Belief Operator in a Universal Space*

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Abstract

In complex informational and strategic situations where individual(s) may exhibit very general preferences, we apply Morris's (JET 69 (1996) 1-23) alternative notion of belief to the universal state space constructed by Epstein and Wang (Econometrica 64 (1996) 1343-1373), and study the logical properties of belief from a decision-theoretic point of view. *JEL Classification: D80.*

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1 Introduction

Following Aumann [2], economists and game theorists usually define the notion of "belief" (or "knowledge") in terms of some *exogenous* information structure. This semantic approach, although very useful in game theory and economics, ignores behavioral or decision-theoretic aspects of belief. In an interesting paper, Morris [19] offered a framework for thinking about information and belief in the context of decision-making under uncertainty, in which the decision maker is assumed to be endowed with preferences, at each state of the world, over acts on the space of states of the world (see also Morris [20]). Without referring to an exogenous information structure, the notion of belief is defined directly by more primitive preferences; accordingly, an event is believed if its complement is null in the sense of Savage [23]. In a *finite* state space, Morris [19] showed how some substantive properties of beliefs can be related to axioms on preferences; among others, Morris showed that if preference relation is a complete ordering, then the belief operator elicited from preferences is "normal" – i.e. the belief operator can be generated by some information correspondence in the standard semantic model. The main purpose of this paper is to further extend this line of research to an infinite "universal" state space where individual(s) may exhibit general preferences.

The primary reason for pursuing the study of this paper is as follows. In complex informational and strategic environments, it is important to consider an infinite regress of a hierarchy of "beliefs about beliefs about beliefs about ...". The full description of a state pertaining to such a decision problem is required to represent this exhaustive subjective uncertainty facing each individual. In doing so, the crucial prerequisite is whether there exists a well-defined state space rich enough so that a state contains a "limit-closure" description of general preferences over the state space. Epstein and Wang [10] (hereafter EW) constructed a well-defined space of types where each type is an infinite regress of a hierarchy of "preferences over preferences over preferences over ..." and has a property that it is homeomorphic to preferences over acts on the product space of the states of nature and types. In this paper we advance the line of research advocated by Morris [19] by working with a universal state space constructed by EW.

Consequently, in this paper we extend Morris's [19] results to the case of an infinite state space. More specifically, in a universal state space constructed by EW, we show that the belief operator defined by preferences associated with states satisfies the most basic property of "normality" (see Theorem 2). This result extends both Morris's [19] "normality" result to an infinite state space and Zamir and Vassilakis's [26] result in terms of the "probabilistic" notion of belief to allow beliefs based upon general preferences, so it enables us to exploit the relatively familiar and simple semantic way of analysis whenever doing so is more convenient. We also study properties of the belief operator and the information structure elicited from the preferences associated with states (see Theorems 1 and 3).

Being carried out in a universal state space, our analysis in this paper is immunized from the "circularity" problem that arises when using Morris's [19] framework where preferences at a state are *exogenously* given.[1] Furthermore, our approach is also significant, because the assumption that the model is commonly known can be

[1] Morris [18] identified the "circularity" problem with his framework. Namely, a decision maker's belief is described in terms of preferences over acts which depend on the state of the world which (implicitly) contains a description of his belief. This problem is closely related to a well-known "self-referential" problem in the conventional semantic framework used in game theory; see, e.g., Aumann [3, p.264] and Fagin *et al.* [11, p.332] for more discussions.

stated formally, whereas this sort of assumption must be understood informally in a meta-sense within the semantic formalism.

The rest of this paper is organized as follows. Section 2 presents the set-up. In Section 3 we study properties of belief operator and information structure which are defined directly from the preferences associated with states in a universal state space. To facilitate reading, the definition of "regular preferences" is summarized in Appendix I, and all the proofs are relegated to Appendix II.

2 Set-up

Consider a compact Hausdorff space X that represents the primitive uncertainty faced by a decision maker. For example, we can take X as the space of all uncertain parameters in a game with incomplete information or as the set of all strategy profiles in a game with complete information. The decision maker faces the space of *states of the world*: $\Omega \equiv X \times T$ where T is the set of Harsanyi's [15] type profiles of *all* the decision maker(s).

The decision maker makes a choice among different *acts*; i.e., Borel measurable functions $f : \Omega \to [0, 1]$ according to his type. We denote by $\mathcal{F}(\Omega)$ the set of the decision maker's acts and by $\mathcal{P}(\Omega)$ the set of the *preferences over* $\mathcal{F}(\Omega)$. Throughout this paper, we restrict ourselves to regular preferences that admit representations by utility functions – i.e., the preferences satisfy **U.1-4** in Appendix I. By EW's Theorems 4.3 and 5.2, we assume $T \sim^{homeomorphic} \mathcal{P}(\Omega)$ and both T and Ω are compact Hausdorff.[2] That is, each decision maker's type identifies, up to a homeomorphism, a utility function

[2] We endow Ω with product topology. For the topology on $\mathcal{P}(\Omega)$, see Appendix I.

in $\mathcal{P}(\Omega)$. Let $\psi : T \to \mathcal{P}(\Omega)$ represent this homeomorphism. We emphasize that the decision maker is allowed to be uncertain about his own type.

For the purpose of this paper, from now on we restrict attention to the case of single-agent decision making problems, although our analysis applies to more general game situations. Let t^{ω} denote the decision maker's type at ω. We write the utility function associated with t^{ω} as $\psi \circ t^{\omega}$ or u^{ω} for convenience. We refer to an arbitrary subset $E \subseteq \Omega$ as an *event*. Given a closed event E, let $\mathcal{P}(\Omega|E)$ denote the set of the decision maker's conditional preferences for which the complement of E is *Savage-null*; i.e. any two acts that agree on E are ranked as being indifferent. Following Epstein [9], we say the decision maker *believes an event E at ω* if there exists a closed subset $\overline{E} \subseteq E$ such that $\psi \circ t^{\omega} \in \mathcal{P}(\Omega|\overline{E})$. Let BE denote the set of all the states where the decision maker believes E; i.e.,

$$BE \equiv \left\{\omega \in \Omega : \psi \circ t^{\omega} \in \mathcal{P}(\Omega|\overline{E}) \text{ for some closed set } \overline{E} \subseteq E\right\}.$$

Clearly, for a closed set E, $BE = \left\{\omega \in \Omega : \psi \circ t^{\omega} \in \mathcal{P}(\Omega|E)\right\}$. The decision maker's information structure generated by the belief operator B is the correspondence $P : \Omega \rightrightarrows \Omega$, such that for all $\omega \in \Omega$,

$$P(\omega) = \bigcap_{\{E \subseteq \Omega:\ BE \ni \omega\}} E.$$

Within the conventional semantic framework, the set $P(\omega)$ is interpreted as: at ω the decision maker knows only that the state is in $P(\omega)$. The information structure P represents *all* information aspects of uncertainty on the part of the decision maker. It constitutes the standard model for "differential" information commonly used in economics.

3 Results

We start with presenting some basic properties of the belief operator B defined directly from the preferences associated with states in an infinite universal state space.

Theorem 1

B1 : $B\varnothing = \varnothing$.

B2 : $B\Omega = \Omega$.

B3 : $E \subseteq F \Rightarrow BE \subseteq BF$.

B4 : $BE \cap BF = B(E \cap F)$ *for any events* E and F.

Remark 1. These properties are common and self-explanatory. The belief operator B may fail to satisfy the other three properties: *the axiom of knowledge*, *the axiom of positive introspection*, and *the axiom of negative introspection* – i.e., $BE \subseteq E$, $BE \subseteq B(BE)$, and $\Omega \backslash BE \subseteq B(\Omega \backslash BE)$.

In accordance with the standard semantic framework of information and knowledge, a belief operator is expressed in terms of an information structure. We may wonder if the belief operator B defined directly from the preferences associated with states can be defined in a semantic fashion. The following Theorem 2 shows that the belief operator defined by preferences associated with states in a universal state space is indeed consistent with the one defined by an information structure. Formally, say a belief operator B is *normal* if, for any event $E \subseteq \Omega$, $BE = \left\{\omega \in \Omega : \widetilde{P}(\omega) \subseteq E\right\}$ where $\widetilde{P}$ is an information structure.

Theorem 2 *B is normal. Moreover,* $BE = \{\omega : P(\omega) \subseteq E\}\ \forall E$.

Remark 2. Morris [19, 20] showed this result in a finite state space following the preference-based approach. In an infinite universal state space, Zamir and Vassilakis [26] among others referred to a similar type of formulation of the "probabilistic" notion of belief.[3] Our result can therefore be viewed as both extending the former to an infinite universal state space and the latter to allow beliefs based upon general preferences.

We next state some properties of the information structure P.

Theorem 3

P1 : $P(\omega) = \{\omega' : \omega \notin B(-\omega')\}$ *is nonempty and closed. Moreover, P is lower-hemicontinuous.*

P2 : $P(\omega) = P(\omega')$ *whenever* $t^{\omega} = t^{\omega'}$.

Remark 3. Note that within the framework of this paper the decision maker is possibly uncertain about his own type or own preferences; see also Heifetz and Samet's [16, p.330] Remark. The approach of this paper can adequately accommodate the analysis of complex economic situations with nonpartitional information structures (see, e.g., Bacharach [4], Dekel and Gul [6], Geanakoplos [12, 13], Luo and Ma [17], Morris [19], Rubinstein [21], Samet [22], and Shin [25] for more extensive discussions on this point).

[3] The normality property of the "probabilistic" belief notion is somewhat trivial: that is, we may consider the only states in the support of a probability measure to be possible. But it is not obvious that there is a natural counterpart of "support" for general preferences.

4 Concluding remarks

We have extended Morris's [19] results to an infinite universal state space constructed by EW. We would like to emphasize several major features in this paper: (1) The framework is perfectly flexible to accommodate general preferences which include the subjective expected utility as a special case. (2) The belief operator defined directly from preferences associated with states would not necessarily satisfy some other axioms of belief, e.g., the axiom of knowledge, the axiom of positive introspection, and the axiom of negative introspection. (3) The information structure defined directly from preferences associated with states may not be partitional. This general approach is suitable for a wide range of informational and strategic situations in economic applications.

As emphasized, this paper has focused on studying the logical properties of beliefs and information structure elicited from preferences associated with states in a universal state space, especially on studying relations with those in the conventional semantic framework for the study of information and knowledge. For this purpose we have adopted Epstein's [9] definition of belief, which is consistent with the definition in a probabilistic setting by using the support of the relevant measure (see, e.g., Dekel and Gul [6] and Zamir and Vassilakis [26]) and can have merits in Choquet and multiple-priors models; see Epstein [9] for more discussions. In this respect Theorem 2 in this paper extends both Morris's [19] "normality" result to an infinite state space and Zamir and Vassilakis's [26] result in terms of the "probabilistic" notion of belief to allow beliefs based upon general preferences.[4]

[4] It is worthwhile to mention that there are other possible definitions of belief in the literature; in particular, an alternative definition of belief is by using "measurable subsets" in place of "closed subsets." The definition of belief used in this paper enables us to show the "normality" result by using the key Lemma 2 in Appendix II that states a variant version of "knowledge continuity" property. However, under the alternative definition by using "measurable sets," normality in general does not

Di Tillio [8] recently provided an alternative construction of type space where even weaker assumptions about preferences are maintained (for example, regularity is not assumed). It is certainly an interesting topic for future study to conduct our analysis in this paper using his framework.

Appendix I: Regular Preferences and Models of Preferences

Let S be an arbitrary compact Hausdorff space. Let $\mathcal{F}(S)$ be the set of Borel measurable functions $f : S \to [0, 1]$. Let

$$\begin{aligned} \mathcal{F}^u(S) &= \left\{ f \in \mathcal{F}(S) : f(S) \text{ is finite; } f^{-1}([r, 1]) \text{ is closed for any } r \in [0, 1] \right\}; \\ \mathcal{F}^l(S) &= \left\{ f \in \mathcal{F}(S) : f(S) \text{ is finite; } f^{-1}((r, 1]) \text{ is open for any } r \in [0, 1] \right\}. \end{aligned}$$

A preference is said to be *regular* if it has a numerical representation $u : \mathcal{F}(S) \to [0, 1]$ satisfying:

U.1. Certainty Equivalence: $u(r) = r$, $\forall r \in [0, 1]$.

U.2. Weak Monotonicity: $f' \geq f \Rightarrow u(f') \geq u(f)$, $\forall f, f' \in \mathcal{F}(S)$.

U.3. Inner Regularity: $u(f) = \sup \{u(g) : g \leq f,\ g \in \mathcal{F}^u(S)\}$, $\forall f \in \mathcal{F}(S)$.

U.4. Outer Regularity: $u(g) = \inf \left\{u(h) : h \geq g,\ h \in \mathcal{F}^l(S)\right\}$, $\forall g \in \mathcal{F}^u(S)$.

Examples of preference models satisfying **U.1-4** include: the subjective expected utility model, the ordinal expected utility model, the probabilistic sophistication model, the Choquet expected utility model, and so on.

The topology on $\mathcal{P}(S)$ is that generated by the subbasis consisting sets of the form: $\{u : \ u(g) < r\}$ and $\{u : \ u(h) > r\}$ where $g \in \mathcal{F}^u(S)$, $h \in \mathcal{F}^l(S)$, and $r \in [0, 1]$.

hold. For example, let $X = [0, 1]$ and the expected utility preference at a state have the marginal Lebesgue measure on X. At that state there is no "minimal (w.r.t. set-inclusion) measurable set with measure 1" because the Lebesgue measure is atomless; see Chen and Luo [5] for more discussion.

Appendix II: Proofs

Proof of Theorem 1. Clearly, **B1** holds by **U.1**. **B2** and **B3** hold by the definition of belief. For **B4** it suffices to show $BE \cap BF \subseteq B(E \cap F)$ for any closed sets $E, F \subseteq \Omega$. Suppose $\omega \in BE \cap BF$. Since E and F are closed, $u^{\omega} \in \mathcal{P}(\Omega|E)$ and $u^{\omega} \in \mathcal{P}(\Omega|F)$. That is, $u^{\omega}(f) = u^{\omega}(g)$ for any f and g which agree on E; $u^{\omega}(f) = u^{\omega}(g)$ for any f and g which agree on F. Therefore, for any f and g which agree on $E \cap F$,

$$u^{\omega}(f) = u^{\omega}(\begin{bmatrix} f & \text{on } E \\ g & \text{on } \Omega \backslash E \end{bmatrix}) = u^{\omega}(\begin{bmatrix} f & \text{on } E \cap F \\ g & \text{on } \Omega \backslash (E \cap F) \end{bmatrix}) = u^{\omega}(g).$$

So $\omega \in B(E \cap F)$. ■

To prove Theorem 2, we need the following Lemmas 1 and 2. For any $f \in \mathcal{F}(\Omega)$ and $E \subseteq \Omega$, let $f1_E$ denote the function such that

$$f1_E(\omega) = \begin{cases} f(\omega), & \text{if } \omega \in E \\ 0, & \text{otherwise} \end{cases}.$$

Lemma 1. *Let $E \subseteq \Omega$ be closed. Suppose that $u(g) = u(g1_E)$ for all $u \in \mathcal{P}(\Omega)$, $g \in \mathcal{F}^u(\Omega)$. Then, $u(f) = u(f1_E)$ $\forall f \in \mathcal{F}(\Omega)$.*

Proof. Note that, if $g \in \mathcal{F}^u(\Omega)$ and E is closed, $g1_E \in \mathcal{F}^u(\Omega)$ and, furthermore, $g1_E \leq f1_E$ whenever $g \leq f$. By **U.3** we have

$$\begin{aligned} u(f) &= \sup\{u(g) : g \leq f, \, g \in \mathcal{F}^u(\Omega)\} \\ &= \sup\{u(g1_E) : g \leq f, \, g \in \mathcal{F}^u(\Omega)\} \\ &= \sup\{u(g1_E) : g \leq f1_E, \, g \in \mathcal{F}^u(\Omega)\} \\ &= \sup\{u(g) : g \leq f1_E, \, g \in \mathcal{F}^u(\Omega)\} \\ &= u(f1_E). \blacksquare \end{aligned}$$

Lemma 2. *Suppose that $\{E_\lambda\}_{\lambda \in \Lambda}$ is a declining family of closed subsets in compact Hausdorff Ω. Then, for each $u \in \mathcal{P}(\Omega)$ and $g \in \mathcal{F}^u(\Omega)$, $\inf_\lambda u(g1_{E_\lambda}) = u(g1_{E^*})$ where $E^* = \cap_\lambda E_\lambda$.*

Proof. (This lemma is an extension of EW's Lemma D.1 for a declining sequence of compact subsets, and its proof is completely similar to that of EW's Lemma D.1.) Clearly, E^* is closed and $g1_{E^*} \in \mathcal{F}^u(\Omega)$ for $g \in \mathcal{F}^u(\Omega)$. By **U.4**, for any $\varepsilon > 0$, there is a simple lsc function $h \geq g1_{E^*}$ such that $u(h) < u(g1_{E^*}) + \varepsilon$. We now show there is some $\overline{\lambda}$ such that $g1_{E_{\overline{\lambda}}} \leq h$. Suppose to the contrary that for each λ there is $\omega_\lambda \in \Omega$ such that

$$g1_{E_\lambda}(\omega_\lambda) > h(\omega_\lambda). \tag{$\bigstar$}$$

Obviously, $\{\omega_\lambda\}_{\lambda\in\Lambda}$ is a net. For $g \in \mathcal{F}^u(\Omega)$ we can express $g = \sum_{m=1}^{M} \alpha_m 1_{F_m}$ where $\alpha_m \geq 0$ and $\Omega = F_1 \supset F_2 \supset ... \supset F_M$ are all closed (see EW [10, p.1366]). Therefore, $g1_{E_\lambda} = \sum_{m=1}^{M} \alpha_m 1_{F_m \cap E_\lambda}$. Since $g1_{E_\lambda}$ has only finitely many possible values, for simplicity we may assume that, for every λ, $g1_{E_\lambda}(\omega_\lambda) = \alpha_1$ and $\omega_\lambda \in (F_1 \cap E_\lambda)$ but $\omega_\lambda \notin (F_m \cap E_\lambda)$ for all $m > 1$.

Likewise, since h has also only finitely many possible values, we may assume $h(\omega_\lambda) = \beta$ for some $\beta \in [0,1]$. By ($\bigstar$), $\alpha_1 > \beta$. Since Ω is compact Hausdorff, without loss of generality, let $\omega_\lambda \to \omega^*$. Since $F_1 \cap E_\lambda$ is closed, it follows $\omega^* \in F_1 \cap E^*$, which implies $g1_{E^*}(\omega^*) \geq \alpha_1$. Since $h \in \mathcal{F}^l(\Omega)$, $\beta = \liminf_\lambda h(\omega_\lambda) \geq h(\omega^*)$. Thus, $g1_{E^*}(\omega^*) > h(\omega^*)$, contradicting the fact that $h \geq g1_{E^*}$. Therefore, there is some $\overline{\lambda}$ such that $g1_{E_{\overline{\lambda}}} \leq h$. Thus, $u(g1_{E_\lambda}) \leq u(h) < u(g1_{E^*}) + \varepsilon$ for all $E_\lambda \subseteq E_{\overline{\lambda}}$. Since ε is arbitrary, by the monotonicity of u, we conclude $\inf_\lambda u(g1_{E_\lambda}) = u(g1_{E^*})$.■

Proof of Theorem 2. Let $\omega \in \Omega$. We proceed to show that there is a closed set $\widetilde{P}(\omega)$ such that $\omega \in B\widetilde{P}(\omega)$ and $\widetilde{P}(\omega) \subseteq E$ for any closed set E satisfying $\omega \in BE$. Consider the following family of closed events

$$\mathcal{E} = \{E \subseteq \Omega : E \text{ is closed and } \omega \in BE\}.$$

By Theorem 1**B2**, $\Omega \in \mathcal{E}$ and hence $\mathcal{E}$ is a nonempty partial ordered set by "inverse" inclusion $\subseteq$. Note that, if $\widetilde{P}(\omega)$ is a "maximal" element in $\mathcal{E}$, then for any given

$E \in \mathcal{E}$, by Theorem 1**B4,** $\left[\widetilde{P}(\omega) \cap E\right] \in \mathcal{E}$ and, hence, $\widetilde{P}(\omega) \subseteq E$. Therefore, by Zorn's Lemma, it remains to show for every chain $\{E_\lambda\}_{\lambda\in\Lambda}$ in $\mathcal{E}$, there is an upper bound E^* in $\mathcal{E}$ – i.e. $E^* \subseteq E_\lambda$ for all λ. To see that $E^* \equiv \cap_\lambda E_\lambda$ is the desired upper bound, it suffices to show that $u^\omega(f) = u^\omega(f1_{E^*})\ \forall f \in \mathcal{F}(\Omega)$. By Lemma 1, we only need to verify that $u^\omega(g) = u^\omega(g1_{E^*})\ \forall g \in \mathcal{F}^u(\Omega)$. However, since $\omega \in BE_\lambda$, $u^\omega(g) = u^\omega(g1_{E_\lambda})$ for all λ. So by Lemma 2, $u^\omega(g) = \inf_\lambda u^\omega(g1_{E_\lambda}) = u^\omega(g1_{E^*})$.

By the definition of belief, $P(\omega) = \bigcap_{\{E\subseteq\Omega:\ E \text{ is closed and } \omega\in BE\}} E$. Since $\widetilde{P}(\omega)$ is closed and $\omega \in B\widetilde{P}(\omega)$, $\widetilde{P}(\omega) \supseteq P(\omega)$. Since $\widetilde{P}$ is generated by B, $\omega \in BE$ implies $\widetilde{P}(\omega) \subseteq E$ and, hence, $\widetilde{P}(\omega) \subseteq P(\omega)$.■

Proof of Theorem 3. We prove **P1** and **P2** in order:

P1 : Nonemptiness of $P(\omega)$ follows from **B1** and Theorem 2; closednesss of $P(\omega)$ follows from the proof of Theorem 2. To see $P(\omega) = \{\omega' :\ \omega \notin B(-\omega')\}$, note that by Theorem 2 $\omega \notin B(-\omega')$ iff $P(\omega) \subseteq \Omega \backslash \{\omega'\}$ iff $\omega' \notin P(\omega)$. Finally, Theorem 2 asserts that belief operator B is the strong inverse of P. For any closed E, by EW's Theorem 4.3, $\mathcal{P}(\Omega|E) \sim^{homeomorphic} \mathcal{P}(E)$. By EW's Theorem 3.1, $BE = \left\{\omega : \psi \circ t^\omega \in \mathcal{P}(\Omega|E)\right\}$ is closed. Consequently, P is lower hemicontinuous (see, e.g., Aliprantis and Border [1, 16.5 Lemma]).

P2 : Since $t^\omega = t^{\omega'}$, $\psi \circ t^\omega = \psi \circ t^{\omega'}$. Therefore, $\omega \in BE$ iff $\omega' \in BE$ for all $E \subseteq \Omega$. Hence, $P(\omega) = P(\omega')$. ■

References

[1] C. Aliprantis, K. Border, Infinite Dimensional Analysis, Springer-Verlag, Berlin, 1999.

[2] R.J. Aumann, Agreeing to disagree, Ann. Statist. 4 (1976), 1236-1239.

[3] R.J. Aumann, Interactive epistemology I: Knowledge, Int. J. Game Theory 28 (1999), 263-300.

[4] M. Bacharach, Some extensions to a claim of Aumann in an axiomatic model of knowledge, J. Econ. Theory 37 (1985), 167-190.

[5] Y.C. Chen, X. Luo, On the relationship of two belief operators, mimeo, Academia Sinica (2006).

[6] E. Dekel, F. Gul, Rationality and knowledge in game theory, in: D. Kreps, K. Wallis (Eds.), Advances in Economics and Econometrics: Theory and Applications, Cambridge University Press, MA, 1997, pp. 87-172.

[7] E. Dekel, B.L. Lipman, and A. Rustichini, Standard state-space models preclude unawareness, Econometrica 66 (1998), 159-173.

[8] A. Di Tillio, Subjective expected utility in games, mimeo, University of Texas at Austin and Universita Bocconi (2006).

[9] L. Epstein, Preference, rationalizability and equilibrium, J. Econ. Theory 73 (1997), 1-29.

[10] L. Epstein, T. Wang, 'Belief about belief' without probabilities, Econometrica 64 (1996), 1343-1373.

[11] R. Fagin, J.Y. Halpern, J. Geanakoplos, M.Y. Vardi, The hierarchical approach to modeling knowledge and common knowledge, Int. J. Game Theory 28 (1999), 331-365.

[12] J. Geanakoplos, Game theory without partitions, and applications to speculation and consensus, mimeo (1989).

[13] J. Geanakoplos, Common knowledge, in: R. Aumann, S. Hart (Eds), Handbook of Game Theory, Vol. 2, North-Holland, Amsterdam, 1994, pp. 1437-1496.

[14] I. Gilboa, D. Schmeidler, Maxmin expected utility with non-unique prior, J. Math. Econ. 18 (1989), 141-153.

[15] J. Harsanyi, Games with incomplete information played by "Bayesian" players, I–III, Manag. Sci. 14 (1967/1968), 159-182, 320-334, 486-502.

[16] A. Heifetz, D. Samet, Topology-free typology of beliefs, J. Econ. Theory 82 (1998), 324-341.

[17] X. Luo, C. Ma, 'Agreeing to disagree' type results: A decision-theoretic approach, J. Math. Econ. 39 (2003), 849-861.

[18] S. Morris, Revising belief: A hierarchical approach, in: R. Fagin (Ed.), Proceedings of the Fifth Conference on Theoretical Aspects of Reasoning about Belief, San Mateo: Morgan Kaufman, 1994, pp. 160-174.

[19] S. Morris, The logic of beliefs and beliefs change: A decision theoretic approach, J. Econ. Theory 69 (1996), 1-23.

[20] S. Morris, Alternative definitions of belief, in: M.O.L Bacharach, L.-A. Gerade Varet, P. Mongin, H.S. Shin (Eds.), Epistemic Logic and the Theory of Games and Decisions, Kluwer Academic Publishers, Amsterdam, 1997, pp. 217-233.

[21] A. Rubinstein, Modeling Bounded Rationality, The MIT Press, MA, 1998.

[22] D. Samet, Ignoring ignorance and agreeing to disagree, J. Econ. Theory 52 (1990), 190-207.

[23] L. Savage, The Foundations of Statistics, Wiley, NY, 1954.

[24] D. Schmeidler, Subjective probability and expected utility without additivity, Econometrica 57 (1989), 571-587.

[25] H.S. Shin, Logical structure of common knowledge, J. Econ. Theory 60 (1993), 1-13.

[26] S. Zamir, S. Vassilakis, Common belief and common belief, J. Math. Econ. 22 (1993), 495-505.