remark 1, we can assume it, without loss of generality.

Remark 4 *If the roommate problem* $(N, (\succ_x)_{x\in N})$ *is solvable then* $\mathcal{A}$ *is an absorbing set if and only if* $\mathcal{A} = \{\mu\}$ *where* μ *is a stable matching.*

In fact, if $\mathcal{A}$ is an absorbing set then, by Lemma 1, $\mathcal{A}$ is the set of all P-stable matchings and those that dominate to them for some stable partition P. But as the roommate problem is solvable then there is an unique P-stable matching which is stable. Hence $\mathcal{A} = \{\mu\}$ where μ is a stable matching. Conversely, if $\mathcal{A} = \{\mu\}$ where μ is a stable matching then it is very easy to see that $\mathcal{A}$ satisfies conditions *i)* and *ii*) of Definition 2.

No all stable partitions determine an absorbing set, that is, no all sets $\mathcal{A}_P$ are absorbing sets. For example, the roommate problem given in Example 1 has three stable partitions:
$P_1 = \{\{1,2,3\},\{4,7\},\{5,8\},\{6,9\},\{10\}\}$, $P_2 = \{\{1,2,3\},\{4,8\},\{5,9\},\{6,7\},\{10\}\}$ and $P_3 = \{\{1,2,3\},\{4,9\},\{5,7\},\{6,8\},\{10\}\}$. It is very easy to verify, using a computer, that this roommate problem has two absorbing sets $\mathcal{A}$ and $\mathcal{A}'$ where $\mathcal{A} = \mathcal{A}_{P_2}$ and $\mathcal{A}' = \mathcal{A}_{P_3}$. However, $\mathcal{A} \neq \mathcal{A}_{P_1}$ and $\mathcal{A}' \neq \mathcal{A}_{P_1}$.

Our purpose is to find the stable partitions which determine the absorbing sets. To it, denote by D_P the set of agents that form part of a blocking pair of some $\mu \in \mathcal{A}_P$ and by $S_P = N \backslash D_P$.

Fix a stable partition P the set D_P can be obtained by an iterative procedure with a finite number of steps. To it, we define recursively a sequence of sets D_0, D_1,..., D_t, ... as follows:
i) for $t = 0$, D_0 is the union of all odd rings of P.
ii) for $t \geq 1$, $D_t = D_{t-1} \cup B_t$ where $B_t = \{b_1, ..., b_l\} \in P$ ($l = 1$ ó 2), for which there exists a $A_t = \{a_1, ..., a_k\} \in P$, $A_t \subseteq D_{t-1}$, such that

$$b_j \succ_{a_i} a_i \text{ and } a_i \succ_{b_j} b_{j-1}, \qquad [2]$$

for some $i \in \{1, ..., k\}$ and $j \in \{1, ..., l\}$.
Given that P contains a finite number of sets then the procedure terminates in a finite number of steps. Let r be the minimun number such that $D_{r+1} = D_r$. Then, by Lemma 8 of the appendix, $D_r = D_P$.

To illustrate how this procedure may be used, let us consider again the roommate problem given in Example 1. Let $P = \{\{1,2,3\},\{4,7\},\{5,8\},\{6,9\},\{10\}\}$. First note that P contains an unique odd ring. Then $D_0 = \{1,2,3\}$. Let

$B_1 = \{4,7\}$ and $A_1 = \{1,2,3\}$. Since $7 \succ_1 1$ and $1 \succ_7 4$, then $D_1 = D_0 \cup B_1 = \{1,2,3,4,7\}$. Consider now the sets $B_2 = \{5,8\}$ and $A_2 = \{4,7\}$. As $8 \succ_4 4$ and $4 \succ_8 5$, then $D_2 = D_1 \cup B_2 = \{1,2,3,4,7,5,8\}$. Finally, let $B_3 = \{6,9\}$ and $A_3 = \{5,8\}$. Since $9 \succ_5 5$ and $5 \succ_9 6$, then $D_3 = D_2 \cup B_3 = \{1,2,3,4,7,5,8,6,9\}$ and the procedure finishes. Hence $D_P = D_3$.

Notice that, for any set $A \in P$, either $A \subseteq D_P$ or $A \subseteq S_P$. Denote by $P \mid_{D_P} = \{A \in P : A \subseteq D_P\}$, $P \mid_{S_P} = \{A \in P : A \subseteq S_P\}$ and by $\mathcal{P} = \{P \mid_{S_P} : P$ is a stable partition$\}$.

We next state and prove one of our main results.

Theorem 2 *Let $(N, (\succ_x)_{x \in N})$ be a roommate problem. Then, $\mathcal{A}$ is an absorbing set if and only if $\mathcal{A} = \mathcal{A}_P$ for some stable partition P such that $P \mid_{S_P}$ is maximal in $\mathcal{P}$*[10]*.*

Proof. ($\Longrightarrow$): Let $\mathcal{A}$ be an absorbing set. Then, by Lemma 1, $\mathcal{A} = \mathcal{A}_P$ for some stable partition P. We prove that $P \mid_{S_P}$ is maximal in $\mathcal{P}$. Suppose, by contradiction, that $P \mid_{S_P}$ is not maximal. Then there exists a stable partition P' such that $P \mid_{S_P} \subset P' \mid_{S_{P'}}$. Let μ and μ' be P-stable and P'-stable matchings, respectively. Then, by Lemma 11 of the Appendix, $\mu' R^T \mu$. Now, as $\mu \in \mathcal{A}$ and μ' dominates to μ, then, by condition *ii*) of Definition 2, $\mu' \in \mathcal{A}$. But then, by condition *i),* $\mu R^T \mu'$ and therefore, by Lemma 11, $P' \mid_{S_{P'}} \subseteq P \mid_{S_P}$, contradicting the fact that $P \mid_{S_P} \subset P' \mid_{S_{P'}}$.
($\Longleftarrow$): We prove that if $P \mid_{S_P}$ is maximal in $\mathcal{P}$ then $\mathcal{A}_P$ is an absorbing set. To it, we must show that $\mathcal{A}_P$ satisfies conditions *i)* and *ii)* of Definition 2. By Lemma 7 of the Appendix, $\mathcal{A}_P = \{\overline{\mu}\} \cup \left\{\mu \in \mathcal{M} : \mu R^T \overline{\mu}\right\}$ where $\overline{\mu}$ is a P-stable matching. Let $\mu \in \mathcal{A}_P$. If $\mu' R \mu$ we have $\mu' R^T \overline{\mu}$. But then, $\mu' \in \mathcal{A}_P$ and Condition *ii)* follows.
Now we show that $\mathcal{A}_P$ satisfies the conditions *ii).* It suffices to prove that $\overline{\mu} R^T \mu$ for any $\mu \in \mathcal{A}_P$ such that $\mu \neq \overline{\mu}$. If μ is not a P'-stable matching for any stable partition P', by Theorem 1 [ ], there exists a P'-stable matching μ' such that $\mu' R^T \mu$. As $\mu R^T \overline{\mu}$ we have $\mu' R^T \overline{\mu}$ (If μ' is a P'-stable matching consider $\mu' = \mu$). But then, by Lemma 11, $P \mid_{S_P} \subseteq P' \mid_{S_{P'}}$, and as $P \mid_{S_P}$ is maximal in $\mathcal{P}$, $P \mid_{S_P} = P' \mid_{S_{P'}}$, hence $\overline{\mu} R^T \mu'$. Since $\mu' R^T \mu$ we have $\overline{\mu} R^T \mu$. ■

[10] We say that $P \mid_{S_P}$ is maximal in $\mathcal{P}$ if there is no a stable partition P' such that $P \mid_{S_P} \subset P' \mid_{S_{P'}}$.

We now determine the number of absorbing sets for a roommate problem.

Theorem 3 *Let $(N, (\succ_x)_{x\in N})$ be a roommate problem. Then, the number of absorbing sets is equal to the number of maximal partitions of $\mathcal{P}$.*

Proof. It follows of Theorem 2 and Lemma 12 of the Appendix. $\blacksquare$

Let $\mathcal{A}$ be an absorbing set. Denote by $D_{\mathcal{A}}$ the set of agents that form part of a blocking pair of some matching $\mu \in \mathcal{A}$ and by $S_{\mathcal{A}} = N \backslash D_{\mathcal{A}}$.

Next, we show that all the absorbing sets share a "similar" structure, in the following sense:

Theorem 4 *Let $(N, (\succ_x)_{x\in N})$ be a roommate problem. Then, for any two absorbing sets $\mathcal{A}$ and $\mathcal{A}'$, it holds that*
i) $D_{\mathcal{A}} = D_{\mathcal{A}'}$ and $S_{\mathcal{A}} = S_{\mathcal{A}'}$.
ii) $\mathcal{A} \mid_D = \mathcal{A}' \mid_D$ where $D = D_{\mathcal{A}} = D_{\mathcal{A}'}$.
iii) $\mathcal{A} \mid_S$ and $\mathcal{A}' \mid_S$ are singletons consisting of a stable matching in $(S, (\succ_x)_{x\in S})$, where $S = S_{\mathcal{A}} = S_{\mathcal{A}'}$.

Proof. From Theorem 2 there exist stable partitions P and P' such that $\mathcal{A} = \mathcal{A}_P$, $\mathcal{A}' = \mathcal{A}_{P'}$ where $P \mid_{S_P}$ and $P' \mid_{S_{P'}}$ are maximals in $\mathcal{P}$.
i) As $S_{\mathcal{A}} = S_P$ and $S_{\mathcal{A}'} = S_{P'}$ and, by Lemma 14, $S_P = S_{P'}$, then $S_{\mathcal{A}} = S_{\mathcal{A}'}$ and therefore $D_{\mathcal{A}} = D_{\mathcal{A}'}$.
ii) It is very easy to verify that $\mathcal{A} \mid_D$ and $\mathcal{A}' \mid_D$ are absorbing sets in the roommate problem $(D, (\succ_x)_{x\in D})$ such that $\mathcal{A} \mid_D = \mathcal{A}_{P|_D}$ and $\mathcal{A}' \mid_D = \mathcal{A}_{P'|_D}$. Now, as $S_{P|_D} = S_{P'|_D} = \oslash$, by Lemma 12, we have $\mathcal{A} \mid_D = \mathcal{A}' \mid_D$.
iii) It follows directly from Lemma 10. $\blacksquare$

Remark 5 *An immediate consequence of this property is that the roommate problem restricted to D is unsolvable and contains an unique absorbing set. However the roommate problem restricted to S is solvable.*

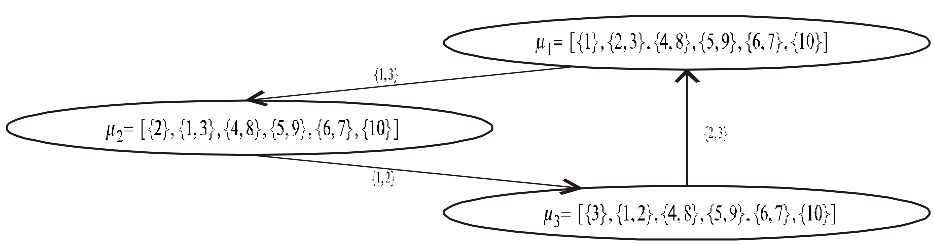
An interesting property of absorbing sets is the property of "outer stability" in the following sense[11].

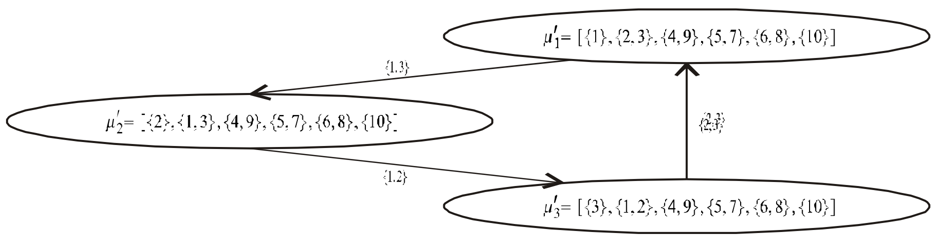
[11]For an arbitrary finite abstract system (X, R) (Kalai, Pazner, and Schmeidler[]) show that the admissible set has this property of "outer stability" We give an easy proof for the roommate problem with strict preferences.

Theorem 5 *Let $(N, (\succ_x)_{x \in N})$ be a roommate problem and let $\mathcal{A}^*$ be the union of the absorbing sets. Then, for any $\mu \notin \mathcal{A}^*$ there exists a $\mu' \in \mathcal{A}^*$ such that $\mu' R^T \mu$.*

Proof. If μ is a P-stable matching for some stable partition P, as $\mu \notin \mathcal{A}^*$ then $\mathcal{A}_P$ is not an absorbing set, i.e., $P\mid_{S_P}$ is not maximal in $\mathcal{P}$. Then, there exists a maximal partition $P'\mid_{S_{P'}}$ of $\mathcal{P}$ such that $P\mid_{S_P} \subset P'\mid_{S_{P'}}$. Let μ' be a P'-stable matching. Since $\mathcal{A}_{P'}$ is an absorbing set then $\mu' \in \mathcal{A}^*$. Moreover, by the Lemma 11, we have $\mu' R^T \mu$ and therefore the result follows. Suposse now that μ is not a P-stable matching for any stable partition P. Then, by Theorem [], there exists a P-stable matching $\overline{\mu}$ such that $\overline{\mu} R^T \mu$. If $\overline{\mu} \in \mathcal{A}^*$ we are done. Otherwise, by the proved before, there exists a $\mu' \in \mathcal{A}^*$ such that $\mu' R^T \overline{\mu}$. But as $\overline{\mu} R^T \mu$ we have $\mu' R^T \mu$. ■

We close this section by showing how these results may be used to determine the absorbing sets. Let us consider the 9-agent roommate given in Example 1. Let $P_1 = \{\{1,2,3\},\{4,7\},\{5,8\},\{6,9\},\{10\}\}$, $P_2 = \{\{1,2,3\},\{4,8\},\{5,9\},\{6,7\},\{10\}\}$ and $P_3 = \{\{1,2,3\},\{4,9\},\{5,7\},\{6,8\},\{10\}\}$ the stable partitions of N. By applying the iterative procedure the set D_P can be obtained for each one of them. We have $D_{P_1} = N$ and $D_{P_2} = D_{P_3} = \{1,2,3\}$. Therefore, $S_{P_1} = \{10\}$ and $S_{P_2} = S_{P_3} = \{4,5,6,7,8,9,10\}$. Then $P_1\mid_{S_{P_1}} = \{\{10\}\}$, $P_2\mid_{S_{P_2}} = \{\{4,8\},\{5,9\},\{6,7\},\{10\}\}$ and $P_3\mid_{S_{P_3}} = \{\{4,9\},\{5,7\},\{6,8\},\{10\}\}$. Hence $P_2\mid_{S_{P_2}}$ and $P_3\mid_{S_{P_3}}$ are the maximal partions of $\mathcal{P}$. Then, by Theorems 2 and 3, this roommate problem has exactly two absorbing sets $\mathcal{A}$ and $\mathcal{A}'$ where $\mathcal{A} = \mathcal{A}_{P_2}$ and $\mathcal{A}' = \mathcal{A}_{P_3}$. Moreover, as $D = \{1,2,3\}$, $\mathcal{A} = \{\mu_1, \mu_2, \mu_3\}$ and $\mathcal{A}' = \{\mu_1', \mu_2', \mu_3'\}$ where μ_1, μ_2, μ_3 are the P_2- stable matchings and μ_1', μ_2', μ_3' the P_3- stable matchings (See figure).





Observe that $\mathcal{A} \mid_D = \{\mu_1 \mid_D, \mu_2 \mid_D, \mu_3 \mid_D\}$ and $\mathcal{A} \mid_D = \{\mu_1' \mid_D, \mu_2' \mid_D, \mu_3' \mid_D\}$ where $\mu_1 \mid_D = \mu_1' \mid_D = [\{1\}, \{2,3\}]$, $\mu_2 \mid_D = \mu_2' \mid_D = [\{2\}, \{1,3\}]$ and $\mu_3 \mid_D = \mu_3' \mid_D = [\{3\}, \{1,2\}]$. Moreover, $\mathcal{A} \mid_S$ and $\mathcal{A} \mid_S$ are singletons consisting of the stable matchings $[\{4,8\},\{5,9\},\{6,7\},\{10\}]$ and $[\{4,9\},\{5,7\},\{6,8\},\{10\}]$, respectively in $(S, (\succ_x)_{x \in N'})$ where $S = \{4,5,6,7,8,9,10\}$.

Appendix

Lemma 6 *For any two distinct P-stable matchings μ and μ', $\mu' R^T \mu$.*

Proof. If P no contains any odd ring then there exists an unique P-stable matching. Suppose that P contains some odd ring. Let $A_1, ..., A_r$ be the odd rings of P and $T = \bigcup_{i=1}^{r} A_i$. Set $A_1 = \{a_1, ..., a_k\}$. As A_1 is a ring, then

$$a_{i+1} \succ_{a_i} a_{i-1} \succ_{a_i} a_i, \qquad [3]$$

for all $i = \{1, ..., k\}$. By Definition 1, since μ and μ' are P-stable matchings, there exist agents a_l, $a_s \in A_1$ such that $\mu(a_l) = a_l$ and $\mu'(a_s) = a_s$. Now, since $\mu(a_l) = a_l$ and $\mu(a_{l-1}) = a_{l-2}$, by condition [3], $\{a_l, a_{l-1}\}$ blocks μ, inducing a P-stable matching μ_1 for which $\mu(a_{l-2}) = a_{l-2}$. By repeating the process, we can consider a sequence of P-stable matchings μ_0, μ_1,..., μ_i, ... as follows:
i) for $i = 0$, $\mu_0 = \mu$.
ii) for $i \geq 1$, μ_i is the P-matching obtained from μ_{i-1} by satisfying the blocking pair $\{a_{l-2(i-1)}, a_{l-2(i-1)-1}\}$.

Let $m_1 \in \{1, ..., k\}$ such that $a_{l-2m_1} = a_s$. Then $\mu = \mu_0, \mu_1, ..., \mu_{m_1}$ is a finite sequence of P-stable matchings such that, for all $i \in \{1, ..., m_1\}$, $\mu_i R \mu_{i-1}$ and $\mu_{m_1} \mid_{A_1} = \mu' \mid_{A_1}$.
Consider now the ring A_2. Reasoning in the same way as before for μ_{m_1} and μ' we obtain a finite sequence of P-stable matchings μ_{m_1}, μ_{m_1+1}, ..., $\mu_{m_1+m_2}$

such that, for all $i \in \{m_1+1,...,m_1+m_2\}$, $\mu_i R \mu_{i-1}$ and $\mu_{m_1+m_2} \mid_{(A_1 \cup A_2)} = \mu' \mid_{(A_1 \cup A_2)}$.

By repeating the same procedure to the remaining odd rings, we obtain eventually a finite sequence of P-stable matchings $\mu = \mu_0, \mu_1, ..., \mu_m$, where $m = \sum_{i=1}^{r} m_i$, and such that, for all $i \in \{1,...,m\}$, $\mu_i R \mu_{i-1}$ and $\mu_m \mid_T = \mu' \mid_T$. Now, as $\mu_m \mid_{(N \setminus T)} = \mu' \mid_{(N \setminus T)}$, then $\mu_m = \mu'$ and the proof is complete. ■

Lemma 7 *Let P be a stable partition and let $\overline{\mu}$ be a P-stable matching. Then,* $\mathcal{A}_P = \{\overline{\mu}\} \cup \{\mu \in \mathcal{M} : \mu R^T \overline{\mu}\}$.

Proof. ($\subseteq$): Let $\mu \in \mathcal{A}_P$. If μ is a P-stable matching and $\mu = \overline{\mu}$ we are done. If $\mu \neq \overline{\mu}$ then, by Lemma 6, we have $\mu R^T \overline{\mu}$. Otherwise, there exists a P-stable matching $\widetilde{\mu}$ such that $\mu R^T \widetilde{\mu}$. If $\widetilde{\mu} = \overline{\mu}$ we are done. If $\widetilde{\mu} \neq \overline{\mu}$ again by Lemma 6 we have $\widetilde{\mu} R^T \overline{\mu}$. Hence $\mu R^T \overline{\mu}$.

($\supseteq$): It follows directly of the definition of $\mathcal{A}_P$. ■

Lemma 8 $Dr = D_P$

Proof. ($\subseteq$): To it, we prove that, for all $t \in \{0,...,r\}$:

a) $D_t \subseteq D_P$ and

b) There exist a $\mu_t \in \mathcal{A}_P$ and a P-stable matching $\overline{\mu}_t$ such that $\mu_t(x) = x$ if $x \in B_t$[12] and $\mu_t(x) = \overline{\mu}_t(x)$, otherwise.

We argue by induction on t. If $t = 0$, let $A = \{a_1, ..., a_k\}$ be an odd ring of P. Let $\widetilde{\mu}$ be the P-stable matching such that $\widetilde{\mu}(a_i) = a_i$. As $\widetilde{\mu}(a_{i-1}) = a_{i-2}$ and $a_i \succ_{a_{i-1}} a_{i-2}$, then $\{a_i, a_{i-1}\}$ blocks $\widetilde{\mu}$ and therefore $a_i \in D_P$. Hence Condition a) follows. Moreover, for $\mu_0 = \overline{\mu}_0 = \overline{\mu}$ where $\overline{\mu}$ is any P-stable matching Condition b) is verified.

For $t \geq 1$, as $D_t = D_{t-1} \cup B_t$, by the inductive hypothesis, it suffices to prove that $B_t \subseteq D_P$. If A_t is not an odd ring, then $A_t = B_{t'}$ for some $t' \in \{1, ..., t-1\}$. Then, by condition b), there exist a $\mu_{t'} \in \mathcal{A}_P$ and a P-stable matching $\overline{\mu}_{t'}$ such that $\mu_{t'}(x) = x$ if $x \in B_{t'}$ and $\mu_{t'}(x) = \overline{\mu}_{t'}(x)$, otherwise. If A_t is an odd ring, consider $\mu_{t'} = \overline{\mu}_{t'} = \widetilde{\mu}$, where $\widetilde{\mu}$ is the P-stable matching such that $\widetilde{\mu}(a_i) = a_i$. Then $\mu_{t'}(a_i) = a_i$ and as $\mu_{t'}(b_j) = \overline{\mu}_{t'}(b_j) = b_{j-1}$, by [2], we have $b_j \succ_{a_i} \mu_{t'}(a_i)$ and $a_i \succ_{b_j} \mu_{t'}(b_j)$. Hence $\{a_i, b_j\}$ blocks $\mu_{t'}$ and therefore $b_j \in D_P$. Let $\widetilde{\mu}_{t'}$ be the matching obtained from $\mu_{t'}$ by satisfying this blocking pair. As $a_i \succ_{b_j} b_{j-1}$,

[12] We assume that $B_t = \emptyset$, if $t = 0$.

by the stability of P, $a_{i-1} \succ_{a_i} b_j$ and as $\widetilde{\mu}_{t'}(a_{i-1}) \in \{a_{i-1}, a_{i-2}\}$ then $\{a_i, a_{i-1}\}$ blocks $\widetilde{\mu}_{t'}$, inducing a matching $\mu^*_{t'}$ for which agents b_j and b_{j-1} are alone. Then $\{b_j, b_{j-1}\}$ blocks $\mu^*_{t'}$. Hence, $b_{j-1} \in D_P$ and therefore Condition a) follows. Moreover, it is very easy to see that for $\mu_t = \mu^*_{t'}$ and $\overline{\mu}_t = \overline{\mu}_{t'}$ Condition b) is satisfied.

($\supseteq$): We must to prove that D_r contains all blocking pairs of all matchings of $\mathcal{A}_P$. Now, by Lemma 7, $\mathcal{A}_P = \{\overline{\mu}\} \cup \{\mu \in \mathcal{M} : \mu R^T \overline{\mu}\}$ where $\overline{\mu}$ is a P-stable matching. Hence it suffices to show that, for any finite sequence of matchings $(\overline{\mu} = \mu_0, \mu_1, ..., \mu_m = \mu)$ where, for all $i \in \{1, ..., m\}$, μ_i is obtained from μ_{i-1} by satisfying the blocking pair $\{x_i, y_i\}$, then $\{x_i, y_i\} \subseteq D_r$.

We argue by induction on m. If $m = 1$, let $A = \{a_1, ..., a_k\}$ and $B = \{b_1, ..., b_l\}$ sets of P such that $x_1 \in A$ and $y_1 \in B$. Then $x_1 = a_i$ and $y_1 = b_j$ for some i and j. As $\{x_1, y_1\}$ blocks $\overline{\mu}$ then $y_1 \succ_{x_1} \overline{\mu}(x_1)$ and $x_1 \succ_{y_1} \overline{\mu}(y_1)$, i.e., $b_j \succ_{a_i} \overline{\mu}(a_i)$ and $a_i \succ_{b_j} \overline{\mu}(b_j)$. Now, if A and B are odd rings, then a_i, $b_j \in D_0$ and we are done. If neither of them is an odd ring then we have $\overline{\mu}(a_i) = a_{i-1}$ and $\overline{\mu}(b_j) = b_{j-1}$. But then $b_j \succ_{a_i} a_{i-1}$ and $a_i \succ_{b_j} b_{j-1}$, contradicting the stability of P. If A is an odd ring but B is not. Then $a_i \in D_0$. As $\overline{\mu}(b_j) = b_{j-1}$ we have $b_j \succ_{a_i} a_i$ and $a_i \succ_{b_j} b_{j-1}$. Hence $b_j \in D_1$. If B is an odd ring and A is not, we argue in similar way.

For $m \geq 2$, let $A' = \{a'_1, ..., a'_{k'}\}$ and $B' = \{b'_1, ..., b'_{l'}\}$ sets of P such that $x_m = a'_i$ and $y_m = b'_j$ for some i and j. As $\{x_m, y_m\}$ blocks μ_{m-1} then $b'_j \succ_{a'_i} \mu_{m-1}(a'_i)$ and $a'_i \succ_{b'_j} \mu_{m-1}(b'_j)$. Suposse, by contradiction, that $\{a'_i, b'_j\} \nsubseteq D_r$. If $\{a'_i, b'_j\} \cap D_r = \oslash$ then $\mu_{m-1}(a'_i) = \overline{\mu}(a'_i)$ and $\mu_{m-1}(b'_j) = \overline{\mu}(b'_j)$. Hence $\{a'_i, b'_j\}$ blocks $\overline{\mu}$. But then, by the inductive hypothesis, $\{a'_i, b'_j\} \subseteq D_r$, contradicting that $\{a'_i, b'_j\} \nsubseteq D_r$. If $a'_i \in D_r$ and $b'_j \notin D_r$ since $\mu_{m-1}(b'_j) = \overline{\mu}(b'_j) = b'_{j-1}$ then $b'_j \succ_{a'_i} a'_i$ and $a'_i \succ_{b'_j} b'_{j-1}$. Hence $b'_j \in D_r$, which contradicts that $b'_j \notin D_r$. If $a'_i \notin D_r$ and $b'_j \in D_r$, we argue in similar way. ∎

Lemma 9 *Let P be a stable partition. Then, there exist a $\mu \in \mathcal{A}_P$ and a P-stable matching $\overline{\mu}$ such that $\mu(x) = x$ if $x \in D_P \backslash D_0$ and $\mu(x) = \overline{\mu}(x)$, otherwise.*

Proof. By Lemma 8 we have $D_r = D_P$. We argue by induction on r. If $r = 0$, consider $\mu = \overline{\mu}$, where $\overline{\mu}$ is any P-stable matching. For $r \geq 1$, by Lemma 8 (to see the proof), there exist a $\mu_r \in \mathcal{A}_P$ and a P-stable matching $\overline{\mu}_r$ such that $\mu_r(x) = x$ if $x \in B_r$ and $\mu_r(x) = \overline{\mu}_r(x)$, otherwise. Let $N' = N \backslash B_r$. Then $P' = P \backslash \{B_r\}$ is a stable partition of N' for which $D_{P'} = D_{r-1}$. Therefore, by the inductive hypothesis, there exist a $\mu' \in \mathcal{A}_{P'}$ and a P'-stable matching $\overline{\mu}'$ such that $\mu'(x) = x$ if $x \in D_{P'} \backslash D_0$ and $\mu'(x) = \overline{\mu}'(x)$, otherwise. Let μ and $\overline{\mu}$ such that $\mu \mid_{N'} = \mu'$, $\mu \mid_{B_r} = \mu_r \mid_{B_r}$, $\overline{\mu} \mid_{N'} = \overline{\mu}'$ and $\overline{\mu} \mid_{B_r} = \overline{\mu}_r \mid_{B_r}$. It is very easy to check that μ and $\overline{\mu}$ satisfy the condition given in the Lemma. ■

Lemma 10 *Let P be a stable partition. Then,*
i) For any $\mu \in \mathcal{A}_P$, $\mu(S_P) = S_P$ and $\mu \mid_{S_P}$ is stable.
ii) For any μ,$\mu' \in \mathcal{A}_P$, $\mu \mid_{S_P} = \mu' \mid_{S_P}$.

Proof. By Lemma 7, we have $\mathcal{A}_P = \{\overline{\mu}\} \cup \left\{\mu \in \mathcal{M} : \mu R^T \overline{\mu}\right\}$ where $\overline{\mu}$ is a P-stable matching.
i) Let $x \in S_P$ and $A \in P$ such that $x \in A$. If $\mu = \overline{\mu}$, as $\overline{\mu}$ is a P-stable matching then $\mu(x) \in A$ and since $A \subseteq S_P$ we have $\mu(x) \in S_P$. If $\mu \neq \overline{\mu}$ then $\mu R^T \overline{\mu}$. But as $\mu(x) = \overline{\mu}(x)$ and $\overline{\mu}(x) \in S_P$ then $\mu(x) \in S_P$. Obviously $\mu \mid_{S_P}$ is stable.
ii) Since $\mu \mid_{S_P} = \overline{\mu} \mid_{S_P}$ for all $\mu \in \mathcal{A}_P$, the result follows directly. ■

Lemma 11 *Let P and P' be two distinct stable partitions and let μ and μ' be any P-stable and P'-stable matchings. Then, $\mu' R^T \mu$ if and only if $P \mid_{S_P} \subseteq P' \mid_{S_{P'}}$.*

Proof. ($\Longrightarrow$): Let $A \in P$ such that $A \subseteq S_P$. We must to prove that $A \in P'$ and $A \subseteq S_{P'}$. As $\mu' R^T \mu$ then $\mu' \in \mathcal{A}_P$ hence $\mathcal{A}_{P'} \subseteq \mathcal{A}_P$. Therefore $S_P \subseteq S_{P'}$. Now, by Lemma 10, we have $\mu' \mid_{S_P} = \mu \mid_{S_P}$, hence $\mu' \mid_A = \mu \mid_A$ and therefore $A \in P'$. Moreover, as $A \subseteq S_P$ and $S_P \subseteq S_{P'}$ then $A \subseteq S_{P'}$.
($\Longleftarrow$): By Lemma 9, there exist a $\mu \in \mathcal{A}_P$ and a P-stable matching $\overline{\mu}$ such that $\mu(x) = x$ if $x \in D_P \backslash D_0$ and $\mu(x) = \overline{\mu}(x)$, otherwise. First we prove that there exist a P'-stable matching $\widetilde{\mu}$ such that $\widetilde{\mu} R^T \mu$. Let $\widetilde{\mu}$ be the P'-stable matching such that $\widetilde{\mu} \mid_{D_0} = \overline{\mu} \mid_{D_0}$. As $\mu(x) = \overline{\mu}(x)$ for all $x \in D_0$ then $\widetilde{\mu} \mid_{D_0} = \mu \mid_{D_0}$. Moreover, since $P \mid_{S_P} \subseteq P' \mid_{S_{P'}}$ we have $\widetilde{\mu} \mid_{S_P} = \overline{\mu} \mid_{S_P}$ and as, by Lemma 10, $\overline{\mu} \mid_{S_P} = \mu \mid_{S_P}$ then $\widetilde{\mu} \mid_{S_P} = \mu \mid_{S_P}$. Then, for every $x \in D_P \backslash D_0$ we have $\widetilde{\mu}(x) \in D_P \backslash D_0$ (Otherwise, $\widetilde{\mu}(x) = \mu(x) = x$ hence $x \notin D_P \backslash D_0$). So we can write $D_P \backslash D_0 = \cup_{i=1}^{s} \{x_i, y_i\}$ where $y_i = \widetilde{\mu}(x_i)$. Now, as agents x_i and y_i are alone under μ we can consider the finite sequence of matchings $(\mu = \mu_0, \mu_1, ..., \mu_s)$ where, for all $i \in \{1, ..., s\}$, μ_i is obtained from μ_{i-1} by satisfying the blocking pair $\{x_i, y_i\}$. Then we have $\mu_s = \widetilde{\mu}$ and $\widetilde{\mu} R^T \mu$. Hence $\widetilde{\mu} R^T \overline{\mu}$ and the result follows directly from Lemma 6. ■

Lemma 12 *Let P and P' be two stable partitions. Then $\mathcal{A}_P = \mathcal{A}_{P'}$ if and only if $P\mid_{S_P} = P'\mid_{S_{P'}}$.*

Proof. From Lemma 7, we have $\mathcal{A}_P = \{\overline{\mu}\} \cup \left\{\mu \in \mathcal{M} : \mu R^T \overline{\mu}\right\}$ and $\mathcal{A}_{P'} = \{\widetilde{\mu}\} \cup \left\{\mu \in \mathcal{M} : \mu R^T \widetilde{\mu}\right\}$ where $\overline{\mu}$ and $\widetilde{\mu}$ are P-stable and P'-stable matchings, respectively. As $\mathcal{A}_P = \mathcal{A}_{P'}$ then $\widetilde{\mu} \in \mathcal{A}_P$ and $\overline{\mu} \in \mathcal{A}_{P'}$. If $\widetilde{\mu} = \overline{\mu}$ then $P = P'$ and we are done. If $\widetilde{\mu} \neq \overline{\mu}$ we have $\widetilde{\mu} R^T \overline{\mu}$ and $\overline{\mu} R^T \widetilde{\mu}$. Hence, by Lemma 11, $P\mid_{S_P} = P'\mid_{S_{P'}}$
The converse is analogous. ■

Lemma 13 *Let P and P' be two stable partitions. Then, for each $A \in P$ either $A \subseteq D_{P'}$ or $A \subseteq S_{P'}$.*

Proof. Let $A \in P$. If A is an odd ring then $A \subseteq D_{P'}$. If A is a singleton we are done. We may, therefore, assume that A is a pair of mutually acceptable agents. Then $A = \{x, y\}$. Suppose, by contradiction, and without loss of generality, that $x \in D_{P'}$ and $y \in S_{P'}$. We may now consider the sequence of agents $x_0, x_1, ..., x_t, ...$ defined recursively as follows:
$i)$ for $t = 0$, $x_0 = x$.
$ii)$ for $t \geq 1$, x_t is the predeccesor of x_{t-1} in A_t where A_t is the set of P (respectively, P') such that $x_{t-1} \in A_t$ if t is odd (respectively, even).

First we prove, by induction on t, that for all $t \geq 1$, $x_t \in S_{P'}$ and $x_{t+1} \succ_{x_t} x_{t-1}$. If $t = 1$, as $A_1 = A$ then $x_1 = y$. Hence $x_1 \in S_{P'}$. Moreover, as $x_0 \in D_{P'}$ and $x_1 \in S_{P'}$, by [2], we have $x_2 \succ_{x_1} x_0$.
For $t \geq 2$. By the inductive hypothesis, $x_{t-1} \in S_{P'}$ and $x_t \succ_{x_{t-1}} x_{t-2}$. As agent x_{t-1} prefers x_t to his predecessor x_{t-2} in A_{t-1}, then by stability of P (respectively, P') if t is odd (respectively, even) agent x_t prefers to his predecessor x_{t+1} in A_t to x_{t-1}. Hence $x_{t+1} \succ_{x_t} x_{t-1}$. Moreover, if t is even, as $A_t \in P'$ and $x_{t-1} \in S_{P'}$ then $A_t \subseteq S_{P'}$ and therefore $x_t \in S_{P'}$. If t is odd and $x_t \in D_{P'}$, as $x_t \succ_{x_{t-1}} x_{t-2}$ then $x_{t-1} \in D_{P'}$, contradicting that $x_{t-1} \in S_{P'}$. Hence $x_t \in S_{P'}$.
Now since the number of agents is finite, some agent appear more than once in this sequence. Let r be the minimun number such that $x_{r+1} = x_t$ for some $t \leq r$. Thus $t = r$ (As $x_t \in S_{P'}$ and $x_0 \in D_{P'}$ then $x_t \neq x_0$ for all $t \geq 1$). Hence $A_t = \{x_{t-1}, x_t\}$ for all $t = 1, ..., r$ and $A_{r+1} = \{x_r\}$. But as $x_{r+1} \succ_{x_r} x_{r-1}$ and $x_{r+1} = x_r$ then $x_r \succ_{x_r} x_{r-1}$, which contradicts that $A_r = \{x_{r-1}, x_r\}$ is a pair of mutually acceptable agents. ■

Lemma 14 *If $P\mid_{S_P}$ and $P'\mid_{S_{P'}}$ are any two maximal partitions of $\mathcal{P}$, then $S_P = S_{P'}$.*

Proof. Suppose that $S_P \neq S_{P'}$. Then, we can assume, without loss of generality that, there exists a $x \in S_P$ such that $x \in D_{P'}$. By Lemma 13, for each $A \in P$ either $A \subseteq D_{P'}$ or $A \subseteq S_{P'}$. We may now consider the partition $\overline{P}$ of N such that $\overline{P} = \{A \in P : A \subseteq D_{P'}\} \cup \{A \in P' : A \subseteq S_{P'}\}$. It is very easy to verify that $\overline{P}$ is a stable partition such that $P'\mid_{S_{P'}} \subseteq \overline{P}\mid_{S_{\overline{P}}}$. Let $\overline{A} \in \overline{P}$ such that $x \in \overline{A}$. As $x \in D_{P'}$ then $\overline{A} \subseteq D_{P'}$. Now, it is easy to see that $\overline{A} \subseteq S_{\overline{P}}$, hence $P'\mid_{S_{P'}} \subset \overline{P}\mid_{S_{\overline{P}}}$, contradicting the maximality of $P'\mid_{S_{P'}}$. ■

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