Common p-belief and uncertainty

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Abstract

This paper generalizes the notion of common p-beliefs to situations of ambiguity or Knightian uncertainty. When players have multiple prior beliefs,we show that Aumann's no-agreement theorem can be approximated. We also provide conditions under which purely speculative trade does not occur in the presence of ambiguity when players preferences are complete or incomplete.

1 Introduction

The notion of common belief was introduced by Monderer and Samet (1989) as a way of approximating common knowledge. They developed this concept and analyzed its implications for interactive decision making when the players are faced with situations of risk only. In scenarios of risk, each player forms beliefs over the possible states of the world using a single probability measure and evaluates actions using their subjective expected utility (Savage, 1954 and Anscombe and Aumann, 1963).

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However, there are situations in which the players distinguish between risk which may be insured against by trading in markets and uncertainty, which may not. In particular Knight's (1921) description of uncertainty suggests that individuals may be averse to situations where a probability distribution over the states of the world is not objectively provided. Ellsberg's (1961) experiments provided evidence of this phenomenon and suggested that Savage's (1954) sure-thing principle might not be applicable in situations such as those described by Knight (1921), where a decision maker faces cognitive or informational constraints that leave him uncertain about what odds apply to the payoff relevant events.

There is now a large literature on individual decision-making under Knightian uncertainty or ambiguity, starting with the multiple-prior models of Gilboa and Schmeidler (1989) and Bewley (1986) and the non-additive probability model of Schmeidler (1989). In the multiple-prior models, individuals use a set of probability measures rather than just a single probability measure to determine the expected utility from actions. The implications of these alternative theories of individual decision-making in interactive and market settings are now being analyzed by, inter alia, Ahn (forthcoming), Bose et al. (2006), Lo (1996, 1999), Rigotti and Shannon (2005), Lopomo et al. (2006).

In this paper, we extend the notion of common belief as an approximation of common knowledge to settings when the players have multiple-prior beliefs. We then examine whether the well known 'agreeing to disagree' theorem of Aumann's (1976) seminal work and the 'no speculative trade' results of Milgrom and Stokey (1982) can be extended to the setting of Knightian uncertainty. Monderer and Samet (1989) and Neeman (1996) generalized Aumann's (1976) result when the asymmetrically informed players have a common singleton prior and the posteriors held by the players are commonly p-believed. Kajii and Ui (2005) established a version of the result that holds when the players share common multiple priors and the set of posteriors is commonly known.

A significant literature has focused on whether purely speculative trade exists in complete markets and the answer is negative even in the presence of asymmetric information. Milgrom and Stokey (1982) initiated this line of investigation by establishing that if such trade was common knowledge and the players were initially at a Pareto-optimal allocation, then the players would be indifferent between trading and not trading at all. Wakai (2002) extended the Milgrom-Stokey result to the Gilboa-Schmeidler (1989) model of multiple-priors. Sonsino (1995) showed that such trade is impossible if the players have singleton priors and it is commonly p-believed that all traders want to trade. We establish that if the players have dynamically consistent preferences, then even in the presence of multiple priors, purely speculative trade is ruled out if the acceptance of trade is commonly p-believed.

The paper is organized as follows. We introduce the notion of common belief in the presence of multiple priors in the next section. Sections 3 and 4 extend Aumann's (1976) no-agreement theorem and the no-speculation result of Milgrom and Stokey (1982) to the case of common belief with multiple priors and section 5 concludes. All proofs are presented in the appendix.

2 Beliefs and common beliefs

(Ω, Σ) denotes a measurable space where Ω is the space of states of the world, with typical element ω, and Σ is the σ-algebra of events. Let $\{\mu_j\}_{j=1}^{J}$ denote a (countable) family of probability measures on (Ω, Σ) which are the extreme points of the closed convex set of measures $\bar{\mu}$. The set of players is finite and is denoted I while Π_i is player $i's$ (measurable) information partition of Ω with countably many elements. All elements in Π_i have positive measure with respect to each μ_j. Denote by $\mathcal{F}_i$ the $\sigma-$algebra generated by Π_i, i.e. $\mathcal{F}_i$ is the collection of all unions of elements of Π_i. We first note the notion that a player p-believes an event according to some measure in the set $\{\mu_j\}_{j=1}^{J}$.

Definition 1 *Player i p-believes $E \in \Sigma$ at ω according to measure j if*

$$\mu_j \left(E|\Pi_i\left(\omega\right)\right) \geq p$$

The event that player i p-believes E according to measure j is given by

$$B_{ij}^p\left(E\right) = \left\{\omega|\mu_j\left(E|\Pi_i\left(\omega\right)\right) \geq p\right\}.$$

When players have multiple priors, the natural notion of p-belief is that a player p-believe any event according to all the measures in the set of priors as noted in the following. This notion of p-belief is also adopted by Ahn (forthcoming) for the specific case of common 1-belief as a definition of common knowledge. In the case of a singleton prior, Brandenburger and Dekel (1987, 1993) provided a detailed analysis of why this 'almost sure' notion of knowledge is the appropriate one.

Definition 2 *Player i p-believes $E \in \Sigma$ at ω if for all j*

$$\mu_j \left(E|\Pi_i\left(\omega\right)\right) \geq p$$

This definition of p-belief is consistent with the Gilboa-Schmeidler (1989) representation for complete preferences and with the Bewley (1986) representation for incomplete preferences under Knightian uncertainty. The event that player i p-believes E at ω is given by

$$B_i^p\left(E\right) = \left\{\omega|\mu_j\left(E|\Pi_i\left(\omega\right)\right) \geq p \text{ for all } j\right\}$$

and clearly

$$B_i^p\left(E\right) = \cap_{j=1}^J B_{ij}^p\left(E\right).$$

We now note some properties of the operator B_i^p that will be used in the results of sections 3 and 4. The following generalizes proposition 2 in Monderer and Samet

(1989) to the case of multiple-priors.

Proposition 1 *For each $p \in [0,1]$ and $E, F \in \Sigma$, the following hold.*

(1) $B_i^p(E) \in \mathcal{F}_i$

(2) *If $E \in \mathcal{F}_i$ then $E \subseteq B_i^p(E)$, with equality if $p > 0$ or if $E = \Omega$*

(3) $B_i^p(B_i^p(E)) = B_i^p(E)$

(4) *If $E \subseteq F$ then $B_i^p(E) \subseteq B_i^p(F)$*

(5) *If (E^n) is a decreasing sequence of events then $B_i^p(\cap_n E^n) = \cap_n B_i^p(E^n)$*

(6) *For all j, $\mu_j(E|B_i^p(E)) \geq p$*

An event $E \in \Sigma$ is called *evident p-belief* with respect to measure μ_j if $E \subseteq B_{ij}^p(E)$ for all i. So, an event E is evident p-belief if $E \subseteq B_i^p(E)$ for all i. The definition of common p-belief for the case of multiple priors is then just the same as the case for singleton prior.

Definition 3 *An event $C \in \Sigma$ is common p-belief at ω if there exists an evident p-belief E such that $\omega \in E$ and $E \subseteq B_i^p(C)$*

Monderer and Samet (1989) show that common p-belief can also be characterized by an iterative prodedure, which is described in the following. First we define the iterative process itself.

Definition 4 $E^p(C) = \cap_{m \geq 1} C^m$ *where* $C^0 = C$ *and for* $m \geq 1$, $C^m = B_i^p(C^{m-1})$

The following result, which is essentially the same as proposition 3 in Monderer and Samet (1989) then shows that common p-belief is also characterized by the above iterative procedure in the multiple-prior case.

Proposition 2 *For any event C and $p \in [0,1]$, (i) $E^p(C)$ is evident p-belief and $E^p(C) \subseteq B_i^p(C)$ for all i and (ii) C is common p-belief at ω iff $\omega \in E^p(C)$.*

The second part of the above result establishes that $E^p(C)$ is the event that 'C is common knowledge' as in the case of a singleton prior.

3 Agreeing to disagree

Aumann (1976) showed that the posteriors formed over an event by players, who share a common singleton prior, must coincide if they are commonly known. Monderer and Samet (1986) and Neeman (1996) generalized this result to case of common beliefs and showed that if the posteriors are common p-belief then they cannot differ by more than $(1-p)$. We now note a generalization of this 'agreeing to disagree' result to the case of multiple prior p-beliefs. Given an event $F \in \Sigma$, let $f_{ij}(\omega) = \mu_j (F|\Pi_i(\omega))$ for all j. Let $r_{ij} \in [0,1]$, we denote the fact that player i has posterior beliefs r_i at ω by setting $\{f_{ij}(\omega)\}_{j=1}^n = \{r_{ij}\}_{j=1}^n$. Let r_i be the closed interval $[\min_j r_{ij}, \max_j r_{ij}] \equiv [r_{*i}, r_i^*]$. Since $\bar{\mu}$ is a closed convex set, for every $x \in r_i$, there exists some $\mu \in \bar{\mu}$ such that $\mu(F|\Pi_i(\omega)) = x$. Denote by $f_i(\omega)$ the closed convex hull of $\{f_{ij}(\omega)\}_{j=1}^n$. Denote by C the event that $\{f_{ij}\}_{j=1}^n = \{r_{ij}\}_{j=1}^n$ or equivalently $f_i = r_i$ for all i, i.e., $C = \cap_{i\in I}\left\{\omega| \left\{f_{ij}(\omega)_{j=1}^n\right\} = \{r_{ij}\}_{j=1}^n\right\} = \cap_{i\in I}\{\omega|f_i(\omega) = r_i\}$.

Proposition 3 *Suppose C is common p-belief at some $\omega \in \Omega$, then there exist $\bar{r}_i \in r_i$ and $\bar{r}_k \in r_k$ such that $|\bar{r}_i - \bar{r}_k| \leq (1-p)$*

The bound can in general not be improved as shown by the example in Neeman (1995). Also, the next example shows that we can not do better than the existence of some $\bar{r}_i \in r_i$ and $\bar{r}_k \in r_k$ such that $|\bar{r}_i - \bar{r}_k| \leq (1-p)$, i.e. it is not in general true that either $|r_{*i} - r_{*k}| \leq (1-p)$ or $|r_i^* - r_k^*| \leq (1-p)$.

Example Let $a, b \in (0, 0.2)$, $a \neq b$.Let $\Omega = \{1,2,3,4,5\}$, $\pm = 2^\Omega$, and define $\bar{\mu} = \{p \in \Delta(\Omega) \mid p = \lambda p_1 + (1-\lambda) p_2\}$, where $p_1(A) = a$ for $A \in \{\{1\},\{2\},\{3\}\}$, $p_1(A) = 2a$ for $A \in \{\{4\},\{1,3\}\}$, $p_1(\{2,4\}) = 3a$, and $p_1(\{5\}) = 1-5a$ and $p_2(A) = b$ for $A \in \{\{1\},\{2\},\{4\}\}$, $p_2(A) = 2b$ for $A \in \{\{3\},\{2,4\}\}$, $p_2(\{1,3\}) = 3b$, and $p_2(\{5\}) = 1-5b$. Let the information partitions for player 1 (Π_1) and player 2 (Π_2) be $\Pi_1 = \{\{1,3\},\{2,4\},\{5\}\}$ and $\Pi_2 = \{\{1,2,3,4\},\{5\}\}$.Then for the event $F = \{1,2\}$, it is common 1-belief at state 1 that $f_1(1) = r_1 = [1/3, 1/2]$ and $f_2(1) = r_2 = \{2/5\}$. So, $r_1 \cap r_2 = \{2/5\}$ and we also have that $|r_{*1} - r_{*2}| = 1/15 > 0$ and

$|r_1^* - r_2^*| = 1/10 > 0$.

In the above example, if the set of priors is not closed, i.e., if we just consider the extreme points $\{p_1, p_2\}$ of $\bar{\mu}$ as the set of priors, then the players have no posterior beliefs in common even when these are commonly 1-believed.

4 Impossibility of speculative trade

We now consider whether it is possible for players to engage in speculative trade after the arrival of private information if it is common p-belief that all traders want to trade. Our presentation follows those of Milgrom and Stokey (1982) and Sonsino (1995).

Consider a (random) pure exchange economy comprised of the I players with L commodities in each state of the world with R_+^L denoting the consumption set of each player $i \in I$ in every state of the world. The (ex-ante) preferences of player i are denoted by $\succsim_i$ and the $\mathcal{F}_i$-measurable function $U_i : \Omega \times R_+^L \rightarrow R$ denotes her von-Neumann-Morgenstern utility.

In what follows, $\succsim_i$ are represented by the Gilboa and Schmeidler (1986) structure with $\bar{\mu}$ as the closed and convex set of prior probabilities which is common across all players. In addition, following the work of Epstein and Schneider (2003) we assume that $\bar{\mu}$ is rectangular with respect to all the elements $A \in \Sigma$ and that the priors are mutually absolutely continuous, i.e., for any $A \in \Sigma \backslash \emptyset$, if there exists $j_1 \in J$ such that $\mu_{j_1}(A) = 0$ then $\mu_j(A) = 0$ for all $j \in J$.

If $\succsim_i$ are represented by the Gilboa and Schmeidler (1986) structure, then for any $\mathcal{F}_i$-measurable consumption contracts $x, y : \Omega \rightarrow R_+^L$,

$$x \succsim_i y \Leftrightarrow \min_{\mu \in \bar{\mu}} E_\mu \left[U_i(x)\right] \geq \min_{\mu \in \bar{\mu}} E_\mu \left[U_i(y)\right].$$

Monderer and Samet (1989) established that in the case of Bayesian games with a singleton prior μ, when the game being played is common p-belief among the players then for any Nash equilibrium strategy profile in the complete information game, there exists an equilibrium strategy profile in the incomplete information game that is very close to equilibrium strategy profile which would be played when the game is common knowledge. In particular Monderer and Samet (1989) prove this when the players behave as ε-maximizers for some $\varepsilon \in \mathbb{R}$ (suitably restricted), i.e., each player i has preferences $\succsim_{i,\varepsilon}$ where

$$x \succsim_{i,\varepsilon} y \Leftrightarrow E_{\mu}\left[U_i\left(x\right)\right] \geq E_{\mu}\left[U_i\left(y\right)\right] + \varepsilon.$$

We adopt the same approach to players preferences in the case of p-beliefs. So, in the case of multiple priors $\bar{\mu}$,

$$x \succsim_{i,\varepsilon} y \Leftrightarrow \min_{\mu \in \bar{\mu}} E_{\mu}\left[U_i\left(x\right)\right] \geq \min_{\mu \in \bar{\mu}} E_{\mu}\left[U_i\left(y\right)\right] + \varepsilon.$$

If $x \succsim_{i,\varepsilon} y$, then player i weakly $\varepsilon-$prefers contract x to contract y.

When players have private information in the form of the partitions $\{\Pi_i\}_{i \in I}$, we denote the conditional (or ex-post) preferences of player i at $\omega \in \Omega$ by $\succsim_{i,\omega}$ so that

$$x \succsim_{i,\omega} y \Leftrightarrow \min_{\mu \in \bar{\mu}} E_{\mu}\left[U_i\left(x\right)|\Pi_i\left(\omega\right)\right] \geq \min_{\mu \in \bar{\mu}} E_{\mu}\left[U_i\left(y\right)|\Pi_i\left(\omega\right)\right].$$

Also, we define $\succsim_{i,\omega,\varepsilon}$ in the obvious way, i.e.

$$x \succsim_{i,\omega,\varepsilon} y \Leftrightarrow \min_{\mu \in \bar{\mu}} E_{\mu}\left[U_i\left(x\right)|\Pi_i\left(\omega\right)\right] \geq \min_{\mu \in \bar{\mu}} E_{\mu}\left[U_i\left(y\right)|\Pi_i\left(\omega\right)\right] + \varepsilon.$$

The endowment of player i is denoted by the $\mathcal{F}_i-$measurable function $e_i : \Omega \rightarrow R^L_+$. An allocation of consumption contracts $x = (x_i)_{i \in I}$ is feasible iff $x_i : \Omega \rightarrow R^L_+$ is $\mathcal{F}_i-$measurable for all i and $\sum_{i \in I} x_i\left(\omega\right) \leq \sum_{i \in I} e_i\left(\omega\right)$ for all $\omega \in \Omega$. An allocation y is ex-ante Pareto optimal iff there is no feasible allocation x such that (a) $\min_{\mu \in \bar{\mu}} E_{\mu}\left[U_i\left(x\right)\right] \geq \min_{\mu \in \bar{\mu}} E_{\mu}\left[U_i\left(y\right)\right]$ for all i and (b) there exists $i' \in I$ with $\min_{\mu \in \bar{\mu}} E_{\mu}\left[U_{i'}\left(x_{i'}\right)\right] > \min_{\mu \in \bar{\mu}} E_{\mu}\left[U_{i'}\left(y_{i'}\right)\right]$.

A feasible allocation x is $\varepsilon-$preferred to the endowment allocation e by the players at state ω, denoted $x \succ_{\omega,\varepsilon} e$ iff (a) $\min_{\mu\in\bar{\mu}} E_\mu \left[U_i(x)\,|\Pi_i(\omega)\right] \geq \min_{\mu\in\bar{\mu}} E_\mu \left[U_i(e)\,|\Pi_i(\omega)\right] + \varepsilon$ for all i and (b) there exists $i' \in I$ such that $\min_{\mu\in\bar{\mu}} E_\mu \left[U_{i'}(x_{i'})\right] > \min_{\mu\in\bar{\mu}} E_\mu \left[U_{i'}(e_{i'})\right] + \varepsilon$. Denote by $A(\varepsilon)$ the event that the players $\varepsilon-$prefer x to e, i.e. $A(\varepsilon) = \{\omega | x \succ_{\omega,\varepsilon} e\}$ and using the result of proposition 2 denote by $E^p[A(\varepsilon)]$ the event that $A(\varepsilon)$ is common p-belief. We assume that $ess \sup_{\mu_j} |U_i(x_i(\omega)) - U_i(e_i(\omega))| \leq M < \infty$ for all $j \in J$ and all $i \in I$, where $M > 0$. The following result shows that purely speculative trade can be ruled out under Knightian uncertainty if p is high enough.

Proposition 4 *Let e be the ex-ante Pareto optimal endowment allocation and let x be a feasible allocation. For $1/2 < p \leq 1$ and $\varepsilon \geq (1-p)M$,*

$$\mu_j\left(E^p\left[A(\varepsilon)\right]\right) = 0 \text{ for all } j. \cap$$

As a consequence of proposition 4 we can also rule out the possibility of speculative trade when players perceive ambiguity in the sense of Bewley (1986) and have incomplete preferences, $\succsim_i^B$, i.e. any $\mathcal{F}_i$-measurable consumption contracts $x, y : \Omega \to R_+^L$,

$$x \succsim_i^B y \Leftrightarrow E_\mu\left[U_i(x)\right] \geq E_\mu\left[U_i(y)\right] \text{ for all } \mu \in \bar{\mu}.$$

We can then define $\left(\succsim_{i,\varepsilon}^B, \succsim_{i,\omega}^B, \succsim_{i,\omega,\varepsilon}^B\right)_{i\in I}$ in the obvious way.

Defining the $\mathcal{F}_i-$measurable function $z = x - y$, we have that

$$x \succsim_i^B y \Leftrightarrow \min_{\mu\in\bar{\mu}} E_\mu\left[U_i(z)\right] \geq 0$$

If we normalize U_i for all i so that $U_i(0) = 0$, this observation allows us to obtain the following result for an exchange economy, called the Bewley economy, where the players preferences are $\left(\succsim_i^B\right)_{i\in I}$ and the endowment allocation is $e' = (e'_i)_{i\in I}$, $e_i : \Omega \to R_+^L$ is $\mathcal{F}_i-$measurable for all i. Let x be a feasible allocation in this economy and denote by $A'(\varepsilon)$ the event that the players $\varepsilon-$prefer x to e', i.e. (a) $E_\mu\left[U_i(x_i)\right] \geq$

$E_\mu [U_i (e_i')] + \varepsilon$ for all $\mu \in \nabla\bar{\mu}$ for all i and (b) $E_\mu [U_i (x_i)] > E_\mu [U_i (e_i')] + \varepsilon$ for all $\mu \in \bar{\mu}$ for some i.

Corollary 1 *Let e' be the ex-ante Pareto optimal endowment allocation in a Bewley economy and let x be a feasible allocation. For $1/2 < p \leq 1$ and $\varepsilon \geq (1-p) M$,*

$$\mu_j (E^p [A' (\varepsilon)]) = 0 \textit{ for all } j$$

5 Conclusion

We extend the notion of common belief to settings of ambiguity and showed that some results from the settings of risk would continue to hold in the former. We do not consider our analysis as suggesting that the distinction between risk and uncertainty is moot. In fact, our analysis shows that when ambiguity exists it can cause significant departures in outcomes of interaction among players unless fairly strict conditions are satisfied.

Our analysis is a first step toward analysing the robustness of equilibrium actions of players under Knightian uncertainity when common knowledge is approximated by common belief. We also propose to further generalize the notion to common belief under Knightian uncertainty when the players' beliefs are smoothly ambiguous, i.e. the players have a probability measure over their set of priors (see Klibanoff et al. (2005) for an analysis of decision making in this setting). This extension will allow for us to analyze a question that arises naturally in the case of multiple priors - what happens when players only p-believe an event with a subset of their set of priors?

6 Appendix

Proof of proposition 1

(1) $B_i^p (E) = \cap_j B_{ij}^p (E) \in \mathcal{F}_i$ since $B_{ij}^p (E) \in \mathcal{F}_i$ for all j. Note that $B_{ij}^p (E) \in \mathcal{F}_i$

follows from the fact that if $\omega \in B_{ij}^{p}(E)$ then $\Pi_i(\omega) \subseteq B_{ij}^{p}(E)$ and Π_i is a countable set.

(2) $E \in \mathcal{F}_i \Rightarrow E = \cup_{\omega\in E}\Pi_i(\omega)$ and since Π_i is countable, E is a union of at most countably many events. Then $\omega \in E \Rightarrow \Pi_i(\omega) \subseteq E$, so $\mu_j(E|\Pi_i(\omega)) = 1 \geq p$ for all j and hence $\omega \in B_i^p(E)$. If $\omega \in B_i^p(E)$ and $p > 0$, then $\omega \in E$ since $\omega \notin E$ $\Rightarrow E \cap \Pi_i(\omega) = \emptyset$ so that $\mu_j(E|\Pi_i(\omega)) = 0$ for all j. If $p = 0$ and $E \neq \Omega$ then for $\omega \in \Omega \backslash E$, $\mu_j(E|\Pi_i(\omega)) = 0$ for all j so that $\omega \in B_i^0(E)$.

(3) Let $\omega \in B_{ij}^{p}\left(B_{ij}^{p}(E)\right)$. Then $\mu_j\left(B_{ij}^{p}(E)|\Pi_i(\omega)\right) \geq p$. If $p > 0$, then $\Pi_i(\omega) \cap B_{ij}^{p}(E) \neq \emptyset \Rightarrow \mu_j(E|\Pi_i(\omega)) \geq p \Rightarrow \omega \in B_{ij}^{p}(E)$. If $p = 0$, then $\mu_j(E|\Pi_i(\omega)) \geq 0 \Rightarrow \omega \in B_{ij}^{0}(E)$. Let $\omega \in B_{ij}^{p}(E)$. Then $\mu_j(E|\Pi_i(\omega)) \geq p \Rightarrow \Pi_i(\omega) \subseteq B_{ij}^{p}(E) \Rightarrow$ $\mu_j\left(B_{ij}^{p}(E)|\Pi_i(\omega)\right) = 1 \geq p \Rightarrow \omega \in B_{ij}^{p}\left(B_{ij}^{p}(E)\right)$. Now consider $\omega \in B_i^p(E)$. Then $\mu_j(E|\Pi_i(\omega)) \geq p$ for all $j \Rightarrow \Pi_i(\omega) \subseteq B_{ij}^{p}(E)$ for all $j \Rightarrow \Pi_i(\omega) \subseteq B_i^p(E) \Rightarrow$ $\mu_j(B_i^p(E)|\Pi_i(\omega)) = 1 \geq p$ for all j. So, $\omega \in B_i^p(B_i^p(E))$. Now, suppose $\omega \in B_i^p(B_i^p(E))$. Then $\mu_j(B_i^p(E)|\Pi_i(\omega)) \geq p$ for all j. If $p > 0$, then $\Pi_i(\omega) \cap B_i^p(E) \neq \emptyset \Rightarrow \mu_j(E|\Pi_i(\omega)) \geq p$ for all j $\Rightarrow \omega \in B_i^p(E)$. If $p = 0$, then $\mu_j(E|\Pi_i(\omega)) \geq 0$ for j $\Rightarrow \omega \in B_i^p(E)$.

(4) $\omega \in B_i^p(E) \Rightarrow \mu_j(E|\Pi_i(\omega)) \geq p$ for all $j \Rightarrow \mu_j(F|\Pi_i(\omega)) \geq p$ for all j since $E \subseteq F \Rightarrow \omega \in B_i^p(F)$.

(5) Let $\omega \in \cap_n B_{ij}^{p}(E^n)$. Then $\mu_j(E^n|\Pi_i(\omega)) \geq p$ for all n. By continuity of the (conditional) measure (and the fact that weak inequalities are preserved in the limit) $\mu_j(\cap_n E^n|\Pi_i(\omega)) = \lim_{n\to\infty}\mu_j(E^n|\Pi_i(\omega)) \geq p$, so $\omega \in B_{ij}^{p}(\cap_n E^n)$. If $\omega \in B_{ij}^{p}(\cap_m E^m)$ then $\mu_j(E^n|\Pi_i(\omega)) \geq p$ for all n since $\cap_m E^m \subseteq E^n$ for all n. Now consider $\omega \in B_i^p(\cap_m E^m)$, then for all j, $\mu_j(E^n|\Pi_i(\omega)) \geq p$, so that $\omega \in \cap_n B_i^p(E^n)$. If $\omega \in \cap_n B_i^p(E^n)$, then for all j, $\mu_j(E^n|\Pi_i(\omega)) \geq p$ and so using continuity we have that for all j, $\mu_j(\cap_n E^n|\Pi_i(\omega)) \geq p \Rightarrow \omega \in B_i^p(\cap_n E^n)$.

(5) $B_{ij}^{p}(E) \in \mathcal{F}_i \Rightarrow B_{ij}^{p}(E) = \cup_{k\in K}\Pi_i(\omega_k)$ where K is countable since Π_i is countable. For any $\Pi_i(\omega_k)$, $\mu_j(E|\Pi_i(\omega_k)) \geq p \Leftrightarrow \mu_j(E \cap \Pi_i(\omega_k)) \geq p\mu_j(\Pi_i(\omega_k))$. So, $\sum_{k\in K}\mu_j(E \cap \Pi_i(\omega_k)) \geq p\sum_{k\in K}\mu_j(\Pi_i(\omega_k)) \Rightarrow \mu_j\left(E|B_{ij}^{p}(E)\right) \geq p$. Now con-

sider $B_i^p(E) \in \mathcal{F}_i$ so that $B_i^p(E) = \cup_{k\in\bar{K}}\Pi_i(\omega_k)$ where $\bar{K}$ is countable. For any $\Pi_i(\omega_k)$, $\mu_j(E|\Pi_i(\omega_k)) \geq p$ for all $j \Leftrightarrow \mu_j(E\cap\Pi_i(\omega_k)) \geq p\mu_j(\Pi_i(\omega_k))$ for all j. So, $\sum_{k\in K}\mu_j(E\cap\Pi_i(\omega_k)) \geq p\sum_{k\in K}\mu_j(\Pi_i(\omega_k))$ for all $j \Rightarrow \mu_j(E|B_i^p(E)) \geq p$ for all j.■

Proof of proposition 2

(i) Note that C^m is a decreasing sequence since for all i and $m \geq 1$, $C^m \subseteq B_i^p(C^{m-1}) \Rightarrow B_i^p(C^m) \subseteq B_i^p(B_i^p(C)) = B_i^p(C)$ so that for all $m \geq 1$, $C^{m+1} = \cap_{i\in I}B_i^p(C^m) \subseteq \cap_{i\in I}B_i^p(C^{m-1}) = C^m$. Now, for all $m \geq 1$, $E^p(C) \subseteq C^{m+1} \subseteq B_i^p(C^m)$ $\Rightarrow E^p(C) \subseteq \cap_{i\in I}B_i^p(C^m)$. So, for all i, $E^p(C) \subseteq \cap_{m\geq 1}B_i^p(C^m) = B_i^p(\cap_{m\geq 1}C^m) = B_i^p(E^p(C))$. Also, $E^p(C) \subseteq C^1 = B_i^p(C)$ by definition.

(ii) If $\omega \in E^p(C)$, then using (i), we have that C is common p-belief at ω. Suppose C is common p-belief at ω. Then there exists an evident p-belief E such that $\omega \in E$ and $E \subseteq \cap_{i\in I}B_i^p(C) = C^1$. Suppose $E \subseteq C^m$, then $B_i^p(E) \subseteq B_i^p(C^m)$ for all i. So, $E \subseteq \cap_{i\in I}B_i^p(C^m) = C^{m+1}$. Hence, $E \subseteq C^m$ for all m, i.e., $E \subseteq E^p(C)$ $\Rightarrow \omega \in E^p(C)$.■

Proof of proposition 3

If C is common p-belief at ω then there exists an evident p-belief E with $\omega \in E$ and $E \subseteq \cap_{i\in I}B_i^p(C)$. Now $B_i^p(E) \in \mathcal{F}_i$, so there exists a countable collection of $\mathcal{F}_i$-measurable sets $\{\Pi_{il}\}_{l\in L}$ such that $B_i^p(E) = \cup_{l\in L}\Pi_{il}$. Also, $E \subseteq B_i^p(C) \Rightarrow B_i^p(E) \subseteq B_i^p(B_i^p(C)) = B_i^p(C)$. So, for every j,

$$
\begin{aligned}
&\mu_j(\cap_{k\in I}B_k^p(E)|\Pi_{il}) \geq \mu_j(E|\Pi_{il}) \geq p \text{ for every } l \\
&\Rightarrow \mu_j((\cap_{k\in I}B_k^p(E))\cap\Pi_{il}) \geq p\mu_j(\Pi_{il}) \text{ for every } l \\
&\Rightarrow \textstyle\sum_{l\in L}\mu_j((\cap_{k\in I}B_k^p(E))\cap\Pi_{il}) \geq p\sum_{l\in L}\mu_j(\Pi_{il}) \\
&\Rightarrow \mu_j((\cap_{k\in I}B_k^p(E)\cap B_i^p(E))) \geq p\mu_j(B_i^p(E)) \\
&\Rightarrow \mu_j(\cap_{k\in I}B_k^p(E)|B_i^p(E)) \geq p.
\end{aligned}
$$

Also, for all j,

$$\begin{aligned}
& r_{*i} \le \mu_j (F|\Pi_{il}) \le r_i^* \text{ for every } l \\
& \Rightarrow r_{*i}\mu_j (\Pi_{il}) \le \mu_j (F \cap \Pi_{il}) \le r_i^* \mu_j (\Pi_{il}) \text{ for every } l \\
& \Rightarrow r_{*i} \textstyle\sum_{l\in L} \mu_j (\Pi_{il}) \le \sum_{l\in L} \mu_j (F \cap \Pi_{il}) \le r_i^* \sum_{l\in L} \mu_j (\Pi_{il}) \\
& \Rightarrow r_{*i}\mu_j (B_i^p (E)) \le \mu_j (F \cap B_i^p (E)) \le r_i^* \mu_j (B_i^p (E)) \\
& \Rightarrow r_{*i} \le \mu_j (F|B_i^p (E)) \le r_i^*.
\end{aligned}$$

Since r_i is a closed interval for all i, the only case we need to focus on is when for $i, k \in I$, $i \neq k$, $r_i^* < r_{*k}$. Let $\mu_* (i) \in \arg\min_j r_i$, i.e., $\mu_* (i) (F|\Pi_i (\omega)) = r_{*i}$ and $\mu^* (k) \in \arg\max_j r_k$, i.e. $\mu^* (k) (F|\Pi_k (\omega)) = r_k^*$.

Now, for all j and any $G \in \Sigma$,

$$\begin{aligned}
\mu_j (G|B_i^p (E)) &\ge \mu_j (G|B_i^p (E) \cap B_k^p (E)) \mu_j (B_k^p (E) |B_i^p (E)) \\
&\ge \mu (G|B_i^p (E) \cap B_k^p (E)) p.
\end{aligned}$$

In particular, $\min \left\{\mu_j (F|B_i^p (E)), \mu_j (F|B_k^p (E))\right\} \ge \mu_j (F|B_i^p (E) \cap B_k^p (E)) p$ for all j. Now, $\mu_* (i) (F|B_i^p (E)) \le r_i^*$ and $\mu_* (i) (F|B_i^p (E)) \ge \mu_* (i) (F|B_i^p (E) \cap B_k^p (E)) p$. Also, $r_{*k} \le \mu_* (i) (F|B_k^p (E)) \le \mu_* (i) (F|B_i^p (E) \cap B_k^p (E)) p + (1-p)$ so that $(r_{*k} - r_i^*) \le (1-p)$.■

Proof of proposition 4

Assume that $\mu_j (E^p [A (\varepsilon)]) > 0$ for some $j \in J$, then given mutual absolute continuity we have $\mu (E^p [A (\varepsilon)]) > 0$ for all $\mu \in \bar{\mu}$. From proposition 2 we know $E^p [A (\varepsilon)] \subseteq \cap_{i\in I} B_i^p (A (\varepsilon))$ and $E^p [A (\varepsilon)] \subseteq \cap_{i\in I} B_i^p (E^p [A (\varepsilon)])$.

Since $\Pi_i (\omega) \cap A (\varepsilon) \neq \emptyset$ for all $\omega \in B_i^p (A (\varepsilon))$, given that $p > 0$, we get that for all $\omega \in B_i^p (A (\varepsilon))$, $x_i \succsim_{i,\omega,\varepsilon} e_i$. Then $E^p [A (\varepsilon)] \subseteq \cap_{i\in I} B_i^p (A (\varepsilon)) \Rightarrow$ for any $\omega \in E^p [A (\varepsilon)]$, $x_i \succsim_{i,\omega,\varepsilon} e_i$ for all i.

We also know that for every $\omega \in E^p [A (\varepsilon)]$ and every $\mu \in \bar{\mu}$, $\mu ((E^p [A (\varepsilon)])^c |\Pi_i (\omega)) \le (1-p)$. Now, consider $\omega \in E^p [A (\varepsilon)]$, such that $\mu_j (\Pi_i (\omega) \backslash E^p [A (\varepsilon)]) > 0$ for all

$j \in J$. Then, for all $\mu' \in \bar{\mu}$

$$[\min_{\mu\in\bar{\mu}} E_\mu [U_i(x_i)|\Pi_i(\omega)\backslash E^p[A(\varepsilon)]] - \min_{\mu\in\bar{\mu}} E_\mu [U_i(e_i)|\Pi_i(\omega)\backslash E^p[A(\varepsilon)]]] \frac{\mu'(\Pi_i(\omega)\backslash E^p[A(\varepsilon)])}{\mu'(\Pi_i(\omega))} \leq (1-p)M$$

However, we also know that

$$\left[\min_{\mu\in\bar{\mu}} E_\mu [U_i(x_i)|\Pi_i(\omega)] - \min_{\mu\in\bar{\mu}} E_\mu [U_i(e_i)|\Pi_i(\omega)]\right] \geq \varepsilon \geq (1-p)M$$

Since $\bar{\mu}$ is rectangular, we get

$$\left[\min_{\mu\in\bar{\mu}} E_\mu [U_i(x_i)|\Pi_i(\omega)\cap E^p[A(\varepsilon)]] \geq \min_{\mu\in\bar{\mu}} E_\mu [U_i(e_i)|\Pi_i(\omega)\cap E^p[A(\varepsilon)]]\right]$$

Then using the rectangularity of $\bar{\mu}$ again, we have for all i

$$\left[\min_{\mu\in\bar{\mu}} E_\mu [U_i(x_i)|E^p[A(\varepsilon)]] \geq \min_{\mu\in\bar{\mu}} E_\mu [U_i(e_i)|E^p[A(\varepsilon)]]\right].$$

Now, if $\mu_j(A(\varepsilon)\cap E^p[A(\varepsilon)]) = 0$ for any $j \in J$, then for any $\omega \in E \subseteq B_i^p(A(\varepsilon))$, $\mu(A(\varepsilon)|\Pi_i(\omega)) \leq (1-p) < 0.5$, which is a contradiction. Then $\mu(A(\varepsilon)\cap E^p[A(\varepsilon)]) > 0$ for all $\mu \in \bar{\mu}$. So, by definition of $A(\varepsilon)$, there exists $\omega \in A(\varepsilon)\cap E^p[A(\varepsilon)]$ such that for some $i' \in I$,

$$\left[\min_{\mu\in\bar{\mu}} E_\mu [U_{i'}(x_{i'})|\Pi_i(\omega)\cap E^p[A(\varepsilon)]] > \min_{\mu\in\bar{\mu}} E_\mu [U_{i'}(e_{i'})|\Pi_i(\omega)\cap E^p[A(\varepsilon)]]\right]$$

and using rectangularity of $\bar{\mu}$ again, we get that

$$\left[\min_{\mu\in\bar{\mu}} E_\mu [U_{i'}(x_{i'})|E^p[A(\varepsilon)]] > \min_{\mu\in\bar{\mu}} E_\mu [U_{i'}(e_{i'})|E^p[A(\varepsilon)]]\right].$$

This contradicts the ex-ante Pareto optimality of e since the allocation $x^* = (x_i 1_{E^p[A(\varepsilon)]})_{i\in I}$ ex-ante Pareto dominates e given the rectangularity of $\bar{\mu}$.$\blacksquare$

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