Ambiguous Act Equilibria

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Abstract

A novel approach for the study of games with strategic uncertainty is proposed. Games are defined such that players' strategy spaces do not only contain pure and mixed strategies but also contain "ambiguous act strategies", in the sense that players can base their choices on subjective randomization devices. Expected utility representation of preferences over strategy profiles consisting of such "ambiguous act strategies" is not assumed. The notions of "independent strategies" as well as "common priors" are relaxed in such a manner that they can be applied to the context of games with strategic uncertainty even though the player's preferences cannot necessarily be represented by expected utility functions. The concept of "Ambiguous Act Equilibrium" is defined. I show that the ambiguous act equilibria of a two player games in which preferences of all players satisfy Schmeidler's uncertainty aversion as well as transitivity and monotonicity are observationally equivalent to the mixed strategy equilibria of that game in the sense that a researcher who can only observe equilibrium outcomes is not able to determine whether the players are uncertainty averse or uncertainty neutral.

Keywords: Uncertainty Aversion, Nash Equilibrium, Ambiguity.

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1 Introduction

There is ample experimental evidence that people treat risky situations in which they know the odds of all relevant outcomes differently from ambiguous situations in which they can only guess these odds. The Ellsberg Paradox is one of the most well-established violations of expected utility theory. It has inspired a large range of different generalizations of expected utility theory. This branch of decision theory continues to thrive.[1]

Uncertainty aversion is deemed particularly relevant in situations that are new to the decision maker. Once a decision is familiar with a situation he should have learned the odds of all relevant outcomes. In this vein, we might expect that someone who is new to gardening would exhibit uncertainty aversion with respect to bets on the growth of her plants. On the other hand, a seasoned gardener should have learned the odds of her plants reaching a certain size. The seasoned gardener should not exhibit any ambiguity aversion when it comes to a bet on the number of leaves on her basil plant by June 15th.

Just as much as this reasoning applies to single person decision problems this reasoning should apply to strategic decision problems. Palacios-Huerta [21], for instance, argues that penalty kicks in professional soccer are a good testing ground for the predictions of mixed strategy equilibrium, as professional soccer players have a large set of experience to draw from when it comes to that particular "game". We should expect that professional goalies view the direction of a penalty kick as the outcome of a lottery with known odds.[2] Conversely we should expect that a player who does not know her opponent (or who just doesn't not know what to expect from the opponent in the context of a particular situation) should not be able to describe the opponent's strategy in terms of a known probability. Following the experimental evidence cited above we would expect that such players would exhibit ambiguity aversion when they face a "new" game.

In short, ambiguity aversion should be at least as relevant for strategic decision making as it is for individual decision making. It is surprising, then, that the literature on games

[1] Some of the seminal contributions are Schmeidler [23], Gilboa and Schmeidler [11], and Bewley [5], for some more recent contributions see Maccheroni, Marinacci, and Rusticchini [17], and Ahn [1]

[2] The same view is expressed in Chiappori, Levitt and Groseclose [7]. These authors argue that penalty kicks are a good case to test the predictions of mixed strategy equilibrium since "the participants know a great deal about the past history of behavior on the part of opponents as this information is routinely tracked by soccer clubs". This argument was first brought forward by Walker and Wooders [24] who initiated the use of data from professional sports to test the predictions of mixed strategy equilibrium

among players that are ambiguity averse stayed comparatively small.[3]

The goal of this paper is first of all to provide a novel approach of a game theory with uncertainty averse players. This novel approach proceeds under the assumption that players can choose to play ambiguous strategies. The players in the present approach can not only pick pure or mixed strategies, they can choose to base their decisions on subjective random devices. Say the gardener of the above example is also a professor that has to decide whether to test her students on topic A or on topic B. She might either pick a pure strategy (test topic A), a mixed strategy (she could role a dice and test A if and only if the dice shows the number 1) or an ambiguous strategy (she could test topic A if and only if her basil plants grew more than 300 leaves by the last day of classes). Assuming that not all of her students are experienced gardeners this makes her strategy ambiguous from the point of view of the students. In terms of the Ellsberg example, the decision of the professor now resembles a draw of an urn with yellow and blue balls in an unknown proportion.

All prior definitions of games with ambiguity averse players that are uncertain about each other's strategies that I am aware of assume that the players either choose pure or mixed strategies.[4] This assumption prevented the authors of these to define equilibrium in the context of such games as a straightforward application of Nash equilibrium. Nash equilibrium would require that all players maximize against a belief and that this belief is true. Given that players' strategy spaces only contain pure and mixed strategies the equilibrium condition that all players beliefs are true eliminates any scope for uncertainty: players would simply know the mixed or pure strategies of their opponents. Consequently the equilibrium concepts in the literature all build on different relaxations of the condition that the equilibrium beliefs are true. These equilibrium concepts all require that players optimize given some belief about the other players strategies and that this belief is "not

[3] For a review see Mukerji and Tallon [19]. It is important to mention that in some of applications of games with uncertainty averse players the players are assumed to be uncertain about the environment rather than about each other's strategies, see Bade [3], Levin and Ozdenoren [15], Bose, Ozdenoren and Pape [20]

[4] This literature was initiated by Klibanoff [12] and Dow Werlang [8]. Lo [16], Marinacci [18], Eichberger and Kelsey [9] proposed variations, extensions and refinements of the equilibrium concepts introduced by Klibanoff and Dow and Werlang. Most recently Lehrer [14] and Eichberger, Kelsey and Schipper [10] have introduced equilibrium concepts for partially specified probabilities and for decision makers that might be ambiguity loving or ambiguity averse.

too far" from the actual strategies of the other players. So the most important question becomes how should "not too far" be interpreted. The papers in the literature all give different answers to this question, I will discuss and compare some of the most prominent approaches in section 7.

With the present approach to games with ambiguity averse players I am able to circumvent this problem. In games which allow for ambiguous strategies, the presence of ambiguity about the actions of the others is not mutually exclusive with the Nash equilibrium requirement that players know the strategies of the others. Before a formal definition of the concept of ambiguous act equilibria I need to tackle the problem that the introduction of *all* kinds of subjective randomization devices goes beyond the goal of a parsimonious deviation from the theory of games with expected utility maximizing agents. I need to rule out correlation devices and grave violations of the notion of common priors. Imagine that the gardening professor of the first example has two friends who play battle of the sexes (they need to decide whether to vacation in Paris or in Rome). Each one of them could condition their choice of a destination on the growth of the professors basil plant. Once it is time to buy the ticket the professor will give each one of them a leaf-count. Here the subjective randomization device basil plant works as a correlation device. Similarly the friends could have extremely divergent views of the world in the sense that one friend would believe that the basil plant always grows more than x leaves whereas the other believes that the plant never grows so many leaves. In this case we can construct strategy profiles such one friend is certain that both meet in Paris, whereas the other friend is certain that both meet in Rome. Sections 2.3 and 2.4 are devoted to ruling out such correlation devices and extremely divergent views of the world. This is somewhat harder than one might initially think as I cannot rely on probabilistic beliefs on the state space to define "independent strategies" and "common priors". Once these hurdles are taken I define an Ambiguous Act Equilibrium in section 2.6 as a profile of ambiguous act strategies such that no player has an incentive to deviate given all other players' strategies. This definition, which does not rely on any particular representation of the players' preferences is the first main contribution of this paper.

This first contribution can be seen as an answer to the first of three questions that Mukerji and Tallon [19] identified as the guiding questions in research on game theory with ambiguity averse players. In their review of applications of David Schmeidler's concept of uncertainty aversion they describe these three questions as follows: "(1)...how should solution concepts... be defined? (2) questions about the general behavioral implications of the new solution concepts (3) questions about insights such innovations might bring to applied contexts". Above I described this papers's contribution to question (1). My main contribution to the remaining two questions is negative. The second main contribution of this paper is a result of observational equivalence between ambiguous act equilibria and mixed strategy equilibria. I ask the following questions: are there any action profiles that can arise in mixed strategy equilibrium but would never arise in ambiguous act equilibrium? Conversely, are there any action profiles that are consistent with the assumption of equilibrium play by ambiguity averse players but are inconsistent with mixed strategy equilibrium. Can the observation of an action profile tell us whether the players are ambiguity neutral or averse? For games between two players with transitive and monotonic preferences the answer is negative. In such games the set of all ambiguous act equilibria is observationally equivalent to the set of all mixed strategy equilibria, in the sense that any action profile that is consistent with equilibrium play among uncertainty averse players is consistent with equilibrium play among uncertainty neutral players. In the present context the answer to Mukerji and Tallon's questions numbers (2) and (3) is that the general behavioral implications of uncertainty aversion are not different from the general behavioral implications of expected utility maximization and consequently that there is little hope for new insights in applied contexts.

This negative result stands in sharp contrast with the existing literature on strategic interactions between uncertainty averse players. The equilibrium predictions following the existing concepts of equilibrium for uncertainty averse players depend on the ambiguity attitude of players.[5] I will use my result on observational equivalence to shed some light on the interpretation of and comparison between these competing concepts. Finally I will turn to games with more than 2 players. After the presentation of an example that shows that things could turn out differently when there are more than 2 players I will conclude with a discussion why I leave the study of games with more than two players for further research.

[5] This excludes Lo [16]

2 Ambiguous Games

2.1 General Ambiguous Games

A *general ambiguous game* G is defined to be any $G = (I, \Omega, A, \succsim)$ that has the following interpretations and properties. The set of players is $I = (1, ..., n)$. There is a finite state space $\Omega = \Omega_1 \times .... \times \Omega_2$.[6] Player i's action space is denoted by A_i and I define $A = \times_{i \in I} A_i$. Action spaces are assumed to be finite, I define $|A_i| = n_i$ for all players i. The set of player i's strategies is the set of all acts $f : \Omega_i \to \mathcal{P}(A_i)$, where $\mathcal{P}(S)$ denotes the set of all lotteries on any (finite) S. The preferences of players are defined over all acts $f : \Omega \to \mathcal{P}(A)$. The preferences of player i are denoted by $\succsim_i$, the preferences of all players are summarized by $\succsim = \times_{i \in I} \succsim_i$.

A strategy profile $\times_{i \in I} f_i$ induces an act $f : \Omega \to \mathcal{P}(A)$ with $f(s)(a) = \prod_{i \in I} f_i(s_i)(a_i)$ for all $a \in A$. So the probability that an action-profile a is being played in state s is determined as the product of the probabilities that all players play action a_i in state s. I denote the act induced by a strategy profile $\times_{i \in I} f_i$ as well as the strategy profile itself by f.

The assumption that player i's action space consists of all acts $f_i : \Omega_i \to \mathcal{P}(A_i)$ contains an assumption on player i's knowledge. In interpret the assumption that player i can only base his actions on the i'th component of the state to mean that player i only knows the i'th component of the state. Player i's knowledge can be described by the event algebra $\mathcal{S}_i$ on Ω such that $E_i \in \mathcal{S}_i$ if $s \in E_i$ implies that $(s_i, s'_{-i}) \in E_i$ for all $s'_{-i} \in \Omega_{-i}$. Player i's strategy space could have been defined equivalently as the set of all $\mathcal{S}_i$-measurable acts $f : \Omega \to \mathcal{P}(A_i)$. It is convenient to define the events $s_i^* := \{s | s_i = s_i^*\}$, and $s_J^* := \{s | s_J = s_J^*\}$.

Another implication of the assumption on the player's strategy spaces is that every player is assumed to have access to an objective, but secret randomizing device, that can generate any lottery on the player's action space A_i. A player that can choose any $f_i : \Omega_i \to \mathcal{P}(A_i)$ is free to generate his strategic choices using roulette wheels, dices, objective computer generators or similar things. In the games under study players are equally free to base their choices on their mood of the day, or on any other subjective

[6] I follow the usual convention and define $x_J := (x_i)_{i \in J}$ and $x_{-J} := (x_i)_{i \notin J}$ for any subset $J \subset \{1, ..., n\}$ for any vector $x = (x_1, ..., x_n)$. So x_{-i} denotes the vector of all but the i-th component of $x = (x_1, ..., x_n)$.

random device to which they have access. I believe that this assumption is natural for the context of game theory, however, the equilibrium concept proposed here is also suitable for acts $f : \Omega \to A$, in which no objective lotteries are assumed.[7]

2.2 Acts

I use the letters f, g, f_i, g_i to denote various acts. Lotteries on action profiles and action spaces are denoted by $p, q \in \mathcal{P}(A)$ or $p_i, q_i \in \mathcal{P}(A_i)$ respectively. As a shorthand I denote a constant act f with $f(s) = p$ for all $s \in \Omega$ and some $p \in \mathcal{P}(A)$ directly by p (and accordingly for f_i). Degenerate lotteries, that is lotteries $p \in \mathcal{P}(A)$ and $p_i \in \mathcal{P}(A_i)$ such that $p(a) = 1$ for some a or $p_i(a_i) = 1$ for some a_i are denoted a or a_i. Finally constant acts with $f(s) = a$ or $f_i(s_i) = a_i$ for all $s \in \Omega$ or $s_i \in \Omega_i$ are denoted by a and a_i respectively. Constant acts a correspond to pure strategy profiles, constant acts a_i correspond to pure strategies. Constant acts p and p_i correspond to mixed strategy profiles and mixed strategies respectively. In short, pure and mixed strategies are naturally embedded in the framework of general ambiguous games.

Often I will want to evaluate an act in which player i plays the constant act that he would play according to act f_i in event s_i in all possible states when all other players play acts f_{-i}. I denote this act by $(f_i(s_i), f_{-i})$, where $f_i(s_i)$ denotes the constant act in which player i chooses the lottery $f_i(s_i)$ in every state. The mixture $\alpha f + (1-\alpha)g$ of two acts f, g is defined component wise, meaning that $(\alpha f + (1-\alpha)g))(s)(a) = \alpha f(s)(a) + (1-\alpha)g(s)(a)$ for all $a \in A$ and all $s \in \Omega$.

For any two acts f, g and any event $E \subset \Omega$ define the act $f_E g$ by $(f_E g)(s) = f(s)$ for $s \in E$ and $(f_E g)(s) = g(s)$ if $s \notin E$. A state s is considered Savage null by a player i if the values that an act f assumes on this state are irrelevant to player i's preference for this act.

Definition 1 *An state $s \in \Omega$ is* i-null *if $f \sim_i g$ for all acts f, g with $f(s') = g(s')$ for $s' \neq s$. If a state s is not i-null then we call this state* i-non null. *We call a state simply*

[7] This could potentially be very interesting. The empirical evidence on mixed strategies suggests that "normal" people are not able to mix objectively. Chiappori Levitt and Groseclose [7], Palacios-Huerrta [21], and Walker and Wooders [24] therefore use "abnormal" people, namely athletes, to test the predictions of mixed strategy equilibrium. Games played by "normal" people could be studied in a framework in which strategies are acts $f : \Omega_i \to A_i$.

null *if it is i*-null *for all players i. An event E is considered null (i-null) if all states that make up the event are null (i-null).*

I identify the set of i-null states as the set of states that are "never" going to happen following player i's belief. If player i prefers an act f to an act g even though these two acts only differ on a state s player i should better believe that this event could possibly happen. Conversely if player i is indifferent between all acts that differ only on a state s this event is irrelevant for player i's payoff, he might as well think that this event will "never" happen.

2.3 Independent Strategies

The goal of the present study is to see how the equilibrium predictions for a game change when the assumption of expected utility maximizing players is replaced by the assumption of ambiguity averse players. In the following two sections I show that general ambiguous games are to general for this purpose: General ambiguous games do not only allow for various ambiguity attitudes, they also allow for correlation devices and wildly diverging beliefs. To see that general ambiguous games allow for correlation devices take the following example of Battle of the Sexes.

Example 1 To save on notation the row player in every example with two players is called Ann, the column player is called Bob. The actions of Ann and Bob are denoted by $a_1, ...a_{n_a}$ and $b_1, ...b_{n_b}$ respectively. I use a and b as subscripts to denote Ann and Bob's strategies and payoffs. Consider a general ambiguous game between Ann and Bob $G = (\{a, b\}, \Omega, A, \succsim)$. Let $\Omega = \{r_a, s_a\} \times \{r_b, s_b\}$ where r_i stands for player i sees rain and s_i for player i sees it shine. Also assume that both players consider the states $\{(r_a, s_b)\}$ and $\{(s_a, r_b)\}$ null, that is they are convinced that they will never disagree on the weather. In this case both players can use the weather to coordinate their actions.

In fact the notion of a general ambiguous game corresponds to the definition of a game that Aumann [2] uses in the article in which he introduces the concept of correlated equilibrium. Aumann starts out with the same general definition of a game and goes on to impose expected utility representation. The present project can be seen as complementary to Aumann's: How would the set of equilibria change if we dropped the assumption that players are expected utility maximizers but retained the assumption that players cannot rely on any correlation devices? With the goal of the most parsimonious deviation from standard theory that allows for the introduction of a new aspect I should proceed by imposing that the strategies of all players are independent.

This is not as easy as it sounds as the standard notion of independent strategies relies on the expected utility representation of the preferences of all players. So I need to develop a behavioral notion of independent strategies. To do so I extend the common notion of state independent preferences to the context of games. Remember that state independence requires that an agent that prefers one option to another in some state should prefer the first option to the second in any other state. For the case of independent strategies I impose that if one player prefers to play one action over another in some event then that player should prefer the first to the second action in any other event. More generally, I impose that the worth of a strategy for a subgroup of players J cannot depend on the event in which it is played. Let's reconsider weather events. If a player i prefers to play the lottery p_i to the lottery q_i when he observes rain, meaning that he compares two strategies that differ only when it rains but are equal for all other kinds of weather given a fixed strategy profile for all other players, then this player should prefer p_i to q_i for any weather. If not this player has to believe that the other players can also peg their actions to the weather, which in turn entails that the weather can be used as a correlation device. I state a weak version of this definition formally as:

Definition 2 *Take a general ambiguous game $G = (I, \Omega, A, \succsim)$. Then Ω_J is called i-independent of Ω_{-J} if the following condition holds for all acts $f_{-J} : \Omega_{-J} \to \mathcal{P}(A_{-J})$ and all $p_J, q_J \in \mathcal{P}(A_J)$*

- *$(p_J, f_{-J}) \succ_i (q_{JE}p_J, f_{-J})$ for some $E \subset \Omega_J$ implies that $(p_J, f_{-J}) \succ_i (q_J, f_{-J})$*
- *$(p_J, f_{-J}) \succsim_i (q_J, f_{-J})$ implies that $(p_{JE}q_J, f_{-J}) \succsim_i (q_J, f_{-J})$ for any $E \subset \Omega_J$.*

If player i's preferences can be represented by an expected utility function then the behavioral notion of independence given in Definition 2 coincides with the standard notion of independence. Observe that the definition of independence is not very restrictive. For instance, in a two player game Ω_1 can be 1-independent of Ω_2, whereas Ω_2 is not 1-independent of Ω_1. Also following this definition $(q_{JE}p_J, f_{-J}) \succ_i (p_J, f_{-J})$ does not imply

$(q_J, f_{-J}) \succ_i (p_J, f_{-J})$.[8] Finally observe that for $J = I$ this definition rules out state dependent preferences: If any player likes the lottery p better than the lottery q in event E he has to like the constant act p better than the constant act q. To see Definition 2 at work let us reconsider example 1.

Example 2 Consider Example 1 and add some information on the players preferences. Let Ann and Bob's preferences and payoffs be given by the following matrix

	b_1	b_2
a_1	$-2, -1$	0,0
a_2	0,0	$-1, -2$

Keep the assumption that Ann considers the events $\{(r_a, s_b)\}$ and $\{(s_a, r_b)\}$ null. To save on notation I define the events $\{(r_a, r_b)\} := r$ and $\{(s_a, s_b)\} := s$. Fix the following acts f, g, h by $f(r) = g(r) = (a_1, b_1)$, $f(s) = h(s) = (a_2, b_2)$, $g(s) = (a_1, b_2)$ and $h(r) = (a_2, b_1)$.[9] These acts can be illustrated by the following table:

f	b_1	b_2
a_1	r	-
a_2	-	s

g	b_1	b_2
a_1	r	s
a_2	-	-

h	b_1	b_2
a_1	-	-
a_2	r	s

Assume that $h \succ_a g \succ_a f$. I will show that Ω_a is not a-independent of Ω_b. Fix Bob's strategy as $f_b(r_b) = b_1, f_b(s_b) = b_2$. And Fix $E = \{s_a\}$. We have that,

$$g = (a_1, f_b) \succ_a (a_{2E}a_1, f_b) = f$$
$$g = (a_1, f_b) \prec_a (a_2, f_b) = h$$

[8]The reason for this asymmetry of the independence requirement is that the goal of this study is to study games with uncertainty averse players. The action profile $(q_{JE}p_J, f_{-J})$ can be seen as a (subjective) mixture between the act (q_J, f_{-J}) and the act (p_J, f_{-J}). So there are potentially two reasons why $(q_{JE}p_J, f_{-J})$ could be preferred to (p_J, f_{-J}), the first is uncertainty aversion the second is that $(p_J, f_{-J}) \succ_i (q_J, f_{-J})$ holds. The notion of uncertainty aversion will be defined and discussed in section 4

[9]I do not specify values of f, g, h on $\{(r_a, s_b)\}$ and $\{(s_a, r_b)\}$. These two events are considered null by Ann and will therefore not matter to her ranking of the 3 acts.

a contradiction to Ω_a being a-independent of Ω_b. So while a_1 is a better response than a_2 to f_b on $E = \{s_a\}$ it is not true that a_1 is a better response than a_2 on Ω_a.

I am aware of two alternative behavioral definitions of independence in the literature by Branderburger, Blume and Dekel [6] and Klibanoff [13]. Brandenburger, Blume and Dekel's definition of independence also builds on the idea that if a constant act p_J is preferred to a constant act q_J on some event $E_J = \{s_J\}$ and some fixed at f_{-J} then the constant act p_J should preferred to a constant act q_J for the fixed at f_{-J} on any event E_J. Their definition differs from the present one insofar as that they use the concept of "conditional preference" to define independence, which has not been defined for the present context. Klibanoff's [13] definition is less restrictive than the present definition. His definition builds on the same condition as mine, however Klibanoff applies this condition is applied to a smaller domain than I do.[10]

2.4 Basic Agreement

I argued above that general ambiguous games are to general for this purpose of the present study as they permit for correlation devices. The main argument of the present section is that even general ambiguous games with the imposition of independent strategies are too general. Such games allow of wildly diverging beliefs. This stands in sharp contrast to games in mixed strategies for which we require the common prior assumption to hold. To see this take the following example:

Example 3 Consider a general ambiguous game defined in example 1. Assume that Ann considers the event r_a null, whereas Bob considers the event s_a null. If Ann follows f_a with $f_a(r_a) = a_1$ and $f_a(s_a) = a_2$ Ann is certain that she plays a_1, whereas Bob is certain that she plays the other action.

Example 3 flies in the face of the common prior assumption. In keeping with the goal of a parsimonious deviation from the theory of mixed strategy equilibrium, I need to impose a condition that would eliminate such games from consideration. The difficulty lies in

[10]Using the terminology defined here it can be said that in Klibanoff's definition the relation has to hold only if $(q_J, f_{-J}) = a$ for some action profile a.

the fact that the common prior assumption, just like strategic independence, is defined in terms of the players expected utility representations. Without expected utility players have no priors on the state space so they cannot be common. I propose the following (weaker) assumption of basic agreement for the present context.

Definition 3 *A general ambiguous game satisfies* basic agreement *if state s is i-null if and only if is j-null for all $i, j \in I$.*

Clearly if the players preferences are representable by expected utilities the common prior assumption implies basic agreement. In the case of representable preferences a state is i-null if and only if i assigns zero probability to this state. The common prior assumption implies that all players assign the same probability to all states, it implies in particular that player i assigns zero probability to a state s if and only if any other player j assigns zero probability to that state. On the other hand the assumption of basic agreement does not imply common priors.

2.5 Ambiguous Games and Ambiguous Act Extensions

In this section I define the main object of this study: ambiguous games are defined as general ambiguous games with independent actions and basic agreement. I impose these two conditions to make sure that any differences between ambiguous act equilibria and mixed strategy equilibria are not generated by hidden violations of independence and/or common priors in the formulation of ambiguous games.

Definition 4 *A general ambiguous game $G = (I, \Omega, A, \succsim)$ is called an* ambiguous game with independent strategies and basic agreement *or simply an* ambiguous game*, if Ω_J is independent of Ω_{-J} for all i, J and if the game satisfies basic agreement.*

It is useful to define ambiguous act extensions in analogy to mixed strategy extensions. To do so I need a notion of restricted preferences. The preferences $\succsim'$ on B' are a restriction of the preferences $\succsim$ on $B \supseteq B'$ if $a \succsim' b$ for $a, b \in B'$ holds if and only if $a \succsim b$, this is denoted by $\succsim' = \succsim|_{B'}$. Games in mixed strategies are defined as triples $(I, A, \succsim)$ where I denotes the set of players, $A = \times_{i \in I} A_i$ the set of action spaces and $\succsim = \times_{i \in I} \succsim_i$ the set of all players preferences over all lotteries $\mathcal{P}(A)$.

Definition 5 *For any game* $G' = (I, A, \succsim')$ *we call the game* $G = (I, \Omega, A, \succsim)$ *an* ambiguous act extension *of* G' *if* $G = (I, \Omega, A, \succsim)$ *is an ambiguous game and if* $\succsim|_{\mathcal{P}(A)} = \succsim'$.

An ambiguous act extension of a game G' is an ambiguous game G such that G reduces to G' when we restrict all players to take only mixed strategies. If G is an ambiguous act extension of G' then the preferences in both games agree on the set of constant acts. It is important to note that for any game $G' = (I, A, \succsim')$ there are many ambiguous act extensions. In contrast, any game in pure strategies has exactly one mixed strategy extension.

2.6 Ambiguous Act Equilibria

The preparations in the prior sections allow me to use the standard notion of Nash equilibrium to define an equilibrium concept for games with ambiguity averse players. An ambiguous act equilibrium of a game G is defined as a Nash equilibrium of an ambiguous act extension of the game.

Definition 6 *Take an ambiguous game* $G = (I, \Omega, A, \succsim)$. *A strategy profile* f *is called an* ambiguous act equilibrium *if there exists no act* $f_i' : \Omega_i \to \mathcal{P}(A_i)$ *for any player* i *such that* $f \prec_i (f_i', f_{-i})$. *We call* f *an* ambiguous act equilibrium (AAE) *of a game* $G' = (I, A, \succsim')$ *if there exists an ambiguous act extension* G *of* G' *such that* f *is an ambiguous act equilibrium in* G. *The sets of all (mixed strategy) Nash equilibria and of all ambiguous act equilibria of a game* G *are denoted by NE(G) and AAE(G) respectively.*

The definition of AAE proposed here differs sharply from the definitions in the literature on games with uncertainty averse players (Klibanoff [12], Dow and Werlang [8], Lo [16], Eichberger and Kelsey [9] and Marinacci [18]). Firstly all definitions mentioned here rely on particular representations of preferences. Equilibrium in these papers is always defined in terms of some features of the utility functions of players. The present definition does not assume any particular representation of preferences. In fact the present definition can be applied to contexts where uncertainty aversion does not hold. Secondly the present definition is a straightforward application of Nash equilibrium. The papers I reference here all assume that players strategy spaces contain only pure or mixed strategies. A direct application of Nash equilibrium to such a game would eliminate any potential for uncertainty. Consequently all the studies referenced here define alternative equilibrium concepts that weaken the consistency assumption of Nash equilibrium. This differs markedly from the present definition. I allow for ambiguous act strategies. Consequently, the use of the Nash equilibrium concept to define AAE does not eliminate uncertainty. The novelty of the present approach lies in the definition of a game, the equilibrium concept itself is not new, I use Nash equilibrium. Prior studies followed an opposite path, they stayed close to the tradition in terms of the formulation of a game and used innovative equilibrium concepts to allow for uncertainty aversion. [11]

2.7 Observational Equivalence

The main claim of this study is that the ambiguous act equilibria and the Nash equilibria of a two player game $G = (\{a, b\}, A, \succsim)$ are observationally equivalent when the preferences of all players satisfy Schmeidler's uncertainty aversion in addition to monotonicity and transitivity. Observational equivalence captures the idea that an outsider who only observes the action profiles that players choose cannot tell whether the players are ambiguity neutral or ambiguity averse. In short, two strategy profiles are considered observationally equivalent if their support coincides. To proceed any further I need to define the notion of the support of an ambiguous act.

Definition 7 *We say that an action profile a is* in the support *of strategy profile f if there exists a non-null state s such that $f(s)(a) > 0$. We denote the set of all actions in the support of f by supp(f).*

Note that the support of a constant act strategy profile p equals the support of the lottery p in the usual sense of the word support.

Definition 8 *Take an ambiguous game $G = (I, \Omega, A, \succsim)$. Two strategy profiles f, g are* observationally equivalent *if they have the same support. Two sets of action profiles*

[11] The current paper is closest to Lo [16]. In section 7 of his paper Lo constructs a a state space and strategies mapping states to actions for all players to ground his equilibrium concept in an environment in which player preferences are defined over fundamentals. Lo acknowledges that the properties of agreement and stochastic independence need to be defined for this environment. Lo goes on to define these properties in terms of the particular representation of preferences he chose. My approach differs in that I provide axioms to capture the named properties. My equilibrium concept does not rely on any particular representation of preferences.

$\mathcal{F}, \mathcal{G}$ are called observationally equivalent if for every $f \in \mathcal{F}$ there is an observationally equivalent $g \in \mathcal{G}$ and vice versa.

It is important to note that "basic agreement" holds in ambiguous games. Without this assumption the notion of "support" would not be well-defined. Without basic agreement there might be some states that are null for some players but not for others. Consequently the present notion of observational equivalence cannot be applied to general ambiguous games.

The question underlying the definition of observational equivalence is: is there any action profile that is consistent with equilibrium play among players with particular attitude towards ambiguity - neutral or averse - without being consistent with equilibrium play among players with a different attitude towards ambiguity. Said otherwise, is there any action profile that "proves" that players are ambiguity averse in the sense that this action profile is in the support of ambiguous act equilibrium but is not contained in the support of any mixed strategy equilibrium?

So one might ask: can't an observer use more than just ONE action profile to determine whether the players are ambiguity neutral or averse? Aren't the frequencies with which all action profiles are being played also observable? Yes they are, however, the question whether some frequencies are consistent with a subjective act is very much open to debate. To see this take an act $f : \Omega \to \mathcal{P}(A)$ with $\Omega = \{s_1, s_2\}$, $A = \{a_1, a_2\}$ and $f(s_1) = a_1, f(s_2) = a_2$ and both s_1 and s_2 non-null. This act entails no prediction about the frequency of the occurrence of a_1 and a_2. I would need to impose further assumptions on players preferences to relate the observed frequencies to the played acts f. I chose to avoid this by using the equality of support as my criterion of observational equivalence.

3 Preferences and Best replies

3.1 Transitivity, Monotonicity and Expected Utility Representation

Until now I have not specified the preferences of the players beyond requiring the properties of independent strategies and basic agreement. To get any results some further requirements will have to be imposed. In this section I define a range of very basic properties of preferences for the context of ambiguous games.

(TR) Preferences are transitive.

(EU) Preferences over constant acts - that is preferences over lotteries - have an expected utility representation; for any player i there exists an affine function $u_i : \mathcal{P}(A) \to \mathbb{R}$ that represents the players preferences over constant acts (lotteries) $\mathcal{P}(A)$.[12]

Finally I define a notion of monotonicity which relies on eventwise comparisons of acts. This notion of monotonicity requires that if an actor i prefers a strategy f_J of a fixed subgroup of players J for all s_J to a strategy g_J, holding the strategy of all other players fixed at f_{-J}, then player i should prefer the strategy profile $f := f_J \times f_{-J}$ to the strategy profile $g := g_J \times f_{-J}$.

(MON) Take two acts f, g and a subset of players $J \subset I$, such that $f_{-J} = g_{-J}$.

- If for all events s_J we have that $(f_J(s_J), f_{-J}) \succsim_i (g_J(s_J)_{s_J} f_J(s_J), f_{-J})$ and if there exists an event s'_J such that $(f_J(s'_J), f_{-J}) \succ_i (g_J(s'_J)_{s'_J} f_J(s'_J), f_{-J})$ then $f \succ_i g$.

- If $f \succ_i g$ then there exists an event s^*_J such that $(f_J(s^*_J), f_{-J}) \succ_i (g_J(s^*_J)_{s^*_J} f_J(s^*_J), f_{-J})$.

To get a better grasp of this concept let me compare (MON) to a more standard definition of monotonicity as given by Gilboa and Schmeidler [11], Maccheroni, Marinacci and Rusticchini [17], and Schmeidler [23].

(standard MON) Take two acts f, g if for all non-null states $s \in \Omega$ we have $f(s) \succsim_i g(s)$ then $f \succsim_i g$.

(MON) and (standard MON) differ with respect to the following four aspects. First of all, (MON) specifically relates to the context of game theory: (MON) does not only rank acts that can be compared for every state s it also ranks acts that can be compared for every event s_J, *for every subset of strategies* J. (MON) looks a lot more similar to (standard MON) for the case of a single agent decision problem. In that case (MON) would only rank acts that can be compared on every state.

[12]Clearly, I could have stated some more basic properties on the player's preferences over constant acts that imply (EU). I chose to summarily state these assumptions as (EU) for the sake of brevity.

Secondly, (standard MON) amalgamates two very different assumptions. These assumptions are one of state independence that could be stated as $p \succ q_E p$ implies $p \succ q$ and one of monotonicity that could be stated as $f(s) \succ g(s)_{\{s\}} f(s)$ for all s implies $f \succ g$. The first assumption rules out state dependent preferences, whereas only the second assumption should be interpreted as a form of monotonicity. In fact Schmeidler [23] interprets (standard MON) as an assumption of state independent preferences. Independence plays a big role in the present study, I not only assume that preferences are state independent, I assume that all players strategies are independent (which can in turn be interpreted as a form of state independence as argued above). Since independence plays such a central role in the present study I chose to disentangle it from the assumption of monotonicity. However, as long as independence holds (which is, of course, the case for ambiguous games as defined here) (MON) can be rewritten as:

(MON') Take two acts f, g and a subset of players $J \subset I$, such that $f_{-J} = g_{-J}$.

- If for all non-null events s_J we have that $(f_J(s_J), f_{-J}) \succsim_i (g_J(s_J), f_{-J})$ and if there exists a non-null event s'_J such that $(f_J(s'_J), f_{-J}) \succ_i (g_J(s'_J), f_{-J})$ then $f \succ_i g$.
- If $f \succ_i g$ then there exists a non-null event s^*_J such that $(f_J(s^*_J), f_{-J}) \succ_i (g_J(s^*_J), f_{-J})$.

Thirdly, (standard MON) and (MON) differ insofar as that (standard MON) is defined for the case of complete preferences whereas (MON) applies to preferences that are potentially incomplete. To fully appreciate the similarity between the two concepts let me restate (MON) for the case of an ambiguous game with a single player with complete preferences (or in other words, for the case of a decision problem of an agent with state independent and complete preferences). In this case we have that $f(s) \succsim_i g(s)$ for all non-null states and $f(s') \succ_i g(s')$ for some non-null state s' implies $f \succ_i g$.

The fourth difference can clearly be seen from this reformulation of (MON) for the context of single person decision making. (MON) is stronger than (standard MON) in the sense that (standard MON) requires one act to be strictly preferred to another for every state whereas (MON) requires the strict preference only for some non-null state. In section 6 I will provide an in depth discussion of alternative concepts of monotonicity that are weaker than (MON) and therefore closer to the definition by authors cited above.

3.2 Mixed Strategy Equilibria

In this subsection I show that the very weak assumptions of (EU) and (MON') suffice to show that a mixed strategy profile p is an NE of G if and only if it is an AAE of G.

Lemma 1 *Take a game $G' = (I, A, \succsim')$ assume (EU) and (MON'). Let $G = (I, \Omega, A, \succsim)$ be an ambiguous act extension of G'. We have that p is an AAE of G if and only if p is a NE of G'.*

The following Lemma on best responses is useful for the proof of this fact.

Lemma 2 *Take an ambiguous game $G = (I, \Omega, A, \succsim)$ satisfying (EU) and (MON'). Let f be an ambiguous act equilibrium in the game G. Fix a non-null state s. There does not exist a $p_i \in \mathcal{P}(A_i)$ such that $(p_i, f_{-i}) > (f_i(s_i), f_{-i})$.*

Proof Suppose such a p_i existed. Define the act g_i by $g_i(s_i) = p_i$ and $g_i(s_i') = f_i(s_i')$ for all $s_i' \neq s_i$. By (MON') we have that $(g_i, f_{-i}) \succ_i f$, so f cannot be an equilibrium. □

Lemma 2 implies that no strictly dominated strategies will be played in equilibrium. The support of any AAE of a game G is a subset of the set of actions that survive iterated elimination of dominated strategies.[13'14] Lemma 2 can now be applied in the proof of Lemma 1

Proof Let p be an AAE of the ambiguous act extension $G = (\Omega, A, I, \succsim)$ of G. Then we have that there exists no deviation f_i for any player i such that $(f_i, p_{-i}) \succ_i p$, in particular there exists no p_i' such that $(p_i', p_{-i}) \succ_i p$, so p is a NE of G'. Next assume that p is a NE of G'. Suppose p was no AAE of G, that is suppose that there exists a deviation f_i for player i such that $(f_i, p_{-i}) \succ_i p$. By (MON') there exists an event s_i such that $(f_i(s_i), p_{-i}) \succ_i p$, a contradiction to the assumption that p is a NE of G'. □

[13]For a definition of the procedure of iterated elimination of dominated strategies see Bernheim [4] and Pearce [22], who introduced this notion.

[14]Other notions of equilibrium for games with ambiguity averse players permit the use of strategies that are not rationalizable. Dow and Werlang [8] provide an example to illustrate that their equilibrium notion does not necessarily describe a subset of all rationalizable profiles. Klibanoff [12] refines his notion of equilibrium using the iterated elimination of dominated strategies as a criterion.

Lemma 1 should not come as a big surprise. For any player with monotonic preferences a deviation from p can only be profitable if it is profitable in some event. Independence then requires that this deviation is profitable in any event. So if p is a Nash equilibrium there cannot be any such profitable deviation. It should be noted thought that Lemma 1 is strong insofar as that is says that if p is a NE of G then p is an AAE of *any* ambiguous act extension of G.

Corollary 1 *Take an ambiguous game $G = (I, \Omega, A, \succsim)$ assume (EU) and (MON'). An AAE exists.*

Proof Direct consequence of Lemma 1 and the fact that a finite game always has an NE. □

4 Uncertainty Aversion

4.1 Definition

Schmeidler [23] defined the "ambiguity aversion" as a preference for randomization: if an agent is indifferent between two uncertain acts then he should like an objective randomization over these two acts at least as much as either one of them. This is the dominant notion of ambiguity aversion in the literature on decision making, I also adopt it here. For the context of incomplete preferences Schmeidler's axiom can formally be stated as.

(UA) Take three acts $f, f', g : \Omega \to \mathcal{P}$ and let neither $g \succ_i f$ nor $g \succ_i f'$ be true. Then it cannot be true that $g \succ_i \alpha f + (1-\alpha) f'$.

I will not attempt to motivate Schmeidler's axiom here and refer the interested reader to extensive literature on uncertainty aversion for a discussion of Schmeidler's axiom.[15]

4.2 Examples of Preferences

[15] The interested reader might consult Gilboa and Schmeidler [11] and Maccheroni, Marinacci and Rustichini [17] as a start.

To illustrate the notions defined above as well as all of the following results I will make use of a range of different examples of preference structures which I define in examples 4 - 9. For the sake of clarity I drop the index i in this subsection.

Example 4 The preferences of a player can be represented by an expected utility function if there exists an affine function $u : \mathcal{P}(A) \rightarrow \mathbb{R}$ and a prior $q \in \mathcal{P}(\Omega)$ such that such that for all f, g

$$f \succsim g \text{ if and only if } \int u(f)dq \geq \int u(g)dq.$$

Example 5 The preferences can be represented by a minimal expected utility (MEU) function following Gilboa and Schmeidler [11] if there exists an affine function $u : \mathcal{P}(A) \rightarrow \mathbb{R}$ and convex and compact set $Q \subset \mathcal{P}(\Omega)$ such that such that for all f, g

$$f \succsim g \quad \text{if and only if} \quad \min_{q \in Q} \int u(f)dq \geq \min_{q \in Q} \int u(g)dq.$$

Example 6 The preferences of a player can be represented by a Choquet expected utility (CEU) function following Schmeidler [23], if there exists a $\mathcal{S}$-measurable capacity $v : \Omega \rightarrow [0, 1]$ such that $v(\emptyset) = 0, v(\Omega) = 1$, $A \subseteq B$ implies $v(A) \leq v(B)$ and $v(A \cup B) - v(A \cap B) \geq v(A) + v(B)$ such that preferences over acts a represented by the following function:

$$CEU(f) = \sum_{k=1}^{K} u_k \left(v \left(\bigcup_{l=1}^{k} E_l \right) - v \left(\bigcup_{l=1}^{k-1} E_l \right) \right)$$

where the events E_k are defined such that $\{E_1, ..., E_K\} = f^{-1}$, $u_k = f(s)$ for $s \in E_k$ and finally $u_k > u_{k+1}$ for all $k = 1, ..., K$.

Example 7 The preferences of a player can be represented by an uncertainty loving utility function using multiple priors if there exists an affine function $u : \mathcal{P}(A) \rightarrow \mathbb{R}$ and convex and compact set $Q \subset \mathcal{P}(\Omega)$ such that such that for all f, g

$$f \succsim g \quad \text{if and only if} \quad \max_{q \in Q} \int u(f)dq \geq \max_{q \in Q} \int u(g)dq.$$

Example 8 The preferences exhibit Knightian uncertainty following Bewley [5] if there exists an affine function $u : \mathcal{P}(A) \rightarrow \mathbb{R}$ and convex and compact set $Q \subset \mathcal{P}(\Omega)$ such that such that for all f, g

$$f \succ g \quad \text{if and only if} \quad \int u(f)dq > \int u(g)dq \text{ for all } q \in Q \text{ and}$$
$$f \sim g \quad \text{if and only if} \quad \int u(f)dq = \int u(g)dq \text{ for all } q \in Q$$

In all other cases the player cannot rank the acts f and g and we write $f \bowtie g$.

Example 9 The preferences are called variational following Maccheroni, Marinacci and Rustichini [17] if there exists an affine function $u : \mathcal{P}(A) \to \mathbb{R}$ and non-negative, convex and lower-semicontinuous function $c : \mathcal{P}(\Omega) \to [0, \infty]$ such that for all f, g

$$f \succsim g \quad \text{if and only if} \quad \min_{q \in \mathcal{P}(\Omega)} \big(\int u(f)dq + c(q) \big) \geq \min_{q \in \mathcal{P}(\Omega)} \big(\int u(g)dq + c(q) \big).$$

Observe that (EU) and (TR) hold for all the given examples. Example 4 satisfies (MON). Examples 5, 7 and 8 satisfy (MON) if and only if all $q \in Q$ have the same support.[16] Schmeidler's assumption on uncertainty aversion (UA) only holds in examples 4, 5, 6 and 9.[17] Example 4 is a special case of examples 5, 7 and 8 (all these examples reduce to example 4 when Q is a singleton). In our context preferences can be represented by a MEU utility function following example 5 if and only if the can be represented by a CEU-utility function following example 6. Example 5 is a special case of example 9. The preferences in all examples but example 8 are complete. An event E is null in examples 4, 5, 7 and 8 if and only if $q(E) = 0$ for all $q \in Q$. And event E is null in example 9 if $c(q) = 0$ for all q with $q(E) > 0$. An event E is null in example 6 if $v(G \cup E) - v(G) = 0$ for all $G \in \Omega$.

5 Observational Equivalence

In this section I will show that for any two player game G we have that the set of Nash equilibria of that game is observationally equivalent to the set of Ambiguous Act equilibria in that game if (UA), (MON), (EU) and (TR) are satisfied (Theorem 1). This is the main

[16] Klibanoff [12] proves this for the case of example 5, his proof can easily be amended to the other two examples.

[17] We can define a condition (UL) that replaces (UA) for example 7 replacing $\succ_i$ by $\prec_i$ wherever it appears in the definition of (UA).

result of this paper. To get there I first show that actions that are in a sense "dominated" will never be used in an ambiguous act equilibrium (Lemma 3). I will then go on to show under this condition a player's belief on the strategy of the other can always be represented by a probability (Lemma 4). These two Lemmata yield the the proof of the main result of this paper, Theorem 1. It is convenient to use matrix algebra to state and prove all these results. Some more notation needs to be introduced.

5.1 Matrices and Vectors

For a fixed $B_i \times B_{-i} \subset A_i \times A_{-i}$ define the matrices $U := (u_i(a)_{a_i \in A_i, a_{-i} \in B_{-i}}), V := (u_i(a)_{a_i \in B_i, a_{-i} \in B_{-i}}), W := (u_i(a)_{a_i \in A_i \setminus B_i, a_{-i} \in B_{-i}})$. So

$$U = \begin{pmatrix} V \\ W \end{pmatrix}$$

The k'th column of the matrix U is denoted by U^k, so we have that $U = (U^k)_{k=1,...,|A_{-i}|}$. A generic vector p is assumed to be a column vector, row vectors are obtained by taking the transpose p'. With this notation we can simply calculate player i's expected utility of a mixed strategy profile p with $supp(p_{-i}) \in B_{-i}$ as $p_i' U p_{-i}$.

For any two vectors x, y of the same length we define the relations " $>$ ", " $\geq$ ", " $\gg$ " and " $=$ " by $x \geq y$ if and only if $x_t \geq y_t$ for all components t, $x \gg y$ if and only if $x_t > y_t$ for all components t, $x = y$ if and only if $x_t = y_t$ for all components t and finally $x > y$ if and only if $x \geq y$ but not $x = y$. Using this notation we can express the following relation between two lotteries p_i, q_i: $(p_i, a_{-i}^k) \succsim_i (q_i, a_{-i}^k)$ for all $k \in \{1, ..., K\}$ and $(p_i, a_{-i}^k) \succ_i (q_i, a_{-i}^k)$ for some $k \in \{1, ..., K\}$ simply as $p_i' U > q_i' U$. I denote the vector $(x, x, ..., x)'$ by $\overline{x}$.

5.2 Ambiguous Act Equilibria and "Dominance"

The next Lemma describes a condition on all actions that might sometimes be played in a best reply. It is shown that there exists no "dominated" mixture over the set of actions played in a best reply in the sense that for any such mixture there does not exist a mixture over all possible actions of the player.

Lemma 3 *Take an ambiguous game $G = (I, \Omega, A, \succsim)$ assume (TR), (EU), (MON') and (UA). Let f be an ambiguous act equilibrium in G. Define an $n_i \times |supp(f_{-i})|$-matrix U as above with $B_i := A_i, B_{-i} := supp(f_{-i})$. There do not exist any $p_i \in \mathcal{P}(A_i), q_i \in \mathcal{P}(B_i)$ such that and $p_i'U > q_i'V$.*

Proof Since f_i is a best reply to f_{-i} we know by Lemma 2, that there is no $\tilde{p}_i$ such that $(\tilde{p}_i, f_{-i}) \succ_i (f_i(s_i), f_{-i})$ for any non-null s_i. An application of (UA) yields that there does not exist an r_i such that $(\tilde{p}_i, f_{-i}) \succ_i (r_i, f_{-i})$ where r_i is a mix over the constant act strategies $f_i(s_i)$ with $supp(r_i) = supp(f_i)$.

Suppose there existed lotteries p_i, q_i such that $supp(q_i) \subset supp(f_i)$ and $p_i'U > q_i'U$. Since $supp(r_i) = supp(f_i)$ we can represent the lottery r_i as a sum $(1-\lambda)\tilde{r}_i + \lambda q_i$ for some $\lambda \in (0, 1]$ and some lottery $\tilde{r}_i$. Now let us compare the lotteries $r_i^* := (1-\lambda)\tilde{r}_i + \lambda p_i$ and r_i. Observe that

$$U = r_i^{*'}U = ((1-\lambda)\tilde{r}_i' + \lambda p_i) > ((1-\lambda)\tilde{r}_i' + \lambda q_i)U = r_i'U$$

So player i weakly prefers (r_i^*, a_{-i}) to (r_i, a_{-i}) for all $a_{-i} \in supp(f_{-i})$ and does so strictly for some $a'_{-i} \in supp(f_{-i})$. This implies that player i weakly prefers $(r_i^*, f_{-i}(s))$ to $(r_i, f_{-i}(s))$ for all non-null states s and does so strictly for some non-null states. We can conclude by (MON') that $(r_i^*, f_{-i}) \succ_i (r_i, f_{-i})$ a contradiction to the non-existence of a $\tilde{p}_i$ such that $(\tilde{p}_i, f_{-i}) \succ_i (r_i, f_{-i})$. □

5.3 Examples of Best Replies and Equilibria

Let me illustrate Lemma 3 at the hand of two examples. The first example demonstrates the strengths of the Lemma

Example 10 Take the following ambiguous game between Ann and Bob with $G = (\{a, b\}, \Omega, A, \succsim)$. Let the following matrix represent the action spaces and preferences of Ann and Bob, let (EU), (TR), (MON') and (UA) be satisfied.

This game does not have an equilibrium with full support (no matter which values we assign to $u_b(a_i, b_j)$). This follows from Lemma 3 and the observation that $pU \gg qU$ for $p = (0, 0, 1)$ and $q = (\frac{1}{2}, \frac{1}{2}, 0)$.

	b_1	b_2	b_3
a_1	$1, u_b(a_1, b_1)$	$0, u_b(a_1, b_2)$	$4, u_b(a_1, b_3)$
a_2	$4, u_b(a_2, b_1)$	$0, u_b(a_2, b_2)$	$1, u_b(a_2, b_3)$
a_3	$3, u_b(a_3, b_1)$	$1, u_b(a_3, b_2)$	$3, u_b(a_3, b_3)$

To see that (UA) is essential for Lemma 3 to hold consider the following variation of the preceding example.

Example 11 Assume that Ann and Bob play the game described in example 10, except for (UA). Assume that Ω_a is a singleton whereas $\Omega_b = \{s_1, s_2, s_3\}$. Let Ann's preferences be representable by multiple priors following Bewley (example 8) with $Q = co((.1, .1, .8), (.8, .1, .1), (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}))$.[18] For simplicity, assume that $u_b(a_i, b_j) = 1$ for $i, j = 1, 2, 3$. Then (p, f_b) with $p = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ and $f_b(s_1) = b_1, f_b(s_2) = b_2, f_b(s_3) = b_3$ is an AAE with full support. To see this observe that for any deviation r from p there is a belief $q \in Q$ such that $rUq < pUq$ where U is defined as the matrix of Ann's payoffs.

5.4 "Dominance" and Mixed Strategy Equilibria

The next Lemma describes a condition under which we can find a probability p_{-i} on all the actions of all other players such that the actions in $B_i \subset A_i$ yield a constant maximal utility given p_{-i}. This condition can again be described as a "dominance" condition for mixtures over the actions that are played in the best reply and mixtures among all other actions.

Lemma 4 *Take a game $G' = (I, A, \succsim')$. Define $B \subset A$ and the matrices U, V, W as above. Suppose there do not exist any $p_i \in \mathcal{P}(A_i), q_i \in \mathcal{P}(B_i)$ such that and $p_i'U > q_i'V$. Then there exists a probability $p_{-i} \in \mathcal{P}(A_{-i})$ with $supp(p_{-i}) = B_{-i}$ and an $x \in \mathbb{R}$ such that $Vp_{-i} = \overline{x}$ and $Wp_{-i} \leq \overline{x}$.*

Proof

($\Rightarrow$) Suppose exists a $p_{-i} \in \mathcal{P}(A_{-i})$ with $supp(p_{-i}) = B_{-i}$ and an $x \in \mathbb{R}$ such that $Vp_{-i} = \overline{x}$ and $Wp_{-i} \leq \overline{x}$. Suppose we also had $p_i \in \mathcal{P}(A_i), q_i \in \mathcal{P}(B_i)$ such that

[18] The set $co(a, b)$ denotes the convex hull of a, b.

$p_i'U > q_i'V$. This yields a contradiction as $x = r_i'\overline{x} \geq p_i'(Up_{-i}) = (p_i'U)p_{-i} > (q_i'V)p_{-i} = q_i'(Vp_{-i}) = q_i'\overline{x} = x$.

($\Leftarrow$) Suppose there exists no $p_{-i} \in \mathcal{P}(A_{-i})$ with $supp(p_{-i}) = B_{-i}$ and $x \in \mathbb{R}$ such that $Vp_{-i} = \overline{x}$ and $Wp_{-i} \leq \overline{x}$. This is equivalent to: $S \cap r = \emptyset$ for $r := \{\overline{x} | x \in \mathbb{R}\}$ and

$$S := \{s | s_V = Vp_{-i} \text{ and } s_W \geq Wp_{-i} \text{ for } p_{-i} \in \mathcal{P}(A_{-i}),\ supp(p_{-i}) = B_{-i}\}.$$

Since S is a convex set there exists a separating hyperplane H such that $r \subset H$ and $H \cap S = \emptyset$. Let this plane H be described by a vector λ such that $\lambda' x = 0$ implies $x \in H$ and $\lambda' x > 0$ for all $x \in S$. Since $r \subset H$ we have that $\sum \lambda_i = 0$.

Next define two vectors κ and ρ by $\kappa_l = \lambda_l$ if $\lambda_l > 0$ and $\kappa_l = 0$ otherwise. Also let $\rho_l = -\lambda_l$ if $\lambda_l < 0$ and $\rho_l = 0$ otherwise. Observe that $\sum \kappa_l = \sum \rho_l > 0$[19]. Define $\widetilde{\lambda}, \widetilde{\kappa}$ and $\widetilde{\rho}$ by

$$\widetilde{\lambda}_l = \frac{\lambda_l}{\sum \kappa_l},\ \widetilde{\kappa}_l = \frac{\kappa_l}{\sum \kappa_l},\ \widetilde{\rho}_l = \frac{\rho_l}{\sum \kappa_l}$$

Observe that $\widetilde{\lambda}$ and λ as normal vectors describe the same plane. Consequently we have that $\widetilde{\lambda}' x > 0$ for all $x \in S$. As $\widetilde{\lambda} = \widetilde{\kappa} - \widetilde{\rho}$ we have that $\widetilde{\kappa}' x > \widetilde{\rho}' x$ for all $x \in S$.

I show next that $\widetilde{\rho}_l = 0$ for all $l > L$. Suppose we had $\widetilde{\rho}_l > 0$ for some $l > L$. Fix an $x \in S$, observe that $\widetilde{\kappa}' x > \widetilde{\rho}' x$ has to hold for this x as this has to hold for all $x \in S$. Next define $\tilde{x}$ by $\tilde{x}_{-l} = x_{-l}$ and $\tilde{x}_l > \frac{\tilde{\kappa}' x - \tilde{\rho}'_{-l} x_{-l}}{\tilde{\rho}_l}$. By our construction of S we can find such an $\tilde{x}$ that is also an element of S. Observe that

$$\tilde{\rho}' \tilde{x} = \tilde{\rho}'_{-l} \tilde{x}_{-l} + \tilde{\rho}_l \tilde{x}_l > \tilde{\rho}'_{-l} x_{-l} + \tilde{\rho}_l \frac{\tilde{\kappa}' x - \tilde{\rho}'_{-l} x_{-l}}{\tilde{\rho}_l} = \tilde{\kappa}' x = \tilde{\kappa}' \tilde{x}$$

Where the very last equality follows from the fact that on the one hand $\tilde{\rho}_l > 0$ implies $\tilde{\kappa}_l = 0$ and on the other hand $x_{-l} = \tilde{x}_{-l}$. But $\tilde{\rho}' \tilde{x} > \tilde{\kappa}' \tilde{x}$ stands in contradiction with $\tilde{\kappa}' x > \tilde{\rho}' x$ holding for all $x \in S$. We conclude that $\tilde{\rho}_l = 0$ for all $l > L$. Observe that the $\widetilde{\rho}, \widetilde{\kappa}$ are by construction elements of $\mathcal{P}(A_i)$. As $\widetilde{\rho}_l = 0$ for $l > L$ we can define $\overline{\rho} \in \mathcal{P}(B_i)$ by $\overline{\rho}_l = \widetilde{\rho}_l$ for $l = 1, ..., L$.

To conclude this proof observe that $\widetilde{\kappa} U^k \geq \widetilde{\rho} U^k = \overline{\rho} V^k$ for all $k = 1, ..., K$ as any U^k can be approached by a sequence $x_n \in S$. Finally it cannot be true that $\widetilde{\kappa} U^k = \widetilde{\rho} U^k$ for all columns k as we could then find $x \in S$ with $\widetilde{\kappa} x = \widetilde{\rho} x$. So it must be true that $\widetilde{\kappa} U^{k'} > \widetilde{\rho} U^{k'} = \overline{\rho} U^k$ for some columns k'. So we found two probabilities $\widetilde{\kappa}$ and $\overline{\rho}$ such that $\tilde{\kappa}' U > \tilde{\rho}' V$. □

[19] The vectors κ and ρ are defined such that $\sum \kappa_l = \sum \rho_l \geq 0$. If we had that $\sum \kappa_l = \sum \rho_l = 0$ we also had $\lambda = 0$, a contradiction with the assumption that λ describes the hyperplane H

5.5 Observational Equivalence: The Main Result

Theorem 1 *Let $G = (\{a, b\}, A, \succsim)$ assume (TR), (EU), (MON') and (UA). The set of AAE of G is observationally equivalent to the set of NE of G.*

Proof

($\Leftarrow$) Let p be an NE of G, then by Lemma 1 p itself is an AAE of G, so G has an AAE with the same support.

($\Rightarrow$) Let f be an AAE of G. Define an $L \times |supp(f_2)|$-matrix U as above with $B := supp(f)$. Following Lemma 3 there do not exist any $p_1 \in \mathcal{P}(A_i), q_1 \in \mathcal{P}(B_i)$ such that $p_1 U > q_1 U$. Applying Lemma 4 we conclude that there exists a probability p_2 on A_2 with $supp(p_2) = supp(f_2)$ such that all $a_1 \in supp(f_1)$ are best replies to p_2. Construct p_1 in the same fashion. Clearly p is an NE of G with $supp(p) = supp(f)$ as all $a_i \in supp(f_i)$ are best replies to p_{-i} for $i = 1, 2$ and $supp(f_i) = supp(p_i)$ for $i = 1, 2$ by construction. $\square$

Theorem 1 is the main result of this study. This result establishes that an outside observer cannot distinguish the behavior of uncertainty averse player from the behavior of uncertainty neutral players when he observes only the outcomes of their play. Of course certain conditions have to hold for this result to apply: it is shown that observational equivalence holds for 2 player games, where both player's are expected utility maximizers with respect to lotteries, have monotonic preferences and satisfy Schmeidler's axiom of uncertainty aversion.

6 Weaker concepts of Monotonicity

6.1 Two concepts of Monotonicity

In section 3.1 I showed that the concept of monotonicity used here differs from the monotonicity axiom standardly used in treatments of uncertainty averse preferences in 4 essential ways. (MON) applies specifically to games, (MON) does not amalgamate an independence axiom with a monotonicity axiom and (MON) does relate to incomplete preferences. However even when one applies (MON) to the case of a single person decision problem of a person with state independent and complete preferences a difference between (MON) and the standard axiom of monotonicity (standard MON) remains: (MON) is stronger, it says that a decision maker prefers an act that is never worse and sometimes better. The standard axiom of monotonicity is weaker, it requires it only ranks two acts if one is *always* better than the other. How would the results of this paper change if I was to replace (MON) by a weaker assumption on monotonicity that is more in keeping with the standard axiom? To answer this question I define the concept of (WMON')

(WMON') Take two acts f, g and a subset of players $J \subset I$, such that $f_{-J} = g_{-J}$.

- If for all non-null events s_J we have that $(f_J(s_J), f_{-J}) \succ_i (g_J(s_J), f_{-J})$ then $f \succ_i g$.
- If $f \succ_i g$ then there exists a non-null event s_J^* such that $(f_J(s_J^*), f_{-J}) \succ_i (g_J(s_J^*), f_{-J})$.

All examples of preferences discussed in section 4.2 satisfy (WMON'). However (WMON') is not enough to even prove Lemma 2. To see this consider the following example:

Example 12

Take the following ambiguous game between Ann and Bob with $G = (\{a, b\}, \Omega, A, \succsim)$ assume that $\Omega_a = \{s_1, s_2\}$ and Ω_b a singleton. Assume that Ann's preferences can be represented following example 7 with $Q = [0, 1]$. Assume that Bob assigns a prior of 1/2 to either one of the states. Let the following matrix represent the action spaces and preferences of Ann and Bob.

	b_1	b_2
a_1	10, 1	0, 0
a_2	11, 0	0, 1

The preferences of Ann and Bob satisfy (EU), (TR) and (WMON'). The action profile (f_a, p_b) with $f_a(s_1) = a_1,\ f_a(s_2) = a_2$ and $p_b(b_1) = 1/2$ is an ambiguous act equilibrium in the game G. Ann is best responding to p_b even though Ann plays a_1 in the non-null state s_1. This can be explained by the Ann's uncertainty lovingness. While Ann does not consider s_1 null, she does not assign any weight to s_1 occurring in this particular situation since the outcome for s_2 is better.

Fortunately a stronger condition of monotonicity is consistent with all representations of uncertainty averse preferences mentioned here (examples 5, 6 and 9). Intuitively I want to define uncertainty averse monotonicity such that an act f is better than an act g if f is in no event worse than g and if f is strictly better in the "worst case". So while under (MON) an act that is never worse is considered better if it is better in *at least one non-null event*, uncertainty aversion considers the act better if it is better in *a particular event, the "worst case"*. To define uncertainty averse monotonicity I first need to find a suitable notion of the "worst case". I do so by singling out a event s_J such that f yields a "worst payoff" on this event. Formally we define (UA-MON) in the spirit of (MON') by

(UA-MON) Take two acts f, g and a subset of players $J \subset I$, such that $f_{-J} = g_{-J}$.

- If for all non-null events s_J we have that $(f_J(s_J), f_{-J}) \succsim_i (g_J(s_J), f_{-J})$ and if there exists a non-null event s'_J such that on the one hand $(f_J(s'_J), f_{-J}) \succ_i (g_J(s'_J), f_{-J})$ and on the other hand $(f_J(s'_J), f_{-J}) \succ_i (f_J(s_J), f_{-J})$ for no non-null s_J then $f \succ_i g$.
- If $f \succ_i g$ then there exists a non-null event s^*_J such that $(f_J(s^*_J), f_{-J}) \succ_i (g_J(s^*_J), f_{-J})$.

Two conditions need to be satisfied to establish a strict preference $f = f_J \times f_{-J} \succ_i g_J \times f_{-J} = g$ following (UA-MON). The first condition says that g cannot be ranked higher than f for any event s_J. The second says that f is ranked strictly better in the worst case, where an event s'_J is defined as a worst case if there exists no non-null event s_J such that $(f_J(s'_J), f_{-J}) \succ_i (f_J(s_J), f_{-J})$. With this the result of Lemma 2 can be recovered.

Lemma 5 *Take an ambiguous game $G = (I, \Omega, A, \succsim)$ satisfying (EU) and (UA-MON). Let f be an ambiguous act equilibrium in the game G. Fix a non-null state s. There does not exist a $p_i \in \mathcal{P}(A_i)$ such that $(p_i, f_{-i}) > (f_i(s_i), f_{-i})$.*

Proof Suppose such a p_i existed. Define the act g_i by

$$g_i(s_i) = \begin{cases} p_i & \text{if } \ (p_i, f_{-i}) \succ_i (f_i(s_i), f_{-i}) \\ f_i(s_i) & \text{otherwise.} \end{cases}$$

Next observe that by the construction of g_i there does not exist any s_i such that $(g_i(s_i), f_{-i}) \prec_i (f_i(s_i), f_{-i}) = (p_i, f_{-i})$. So we can conclude by (UA-MON) that $(g_i, f_{-i}) \succ_i (f_i, f_{-i})$ and f_i cannot have been a best reply in the first place. $\square$

However (TR), (EU), (UA) and (UA-MON) are not enough to obtain the result of observational equivalence of Theorem 1. To see this consider the following example:

Example 13

Take the following ambiguous game between Ann and Bob with $G = (\{a, b\}, \Omega, A, \succsim)$ assume and assume that $\Omega_b = \{s_1, s_2\}$ and Ω_a a singleton. Let Bob be an expected utility maximizer that believes both states are equally likely. Let Ann's preferences be representable by a MEU function (example 5) with $Q = [0, \frac{1}{2}]$ where $q \in Q$ is a probability of state s_1. Let the payoffs be given by the matrix in the prior example 12. Then (p_a, f_b) with $p_a = (\frac{1}{2}, \frac{1}{2})$ and $f_b(s_1) = b_1, f_b(s_2) = b_2$ is an AAE with full support. To see this observe that according to the most pessimistic belief in Ann's set of beliefs s_1 has zero probability. Consequently Ann will disregard the payoff difference between a_1 and a_2 even though the event s_1 is non-null.

The equilibrium constructed in the prior example strikes me as particulary unappealing. Why would Ann play a_1 when playing a_2 is never worse for Ann and strictly better in some non-null event? The preceding example proves that a theory of games with ambiguity averse players can yield different predictions than standard game theory: the game defined above does not have a mixed strategy equilibrium with full support. It has to be said however, that the differences between these two theories should not depend on such shaky examples in which some player uses a strategy that is "dominated" in the sense that this player has another strategy available that is never worse and strictly better in some non-null event. In the following section I define a refinement that rules out such peculiar behavior.

6.2 Solid Ambiguous Act Equilibria

This section is devoted to the definition of a refinement of AAE that rules out examples such as example 13.[20]

[20] Klibanoff [12] already discussed this unappealing feature of the theory of uncertainty aversion at the hand of preferences that can be represented following Gilboa and Schmeidler (example 5). His way to remedy the problem is to derive a representation of preferences that does not have this feature, these preferences violate the continuity axiom. My approach can be seen as complementary to Klibanoff's:

Definition 9 *Take an ambiguous game* $G = \{I, \Omega, A, \succsim\}$. *We say that an AAE* f *is a solid ambiguous act equilibrium (SAAE) if there does not exist any* f_i' *such that* $(f_i', f_{-i})(s) \succsim_i f(s)$ *for all non-null states* s *and* $(f_i', f_{-i})(s^*) \succ_i f(s^*)$ *for some non-null state* s^*.

Remark 1 For all preferences that satisfy (MON) the set of AAE coincides with the set of all SAAE. This is important insofar as that the refinement proposed here does not have any bite for games with expected utility maximizing agents as their preferences always satisfy (MON). Consequently the present refinement is not equivalent to any other refinement proposed for the context of mixed strategy equilibria.

The results on the relation between NE's and AAE's of section 3.2 transfer to the case of SAAE's. To see this I state and prove the folllowing variants of Lemma 1 and Corollary 1 next.

Lemma 6 *Take a game* $G' = (I, A, \succsim')$ *assume (EU). Let* $G = (I, \Omega, A, \succsim)$ *be an ambiguous act extension of* G' *satisfying (WMON'). We have that* p *is an SAAE of* G *if and only if it is a NE of* G'.

Proof We know from Lemma 1 that any p is an AAE of G if and only if it is an NE of G'. So we only need to show that any NE of G' is a SAAE in G. So suppose that the AAE p is not solid. That is suppose there exists a player i and a strategy f_i such that $(f_i, p_{-i})(s) \succsim_i p(s)$ for all non-null states and $(f_i, p_{-i})(s^*) \succ_i p(s^*)$ for some non-null s^*. Observe that $(f_i, p_{-i})(s) = (f_i(s_i), p_{-i})$ and $p_{-i}(s_{-i}) = p_{-i}$ as p_{-i} is a constant act. So we conclude that $(f_i(s_i^*), p_{-i}) \succsim_i p$ which stands in contradiction that p being an NE of G'.
□

Corollary 2 *Take an ambiguous game* $G = (\Omega, A, I, \succsim)$ *assume (EU) and (WMON'). A SAAE exists.*

Proof The proof follows as a direct consequence of Lemma 6 and the fact that any finite game has an NE. □

I do keep Gilboa and Schmeidler's preferences in the set of preferences to be considered, however I do strengthen the equilibrium concept.)

An amendment of Lemma 3 to the case of SAAE yields the most important step in showing the observational equivalence between NE and SAAE for two player games that satisfy the weaker notion of monotonicity (UA-MON).

Lemma 7 *Take an ambiguous game $G = (I, \Omega, A, \succsim)$ assume (TR), (EU), (UA) and (UA-MON). Let f be an SAAE in G. Define an $n_i \times |supp(f_{-i})|$-matrix U as above with $B := supp(f)$. There do not exist any $p_i \in \mathcal{P}(A_i), q_i \in \mathcal{P}(B_i)$ such that and $p_i'U > q_i'V$.*[21]

Proof The proof follows mutatis mutandis, replacing (MON') in the last conclusion by the requirement that the equilibrium hast to be solid. □

Lemma 7 together with Lemma 4 yield the proof of the following variation of Theorem 1:

Theorem 2 *Let $G = (\{a, b\}, A, \succsim)$ assume (TR), (EU), (UA) and (UA-MON). The set of SAAE of G is observationally equivalent to the set of NE of G.*

I conclude that a weakening of the monotonicity axiom to stay closer to the prefernece representations of Gilboa and Schmeidler [11], Schmeidler [23] and Maccheroni, Marinacci and Rusticchini [17] does not significantly increase the set of ambiguous act equilibria. In two player games there are some ambiguous act equilibria that are not observationally equivalent to any mixed strategy equilibrium. However none of these equilibria is solid. Said otherwise, in any one of these equilibria at least one player plays an action that is never better and sometimes strictly worst than some other available action.

7 Other Equilibrium Concepts

7.1 Four Different Definitions

Prior definitions of equilibrium for games with uncertainty averse players considered mixed or pure strategies as the objects of choice of the players. The different equilibrium concepts vary by their different relaxations of the assumption that players know the strategies of all

[21]Without the requirement that the AAE be solid the weaker implication $p_i'U \gg q_i'V$ holds.

opponents. I summarize four different equilibrium notions in the next definition.[22] I state all these definitions in the framework of Gilboa and Schmeidler's (example 5) minimal expected utility representation with $u_i : \mathcal{P}(A) \to \mathbb{R}$ an affine function that represents player i's utility of all constant acts (or lotteries). I also restrict attention to two player games. All four definitions have the assumption that players maximize their utility given their "belief" on the other player in common. They also share the assumption that players have convex and compact belief sets on the strategy of the other and that the utility of a player is the minimal expected utility with respect to all the beliefs in these convex sets. The concepts differ in their assumption on the relation between a players' set of beliefs on the strategy of the other and the actual strategy played by that player.

Definition 10 *Take a game $G = (\{a, b\}, A, \succsim)$. Consider a profile of mixed strategies p^* and two convex and compact belief sets $Q_a \subset \mathcal{P}(A_a)$, $Q_b \subset \mathcal{P}(A_b)$, such that p_i^* maximizes $\min_{q_{-i} \in Q_{-i}} u^i(p_i \times q_{-i})$ for $i = a, b$. Then p^* is called*

- *a Klibanoff equilibrium (KE) if $p_i^* \in Q_i$ for $i = a, b$;*
- *a Dow-Werlang equilibrium (DWE) if $p_i^* \in Q_i$ and there does not exist a $q_i \in Q_i$ such that $supp(q_i) \subsetneq supp(p_i^*)$ for $i = a, b$;*
- *a Marinacci equilibrium (ME) if $p_i^* \in Q_i$ and $supp(p_i^*) \subset supp(q_i)$ for all $q_i \in Q_i$ for $i = a, b$;*
- *a Lo equilibrium (LE) if $p_i^* \in Q_i$ and $a_i \in supp(q_i)$ for any $q_i \in Q_i$ implies that a_i maximizes $EU_i(p_i, q_{-i})$ for $i = a, b$.*

Let me state without proof that the following subset relation between the different concepts holds: (NE)⊂(LE)⊂(ME)⊂(DWE)⊂(KE). It is easy to see that the last three inclusions hold as the first three equilibrium concepts only differ in their increasingly strict requirements on the relation between beliefs and strategies. The difference between the set of NE and the set of LE, (the first two sets in this chain) arises since players might

[22]The equilibrium notions do not only differ with respect to their different requirements for consistency between a players strategy and all other player's beliefs about this strategy. I chose to abstract from all other differences to make the comparison as easy as possible. Consequently the following definition would appear as an oversimplification for any other purpose

use strategies that are never better and sometimes worse in LE. If one where to apply a refinement similar to solidity of section 6.2 the difference would disappear. Let me use the following example of Klibanoff [12] to show that the difference between NE and the three other concepts is more substantial.

Example 14 Take the following normal form game between Ann and Bob $G = \{\{a, b\}, A, \succsim\}$ with

	b_1	b_2
a_1	3,0	1,2
a_2	0,4	0,-100

Klibanoff shows that (a_1, b_1) is a KE of this game but not an NE. To see that (a_1, b_1) is a KE let $Q_a = [.1, 1]$ and $Q_b = \{1\}$. Bob's utility of his strategy p^b can be written as $u^b(p_b) = \min_{q_a \in [.1,1]} 2(1-p_b)q_a + 4p(1-q_a) - 100(1-p_b)(1-q_a)$. Bob's utility is maximized for $p_b = 1$. On the other hand a_1 is Ann's best reply to the pure strategy b_1. Next observe that (a_1, b_1) and Q satisfy the consistency requirement for KE, DWE and ME. So (a_1, b_1) is an an equilibrium following any of the three concepts.

This example raises an important question: The preferences used to define the KE, DWE and ME satisfy (TR), (EU), (UA) and (UA-MON). So how can there the KE, DWE and ME differ so starkly from the NE of a game? Do these concepts violate independence and/or basic agreement? I will argue in the sequel that the difference between these equilibrium notions and NE lies not so much in their allowance for ambiguity averse players but rather in a violation of "basic agreement". To do so I need some more definitions.

7.2 Standard Games without Agreement

I first define a class of games that differs from games in mixed strategies only insofar as that "basic agreement" might be violated in these games.

Definition 11 *A general ambiguous game $G = (I, \Omega, A, \succsim)$ is called a* standard game without agreement *if Ω_J is independent of Ω_{-J} for all i, J and if all players preferences can be represented by expected utilities, so that $\int u_i(f)dq^i$ with some $q^i \in \mathcal{P}(\Omega)$ and $u_i : \mathcal{P}(A) \to \mathbb{R}$ an affine function represents player i's utility of f.*

Observe that ambiguous games and standard games without agreement are two different generalizations of standard games $G = (I, A, \succsim)$ and two different specializations of general ambiguous games. Ambiguous games adopt "all" features of standard games except for the expected utility representation, standard games without agreement adopt "all" features of standard games except for basic agreement. Next I define the notion of a standard extension without agreement and the notion of an equilibrium without agreement paralleling the definitions ambiguous act extensions and of AAE.

Definition 12 *For any game $G' = (I, A, \succsim')$ I call the game $G = (I, \Omega, A, \succsim)$ a* standard extension without agreement *of G' if $G = (I, \Omega, A, \succsim)$ is an standard game without agreement and if $\succsim|_{\mathcal{P}(A)} = \succsim'$. Take a standard game without agreement $G = (I, \Omega, A, \succsim)$. A strategy profile f is called an* equilibrium without agreement *(Ew/oA) if there exists no act $f'_i : \Omega_i \to \mathcal{P}(A_i)$ for any player i such that $f \prec_i (f'_i, f_{-i})$. A strategy profile $f : \Omega \to \mathcal{P}(A)$ is called a Ew/oA of $G' = (I, \Omega, A)$ if there exists a standard extension without agreement $G = (I, \Omega, A, \succsim)$ of G' such that f is an an Ew/oA in G.*

The goal of this section is to compare the set of Ew/oA of a game $G = (I, A, \succsim)$ to the set of KE, DWE and ME of this game. In section 2.7 I developed the notion of observational equivalence to relate ambiguous act profiles to mixed strategy profiles: two profiles are called "observationally equivalent" if they have the same support. Unfortunately, this notion of observational equivalence cannot be applied here, since without the assumption of "basic agreement" the notion of the support of an ambiguous act profile is not well-defined. An alternate notion of "equivalence" needs to be developed. To do so I use the fact that every player has a prior on the state space in a standard game without agreement. I say that player j considers strategies f_i and p_i equivalent if every action a_i is played with the same probability under p_i and f_i given j's prior on the state space.

Definition 13 *A mixed strategy p_i is called j-equivalent to a strategy f_i if $p_i(a_i) = \int f_i(s)(a_i)dq^j$ for all a_i. An ambiguous act profile f is called $*$-equivalent to a mixed strategy profile p if f_i is i-equivalent to p_i for all i.*

Clearly no two players need to agree on the equivalence between two strategies. If two players hold different priors on the state space, they might disagree on equivalence statements. Lacking a neutral or view-point free notion of equivalence I chose to call to profiles f and p $*$-equivalent if f_i is i-equivalent to p_i for all i. Two strategies of player i

are considered $*$-equivalent if they are considered equivalent in the eyes of player i. Next I'll relate the equilibrium constructed in example 14 to an Ew/oA in the same game.

Example 15 Take the normal form game defined in example 14. Construct a standard extension without agreement $G = (I, \Omega, A, \succsim)$ of G such that $\Omega_a = \{s_1, s_2\}$ and Ω_b a singleton. Let $q^b(s_1) = .1, q^a(s_1) = 1$. The strategy profile f with $f_a(s_1) = a_1, f_a(s_2) = a_2, f_b = b_1$ is an Ew/oA. Ann's best reply to b_1 is a_1, according to her belief $q^a(s_1) = 1$ she is always playing a_1. Bob on the other hand believes that Ann is playing a_1 only in a tenth of all cases $q^b(s_1) = .1$, so his playing b_1 is a best reply to this belief. Observe that in this strategy profile Ann believes that she always plays a_1, and Bob always plays b_1, therefore (a_1, b_1) and f are $*$-equivalent.

7.3 Ambiguity Neutrality versus Basic Agreement

In the next theorem I show that the relationship between the equilibrium constructed in example 14 and the Ew/oA in example 15 is not accidental.

Theorem 3 *Take a game $G' = (\{a, b\}, A, \succsim')$ with two players. Let p be a KE of G'. Then p is $*$-equivalent to an Ew/oA of G'.*

Proof Let p^* be a KE. So p_a^* maximizes $\min_{q_b \in Q_b} p_a' U q_b$. Fan's theorem implies that the order of minimization and maximization can be exchanged. So p_a^* is a solution to $\min_{q_b \in Q_b} \max_{p_a \in [0,1]} p_a' U q_b = \min_{q_b \in Q_b} p_a'(q_b) U q_b$ where $p_a(q_b)$ denotes $argmax_{p_a \in [0,1]} p_a' U q_b$ for any q_b. Let q_b^* be the minimizer (in Q_b) of the function $p_a'(q_b) U q_b$. So p_a^* maximizes $p_a' U q_b^*$. This implies that p_a is a best reply for Ann if Bob plays $q_b^* \in Q_b$. Analogously derive q_a^* such that p_b is a best reply to q_a^* for Bob.

Now let $G = (\{a, b\}, \Omega, A, \succsim)$ be a standard extension without agreement such that $\Omega_i = \{s_1^i, s_2^i\}$ for $i = a, b$. Let s_1^a and s_1^b be null for Ann and let s_2^a and s_2^b be null for Bob. Define f by $f_a(s_1^a) = q_a^*, f_a(s_2^a) = p_a^*, f_b(s_1^b) = p_b^*$ and $f_b(s_2^b) = q_b^*$. Observe that in f Ann believes she always plays p_a^* and Bob believes that he always plays p_b^*, so f is $*$-equivalent to p^*. Next observe that both players are best replying: Ann believes that Bob always plays q_b^* which was picked such that p_a^* is a best reply to q_b^* and conversely for Bob. So f is an Ew/oA in G and is $*$-equivalent to p. $\square$

The converse does not hold true: not every Ew/oA in a game $G = (I, A, \succsim)$ is $*$-equivalent to a KE. A simple variation of Example 14 illustrates this.

Example 16 Take the following normal form game between Ann and Bob $G = \{\{a,b\}, A, \succsim\}$ with

	b_1	b_2
a_1	3,0	1,2
a_2	0,4	0,1

The strategy profile (a_1, b_1) is not a KE. To see this suppose to the contrary that it would be a KE. This implies that there exists a set Q^a with $1 \in Q^a$ such that playing b_1 is a best reply for Bob. Since $1 \in Q^a$ we have that Bob's utility of playing b_1 is 0. On the other hand Bob's utility of playing b_2 is not smaller than 1 since Bob receives a utility of at least 1 whether Ann plays a_1 or a_2.

However, it is easy to find a standard extension without agreement such that (a_1, b_1) is $*$-equivalent to an equilibrium in that standard extension: To see this observe that the strategy profile constructed in example 15 is also an Ew/oA in this game.

Corollary 3 *Take a game $G' = (\{a,b\}, A, \succsim')$ with two players. Let p be a ME or a DWE of G'. Then p is $*$-equivalent to an Ew/oA of G'. The converse does not hold true.*[23]

Proof The proof follows from the observation that the set of all ME is a subset of the set of all DWE which in turn is a subset of the set of all KE. □

Theorems 1 and 3 imply that the difference between mixed strategy equilibria and the existing equilibrium concepts for games with ambiguity averse players is not so much owed to the relaxation of the assumption of ambiguity neutrality as it is owed to a relaxation of "basic agreement" (or common knowledge of rationality). However such disagreement on the events which might possibly happen is not sufficient to describe the set of all KE, DWE or ME. These equilibria are strict subsets of the equilibria of Ew/oA. Ambiguity aversion enters only insofar as that a player i would only take deviations from the opponent's actual

[23] The same applies to partially specified equilibria following Lehrer [14], every such equilibrium is a KE. The route to show that every Equilibrium under Ambiguity following Eichberger, Kelsey and Schipper [10] is also $*$-equivalent to an Ew/oA is similar but slightly different.

strategy p^*_{-i} into account if this deviation lowers the payoff of i. Optimistic deviations are not considered.

I do find it problematic that the KE and DWE concepts do not restrict the weight that an ambiguity averse player assigns to these actions which are "never" chosen by the opponent. In these concepts beliefs equilibrate freely. Some strategy profiles might only be supported by very different beliefs: in some cases Ann might need to believe that s_1 happens with probability close to 1 while Bob needs to believe that s_1 happens with probability close to 0 to support a strategy profile p as an equilibrium.

I suppressed one major aspects of ME in my definition of the 4 equilibrium concepts to make ME more comparable to the other notions of equilibrium. To be fair this aspect needs to be discussed here: Marinacci's definition of ME contains a parameter that describes the ambiguity level in a game. Beliefs are not freely equilibrating, to the contrary the gap between Ann's belief and Bob's actual strategy is determined by the parameter that describes the ambiguity level of the game. This parametrization imposes the necessary discipline to structure a limited deviation from "basic agreement".

Secondly it is unclear what the empirical predictions of KE (and DWE and ME) should be. In Theorem 3 as well as in Corollary 3 I argue that these concepts implicitly assume that players might disagree on the set of null events. To generate different equilibria there need to be some event E that is considered null by one player, say player 1, and non-null by the other player. Reconsider examples 14 and 15. In the equilibrium constructed there Ann plays a_2 in s_2 which she considers to be null. On the other hand Bob believes that s_2 might occur. What is an empiricist to do if he observes the action a_2 being played? If he uses KE, DWE and ME he can use the observation of a_2 being played to prove that the players are not playing the equilibrium under consideration, Ann herself believes that she never plays a_2. But why should the empiricist side with Ann on this matter. Bob considers it very much possible that a_2 is being played in equilibrium. Shouldn't the empiricist also take Bob's opinion into account? If he does take Bob seriously observing Ann play a_2 is consistent with the conjectured equilibrium. The absence of "basic agreement" makes this situation hard to interpret. It is unclear how an empiricist should approach a situation in which some players believe that s happens sometimes whereas others belief that s never happens.

Finally let me say that Theorem 1 can be used to justify Lo's concept - for the two player case. Lo's equilibrium concept is closest to the concept proposed here. The two main differences between Lo's concept and the concept proposed here are that Lo uses Gilboa and Schmeidler's representation of preferences to define equilibrium and that Lo does not allow for ambiguous act strategies. However, while Lo works with a particular representation of preferences over acts, he also provides some intuition that applies to the preferences over acts. Lo's justifies his equilibrium requirement that $a_i \in supp(q_i)$ for any $q_i \in Q_i$ implies that a_i maximizes $EU_i(p_i, q_{-i})$ for $i = a, b$ with an intuition that is similar to my "basic agreement". Lo's requirement could be restated as: any action that Bob plays in a state that is considered non-null by Ann has to be an optimal action for Bob. So Ann and Bob might as well agree on the set of null states. On the other hand there is no equivalent concept for "strategic independence" for two player games in Lo's work: such a requirement is rendered obsolete by the assumption that players cannot use subjective random devices to generate strategies. Theorem 1 can be interpreted as a generalization of Lo's work. I derive observational equivalence from a small and straightforward set of axioms that encompasses a wide set of preferences, Lo already derived a similar equivalence result in a more restrictive setup.

8 Games with More than Two Players

The definitions in this paper all apply to n-player games. The main result of this paper, Theorem 1, only pertains to 2 player games. Does this result extend to n-player games? In this section I will first provide an example that the answer is negative. A theory of games with more than two ambiguity averse players carries the potential to yield substantially different predictions from standard theory of mixed strategy equilibrium. I will then provide some reasons why a detailed study of this question lies beyond the scope of this paper. I will claim that the basic understanding of "common priors" and "independent strategies" in an environment without priors developed here does not suffice to tackle the case of n-players. A better grasp of these concepts is needed to fully understand the case of games with more than two players. The following example builds on Example 2.3 in Aumann [2].

Example 17 Take the following ambiguous game $G = (\{1, 2, 3\}, \Omega, A, \succsim)$. Let $\Omega_3 =$

$\{s, r\}$ and Ω_1 and Ω_2 be singletons. Player 3 is an expected utility maximizer that assigns probability $\frac{1}{2}$ to either state. The first two players' preferences can be represented by MEU functions following example 5 with $Q = [\frac{1}{4}, \frac{3}{4}]$ for both. Let the following matrix represent the action spaces and the payoffs of all pure strategy profiles.

l

	L	R
T	$0, 8, 0$	$3, 3, 3$
B	$1, 1, 1$	$0, 0, 0$

r

	L	R
T	$0, 0, 0$	$3, 3, 3$
B	$1, 1, 1$	$8, 0, 0$

The strategy profile f with $f_1 = T, f_2 = R$ and $f_3(r) = l, f_3(s) = r$ is a AAE of this game[24]. To see this observe that player 3 does not have an incentive to deviate as his utility is 3 no matter which of the two boxes he picks. Secondly player 1 and 2' utilities from playing all their possible actions (keeping the strategies of all other players fixed) can be calculated as:

$$u_1(T, f_{-1}) = \min_{q \in [\frac{1}{4}, \frac{3}{4}]} q \times 3 + (1 - q) \times 3 = 3$$

$$u_1(B, f_{-1}) = \min_{q \in [\frac{1}{4}, \frac{3}{4}]} q \times 0 + (1 - q) \times 8 = \frac{3}{4} \times 0 + \frac{1}{4} \times 8 = 2 < 3$$

$$u_2(R, f_{-2}) = \min_{q \in [\frac{1}{4}, \frac{3}{4}]} q \times 3 + (1 - q) \times 3 = 3$$

$$u_2(L, f_{-1}) = \min_{q \in [\frac{1}{4}, \frac{3}{4}]} q \times 8 + (1 - q) \times 0 = \frac{1}{4} \times 8 + \frac{3}{4} \times 0 = 2 < 3$$

Aumann [2] shows that the game in example 17 has no NE with (TRr) or (TRl) in its support. At the same time he shows that the game has a "correlated equilibrium"[25] in which the first two players play T, L. A necessary condition for the existence of such a "correlated equilibrium" is that player 1 assigns a higher probability to player 3 picking l than player r does. Aumann shows in particular that there is such an equilibrium if player 1 believes that player 3 chooses the left matrix with a probability of $\frac{3}{4}$ whereas player 2 believes that this probability is $\frac{1}{4}$. The set $Q = [\frac{1}{4}, \frac{3}{4}]$ used in the example, implies that the

[24] The preferences given here satisfy (MON), consequently the given profile is also an SAAE

[25] Aumann [2] uses the term "correltated equilibrium" to designate two different deviations from standard theory: 1. players can use correlation devices, 2. Players do not need to have common priors. The present correlated equilibrium only deviates with respect to the second criterion from standard theory.

two first players will use exactly these priors when calculating their respective *minimal* expected utility of strategy profile f in the above example.

So is a game theory with ambiguity averse players going to herald a revival of game theory without common priors? Is any NE without common priors observationally equivalent to an AAE with ambiguity averse players? Yes it is - in a very unsatisfactory way. To see this observe that the assumption of “basic agreement” only imposes that players agree on the set of non-null states. This assumption does not preclude a scenario in which all players are expected utility maximizers but do not have common priors. Such an equivalence result would not be driven by the players ambiguity aversion (in fact players would be assumed to be expected utility maximizers) but would instead be driven by the fact that the assumption of “basic agreement” is a very weak one. This assumption was strong enough for the purpose of the present paper: the observational equivalence result could be derived using only this weak assumption on the players agreement of beliefs. The condition of “basic agreement” would have to be strengthened considerably for a study of games with more than 2 players.

In a similar vein the a deeper understanding of the independence assumption is needed to understand n-player games. The above example assumes that no players are uncertain about the strategies of players 1 and 2. Consequently it is easy to see that Ω_J is independent of Ω_{-J} for all $J \subset \{1, 2, 3\}$. Matters look a little differently when players are uncertain about the strategies of multiple other players. These questions on “common priors” and “independent strategies” merit careful attention, they have to be solved before the study of games with more than two players can be continued.[26]

[26]Lo [16] covers games with more than 2 players. He assumes that preferences can be represented following Gilboa and Schmeider [11] (example 5). He solves the two questions by assuming Gilboa and Schmeidler’s representation of independent strategies and replacing the common priors assumption by the assumption that the belief sets of all players have to be equal. This is unsatisfactory as neither of these answers is axiomatically founded. Eichberger and Kelsey [9] acknowledge that independence and common priors matter for the context of games with more than 2 players. They do not attempt to tackle these questions in their article.

9 Conclusion

The first contribution of this paper is to provide a novel framework (or rather to revive Aumann's framework of 1974) to analyze strategic interactions between ambiguity averse agents. This framework allows me define an equilibrium notion for this context using the standard notion of Nash equilibrium.

The second main contribution is my proof that an outside observer cannot distinguish whether a game is played by two uncertainty averse players or two uncertainty neutral ones. The third main contribution concerns the the different predictions of alternative equilibrium concepts for uncertainty averse players. I show that they are not so much a result of the assumption of uncertainty aversion but rather a result on the players disagreements on the possible occurrence of all events in the state-space. Is there any hope for a manageable theory of games with uncertainty averse players that yields predictions that differ from standard theory?

For me, the answer is a clear yes. I see the following three avenues for future research. First we might want to give up on basic agreement. If this is the case then we should do so in a controlled manner. As already discussed in section 7 Marinacci [18] does exactly that. The advantage of his concept is that he parameterizes the uncertainty of players in a game. In the light of the present study such a parametrization seems very important as it allows us to gradually relax the condition of "basic agreement". Marinacci's approach allows us to find equilibrium predictions for ambiguity averse players that differ from the equilibrium predictions of mixed strategy equilibrium while retaining control over the gap between the player's actual strategies and other players beliefs on these strategies.[27] Marinacci's main contribution is a proof of existence of ME for any level of uncertainty. The concept has yet to prove its merits in applied studies.

Even if we insist on basic agreement a game theory with uncertainty averse players might yield observationally different results. Theorem 1 crucially depends on the assumption of Schmeidler's [23] uncertainty aversion (UA). I showed in example 11 that a game theory with ambiguity averse players who are modelled following Bewley [5] (example 11) can yield observationally different predictions from the standard theory of mixed strategy equilibrium.

[27]Eichberger and Kelsey [9] provide an alternative parametrization of the degree of uncertainty in a game.

Finally, I showed with example 17 that a game theory with more than 3 ambiguity averse players carries the potential to yield observationally different results from the theory of mixed strategy equilibrium. The development of such a theory has to be preceded by a deeper investigation of "independent strategies" and "common priors" in a context where there are no priors.

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