Divide-and-Conquer:
A Proportional, Minimal-Envy Cake-Cutting Procedure

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Abstract

Properties of discrete cake-cutting procedures that use a minimal number of cuts ($n - 1$ if there are n players) are analyzed. None is always envy-free or efficient, but divide-and-conquer (D&C) minimizes the maximum number of players that any single player may envy. It works by asking $n \geq 2$ players successively to place marks on a cake that divide it into equal or approximately equal halves, then halves of these halves, and so on. Among other properties, D&C (i) ensures players of more than $1/n$ shares if their marks are different and (ii) is strategyproof for risk-averse players. However, D&C may not allow players to obtain proportional, connected pieces if they have unequal entitlements. Possible applications of D&C to land division are briefly discussed.

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1. Introduction

A cake is a metaphor for a heterogeneous good, whose parts each of n players may value differently. A *proportional division* of a cake is one that gives each player, as it values the cake, at least a $1/n$ portion, which we call a *proportional share*.

We represent a cake by the interval [0, 1], over which players' preferences are given by probability density functions. There exist several procedures for cutting this cake into pieces such that each player receives a proportional share. But we know of only one procedure, due to Dubins and Spanier (1961), that does so using only $n - 1$ parallel, vertical cuts (the minimal number). However, this procedure, which we will describe later, is not discrete but instead uses a continuous "moving knife."[1]

Almost all the proportional procedures we know of have another limitation: They restrict at least one player to receiving exactly $1/n$ of the cake. By contrast, the minimal-cut procedures we analyze here carry no such restriction—they allow all players to receive more than $1/n$ of a cake if the marks they make on a cake (to be described later) are all different.

Because these procedures do not guarantee a player a most-valued piece, some players may envy others for receiving what they perceive to be more valuable pieces. Thus, while proportional, these discrete procedures are not *envy-free*. (When we say "not

[1] The equitability procedure (EP) (Brams, Jones, and Klamler, 2006), which ensures that each player receives exactly the same amount in its eyes, is discrete and uses only $n - 1$ cuts. But EP requires players to provide a referee, who makes the cuts, with complete information about their valuations of the cake, whereas the proportional procedures discussed here do not require the help of such a third party. For a description of different proportional procedures, see Brams and Taylor (1996) and Robertson and Webb (1998).

envy-free" (or "not efficient" later), we mean that there are examples in which, using these procedures, these properties can *never* be satisfied.)

Discrete procedures do, however, put an upper bound on the number of envies of all players may have without the possibility of making trades that would reduce this number. We show that a discrete procedure we call *divide-and-conquer* (D&C) minimizes the *maximum* number of players that any single player may envy.[2]

D&C is a take-off on *divide-and-choose*, the well-known 2-player cake-cutting procedure in which one player cuts a cake into two pieces, and the other player chooses one piece.[3] We substitute the stronger "conquer" for "choose" to emphasize that n players can, in general, do better than $1/n$ shares under D&C.

Besides not being envy-free, a D&C allocation need not be *efficient* (or Pareto-optimal)—there may be another allocation that is better for one or more players and at least as good for all the others. By contrast, an envy-free allocation that uses $n - 1$ cuts is always efficient (Gale, 1993; Brams and Taylor, 1996, pp. 150-151). But we know of no procedure that gives such an allocation if $n > 3$, although there is an algorithm that gives an approximate envy-free division (Su, 1999).[4]

On the positive side, D&C is relatively simple to apply: It does not require that the players know the valuations of the other players, nor does it require a referee to implement it, although such a person could be helpful. Also, D&C is *strategyproof* for

[2] The computational complexity of D&C and related cake-cutting procedures is analyzed in, among other places, Even and Paz (1984) and Magdon-Ismail, Busch, and Krishnamoorthy (2006). A somewhat different definition of D&C than the one given in section 2 is given in Robertson and Webb (1998, pp. 25-28), wherein "cuts" are used for what we later call "marks."

[3] *What* is divided need not be a cake but could, for example, be separate items that the divider puts into two piles, one of which the chooser selects.

[4] If $n = 3$, there are two known envy-free procedures, one that uses two moving knives (Barbanel and Brams, 2004) and the other that uses four (Stromquist, 1980).

risk-averse players: If a player is truthful, D&C guarantees it a proportional share, whereas this guarantee does not hold if a player is not truthful.

Players that are risk-averse, therefore, have good reason not to try to manipulate D&C. Should they try to gain an edge over other players, they may only succeed in hurting themselves and *not* obtaining a proportional share.

The paper proceeds as follows. In section 2 we describe D&C first with an example and then formally define it by giving six rules of play.

In section 3 we count the maximum number of envies under D&C, beginning with a 7-player example. We then show that *all* discrete procedures give the same maximum sum of envies of all players, $[(n-1)(n-2)]/2$, but D&C minimizes the maximum number of players that any player may envy.

In section 4 we show with a 3-player example that D&C is not envy-free. A different 3-player example establishes that D&C is not efficient. For the latter example, we show that there are efficient allocations that are envy-free or equitable, but they are quite different.

In section 5, we show that D&C is strategyproof for risk-averse players, but it may not allow players to obtain proportional, connected pieces if they have unequal entitlements. This problem can be circumvented, however, by introducing additional fictitious players, or clones, but the union of the pieces that the clones receive may be disconnected.

In the absence of an envy-free cake-cutting procedure, we conclude in section 6 that D&C provides a compelling alternative that ensures proportionality while limiting the amount of envy that any player can experience. Coupled with its economy and practicality, it seems applicable to the division of land and other divisible goods among two or more players.

2. Divide-and-Conquer (D&C)

We assume that player measures over [0, 1] are finitely additive, nonatomic probability measures. Finite additivity ensures that the value of a finite number of disjoint pieces is equal to the value of their union. It follows that no subpieces have greater value than the larger piece that contains them. Nonatomic measures imply that a single cut, which defines the border of a piece, has no area and so contains no value. We also assume that the measures of the players are *absolutely continuous*, so no portion of cake is of positive measure for one player and zero measure for another player.

To introduce D&C, it is useful to begin with a simple example. It illustrates how D&C can be defined recursively, starting with $n = 2$ players and moving up to $n = 5$. For each n, the players A, B, C, . . ., are asked only to put marks *a, b, c, . . .* that indicate some fraction of the cake that lies to the left of the mark:[5]

***n* = 2**. Players A and B independently put marks, *a* and *b*, at their 1/2 points, and a cut is made strictly between them, indicated by the vertical bar, | (given in boldface below to stand out):[6]

0-----------*a*-**|**-*b*------------1

---

[5] The use of marks in cake division is discussed in Shishido and Zeng (1999). As an example of a procedure that uses marks, Lucas's "method of markers" requires that players mark 1/*n* points across a cake (Brams and Taylor, 1996, pp. 57-62), which asks more of players than D&C, as we will show. Although this procedure ensures that each player receives a proportional share, it may leave pieces of cake unassigned and offers no way to award them, or parts thereof, to the players.

[6] If the players' 1/2 points are exactly the same, then players A and B will put their marks at the same point, making it impossible to make a cut strictly between them; instead, | will be this point. With this exception, however, which still gives each player a proportional share (exactly 1/2 the cake), D&C gives each player more than 1/2, as each values it.

Player A gets [0, |], and player B gets (|, 1], that include their marks (*a* and b, respectively).

***n* = 3**. Players A, B, and C independently put their marks at their 1/3 points, and a first cut is made at | between the first two marks—or at these marks if they coincide:

0----------*a*-|-*b*-*c*--------------1

Player A gets [0, |] that includes its mark; the *n* = 2 procedure is then applied to (|, 1] for players B and C.

***n* = 4**. Players A, B, C, and D independently put marks at their 1/2 points, and a first cut is made at | between the 2nd and 3rd marks—or at these marks if they coincide:

0--------- *a*-*b* |-*c*-*d*----------------1

The *n* = 2 procedure is applied to [0, |] for players A and B, and to (|, 1] for players C and D.

***n* = 5**. Players A, B, C, D, and E independently put marks at their 2/5 points, and a first cut is made at | between the 2nd and 3rd marks—or at these marks if they coincide:

0----------*a*-*b*-|-*c*-*d*-*e*----------------1

The *n* = 2 procedure is applied to [0, |] for players A and B, and the *n* = 3 procedure is applied to (|, 1] for players C, D, and E.

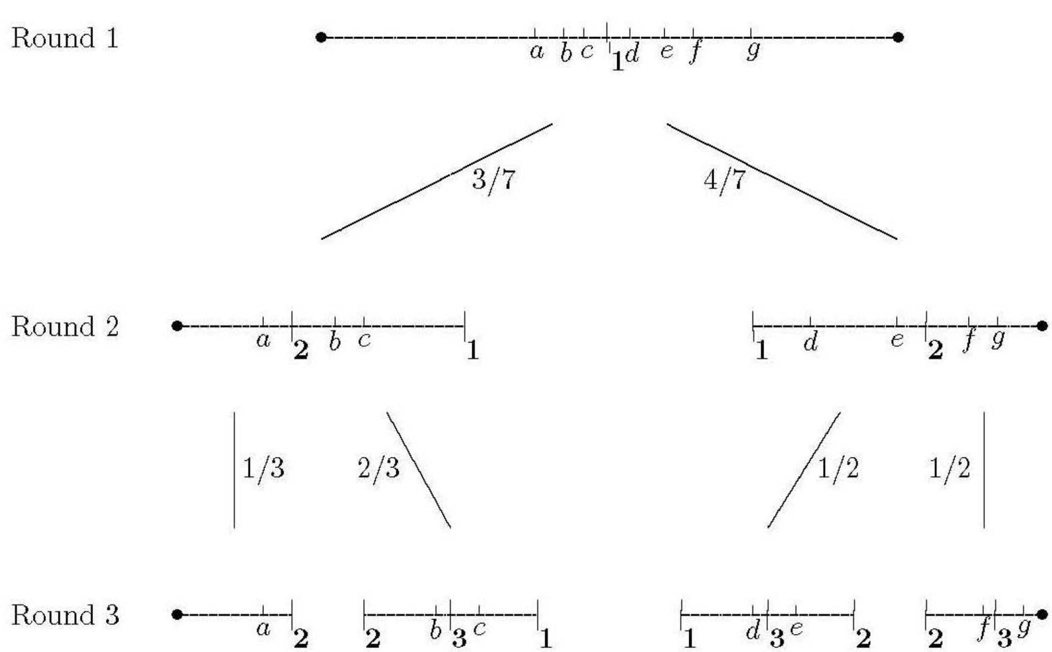
Clearly, each player receives a proportional share by getting a piece that it values at $1/n$ or more. We next define D&C formally by specifying its rules of play for $n \geq 2$ players.

1. Each player independently places a mark at a point such that
   (i) if n is even, 1/2 the cake lies to the left and 1/2 to the right;
   (ii) if n is odd, $[(n - 1)/2]/n$ proportion of the cake lies to the left, and $[(n + 1)/2]/n$ proportion lies to the right.
2. The cake is cut at point | strictly between the $(n/2)^{\text{th}}$ and the $(n/2 + 1)^{\text{st}}$ marks in case (i), and strictly between the $[(n - 1)/2]^{\text{th}}$ and the $[(n + 1)/2]^{\text{st}}$ marks in case (ii). If these marks are the same, the cake is cut at this point.
3. If $n = 2$ in case (i), stop. If $n = 3$ in case (ii), cut the subpiece containing 2 marks according to case (i), and stop.
4. If $n \geq 4$, cut the subpiece on the left and the subpiece on the right of | according to rules 1 and 2, changing n in each case to the number of players that make marks on the left and the right subpieces.
5. Apply rule 4 repeatedly to the smaller and smaller subpieces that remain after cuts are made. When subpieces are reached where $n = 2$ or $n = 3$, apply rule 3.
6. After all cuts are made, assign pieces to the players that include their marks when individual pieces are allocated so as to give each player a proportional share. If there is mutual envy or envy cycles (to be discussed in section 3), have players make trades that eliminate them.

The sequence of cuts under D&C can be described by a binary tree. Each subpiece of the cake is divided in two, according to (i) or (ii), in successive rounds until there are n individual pieces that can be assigned to each player.

The depth of the tree is the number of rounds that are needed before each player receives an individual piece. This number is $\lceil \log_2(n) \rceil$ if n is even and $\lceil \log_2(n+1) \rceil$ if n is odd, because on each round the cake is divided into two subpieces.[7] Each subpiece contains the same number of marks on each side of | if n is even; if n is odd, the numbers differ by 1. For example, if $n = 20$, $\lceil \log_2(20) \rceil = \lceil 4.33 \rceil = 5$, so the depth of the tree is 5. If $n = 15$, $\lceil \log_2(16) \rceil = \lceil 4 \rceil = 4$, so the depth of the tree is 4.

To illustrate the successive division of a cake, assume $n = 7$, so the depth of the tree is $\lceil \log_2(8) \rceil = \lceil 3 \rceil = 3$ (see Figure 1). On the 1st round, the division, at 3/7, is into a left subpiece containing 3 marks and a right subpiece containing 4 marks. The first cut is shown by a boldface "1."



[7] $\lceil k \rceil$ indicates k rounded up to an integer if k is not already an integer.

Figure 1. Binary Tree Illustrating the Division of a Cake into 7 Pieces

On the 2nd round, the division of the left subpiece at 1/3 (of 3/7) is into two sub-subpieces containing 1 and 2 marks each; the division of the right subpiece at 1/2 (of 4/7) is into two sub-subpieces containing 2 marks each. The 2nd-round cuts are shown by two boldface "2's"; the 1st-round cut is now separated, becoming an endpoint for two subpieces. On the 3rd round, each of the three sub-subpieces containing 2 marks each (the third contains 1 mark) is divided. The three 3rd-round cuts are shown by three boldface "3's"; the 1st and 2nd-round cuts now become endpoints. Thereby the cake is cut into a total of 7 individual pieces. Note that *a*, *b*, . . ., *g* may represent different players' cutpoints on each round.

3. Maximum Number of Envies

To count the maximum number of players that players may envy under D&C, which we call the *envies of players*, we count only envies that cannot be alleviated by trades. To illustrate this count, assume there is a set of 8 players, {A, B, C, D, E, F, G, H}. Assume the 1/2 points of players A – D lie to the left of the 1/2 points of players E – H, so the first cut under D&C, indicated by $|_1$, is made between the 1/2 points of players D and E:

0------------------*a*--*b*--*c*--*d*-$|_1$-*e*--*f*--*g*--*h*------------------1

Now apply D&C to divide the cake into 8 individual pieces. It is possible that the 4 players to the left of $|_1$ may envy up to 3 of the 4 players to the right (say, E, F, and G) if player H's piece is sufficiently small in the eyes of players A – D. (A player on the left cannot envy all 4 players on the right, because not all 4 on the right can each receive more than 1/8 in the eyes of a left player.) By the same token, player H may envy players B, C, and D on the left if player A's piece is sufficiently small in the eyes of player H. Thus, 5 players (A, B, C, D, H) may envy up to 3 other players across $|_1$ without the possibility of trades (total possible envies: 15).[8]

More envies are possible. On the 2nd round, assume the cuts on the left and the right, both indicated by $|_2$, are as follows:

0----------*a*--*b*-$|_2$-*c*--*d*-----------$|_1$-----------*e*--*f*-$|_2$-*g*--*h*---------1

To the left of $|_1$, players A and B may envy up to 1 player (say, C) across $|_2$ on the left, and player D may envy up to 1 player (say, B) in the other direction. To the right of $|_1$, players E and F may envy player G across $|_2$ on the right, and player H may envy player F in the other direction. Thus, 6 players (A, B, D, E, F, H) may envy up to 1 other player across each of the two $|_2$'s without the possibility of trades (total possible envies: 6).

Across the four cuts $|_3$ made on the 3rd round that divide the cake into 8 individual pieces, no new envies are created. For example, the cut $|_3$ dividing the pieces that A and B receive does not cause one of these players to envy the other.

Under this construction, each player I's envy of some player J is strictly one-way, which we indicate by I $\triangleright$ J. Hence, there is no *mutual envy*—in which I $\triangleright$ J and J $\triangleright$ I—which would allow players I and J, by trading pieces, to rid themselves of envy. Moreover, there are no *envy cycles,* whereby, for example, I $\triangleright$ J, J $\triangleright$ K, and K $\triangleright$ I, in which case a three-way trade would rid the players of envy.[9]

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[8] More envies across $|_1$ would lead to mutual envy—and the possibility of trades—which we discuss below.

[9] Altogether, we have

A, B $\triangleright$ C, E, F, G;   C $\triangleright$ E, F, G;   D $\triangleright$ B, E, F, G;   E, F $\triangleright$ G;   H $\triangleright$ B, C, D, F.

D&C is an example of a *discrete procedure*. This is a procedure in which the first cut divides the cake into two pieces such that m players lie to the left and $n - m$ players lie to the right of the m/n division of the cake, where m is an integer satisfying $1 \leq m < n$. Subsequent cuts are made to divide and subdivide these pieces, with the process terminating when each player receives exactly one piece.

Thus, in the preceding 8-player example, assume $m = 2$ for the first cut, which produces a division at 1/4. Two players then must divide what they consider to be at least 1/4 of the cake on the left of their two marks; and six players must divide at least 3/4 of the cake that they consider on the right of their six marks. Now the two players on the left must make marks at the 1/2 point of their piece, whereas the six players on the right may make marks at any one of the five possible divisions, {1/6, 1/3, 1/2, 2/3, 5/6}, of their piece. Further division of the right piece will be required to give each of the six players individual pieces. We next count how many envies are possible.

Theorem 1. *For $n \geq 2$, the maximum number of envies that all players may have under a discrete procedure without the possibility of trades is $[(n - 1)(n - 2)]/2$.*

Proof. We use induction, making the base case $n = 2$. It satisfies the formula $(n - 1)(n - 2)/2$, because there are zero envies when there are two players. Under the strong induction hypothesis, assume the formula holds for any number of players less than n.

Thus, 4 players (A, B, D, H) may envy up to 4 others, 1 player (C) may envy up to 3 others, 2 players (E, F) may envy up to 1 other, and 1 player (G) may envy no others, making for a total of 21 one-way envies, or an average of 21/8 = 2.625 envies per player without the possibility of trades. If there are 4 or more players, D&C does not preclude mutual envy and thus a trade that would give the traders preferred pieces. In the case of 4 players, for example, assume the first cut gives players A and B the left portion of a cake and players C and D the right portion. Then after the left and right portions are divided, it is possible that A would envy C and C would envy A. In that case, we assume that A and C would trade their pieces to eliminate their mutual envy. The construction in the 8-player example, and in the proof of Theorem 1, precludes such envy.

Suppose there are n players. Under a discrete procedure, the cake is cut initially so that m players lie to the left and $n - m$ players lie to the right. Applying the induction hypothesis, the maximum number of envies on the left is $(m - 1)(m - 2)/2$, and the maximum number of envies on the right is $(n - m - 1)(n - m - 2)/2$. Summing these numbers gives the maximum number of possible envies on both the left and right sides of the initial cut:

$$[(m^2 - 3m + 2) + (n^2 - nm - 2n - mn + m^2 + 2m - n + m + 2)]/2$$
$$= [2m^2 - 2mn - 3n + n^2 + 4]/2. \qquad (1)$$

Next we count the maximum number of envies of (i) the right players by the left players and of (ii) the left players by the right players. Because each player must receive a proportional piece, each of the m left players can envy a maximum of $(n - m - 1)$ right players. Thus in case (i), there are a maximum of $(m)(n - m - 1)$ envies caused by the initial cut if the envies of all the players on the left leave one player unenvied on the right. This right player, in turn, may envy a maximum of $(m - 1)$ players on the left. This construction prohibits two-way envy as well as envy cycles across stages, because rightward envy never changes to leftward envy, or vice versa, across stages.[10]

Altogether, the maximum number of possible envies caused by the initial cut is

$$(m)(n - m - 1) + (1)(m - 1)$$
$$= mn - m^2 - 1. \qquad (2)$$

[10] As an example, observe that the envies of players A, B, and C in note 9 are only rightward; although player D has leftward envy of player B, it is not reciprocated. Likewise, players E and F have only rightward envies, whereas player H has only leftward envies.

Adding (1) and (2), we obtain the maximum total (T) number of envies without the possibility of trades:

$$T(n, m) = (m^2 - mn - 3/2n + n^2/2 + 2) + (mn - m^2 - 1)$$

$$= n^2/2 - 3/2n + 1. \qquad (3)$$

But (3) is $[(n-1)(n-2)]/2$, which validates the formula for n players. Q.E.D.

Theorem 1 establishes that we can drop the argument m in $T(n, m)$ and write the maximum total number of envies of a discrete cake-cutting procedure as $T(n)$. Because one of the two factors in the numerator of $[(n-1)(n-2)]/2$ must be even, the numerator must also be even, rendering $T(n)$ always an integer.

If every discrete procedure gives the same maximum total number of envies, what is special about D&C?

Theorem 2. *Assume $n \geq 2$, where $2^k \leq n < 2^{(k+1)}$. Under D&C, the maximum number of envies of an individual player (I) without the possibility of trades is*

$$I_{D\&C}(n, k) = n - k - 1.$$

This number is minimal among all possible ways of cutting the cake into individual pieces for each player.

Proof. First we prove this result for the lower bound, $n = 2^k$, of the theorem's double inequality for n. After D&C has been applied, imagine that the player in question, X, receives the left-most piece of cake. X will not envy the player to its immediate right, but it may envy up to 1 of the 2 players that are next on the right, up to 3 of the 4 players that are next on the right of these 2 players, . . ., and up to $2^{(k-1)} - 1$ of the $2^{(k-1)}$ players that are on the right half. This sum, which is the same whichever piece X receives, is

$$
\begin{aligned}
I_{\mathrm{D\&C}}(n, k) &= \sum_{i=1}^{k-1} (2^i - 1) \\
&= \sum_{i=1}^{k-1} 2^i - (k-1) \\
&= (2^k - 2) - (k - 1) \\
&= 2^k - k - 1 \\
&= n - k - 1. \qquad (4)
\end{aligned}
$$

In addition to the case of $n = 2^k$, consider all values of n such that $2^k < n < 2^{(k+1)}$. Then X may envy each additional player beyond 2^k, up to but not including player $2^{(k+1)}$. Thus, (4) is valid not only for $n = 2^k$ but also for any n allowed by $2^k < n < 2^{(k+1)}$.

That $I_{\mathrm{D\&C}}(n, k)$ minimizes the maximum number of envies of a player without the possibility of trades follows from the fact that its cuts are made at the (approximate) halfway points on each round. Suppose they are not. If there are fewer players on the left and more on the right than the number given by D&C on any round, then a player on the left can envy more players than the maximum given by D&C. Thus, $I_{\mathrm{D\&C}}(n, k)$ is minimal. Q.E.D.

It is useful to compare the values of $T(n)$ and $I_{\mathrm{D\&C}}(n, k)$ for the values of n shown in the table below:

n	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
***T*(*n*)**	0	1	3	6	10	15	21	28	36

$\boldsymbol{I_{D\&C}(n)}$	0	1	1	2	3	4	4	5	6

Note that we now write $I_{\text{D\&C}}(n, k)$ as $I_{\text{D\&C}}(n)$ in this table, because $k = \lceil \log_2(n) \rceil$ is not an independent variable but instead a function of n. Observe that whereas $T(n)$ increases rapidly with n, $I_{\text{D\&C}}(n)$ increases more slowly than n.

Desirably, $I_{\text{D\&C}}(n)$ is less than the maximum number of envies a player may have under the well-known moving-knife procedure of Dubins and Spanier (1961), which also uses $n - 1$ cuts.[11] Under the Dubins-Spanier (DS) procedure, a referee moves a knife slowly across a cake from left to right. A player that has not yet received a piece calls "stop," and makes a cut, when the knife reaches a point that gives it exactly $1/n$ of the cake rightward of the last point at which a player called stop—or from the left edge for the first player to call stop.

The first player to call stop may envy as many as all the other players except one, or $n - 2$ other players, so

$$I_{\text{DS}}(n) = n - 2.$$

This is the maximum number of envies of an individual player. The second player to call stop may envy as many as $n - 3$ other players, . . ., and the n^{th} player, which receives the cake from the point at which the $(n - 1)^{\text{st}}$ player called stop up to the right edge, will envy no other players. Altogether, the n players may envy as many as

$$T_{\text{DS}}(n) = (n - 2) + (n - 3) + \ldots + 0$$

[11] Moving-knife procedures are discussed in, among other places, Brams, Taylor, and Zwicker (1995), Brams and Taylor (1996), and Robertson and Webb (1998). For nonconstructive results on cake-cutting, which address the existence but not the construction of fair divisions that satisfy different properties, see Barbanel (2005).

$$
\begin{aligned}
&= \sum_{i=0}^{n-2}(n-2-i) \\
&= [(n-1)(n-2)] - \sum_{i=0}^{n-2} i \\
&= [(n-1)(n-2)](1-1/2) \\
&= [(n-1)(n-2)]/2
\end{aligned}
$$

other players under DS, which duplicates $T_{\mathrm{D\&C}}(n)$.[12] Obviously, no trades are possible, because envy goes only from left to right.

We note in passing that D&C minimizes not only the maximum total number of envies of players but also the number of rounds on which the players must indicate their marks. As a case in point, the 7-player example at the beginning of this section requires that the players make marks on three rounds under D&C, but more rounds would be required if 50-50 cuts were not made on each round.

We give below values for $T_{\mathrm{DS}}(n)$ and $I_{\mathrm{DS}}(n)$ that are analogous to the values given in the previous table:

n	2	3	4	5	6	7	8	9	10
$T_{\mathrm{DS}}(n)$	0	1	3	6	10	15	21	28	36
$I_{\mathrm{DS}}(n)$	0	1	2	3	4	5	6	7	8

Comparing values of $I_{\mathrm{DS}}(n)$ with $I_{\mathrm{D\&C}}(n)$, observe that $I_{\mathrm{DS}}(n)$ jumps ahead of $I_{\mathrm{D\&C}}(n)$ at $n = 4$, slowly increasing its lead in envies as n increases. Consequently, D&C in general does better than DS in preventing any player from being too aggrieved.

[12] DS is a continuous version of the discrete Banach-Knaster "last-diminisher" procedure (Brams and Taylor, 1996, pp. 35-36), whereby each player receives exactly a $1/n$ share on each round. A discrete procedure in which $m = 1$ is arguably better than DS and last diminisher, because it does not limit players to exactly $1/n$ pieces.

Because DS requires that a referee continuously move a knife across a cake, it would seem less practical than a discrete procedure, like D&C, which requires each player to make at most $\lceil \log_2(n) \rceil$ (n even) or $\lceil \log_2(n+1) \rceil$ (n odd) marks. While players under D&C would not necessarily need a referee to record their marks and keep them secret from other players, a trustworthy referee may facilitate this process.

4. Envy-Freeness and Efficiency

In this section we show that, with one possible exception, a discrete procedure may not yield an envy-free or efficient allocation. We will use D&C, which is a discrete procedure, to illustrate these largely negative results.

Theorem 3. *For $n \geq 3$ players, a discrete procedure may not yield an envy-free allocation, whether or not it is applied from the left edge or the right edge of a cake.*

Proof. Assume that players A and B have piecewise linear value functions over the cake that are symmetric and V-shaped:

$$v_A(x) = \begin{cases} -4x + 2 \text{ for } x \in [0, \frac{1}{2}] \\ 4x - 2 \text{ for } x \in [\frac{1}{2}, 1] \end{cases}$$

$$v_B(x) = \begin{cases} -2x + \frac{3}{2} \text{ for } x \in [0, \frac{1}{2}] \\ 2x - \frac{1}{2} \text{ for } x \in [\frac{1}{2}, 1] \end{cases}$$

Whereas both functions have maxima at $x = 0$ and $x = 1$ and a minimum at $x = 1/2$, A's function is steeper (higher maximum, lower minimum) than B's, as illustrated in Figure

2. In addition, suppose that a third player, C, has a uniform value function, $v_C(x) = 1$, for $x \in [0, 1]$.

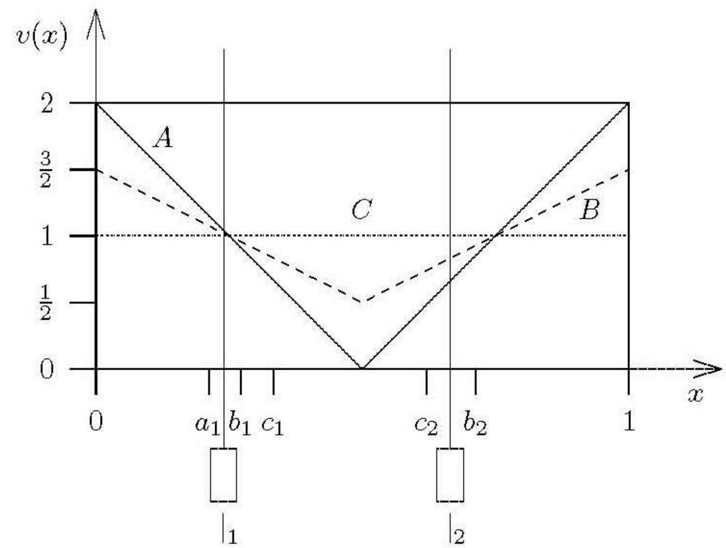

Figure 2. Impossibility of an Envy-Free Division for Three Players under D&C

We show in Figure 2 player A, B, and C's 1/3 marks (a_1, b_1, and c_1), where

$$a_1 = \frac{1}{2} - \frac{\sqrt{3}}{6} \approx 0.211;\ b_1 = \frac{3}{4} - \frac{\sqrt{33}}{12} \approx 0.271;\ c_1 = \frac{1}{3} \approx\ 0.333.$$

The first cut under D&C is between a_1 and b_1, which we denote by $|_1$. Player A receives the piece to the left of this cut.

Players B and C, which value the remaining piece at more than 2/3, place 1/2 marks on this remainder, which we denote by b_2 and c_2. Let $y = |_1$. To ensure that players B and C value the middle piece and the right piece equally, b_2 and c_2 must satisfy the following two equations:

$$\int_y^{b_2} v_B(x)dx = \int_{b_2}^1 v_B(x)dx \text{ and } \int_y^{c_2} v_C(x)dx = \int_{c_2}^1 v_C(x)dx.$$

Solving these equations for b_2 and c_2, we obtain

$$b_2 = \frac{1 + \sqrt{2y - 2y^2}}{2} \text{ and } c_2 = \frac{y + 1}{2}.$$

The second cut, $|_2$, is made between b_2 and c_2.

As functions of y, $b_2 > c_2$ for all positive $y < 2/3$. Because y cannot exceed $b_1 \approx 0.271$, it follows that $b_2 > c_2$, as shown in Figure 2.

We now show that the right piece that B receives is bigger, lengthwise, than the left piece that A receives, so A will envy B. Consider the position of b_2. It must be farther from 1 than b_1 is from 0 in order to give B more than 1/3. But because the length of A's piece on the left is less than b_1, A must think that the length of B's piece on the right is greater—that is $[|_2, 1] > [0, |_1]$. Hence, A will envy B no matter where $|_1$ and $|_2$ lie in their respective intervals.[13]

If D&C is applied from the right so that A receives the right piece, A will envy B for the piece that B receives on the left for analogous reasons. This example can be generalized to more than 3 players by assuming that the additional players have, like C, uniform distributions. Hence, they will receive non-end pieces while A and B receive the left and right pieces, creating a situation in which A envies B, whether or not D&C is applied from the left edge or the right edge of the cake. Q.E.D.

We turn next to analyzing the efficiency of allocations using a discrete procedure.

Theorem 4. *For $n \geq 3$, a discrete procedure may not yield an efficient allocation, whether or not it is applied from the left edge or the right edge of a cake.*

Proof. Consider the example shown below, where the numbers that players A, B, and C place on three equal-length portions of the cake (demarcated by slashes) indicate

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[13] In the Figure 2 example, the equitability procedure (EP) mentioned in note 1 also causes A to envy B by giving all three players exactly the same value (0.393); it gives a left cutpoint of 0.269 and a right cutpoint of 0.662 (Brams, Jones, and Klamler, 2006, p. 1318), which fall in the D&C intervals of $[a_1, b_1]$ and $[c_2, b_2]$. Whereas EP always gives an efficient allocation, however, D&C may not, as we next show.

the values that the players receive from obtaining that entire portion (total: 100 points). We assume these values are uniformly distributed over each portion. The decimal portions of these values are 1/3, which for convenience we round to 0.33.

A: /-----34.33-----/-----24.33-----/-----41.33-----/

B: /-----16.33-----/-----41.33-----/-----42.33-----/

C: /-----33.33-----/-----33.33-----/-----33.33-----/

We can also express these distributions as value functions,

$$v_A(x) \approx \begin{cases} 1.03 \text{ on } [0,\frac{1}{3}) \\ 0.73 \text{ on } [\frac{1}{3},\frac{2}{3}) \\ 1.24 \text{ on } [\frac{2}{3},1] \end{cases}, \; v_B(x) \approx \begin{cases} 0.49 \text{ on } [0,\frac{1}{3}) \\ 1.24 \text{ on } [\frac{1}{3},\frac{2}{3}) \\ 1.27 \text{ on } [\frac{2}{3},1] \end{cases}, \; v_C(x) = 1 \text{ on } [0,1],$$

but it is more convenient to use the “slash” representation.

When D&C is applied from the left, A obtains the left piece, C the middle piece, and B the right piece, which we call order ACB in this representation. Assume points go from 0 on the left border to 100 on the right border. It is not difficult to show that

- A gets a maximum of 34.33 (between 0 on the left and a 1st cut at 33.33)
- C gets a maximum of 34.51 (between a 1st cut at 32.37 and a 2nd cut at 66.87)
- B gets a maximum of 42.94 (between a 2nd cut at 66.20 and 100 on the right).

When D&C is applied from the right, the order ACB (BCA from the right) stays the same, but now D&C starts by giving B the piece on the right, C the middle piece, and A the left piece. It is not difficult to show that

• A gets a maximum of 36.92 (between 0 on the left and a 2nd cut at 36.87)

• C gets a maximum of 41.04 (between a 2nd cut at 32.73 and a 1st cut at 73.77)

• B gets a maximum of 34.14 (between a 1st cut at 73.12 and 100 on the right).

Whether D&C is applied from the left or from the right, we next show that there are cuts that give all three players more than their maxima, which cannot be realized simultaneously under D&C. But unlike D&C, players receive pieces from the left in the order CBA instead of ACB.[14] More specifically,

1. If cuts are made at 34.52 and 69.10, C obtains 34.52 from the left piece, B obtains 42.96 from the middle piece, and A obtains 38.30 from the right piece, which exceed the three players' maxima when D&C is applied from the left.
2. If cuts are made at 41.05 and 69.11, C obtains 41.05 from the left piece, B obtains 35.05 from the middle piece, and A obtains 38.29 from the right piece, which exceed the three players' maxima when D&C is applied from the right.

In sum, there are allocations that Pareto-dominate the D&C allocations, whether D&C is applied from the left or from the right, rendering the D&C allocations inefficient.

This example can readily be extended to 4 or more players. For example, if there are 4 players, assume that the 4th player has virtually all its value concentrated in the 2nd quarter of a 4-way division of the cake into equal-length portions, whereas the other three players have almost all their values distributed in the 1st, 3rd, and 4th quarters in the same manner given in the 3-player example. Then the D&C allocations will be as in the 3-

[14] If each player receives a different 1/3 equal-length piece in the slash representation, CBA, or the allocation along the off-diagonal, maximizes the sum of the values that the three players can receive (total: 116 points), whereas ACB yields the second-largest sum (109 points). The extra "wiggle room" of CBA is what enables us to find CBA allocations that Pareto-dominate the ACB allocations of D&C.

player example, except that the 4th player will obtain the 2nd piece from the left. But as in the 3-player example, there will be an allocation that Pareto-dominates each of the D&C allocations, giving the 4th player the 2nd piece from the left and changing the order for the other 3 players from ACB to CBA. Q.E.D.

The allocations of (1) and (2) above are efficient, but they are not envy-free. In the case of (1), C envies B for obtaining what it thinks is 34.58, which exceeds its allocation of 34.52. In the case of (2), B envies A for obtaining what it thinks is 38.32, which exceeds its allocation of 35.05. But there is no mutual envy or an envy cycle, which would render trades possible that could lead to an efficient allocation.

Trades are not possible under D&C for 3 players, even if, as in the above example, the allocation is inefficient. While the 1st player to make a cut may envy one of the other two players, they will never envy the 1st player or each other, no matter where the two D&C cuts made.

If neither D&C nor an allocation that Pareto-dominates it is envy-free, can one be assured that there always is an efficient, envy-free allocation? The answer is "yes," though there is only an approximate *n*-player algorithm for finding such an allocation (Su, 1999). In the 3-player case, however, there are two moving-knife procedures that yield exact envy-free allocations (Stromquist, 1980; Brams and Barbanel, 2004), which are always efficient in the minimal-cut case (Gale, 1993; Brams and Taylor, 1996, pp. 150-151).

To illustrate, we apply the "squeezing procedure" of Brams and Barbanel (2004) to the previous 3-player example, wherein player C is the squeezer.[15] This produces a 1st cut at 33.60 and 2nd cut at 67.20, yielding the following efficient, envy-free allocation:

- C obtains the left piece, which it values at 33.60 (same as the middle piece)
- B obtains the middle piece, which it values at 41.65 (same as the right piece)
- A obtains the right piece, which it values at 40.67.

In general, there will be an infinite number of envy-free allocations if the players' value functions are continuous.

By contrast, there is usually a unique EP allocation (see notes 1 and 13), which in our example gives each player an allocation of 38.29. C obtains the left piece (1st cut at 38.29), B obtains the middle piece (2nd cut at 69.11), and A obtains the right piece. This allocation, however, is not envy-free: B envies A for getting a piece that it thinks is worth 39.19.

To recapitulate, D&C allocations may not be either envy-free or efficient. While there are efficient allocations that are envy-free or equitable, they may not be both.[16]

In the next section, we consider two other properties of a discrete envy-free procedure like D&C. The first—strategyproofness—is satisfied, and the second—giving

[15] The idea is that player C, using two moving knives, continuously increases its left and middle 33.33 pieces equally so as to diminish its right 33.33 piece until one of A and B, both of which initially prefer the right piece to the other two, calls "stop" when this diminished piece ties one of the other two (now enlarged) pieces. The first player to call stop will be B, when the middle piece ties with the right piece for it (at the cutpoints given in the text), so B gets this piece, C gets the left piece, and A gets the right piece. Because of the two ties (see text) that this procedure creates, each player thinks it receives at least a tied-for-largest piece and so is not envious of anybody else.

[16] That envy-free and equitable allocations may be different is illustrated by the previous example, but that these two properties cannot always be satisfied simultaneously is proved in Brams, Jones, and Klamler (2006) using the Figure 2 example.

players proportional shares that reflect their entitlements—is in general impossible to satisfy unless clones are allowed and players can receive disconnected pieces.

5. Strategyproofness and Entitlements

Recall from section 1 that a strategyproof procedure is one that guarantees a risk-averse player a proportional share if and only if that player is truthful. A *risk-averse* player is one that would not choose a strategy that could yield it less than a proportional share.

Theorem 5. *For $n \geq 2$, D&C is strategyproof for risk-averse players.*

Proof. We showed in section 2 that D&C guarantees a player a proportional share if it is truthful. We now consider the case when two players, A and B, must divide a cake, but one may not be truthful. If their truthful 1/2 points are as shown below, then cutting the cake at | gives each player more than 1/2:

0-----------*a*-|-*b*------------1

But if player A should report that its 1/2 point is either to the left or right of *a*, it risks getting less than 1/2 the cake if (i) | is to the left of *a* or (ii) | is to the right of *b*. This argument for the vulnerability of D&C—that it does not guarantee a player a proportional share if it is not truthful—can readily be extended to $n > 2$ players. Q.E.D.

Finally, we consider a situation in which players are not equally entitled to portions of a cake. If the players have different entitlements, then D&C does not guarantee a player a proportional share.

Theorem 6. *For $n \geq 2$, D&C may not give players proportional shares, using $n - 1$ cuts, if they do not have equal entitlements.*

Proof. Assume that player A is entitled to 2/3 of the cake and player B to 1/3. Then if their 1/3 and 2/3 points are a_1 and a_2 and b_1 and b_2, respectively,

0----------a_1--b_1----------b_2--a_2----------1

it is not hard to see that no single cut can give *both* players at least their entitlements. Q.E.D.

But now suppose that player A is really two players, clones A1 and A2, each of which has the same preference and is entitled to 1/3 each. Then the application of D&C would result in the following cuts:

0----------$a_1|_1$--b_1----------b_2-$|_2$-a_2----------1

Because clones A1 and A2 would put the same mark at a_1, $|_1$ would be the cutpoint (though it is shown as adjacent in the above diagram), giving the clone that gets $[0, a_1]$ (say, A1) exactly 1/3. In comparison, $|_2$ would be strictly between between b_2 and a_2, so A2 would receive more than 1/3 from $[|_2, 1]$. Finally, player B would receive $(|_1, |_2)$, a more-than-1/3 share in the middle.

Thus, by "breaking up" the more entitled players—or those with more resources—into clones in such a way that the clones and players have equal entitlements, D&C can be applied as before to give both the clones and players proportional shares. But note that the clones that constitute a single player may not get a single connected piece (clones A1 and A2 get separated pieces in the example). Furthermore, one of the clones (A2 in the example) may get a larger proportional share than another (A1), which would cause A1 to envy A2.

If these clones are treated as the single player A, however, their combined 2/3+ share is greater by a factor of 2 than player B's less-than-1/3-share in player A's eyes. Likewise, player B's 1/3+ share is greater than 1/2 player A's (combined) share in player B's eyes. Consequently, neither player A nor player B would envy the other for getting what it thinks is a disproportionally large piece, given their entitlements.[17]

6. Conclusions

D&C is not perfect—its allocations may not be envy-free or efficient—but there is no exact procedure, using $n - 1$ cuts, that guarantees these properties for more than three players. This is perhaps a price one pays for its simplicity. It has, however, several positive features:

1. It requires n players to make at most $\lceil \log_2(n) \rceil$ (n even) or $\lceil \log_2(n+1) \rceil$ (n odd) marks on a cake.
2. It requires the minimal $n - 1$ cuts to divide the cake into n pieces, which matches the Dubins-Spanier (DS) moving knife-procedure.
3. While giving the same upper bound as DS and all discrete cake-cutting procedures on the total number of envies of all players without the possibility of trades, it gives a lower upper bound on the number of players that any individual player may envy.

[17] For a discussion of the use of clones to define envy-freeness in terms of entitlements, see Brams and Taylor (1996, pp. 48-49, 152-153) and Robertson and Webb (1998, pp. 35-36).

4. Its proportional shares are generally greater than $1/n$, which DS and some other proportional procedures do not guarantee for all players.
5. It is strategyproof, guaranteeing truthful players proportional shares whatever the choices of the other players.
6. It is applicable to players with unequal entitlements, but this may require the creation of fictitious players, or clones, that get disconnected pieces.

We think D&C would not be difficult to apply to a divisible good like land if it was feasible to divide it with parallel, vertical cuts. Unlike DS, it does not require players to make instantaneous decisions about when to stop a moving knife sweeping across the land, which is likely to make real-life people quite anxious about when to call stop. D&C players, by comparison, make decisions about how to divide smaller and smaller parcels of the land—exactly in half if there is an even number of players, approximately in half if there is an odd number—without being under extreme time pressure.

Although players can implement D&C on their own, it may be helpful to use a referee to record players' marks unbeknownst to the other players. Of course, this could also be done by a computer if there were safeguards to ensure that the players' marks, when submitted, cannot be read or inferred by the other players.

A question that we think is worth exploring further is exactly where should the cuts between players' marks be made? For example, should the goal be to give players *equitable shares*, so that every player gets the same proportion greater than $1/n$? Although the equitability procedure (EP) is an efficient, strategyproof procedure for doing this (Brams, Jones, and Klamler, 2006), EP is not envy-free. Additionally, it requires much more information from the players than does D&C as well as possibly complex calculations by a referee.

EP will generally give a unique division, but an envy-free division will generally not be unique. An "ideal" division might be an envy-free division that is as close to being equitable as possible, but there is no general *n*-person procedure that yields even an envy-free division, much less one that is as equitable as possible. Approximate procedures of the kind Su (1999) discusses, however, might be feasible.

We conclude that D&C is a parsimonious and practical procedure for dividing a divisible good like land. The question of complicating it to allow for envy-free or equitable shares requires further study.

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