THE MERTENS AND NEYMAN VALUES ARE NOT EQUAL

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Abstract. It was asked in [3] whether or not the Mertens and the Neyman values coincide on a certain large space of games. We answer to the negative. Surprisingly, the counterexample is a game of the form $f \circ \mu$ where f is Lipschitz, i.e. - the Neyman and Mertens values already different on "reasonable" games. This answer suggests that given a non-differentiable game $f \circ \mu$ we may use different "marginals" of f to construct different price mechanisms. Finally we extend the value formula derived in [3, section 6.1] and use it to prove that these values coincide on a large space of games.

1. Introduction

A solution concept for a set Q of cooperative games with an underlying coalitional structure (I, Σ) is a correspondence $\mathcal{S} : Q \to \mathbb{P}(\mathcal{M}(I))$ between games in Q and sets of measures over I, which follows a list of desirable axioms. During the years many different axioms were formulated. Among the most common axioms we may find efficiency (i.e.- $\forall v\ (\mathcal{S}v)(I) = \{v(I)\}$), symmetry (i.e. - $\forall \theta \in Aut(I, \Sigma) \forall v \in Q,\ \mathcal{S}(\theta^* v) = \theta^*(\mathcal{S}v)$ monotonicity (i.e.- $v \in Q^+ \Rightarrow \mathcal{S}v \subset \mathcal{M}^+(I)$) etc'. One of the most basic solution concepts of cooperative game theory is the Shapley value. It was first introduced in the setting of n-players games, where it can be viewed as the players' expected payoffs. It has a wide range of applications in various fields of economics and political science. In many such applications it is necessary to consider games that involve a large number of players s.t. most of them are "insignificant". Among the typical examples we find voting among stockholders of a corporation and markets with perfect competition. In such cases it is fruitful to model the game as a cooperative game with an underlying coalitional structure $([0, 1], \mathcal{C})$, i.e.- a game with a "continuum of players". [1] expanded the definition of value to games with a continuum of players. The value was defined using the axioms of efficiency, symmetry and monotonicity.

When we define a solution concept $\mathcal{S}$ on a set of games Q, it is natural to ask wither this solution is unique. The core of a cooperative game, the nucleus and the Shapley value are examples for solution concepts of cooperative games which are unique under certain conditions. In [1] it is shown that the value exist and is indeed unique on some spaces of games with a continuum of players (e.g. - the space of all "differentiable" market games). It was unknown for quite a long time wither there exist a value on the spaces of games which are not "smooth". An example is the space of all market games. [2] introduced a value on a very large set of "non-differentiable" games. [3] introduced a value on yet another very large set of "non-differentiable" games. Both values are obviously not unique, due the use of Banach limits in the construction. Yet, [3] asked wither "modulo Banach limits", do these values coincide on the intersection of their domains? This work answers this question to the negative, thus proving that the value on the intersection of domains is not unique. This demonstrates another nice property - the Shapley, Aumann-Shapley, Mertens and Neyman values were constructed as some average over "marginal values"; This work shows that in the setting of "non-differentiable" games there is great importance to the question "which marginal value?".

2. Preliminaries

2.1. **The Mertens Value.**

Definition 2.1. An *extension operator* is a linear and symmetric map ψ from a linear symmetric space of games to real-valued functions on $B_1^+(I,\mathcal{C})$, with $(\psi v)(0)=0$, s.t. $(\psi v)(1)=v(I)$, $||\psi v||_{IBV}\leq\|v\|$, ψv is finitely additive whenever v is finitely additive and a constant sum whenever v is constant sum.

Mertens [2] proves the existence of an extension operator on a large symmetric space Q. In the same paper he defined the following spaces; 1. Let $Q_D\subset Q$ the space of all games for which the following integral and limit exist:

$$(\varphi_D v)(\chi)=\lim_{\tau\to 0}\int_0^1\frac{\bar{v}(t+\tau\chi)-\bar{v}(t-\tau\chi)}{2\tau}dt \tag{2.1}$$

2. For every w in the range of φ_D and every $\xi,\chi\in B(I,\mathcal{C})$ let

$$[w]_\xi(\chi)=\lim_{\tau\to 0}\frac{\bar{v}(\chi+\tau\xi)-\bar{v}(\chi-\tau\xi)}{2\tau}. \tag{2.2}$$

Let Q_M be the closed symmetric space generated by all games v in Q_D s.t. either $\varphi_D(v)\in FA$ or $\varphi_D(v)$ is a function of finitely many non-atomic measures.

Theorem 2.2 (Mertens [2], Section 2)**.** *Let $v\in Q_M$. Then for every $\xi\in B(I,\mathcal{C})$ $[\varphi_D]_\xi(\chi)$ exists for P-almost every χ and is P-integrable in χ. In particular the map $\varphi_M : Q_M\longrightarrow FA$ given by*

$$(\varphi_M v)(S)=\int[\varphi_D(v)]_S(\chi)dP(\chi)$$

is a value of norm 1 on Q_M.

2.2. **The Neyman Value.**

Let $Q(\mu)$ be the space of all bounded variation games of the form $f\circ\mu$, where μ is a vector measure in $\left(NA^1\right)^k$ for some k, and f is continuous in 0 and in $\mathbf{1}_k$ For any $\mathbb{R}^k$ valued non-atomic measure μ define a map φ_μ^δ from $Q(\mu)$ to BV as follows:

Let $I_\delta(t)=I(3\delta\leq t<1-3\delta)$ and $x\in 2\mathcal{R}(\mu)-\mu(I)$. Define,

$$F_{f,\mu}(\delta,x,S)=\int_0^1 I_\delta(t)\frac{f(t\mu(I)+\delta^2x+\delta^3\mu(S))-f(t\mu(I)+\delta^2x)}{\delta^3}dt. \tag{2.3}$$

Let P_μ^δ the restriction of P_μ to the set $\{x\in\mathbb{R}^k\ :\ \delta x\in 2\mathcal{R}(\mu)-\mu(I)\}$. The function $x\mapsto F_{f,\mu}(\delta,x,S)$ is continuous and bounded and therefore,

$$\varphi_\mu^\delta(f\circ\mu,S)=\int_{AF(\mu)}F_{f,\mu}(\delta,x,S)dP_\mu^\delta, \tag{2.4}$$

where $AF(\mu)$ is the affine space spanned by $\mathcal{R}(\mu)$, is well defined.

$Q(\mu)$ is not symmetric and φ_μ^δ doesn't map it into FA. φ_μ^δ is also not efficient nor symmetric and its restriction to $Q(\mu)\cap Q(\nu)$ isn't necessarily equal to φ_ν^δ. However, these violations of the value axioms diminishes as $\delta \longrightarrow 0$, and $\varphi_\mu^\delta(f\circ\mu) - \varphi_\nu^\delta(g\circ\nu) \longrightarrow 0$ as $\delta \longrightarrow 0$ whenever $f\circ\mu = g\circ\nu$. This remains true even if the limit exists only as some Banach limit (see [3], section 3.2 for details).

Let $Q_N = \bigcup Q(\mu)$, and define $\varphi_N : Q \longrightarrow \mathbb{R}^{\mathcal{C}}$ by

$$\varphi_N(v)(S) = L(\varphi_\mu^\delta(v,S)) \tag{2.5}$$

whenever $v \in Q(\mu)$. Then φ_N is a value of norm 1 on Q_N ([3], Proposition 1).

We denote by Q'_N the set of all games $f\circ\mu \in Q_N$ s.t. the limit $\phi_\mu^\delta(f\circ\mu,\cdot)$ exists in the usual sense (i.e. - not as a Banach limit).

3. Statement of Results

Theorem 3.1. *There is a game $v = f\circ\mu \in Q_M \cap Q'_N$, where f is Lipschitz and $\varphi_N(v) \neq \varphi_M(v)$.*

Lemma 3.2. *Let $f\circ\mu \in Q'_N$ be a game s.t. $f : [0,1]^n \mapsto \mathbb{R}$ and $\mu = (\mu_1, ..., \mu_n)$ is a vector of probability measures. Then for each coalition $S \in \mathcal{C}$ and $\epsilon > 0$ there is some open set $\Omega_\epsilon \subset [0,1]$ of measure at most ϵ s.t. the directional derivative $f_y(x)$ exist for almost each x in a neighborhood of $[0,\mu(I)] \setminus (\Omega_\epsilon \cdot \mu(I))$, and for any sufficiently small $\delta > 0$*

$$\psi_\mu^\delta(f\circ\mu, S) = \int\int I_\delta(t)\cdot \chi_{\Omega_\delta} f_{\mu(S)}(t\mu(I) + \delta^2 x)dtdP_\mu^\delta(x)$$

is well defined and

$$\varphi_N(f\circ\mu)(S) = \lim_{\delta\to 0^+} \psi_\mu^\delta(f\circ\mu, S).$$

Theorem 3.3. *Let $f\circ\mu \in Q'_N \cap Q_M$ be a game s.t. $f : [0,1]^n \mapsto \mathbb{R}$ and μ is a probability vector measure and for each coalition $S \in \mathcal{S}$ and each $\epsilon > 0$,*

$$\liminf_{\delta\to 0^+} f_\mu(S)(t\mu(I) + \delta^2 x) \geq \left(f_x(t\mu(I))\right)_{\mu(S)},$$

for almost each $t \in [\epsilon\mu(I), (1-\epsilon)\mu(I)]$ and almost each $x \in AF(\mu)$ then

$$\varphi_N(f\circ\mu) = \varphi_M(f\circ\mu)$$

Corollary 3.4. *If $v = f\circ\mu \in BV$ is a game as above s.t. there is a partition of $[0,1]^n$ into k polygonal areas $P_1, P_2, ..., P_k$ where $f|_{P_i}$ is a polynomial then*

$$\varphi_N(v) = \varphi_M(v)$$

References

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