On the Optimal Number of Licensees for a Technology

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Abstract

A framework is proposed to analyze the sale of multiple licenses to use a cost-reducing technology. Firms differ in their ability to reduce their costs by purchasing a license. A purchaser imposes a negative externality on others. The payoff of each firm depends on the number and abilities of the licensees. The seller maximizes her revenue by optimally choosing the licensees. The optimal mechanism is determined both when each firm's ability to reduce its cost is publicly observable and when it is not. In the optimal mechanism, sometimes non-licensees make a payment to the seller.

1 Introduction

This paper considers the problem of a seller who has developed a cost-reducing technology and wants to earn revenue by selling multiple licenses to use the technology. The licenses are identical in the sense that they endow the licensee with the right to use the same technology. The buyers of the licenses are the firms in the relevant industry that compete in the product market.

The payoff of each firm depends on the marginal costs of production of every firm in the industry; there is no fixed cost of production. Each firm has access to an existing publicly available technology that allows it to produce at a marginal cost of one. The marginal cost of production of any firm that uses the new technology is at most one. However, it is assumed that different firms in the industry have different abilities to exploit the technology and, consequently, there is heterogeneity in the marginal costs of the firms that use the new technology. Each firm receives a one-dimensional signal that is a measure of the cost reduction that firm can achieve by using the new technology. A firm that receives a higher signal than another firm achieves a lower marginal cost than the other firm and hence has a higher payoff. If a firm cannot purchase a license to the new technology, it uses the existing technology and achieves a marginal cost of one. Furthermore, the payoff of a firm also depends negatively on the signal of any other licensee of a license. This has the implication that even though a firm obtains the user rights to the technology by purchasing just one license, it still has a positive value for the second license because, by purchasing the second license, it prevents a competitor from acquiring a license. Moreover, the payoff of a firm depends on the signals of the firms that purchase a license, not on how many licenses each of these firms has. Hence, given a profile of signals, the seller maximizes her revenue by optimally choosing the number of licensees. I provide a framework for the analysis of the optimal number of licensees. Because the payoff of a licensee depends on the nature of the product market, I also analyze the role of different product market factors in the determination of the optimal number of licensees.[1]

In the paper, I consider the simplified problem of determining the seller's revenue when the seller sells two identical licenses and there are three firms. The analysis can be extended to the sale of k licenses to n firms with its attendant combinatorial complexity. The payoffs of the firms are taken to have the same properties as the payoffs of firms that compete in quantities or prices using differentiated products. First, I determine the seller's revenue when the signal of each firm is publicly observable. I find that in the optimal selling mechanism

[1] Other situations in which special cases of the model apply are the sale of production rights in an industry (as in Dana and Spier (1994)), the sale of airport takeoff and landing rights (as in Gale (1994)), and the sale of franchise rights (as in Degraba and Postlewaite (1992)).

when the signal of each firm is publicly observable, the seller selects the allocation rule that maximizes the industry (gross) payoff. The optimal mechanism of the seller for the case in which all licensees have the same signal was solved in Kamien ((1992), pp. 348-352) and in Kamien, Oren, and Tauman (1992). I extend their analysis to the case in which licensees can have different signals.

Next, I analyze the case wherein the signal of each firm is its private information. In order to analyze the problem, I define an allocation rule as a specification of the licensees, and consider the allocation in a truth-telling equilibrium of a direct mechanism. The seller's problem is to choose the allocation rule that maximizes her revenue from the sale of two licenses, provided the allocation rule satisfies the incentive compatibility and individual rationality constraints. Therefore, I first derive the necessary and sufficient conditions for incentive compatibility. I then show that some well-known allocation rules satisfy the necessary and sufficient conditions for incentive compatibility. Because of the assumption of independence of signals, I prove that two selling procedures that have the same allocation rule and the same payoff of a firm with the worst possible signal yield the same revenue to the seller. This is known as the *revenue equivalence theorem,* first established in a simpler context by Riley and Samuelson (1981).

I then determine the seller's revenue in an incentive compatible mechanism (that is, in a mechanism in which the allocation rule satisfies the necessary and sufficient conditions for incentive compatibility). In order to determine the seller's revenue, I define the concept of *industry virtual payoff*, which is the analog, for the situation in which there are externalities, of the concept of virtual value (Myerson (1981), Bulow and Roberts (1989), and Bulow and Klemperer (1996)). The industry virtual payoff is the industry profit, less the information rents of the licensees; hence, the industry virtual payoff, given any profile of signals, depends on the allocation rule. The revenue of the seller, given any allocation rule, is the expected industry virtual payoff, less the product of the payoff of a firm with the lowest possible signal and the number of firms in the industry. In models of sales with externalities, the payoff of the firm with the lowest possible signal depends on the mechanism (Kamien (1992), Jehiel, Moldovanu, and Stachetti (1996) and (1999)). If the seller can credibly commit to punish a firm that refuses to participate in the mechanism by allocating a license to the other firms, then the payoff of the firm with the lowest possible signal is minimized and is independent of the equilibrium allocation. Hence, in the optimal mechanism, the seller chooses the allocation mechanism that maximizes the industry virtual payoff for any arbitrary profile of signals, and threatens to punish a firm that refuses to participate in the mechanism by allocating a license to the other firms. I determine the number of licensees in the optimal mechanism, both when the signal of each firm is publicly observable, and when the signal of each firm is its private information.

I then illustrate the role of several product market factors in the determination of the optimal number of licensees. First, I show that the presence of significant externalities along with private information may cause the seller to select a fewer number of licensees compared to the situation in which the signal of each firm was publicly observable. Next, I show that an increase in the magnitude of externalities may lead to a decrease in the expected number of licensees. Finally, I show that when the firms are likely to be more efficient users of the technology, then the expected number of licensees increases.

1.1 Other Related Literature

This paper is closely related to the literature on sales with externalities. One of the earliest analyses of sales of licenses in the presence of externalities is Katz and Shapiro (1986). Their analysis assumes that the signal of a firm is publicly observable, and that each firm can purchase at most one license. Another article in the same spirit is Hoppe, Jehiel, and Moldovanu (2004). I relax both of the assumptions mentioned above. There has been work on sale with externalities, in which the signal of each firm is its private information. Jehiel, Moldovanu, and Stachetti ((1996) and (1999)) show that the payoff of the firm with the worst signal is endogenous to the mechanism, when the licensee of the license imposes an externality on the others. I find such a result in my model. Other examples of articles that analyze sales with externalities are Jehiel and Moldovanu (2000), Moldovanu and Sela (2003), DasVarma (2003), Katzman and Rhodes-Kropf (2002) and Goeree (2003). However, unlike my paper, none of these papers consider multiple licenses.

There has also been some work on sales of multiple licenses in the presence of externalities. Jehiel and Moldovanu (2001 and 2004) show the impossibility of implementing efficient allocations when the signals are multi-dimensional. In my paper, the signals are unidimensional. My paper is also closely related to Dana and Spier (1994). In their article, Dana and Spier consider the problem of auctioning production rights to firms in an industry. Dana and Spier assume that "a firm earns zero profits if it is not awarded a production right" (Dana and Spier (1994), p. 129) while I assume that if a firm does not purchase a license, its payoff is lower (and depends on the signals of the other licensees) compared to its payoff before the sale. In Dana and Spier, the seller (which is the government) maximizes social welfare which is a function of the revenue from the sale, profits of firms, and consumer surplus, while in my model, the seller maximizes her revenue from the sale.

Schmitz (2002) has analyzed revenue-maximizing allocations from a sale of multiple licenses when the signal of each firm is its private information. He has shown that the optimal number of licensees under private information can be two even when the optimal number of licensees under complete information is one. There are two major difference between Schmitz's model and mine. First, in Schmitz's model, each firm can win at most one license but I impose no such restriction. Second, Schmitz assumes that with positive probability a licensee is not able to commercially exploit the technology whereas, in my model, this is not the case. It can be shown that if this assumption is relaxed in Schmitz's model, it is always optimal for the seller to sell both licenses to one firm. In contrast, I do not make such an assumption but show that it can be optimal for the seller to choose multiple licensees. Brocas (2005) also analyzes a model of sale of k licenses in which each firm can win at most one license but her payoff functions are not motivated by standard models of market competition. In simultaneous but independent work, Figueroa and Skreta (2005) have considered a general model of sale of multiple objects in the presence of externalities whereas my model deals only with the sale of licenses. They derive the optimal mechanism both when the non-participation payoffs are own-type dependent and when they are own-type independent; I consider the optimal mechanism when the non-participation payoffs are own-type independent. In my model, the payoff function of firms have the properties of payoff functions that arise in equilibrium when firms compete in quantities or in prices using differentiated products; in Figueroa and Skreta the payoff functions are not derived from an oligopoly model. Because I model the nature of competition in the product market more explicitly, I show the relationship between the level of product differentiation and the optimal number of licensees. Moreover, because I assume that the marginal payoff of a firm is decreasing in a competitor's signal (as in many oligopoly models), I derive a different regularity condition from the one in Figueroa and Skreta. Further, I analyze the problem both when the signal of every firm is publicly observable and when they are private information. This allows me to highlight the effect of negative externalities alone and the effect of both negative externalities and private information.

There is another related literature that analyzes auctions of heterogeneous objects. Palfrey (1983), Armstrong (2000) and Avery and Hendershott (2000) analyze auctions of heterogeneous objects when buyers have an exogenously specified private value for each object. In contrast, in my model, the licenses are identical and the value of winning a license is determined only after the sale. In Palfrey's model, the seller, who is the owner of many heterogeneous objects, decides to partition the objects into separate bundles and sell each bundle separately using auctions. In comparison, in my model, the seller makes the bundling decision *ex post.* Palfrey finds that the desirability of selling all the objects as a bundle depends on the number of bidders. Armstrong (2000) and Avery and Hendershott (2000) extend Palfrey's analysis to determine the revenue-maximizing auction, under different assumptions about the buyers.[2]

I determine the seller's revenue when the signal of each firm is publicly observable in Section 2. In Sections 3 and 4, I describe the seller's mechanism for the sale of two licenses when the signal of each firm is its private information, and use the mechanism to determine the seller's revenue. The findings have been summarized in Section 5. Most of the proofs are in the Appendix (and the others are in the text).

2 Seller's Revenue under Complete Information

2.1 Model

Consider a seller who wants to maximize revenue by selling two licenses to use a cost-reducing technology (process innovation). There are three potential buyers for the licenses, labelled firms 1, 2, and 3. Initially, the firms produce with a marginal cost of 1; there are no fixed costs of production. Before the sale occurs, each firm i $(i = 1, 2, 3)$ receives a signal $s_i \in [0, 1]$ that determines its marginal cost of production c_i if it purchases a license to the technology, in the following way:

$$c_i = 1 - s_i. \tag{1}$$

Otherwise, the firm continues to produce at a marginal cost of 1. It is assumed that no firm exits the industry after the sale of licenses to the new technology.

The signals are assumed to be identically and independently distributed across firms, with $G(s)$ (resp., $g(s)$) as the distribution function (resp., density function). In this section, it is assumed that the signal of each firm is publicly observable. Let $s^3_{(k)}$ be the kth highest statistic from a sample of size 3 where the sample includes all the potential buyers for the licenses and assume that $s^3_{(k)}$ is distributed as $F^3_k(\cdot)$ with the associated density function $f^3_k(\cdot)$. A typical firm has two competitors and the expected payoff of a firm depends on the signals of the two competitors. Therefore, when a sample refers to a firm's competitors, I work with a sample size of 2 and in these cases, I replace 3 with 2 in the superscripts of the expressions above. In such a case, the sample consists of all the competitors of firm i. I denote the joint density of $s^j_{(1)}, \ldots, s^j_{(k)}$ by $f^j_{1\ldots k}\left(s_{(1)}, \ldots, s_{(k)}\right)$ where $j, k = 1, 2, 3$ and $k \leq j$.

A firm's payoff depends upon its own signal as well as on the signals drawn by the other firms. If firm i purchases a license and, if the firm with signal $s^2_{(p)}$ also purchases a license, then firm i 's payoff is given by $\pi(s_i; s^2_{(p)}, 0)$ where p is either 1 or 2. If firm i does not

[2] In Armstrong (2000), all buyers gain a positive payoff from winning any object. In contrast, in Avery and Hendershott (2000), only some buyers gain a positive payoff from winning any object, while the others gain a positive payoff from winning only some of the objects.

purchase a license and, if the firms with signals $s^2_{(1)}$ and $s^2_{(2)}$ purchase one license each, then firm i' s payoff is $\pi(0; s^2_{(1)}, s^2_{(2)})$. Finally, if firm i is not a licensee and the firm with signal $s^2_{(p)}$ purchases both licenses, then the payoff of firm i is $\pi(0; s^2_{(p)}, 0)$; $p = 1, 2$. The payoff function $\pi(\cdot; \cdot, \cdot)$ is also assumed to be symmetric across firms; that is, if the signals amongst any two firms are permuted, their payoffs are permuted as well.

The exact specification of the payoff function $\pi(\cdot; \cdot, \cdot)$ depends on the nature of competition among the firms and other market parameters such as the demand function. However, regardless of the functional form, I assume that the payoff function $\pi(\cdot; \cdot, \cdot)$ is twice continuously differentiable in all its arguments and has the following properties:

$$\pi_1(s_i; \cdot, \cdot) > 0 , \quad \pi_{11}(s_i; \cdot, \cdot) \geq 0, \tag{2}$$

$$\pi_j(\cdot; \cdot, \cdot) < 0, \; j = 2, 3, \tag{3}$$

$$\pi_{12}(\cdot; \cdot, \cdot) < 0, \quad \pi_{13}(\cdot; \cdot, \cdot) < 0, \;\; \pi_1(\cdot; \cdot, \cdot) > -\pi_2(\cdot; \cdot, \cdot), \tag{4}$$

where $\pi_j(\cdot; \cdot, \cdot)$ refers to the partial derivative of $\pi(\cdot; \cdot, \cdot)$ with respect to the jth argument. The inequalities in (2) imply that the payoff of firm i, when it purchases a license, is increasing and convex in its own signal. The inequality in (3) captures the effect of negative externalities in this model because it implies that when a competitor of firm i purchases a license, then the payoff of firm i is strictly decreasing in that competitor's signal. The first two inequalities in (4) imply that the marginal payoff of a firm's signal is decreasing in another firm's signal, while the third inequality in (4) implies that the payoff of a firm is more sensitive to its own signal than to another firm's signal.

Below, I illustrate the payoff function for different specifications about the nature of competition and show that the payoff function in each of these examples satisfies the properties described in (2), (3), and (4). In the first example, I consider a market in which the firms compete in quantities.

Example 1 (Sale of Licenses to use a Process Innovation): Suppose an independent research lab wants to sell two licenses to use a cost-reducing technology. There are three potential buyers, who are firms that compete in quantites producing differentiated products. The inverse demand function for firm i is given by:

$$p_i = \tau - q_i - \mu q^2_{(1)} - \mu q^2_{(2)}; \;\; \mu \in [0, 1] .$$

In this demand function, $q^2_{(j)}$ is the output of the firm with cost $1 - s^2_{(j)}$.[3] Also, μ is the

[3]Recall that $s^2_{(1)} \geq s^2_{(2)}$.

externality parameter that captures the effect of the other firms' decisions on firm i's payoff.[4] Given $\mu \in [0,1]$, the payoff function is given by:

$$\pi\left(s_i; s_{(1)}^2, s_{(2)}^2\right) = \left[\frac{(\tau-1)(2-\mu) + (2+\mu)s_i - \mu\left(s_{(1)}^2 + s_{(2)}^2\right)}{2(1+\mu)(2-\mu)}\right]^2. \tag{5}$$

Because I am considering the problem of allocating two licenses, at least one of s_i, $s_{(1)}^2$, or $s_{(2)}^2$ is 0. It can be verified that the payoff function (which is the reduced form equilibrium profit function) satisfies (2), (3), and (4). ■

Next, I consider another example in which the firms compete in prices instead of in quantities.

Example 2 This example is similar to the example above, except for the nature of competition. Suppose the three firms compete in prices selling differentiated products, and let the demand function be given by

$$q_i = \tau - p_i + \mu p_{(1)}^2 + \mu p_{(2)}^2; \;\; \mu \in [0, 0.5].$$

Analogous to the example above, $p_{(j)}^2$ is the price of the firm with cost $1 - s_{(j)}^2$ and μ is the externality parameter. If all the firms were able to use the new technology, the payoff function of firm i would be given by

$$\pi\left(s_i; s_{(1)}^2, s_{(2)}^2\right) = \left[\frac{(\tau+2\mu-1)(2+\mu) + (2-\mu-2\mu^2)s_i - \mu\left(s_{(1)}^2 + s_{(2)}^2\right)}{2(1+\mu)(2+\mu)}\right]^2.$$

It can be verified that the payoff function (which is the reduced form equilibrium profit function) satisfies (2), (3), and (4). ■

Dana and Spier (1994) consider the sale of production rights in an industry and their problem has a similar flavor to my paper. One important way in which their paper is different is the nature of the payoff function of firm i when firm i does not win a license. Dana and Spier assume that

$$\pi_2(0;\cdot,\cdot) = \pi_3(0;\cdot,\cdot) = 0, \tag{6}$$

while I assume that

$$\pi_2(0;\cdot,\cdot) < 0 \text{ and } \pi_3(0;\cdot,\cdot) < 0. \tag{7}$$

[4] Another interpretation of μ is that it captures the level of product differentiation in the industry.

A payoff function that satisfies (6) is said to have *fixed externalities,* while a payoff function that satisfies (7) is said to have *signal-dependent externalities.* The nature of the externality depends on the context of the problem. In Dana and Spier, a firm that does not win a production right cannot enter the industry and has a payoff of 0, thereby implying that the payoff function has fixed externalities. In my model, I assume that the number of firms in the industry is fixed and the seller can only determine the number of firms that can use its technology. Hence, in my model, I have signal-dependent externalities.

It is also important to note the relationship between firm i's payoff from purchasing a license when it has a signal of 0, and its payoff from not purchasing a license. First, consider the case that only one of firm i's competitors—the firm with signal $s^2_{(p)}$—purchases a license. Then firm i's payoff is $\pi(0; s^2_{(p)}, 0)$ when either it purchases a license or when the firm with signal $s^2_{(p)}$ purchases both the licenses. Second, in the case when both of firm i's competitors with signals $s^2_{(1)}$ and $s^2_{(2)}$ purchase a license, firm i's payoff is $\pi(0; s^2_{(1)}, s^2_{(2)})$. However, if firm i instead purchases a license by displacing one of its rivals (say the firm with signal $s^2_{(1)}$), then its payoff is $\pi(0; s^2_{(2)}, 0)$ which is different from $\pi(0; s^2_{(1)}, s^2_{(2)})$. It follows from (3) that firm i with signal 0 may be better off by purchasing a license if it displaces a competitor; even though it cannot obtain a reduction in its own marginal cost, it can prevent a rival from doing so.

2.2 Revenue

In the rest of this section, I determine the revenue of the seller in the optimal mechanism, when the signal of each firm is publicly observable.[5] In order to do so, I fix a profile of signals $\left(s^3_{(1)}, s^3_{(2)}, s^3_{(3)}\right)$. Each allocation is denoted by the vector $a = [a1, a2, a3]$ where ai is the number of licenses that the firm with signal $s^3_{(i)}$ purchases in equilibrium; $i = 1, 2, 3$.[6] Suppose the seller commits to allocate the two licenses according to a in exchange for a payment of $m^c_i(a)$ from firm i; $i = 1, 2, 3$. If firm i does not accept the seller's offer (that is, if firm i does not participate in the mechanism), the payoff of firm i is denoted by $\underline{\pi}^c_i$. Let $I^c(ai)$ be an indicator variable that takes a value 1 if $ai > 0$ and 0 otherwise. Therefore, the seller's problem is the following:

$$\max_a \sum_{i=1}^3 m^c_i(a)$$
$$\text{s.t. } \pi\left(I^c(ai)\, s_i; \cdot, \cdot\right) - m^c_i(a) \geq \underline{\pi}^c_i; \quad i = 1, 2, 3. \tag{8}$$

[5]The mechanism is an extension of the "chutzpah mechanism" (described in Kamien ((1992), pp. 348-352) or in Kamien, Oren, and Tauman (1992)) to the case in which licensees achieve different marginal costs of production with the new technology.

[6]In any allocation, $a1$, $a2$ and $a3$ are integers that must sum to 2, because the seller sells two licenses.

Observe that the seller can extract a higher payment from firm i if she can reduce the payoff of firm i from not participating in the mechanism. In the lemma below, $\underline{\pi}_i^c$ has been determined for every firm i in the optimal mechanism.

Lemma 1 *In the optimal mechanism,*

$$\underline{\pi}_i^c = \pi(0; s_{(1)}^2, s_{(2)}^2) \text{ for } i = 1, 2, 3, \tag{9}$$

where $s_{(1)}^2$ and $s_{(2)}^2$ are the signals of firm i's competitors such that $s_{(1)}^2 \geq s_{(2)}^2$.

Proof. See the Appendix. ■

If a firm does not participate in the mechanism, then its payoff depends on how the licenses are allocated in such an eventuality. The maximum punishment that the seller can credibly threaten to inflict on a non-participating firm is to allocate the licenses to both of its competitors, and the payoff of firm i in such a case is given by $\pi(0; s_{(1)}^2, s_{(2)}^2)$; note that $\pi(0; s_{(1)}^2, s_{(2)}^2)$ is less than $\pi(0; 0, 0)$ which is the profit a firm makes before the innovation is introduced. In the optimal mechanism, the seller makes each firm indifferent between participating and non-participating and hence we have the above lemma.

Under complete information, the seller can extract the entire surplus of the three firms and hence, in the optimal mechanism, all the constraints in (8) must be tight. Therefore, the seller's problem can be re-stated as

$$\max_a \left\{ \sum_{i=1}^3 \pi\left(I^c\left(ai\right) s_i; \cdot, \cdot\right) - \sum_{i=1}^3 \underline{\pi}_i^c \right\}, \tag{10}$$

and it follows from (9) that

$$\sum_{i=1}^3 \underline{\pi}_i^c = \pi(0; s_{(2)}^3, s_{(3)}^3) + \pi(0; s_{(1)}^3, s_{(3)}^3) + \pi(0; s_{(1)}^3, s_{(2)}^3). \tag{11}$$

Notice that $\sum_{i=1}^3 \pi\left(I^c\left(ai\right) s_i; \cdot, \cdot\right)$ is the industry gross payoff from the mechanism. Hence, the seller's optimal allocation a^* maximizes the industry gross payoff. This is stated formally in the following proposition.

Proposition 1 *Suppose the signal of each firm is publicly observable. Then, in the optimal allocation a^*, the seller maximizes the industry gross payoff, that is,*

$$a^* \in \arg\max \sum_{i=1}^3 \pi\left(I^c\left(ai\right) s_i; \cdot, \cdot\right),$$

and the seller's revenue in the optimal mechanism is

$$\sum_{i=1}^{3} \pi\left(I^{c}\left(ai^{*}\right) s_{i}; \cdot, \cdot\right) - \sum_{i=1}^{3} \underline{\pi}_{i}^{c},$$

where $\sum_{i=1}^{3} \underline{\pi}_{i}^{c}$ *is given by (11).*

Proof. Follows from the discussion above. ■

In later sections, I show that when the signal of each firm is its private information, then the seller cannot extract all the surplus of the firms and hence the optimal allocation in such a case may not be the allocation that maximizes the industry gross payoff. In the presence of private information, the seller sometimes sells licenses to a smaller number of licensees compared to what she would have done if the signals were publicly observable. In order to determine the seller's revenue when the signal of each firm is its private information, in the section below I describe the seller's problem when the signal of each firm is its private information.

3 The Mechanism under Incomplete Information

From now on, unless otherwise mentioned, the signal of each firm is assumed to be its private information and the seller is assumed to have the power to credibly commit to a mechanism for the sale of two licenses. By the revelation principle, there is no loss of generality in restricting our attention to direct mechanisms. In a direct mechanism, the seller asks each firm to give a report about its signal, and implements some outcome depending on the profile of reports.

3.1 Direct Mechanism

Before I proceed to set up the direct mechanism, I define its building blocks. The first building block is the ***signal space*** of the firms. I denote it by Ω. Because the signal of each firm is distributed on the unit interval, $\Omega = [0,1] \times [0,1] \times [0,1]$.

The second building block of a direct mechanism is the payment rule and it is defined below:

A ***payment rule*** *is a mapping that specifies the payments of the firms as a function of the profile of reports. Let* $r = \{r_1, r_2, r_3\} \in \Omega$ be the reports of the three firms about their signals. Then, the payment rule $M(r)$ *is specified by the following vector:*

$$M(r) = \left(M_1(r),\ M_2(r),\ M_3(r)\right),$$

where $M_j(r)$ is the payment of firm j when the profile of reports is r. The third building block of a direct mechanism is the allocation rule which is specified below:

An (ex post) ***allocation rule*** *is a mapping that prescribes the distribution of licenses among the firms as a function of the firms' reports $r = \{r_1, r_2, r_3\} \in \Omega$. Let $q_j(r)$* be the number of licenses firm j obtains when the reported profile is r, where $j = 1, 2$, or 3. *The allocation rule $Q(r)$ is then defined to be the vector $Q(r) = (q_1(r), q_2(r), q_3(r))$. Because the seller commits to sell two licenses, the following relations must also hold:*

$$\text{For all } r \in \Omega,\ q_j(r) \in \{0, 1, 2\} \text{ and } \sum_{j=1}^{3} q_j(r) = 2; \qquad j = 1, 2, 3.$$

I restrict consideration to allocation rules that are symmetric and with one exception, ex post deterministic.[7] The exception is when $r_1 = r_2 = r_3$; in this case, because there are only two licenses to be allocated, the seller can allocate them in any random way that does not depend on the identites of the would-be licensees. The probability of such an event occuring in a truth-telling equilibrium is zero. For all other report profiles, the allocation rule is deterministic and, if r_i and r_j are permuted in the report profile for $i \neq j$, then $q_i(\cdot)$ and $q_j(\cdot)$ are permuted as well.[8]

I now define the direct mechanism as follows:

A ***direct mechanism*** *is the set $\Gamma = \{\ \Omega,\ Q(\cdot),\ M(\cdot)\}$.*

I denote the *ex ante* expected revenue of the seller when she commits to implement an allocation rule $Q(\cdot)$ by R_Q. The seller's problem is to choose the allocation rule that maximizes her expected revenue. In a symmetric allocation rule, all that matters for the *ex post* allocation is the signal of a firm and not its identity. Hence, from now on, only the non-decreasing permutation of the reports $\hat{r} \equiv \left(r^3_{(1)}, r^3_{(2)}, r^3_{(3)}\right)$ is considered, where $r^3_{(1)} \geq r^3_{(2)} \geq r^3_{(3)}$.

3.2 Expected Payoff of a Firm

A revenue-maximizing seller extracts the maximum possible amount from each firm, under the condition that each firm's signal is its private information and the seller has two licenses to sell. The amount that the seller extracts from each firm depends on the expected payoff of each firm, which in turn depends on the seller's allocation rule. In the rest of this section, I show the relationship between the seller's allocation rule and the expected payoff

[7] In other words, given any profile of reports, the allocation rule specifies the number of licenses each firm obtains, with certainty, in all but one case.

[8] It is possible to analyze the problem with any arbitrary allocation rule. However, for the sake of exposition, I have considered only symmetric allocation rules throughout the paper.

of a firm.

In order to do so, first notice that given any symmetric allocation rule Q, the signal space can be partitioned into six (mutually exclusive and exhaustive) sets

$$A(\hat{r}|Q) \equiv \{A1(\hat{r}|Q), A2(\hat{r}|Q), \ldots, A6(\hat{r}|Q)\}$$

such that each set maps an ordered profile of reports $\hat{r}$ to a particular *ex post* allocation of licenses. The allocation corresponding to each set is presented in the table below. As an example, suppose that under an allocation rule Q, a particular ordered profile of reports $\hat{r}$ belongs to the set $A6(\hat{r}|Q)$. Then, the firm reporting $r^3_{(1)}$ purchases two licenses and the others purchase nothing. Similarly, depending on other profile of reports, a different *ex post* allocation of licenses is obtained. *Also notice that any allocation rule is associated with a unique partition of the signal space and vice versa.* It is assumed that the seller can commit to allocate the licenses according to the allocation rule Q.

Set	Allocation of $r^3_{(1)}$	Allocation of $r^3_{(2)}$	Allocation of $r^3_{(3)}$
$A1(\hat{r}|Q)$	0	2	0
$A2(\hat{r}|Q)$	0	0	2
$A3(\hat{r}|Q)$	0	1	1
$A4(\hat{r}|Q)$	1	1	0
$A5(\hat{r}|Q)$	1	0	1
$A6(\hat{r}|Q)$	2	0	0

Table 1: Partition of the report space induced by an allocation rule

Next, in order to relate the allocation rule to firm i's expected payoff, I determine the *ex post* distribution of licenses from firm i's perspective, when the seller commits to the allocation rule Q, firm i reports r_i, and the other firms report truthfully. Let $s^2_{(1)}$ (resp., $s^2_{(2)}$) be the highest (resp., lowest) of the signals of firm i's competitors and denote

$$\hat{s}_{-i} = \left(s^2_{(1)}, s^2_{(2)}\right)$$

as the non-decreasing permutation of the signals of firm i's competitors. Then, given the profile of reports $(r_i, \hat{s}_{-i})$ and the partition $A(\hat{r}|Q)$, there also exists a partition

$$B(r_i) \equiv \{B1(r_i), \ldots, B6(r_i)\}$$

of the signal space of firm i's competitors, as a function of firm i's report r_i. Each subset of the partition $B(r_i)$, given r_i, is the set of ordered signals of firm i's competitors $\hat{s}_{-i}$ that result in the same allocation. In the table below, I list the allocation corresponding to each subset of the partition $B(r_i)$. For example, it follows from the table that $B3(r_i)$ is defined as follows:

$$B3(r_i) = \{\hat{s}_{-i} | \text{firm } i \text{ is not a licensee and its competitors purchase one license each}\}.$$

Notice that there are six subsets of $B(r_i)$ because there are six ways of allocating the two licenses.

	Number of licenses allocated to each firm		
Set	Firm i	Firm with signal $s^2_{(1)}$	Firm with signal $s^2_{(2)}$
$B1(r_i)$	0	2	0
$B2(r_i)$	0	0	2
$B3(r_i)$	0	1	1
$B4(r_i)$	1	1	0
$B5(r_i)$	1	0	1
$B6(r_i)$	2	0	0

Table 2: Allocation of Licenses as a function of firm i's report r_i and the partition $B(r_i)$

The relationship of the partition $B(r_i)$ and the partition $A(\hat{r}|Q)$ is explained in the Appendix. The partition $B(r_i)$ can be used to determine the probability $\Phi_k(r_i)$ that firm i obtains k licenses ($k = 0, 1, 2$) and its expected payoff conditional on obtaining k licenses $\Pi(r_i, s_i|k)$ when firm i reports r_i and the others report truthfully. Hence, I can determine the expected payoff of firm i from reporting r_i when its competitors report truthfully, and this is discussed below.

Let $\phi_{Bk}(r_i)$ be the probability that the ordered profile of signals of firm i's competitors belong to the set $Bk(r_i)$ when firm i reports its signal as r_i, the rivals report truthfully, and the seller commits to the allocation rule Q, that is,

$$\phi_{Bk}(r_i) \equiv P\{\hat{s}_{-i} \in Bk(r_i)\}; \quad k = 1, \ldots, 6.$$

The expected payment of firm i, denoted by $m_i(r_i)$, can be similarly defined when firm i reports r_i, the other firms report truthfully, and the seller commits to the allocation rule Q.[9] Therefore,

$$m_i(r_i) = \int_{\hat{s}_{-i}} M_i(r_i, \hat{s}_{-i}) f^2_{12}(\cdot) \, d\hat{s}_{-i}.$$

Using the definitions of $\phi_{Bk}(r_i)$ and $m_i(r_i)$, the interim expected payoff of firm i, denoted

[9] Note that, while I consider only symmetric allocation rules, I allow the payment rule to be asymmetric across firms.

by $V_{iQ}(r_i, s_i)$, is defined below, when firm i with a signal of s_i reports r_i, the others report truthfully, and the seller chooses Q.

$$\begin{aligned} V_{iQ}(r_i, s_i) = {} & \phi_{B1}(r_i)\, E\left\{\pi\left(0; s_{(1)}^2, 0\right) | \hat{s}_{-i} \in B1(r_i)\right\} \\ & + \phi_{B2}(r_i)\, E\left\{\pi\left(0; s_{(2)}^2, 0\right) | \hat{s}_{-i} \in B2(r_i)\right\} \\ & + \phi_{B3}(r_i)\, E\left\{\pi\left(0; s_{(1)}^2, s_{(2)}^2\right) | \hat{s}_{-i} \in B3(r_i)\right\} \\ & + \phi_{B4}(r_i)\, E\left\{\pi\left(s_i; s_{(1)}^2, 0\right) | \hat{s}_{-i} \in B4(r_i)\right\} \\ & + \phi_{B5}(r_i)\, E\left\{\pi\left(s_i; s_{(2)}^2, 0\right) | \hat{s}_{-i} \in B5(r_i)\right\} \\ & + \phi_{B6}(r_i)\, \pi(s_i; 0, 0) - m_i(r_i). \end{aligned} \tag{12}$$

As mentioned above, let $\Phi_k(r_i)$ be the probability that firm i purchases k licenses when it reports r_i, the other firms report truthfully, and the seller commits to the allocation rule Q. In particular, the probabilities $\Phi_k(r_i)$ for $k = 1, 2, 3$ are related to the probabilities $\phi_{Bj}(r_i)$ for $j = 1, 2, \ldots, 6$ as follows:

$$\begin{aligned} \Phi_0(r_i) & \equiv \phi_{B1}(r_i) + \phi_{B2}(r_i) + \phi_{B3}(r_i), \\ \Phi_1(r_i) & \equiv \phi_{B4}(r_i) + \phi_{B5}(r_i) \\ \text{and } \Phi_2(r_i) & \equiv \phi_{B6}(r_i). \end{aligned} \tag{13}$$

Observe that the probability of purchasing k licenses depends on firm i's reported signal r_i and not on its true signal s_i. Moreover, the expected gross payoff of firm i from purchasing k licenses (when its true signal is s_i and its reported signal is r_i) denoted by $\Pi(r_i, s_i|k)$ is described below for $k = 0, 1,$and 2.

$$\begin{aligned} \Pi(r_i, s_i|0) \equiv {} & \frac{\phi_{B1}(r_i)}{\Phi_0(r_i)} E\left\{\pi\left(0; s_{(1)}^2, 0\right) | \hat{s}_{-i} \in B1(r_i)\right\} \\ & + \frac{\phi_{B2}(r_i)}{\Phi_0(r_i)} E\left\{\pi\left(0; s_{(2)}^2, 0\right) | \hat{s}_{-i} \in B2(r_i)\right\} \\ & + \frac{\phi_{B3}(r_i)}{\Phi_0(r_i)} E\left\{\pi\left(0; s_{(1)}^2, s_{(2)}^2\right) | \hat{s}_{-i} \in B3(r_i)\right\}, \end{aligned} \tag{14}$$

$$\begin{aligned} \Pi(r_i, s_i|1) \equiv {} & \frac{\phi_{B4}(r_i)}{\Phi_1(r_i)} E\left\{\pi\left(s_i; s_{(1)}^2, 0\right) | \hat{s}_{-i} \in B4(r_i)\right\} \\ & + \frac{\phi_{B5}(r_i)}{\Phi_1(r_i)} E\left\{\pi\left(s_i; s_{(2)}^2, 0\right) | \hat{s}_{-i} \in B5(r_i)\right\}, \end{aligned} \tag{15}$$

and

$$\Pi\left(r_i, s_i|2\right) \equiv \pi\left(s_i; 0, 0\right). \tag{16}$$

Furthermore, $\Pi_j\left(r_i, s_i|k\right)$ denotes the partial derivative of $\Pi\left(r_i, s_i|k\right)$ with respect to the jth argument where j is either 1 or 2.

Remark: It is interesting to observe that, when the payoff function $\pi\left(\cdot;\cdot,\cdot\right)$ exhibits fixed externalities, the expected gross payoff of firm i, given that it has not won any license (denoted by $\Pi\left(r_i, s_i|0\right)$), is a constant and does not depend on either its reported signal or its true signal. However, in the presence of signal-dependent externalities, $\Pi\left(r_i, s_i|0\right)$ depends only on firm i's reported signal r_i and not on its true signal s_i. Moreover, firm i's expected gross payoff from purchasing one license (denoted by $\Pi\left(r_i, s_i|1\right)$), depends both on its reported signal and on its true signal. Finally, firm i's expected gross payoff from purchasing both licenses (denoted by $\Pi\left(r_i, s_i|2\right)$), depends only on its true signal s_i.

3.3 The Seller's Problem

I now state the seller's problem formally. In order to do so, I use (14), (15) and (16) to rewrite the interim expected payoff of firm i as given in (12) as follows:

$$V_{iQ}\left(r_i, s_i\right) = \Phi_0\left(r_i\right)\Pi\left(r_i, s_i|0\right) + \Phi_1\left(r_i\right)\Pi\left(r_i, s_i|1\right) + \Phi_2\left(r_i\right)\Pi\left(r_i, s_i|2\right) - m_i\left(r_i\right). \tag{17}$$

In the truth-telling equilibrium, it is a best response of firm i to report its signal truthfully, given that other firms report truthfully. This is known as *Bayesian Incentive Compatibility* and is defined formally below:[10]

The allocation and payment rule satisfy ***Bayesian Incentive Compatibility*** *(henceforth BIC) if, for every firm i,*

$$V_{iQ}\left(s_i, s_i\right) \geq V_{iQ}\left(r_i, s_i\right) \text{ for all } r_i \text{ and } s_i \in [0,1]. \tag{18}$$

Moreover, no firm can be forced to participate in the mechanism. This can be ensured if no firm becomes worse off participating in the mechanism than by staying out. This is known as *Individual Rationality.* Suppose that if firm i stays out of the mechanism, it gets a payoff of $\underline{\pi}_i$. Then, the Individual Rationality constraint is formally as follows.

The allocation and payment rule satisfy ***Individual Rationality*** *(henceforth IR) if, for every firm i,*

$$V_{iQ}\left(s_i, s_i\right) \geq \underline{\pi}_i. \tag{19}$$

[10] The term "Bayesian" has been used because the equilibrium concept used is Bayes-Nash. See Krishna (2002, p. 280).

I now define the seller's problem as follows:

$$\begin{aligned} &\text{Select } \{B1(\cdot), \ldots, B6(\cdot), m_1(\cdot), m_2(\cdot), m_3(\cdot)\} \text{ to} \\ &Max \sum_{i=1}^{3} \int_0^1 m_i(s_i)\, g(s_i)\, ds_i \text{ s.t. } BIC \text{ and } IR. \end{aligned} \tag{20}$$

The optimal mechanism solves the problem for all possible $\{Q, M_i\}$. In principle, this problem can be solved in two steps. First, fix the allocation rule $Q(\cdot)$ arbitrarily.[11] Then, determine $m_i(\cdot)$ to satisfy BIC and IR and use this function to obtain R_Q where

$$R_Q = \sum_{i=1}^{3} \int_0^1 m_i(s_i)\, g(s_i)\, ds_i. \tag{21}$$

In the second step, select $Q(\cdot)$ to maximize R_Q.

In the next section, I use incentive compatibility to narrow down the search to a subset of all possible allocation and payment rules. Then, I show that the expected payment function $m_i(\cdot)$, in an incentive compatible direct mechanism, is a function of the allocation rule and the equilibrium payoff of the firm with signal 0, given by $V_{iQ}(0,0)$. I then discuss how $V_{iQ}(0,0)$ is determined in the optimal mechanism.

4 Seller's Revenue under Incomplete Information

In this section, I determine the seller's revenue in an incentive compatible direct mechanism, as a function of the allocation rule. In order to do so, I first provide an alternative characterization of incentive compatibility in the following proposition:

Proposition 2 *The direct mechanism is incentive compatible if and only if*

$$V_{iQ}(s_i, s_i) = V_{iQ}(0,0) + \int_0^{s_i} [\Phi_1(s)\, \Pi_2(s, s|1) + \Phi_2(s)\, \Pi_2(s, s|2)]\, ds, \tag{22}$$

and

$$\Phi_1(r_i)\, \Pi_2(r_i, s_i|1) + \Phi_2(r_i)\, \Pi_2(r_i, s_i|2) \text{ is non-decreasing in } r_i \text{ for all } r_i \in [0,1]. \tag{23}$$

Proof. See the Extended Appendix. ■

[11] Recall that with a Q, we can associate a unique partition A and hence a resulting partition $B(\cdot)$.

From (22) it follows that the marginal change in the equilibrium payoff $V_{iQ}(s_i, s_i)$ with respect to the signal s_i is given by

$$\Phi_1(s_i)\,\Pi_2(s_i, s_i|1) + \Phi_2(s_i)\,\Pi_2(s_i, s_i|2). \tag{24}$$

Notice that under the assumptions of the model, the expression in (24) is positive. In addition, I also show that, under the assumptions of the model, the expression in (24) is non-decreasing in the signal s_i.

Corollary 1 *Suppose the payoff of a firm is strictly convex in its signal s_i. Then (23) implies that the marginal change in the equilibrium payoff (given in (24)) with respect to s_i is non-decreasing in s_i.*

Proof. See the Extended Appendix. ■

The above corollary and (24) implies that, in my model, the equilibrium payoff function $V_{iQ}(s_i, s_i)$ is positively sloped and convex in s_i. Hence, incentive compatibility ensures that if the IR constraint is satisfied for a firm with signal 0, then it is satisfied for a firm with any arbitrary signal.

4.1 Incentive Compatible Allocations

It is instructive at this point to consider the kind of allocations that satisfy (23) and hence are implementable. First, I consider allocations in which the number of licensees is known with certainty when the firms report their types. There are six possible allocation rules with such a feature and they are the following: *(i)* The firm with the highest report purchases both the licenses, *(ii)* the firm with the second highest report purchases both the licenses, *(iii)* the firm with the third highest report purchases both the licenses, *(iv)* the firms with the highest and second highest reports purchase a license each, *(v)* the firms with the second highest and third highest reports purchase a license each, and *(vi)* the firms with the highest and third highest reports purchase a license each. Suppose the seller commits to sell both the licenses to the firm with the highest report, that is, it commits to implement the first allocation rule mentioned above. In this case,

$$\begin{aligned}&\Phi_1(r_i)\,\Pi_2(r_i, s_i|1) + \Phi_2(r_i)\,\Pi_2(r_i, s_i|2)\\ = &\int_0^{r_i} \pi_1(s_i; 0, 0)\, f_1^2\left(s_{(1)}^2\right) ds_{(1)}^2\end{aligned}$$

and the above expression is non-decreasing in the report r_i. Hence, the allocation in which the firm with the highest report purchases both licenses is implementable. One can check that the allocations mentioned in (ii) and (iii) are not implementable. Next consider (iv), that is, the allocation in which the seller commits to sell one license each to the firms with the highest two reports. In this case,

$$\begin{aligned}&\Phi_1(r_i)\,\Pi_2(r_i,s_i|1)+\Phi_2(r_i)\,\Pi_2(r_i,s_i|2)\\ =&\int_0^{r_i}\int_{s^2_{(2)}}^1 \pi_1\left(s_i;s^2_{(1)},0\right) f_1^2\left(s^2_{(1)}|s^2_{(2)}\right) f_2^2\left(s^2_{(2)}\right) ds^2_{(1)}ds^2_{(2)}\end{aligned}$$

and this is also non-decreasing in r_i. Hence, the allocation in which the two firms with the highest signals purchase one license each is implementable. One can check that the allocations in (v) and (vi) are not implementable.

Next, I consider allocations that have the feature that the number of licensees is uncertain before the sale. There are potentially many allocation rules that have this feature. Below, I consider a particular class of such allocation rules. I call each rule in this class the *Non-decreasing cutoff (NDC) rule*. These allocation rules have the feature that if the reports $r^3_{(1)}$ and $r^3_{(2)}$ are "close" to each other, then the firms with the reports $r^3_{(1)}$ and $r^3_{(2)}$ purchase one license each; else, the firm with the report $r^3_{(1)}$ purchases both licenses. In particular, corresponding to each allocation Q, there is a non-decreasing function $\underline{s}(r;Q)\leq r$, such that if $r^3_{(2)}\geq \underline{s}\left(r^3_{(1)};Q\right)$, then $\left(r^3_{(1)},r^3_{(2)},r^3_{(3)}\right)\in A4(\hat{r}|Q)$ and the firms with the highest and second highest reports purchase one license each, while if $r^3_{(2)}<\underline{s}\left(r^3_{(1)};Q\right)$, then $\left(r^3_{(1)},r^3_{(2)},r^3_{(3)}\right)\in A6(\hat{r}|Q)$ and the firm with the highest report purchases both licenses.[12] Formally,

$$\left(r^3_{(1)},r^3_{(2)},r^3_{(3)}\right)\in\begin{cases} A4(\hat{r}|Q) & \text{if } 0\leq \max\left\{r^3_{(3)},\underline{s}\left(r^3_{(1)};Q\right)\right\}\leq r^3_{(2)}\leq r^3_{(1)}\leq 1,\\ A6(\hat{r}|Q) & \text{if } 0\leq r^3_{(3)}\leq r^3_{(2)}<\underline{s}\left(r^3_{(1)};Q\right)\leq r^3_{(1)}\leq 1.\end{cases} \tag{25}$$

I now determine the expected payoff of firm i under the NDC rule, when it reports r_i. It follows from (25) that under any allocation rule that belongs to the NDC class, if $r^2_{(1)}<\underline{s}(r_i;Q)$, then firm i purchases both licenses, and if $\underline{s}(r_i;Q)\leq r^2_{(1)}<r_i$, firm i purchases one license.[13] In order to specify firm i's allocation under the NDC rule in the case that $r^2_{(1)}$ is greater than r_i, define

$$\overline{s}(r_i;Q)\equiv\max\left\{r^2_{(1)}|\underline{s}\left(r^2_{(1)};Q\right)\leq r_i\right\}.$$

[12]The optimal allocation rule for fixed externalities belongs to this class. See Dana and Spier (1994) and Schmitz (2002).

[13]Observe that in this case firm i has the highest of the three signals.

Given firm i's report r_i, $\overline{s}(r_i;Q)$ is, by construction, the maximum value of $r^2_{(1)}$ such that firm i can purchase a license. The above statement implies that if $r^2_{(2)} < r_i < r^2_{(1)} \leq \overline{s}(r_i;Q)$, then firm i can purchase exactly one license while, if either $r^2_{(2)} > r_i$ or if $r^2_{(1)} > \overline{s}(r_i;Q)$, then firm i cannot purchase any license. Observe that $\overline{s}(r_i;Q)$ is non-decreasing in r_i because of the fact that $\underline{s}(r_i;Q)$ is non-decreasing in r_i.

Suppose firm i reports r_i and the other firms report truthfully. Under an allocation rule that belongs to the NDC class described above,

$$\Phi_1(r_i)\,\Pi(r_i,s_i|1) = \int_{\underline{s}(r_i;Q)}^{\overline{s}(r_i;Q)} \int_0^{\min\left\{r_i,s^2_{(1)}\right\}} \pi\left(s_i;s^2_{(1)},0\right) f^2_{12}\left(s^2_{(1)},s^2_{(2)}\right) ds^2_{(2)} ds^2_{(1)} \tag{26}$$

and

$$\Phi_2(r_i)\,\Pi(r_i,s_i|2) = \pi(s_i;0,0) \int_0^{\underline{s}(r_i;Q)} \int_0^{s^2_{(1)}} f^2_{12}\left(s^2_{(1)},s^2_{(2)}\right) ds^2_{(2)} ds^2_{(1)}. \tag{27}$$

I now check whether this class of allocations satisfy (23). Notice that, from (26) and (27),

$$\begin{aligned} & \frac{\partial}{\partial r_i}\left\{\Phi_1(r_i)\,\Pi_2(r_i,s_i|1) + \Phi_2(r_i)\,\Pi_2(r_i,s_i|2)\right\} \\ = \; & \overline{s}'(r_i;Q)\,\pi_1(s_i;\overline{s}(r_i;Q),0)\,F^2_2\left(r_i|s^2_{(1)} = \overline{s}(r_i;Q)\right) f^2_1(\overline{s}(r_i;Q)) \\ & + \underline{s}'(r_i;Q)\left\{\int_0^{\underline{s}(r_i;Q)} -\pi_{12}(s_i;s,0)\,ds\right\} f^2_1(\underline{s}(r_i;Q)). \end{aligned} \tag{28}$$

It follows that, given (2) and (4), an allocation rule in this class satisfies (23), and is hence implementable.

4.2 Revenue

I now determine the expected payment of a firm in an incentive compatible direct mechanism. Below, I prove a version of the revenue equivalence theorem by showing that the expected payments of firm i with signal s_i in two mechanisms that have the same allocation rule[14] and the same net payoff for a firm with signal 0 are equal.

I define $\alpha_k(r_i)$ for $k = 1,2,\ldots,6$ as follows:

$$\alpha_k(r_i) = \begin{cases} E\left\{\pi(\cdot;\cdot,\cdot)\,|\hat{s}_{-i} \in Bk(r_i)\right\} \text{ if } \hat{s}_{-i} \in Bk(r_i), \\ 0 \text{ if } \hat{s}_{-i} \notin Bk(r_i). \end{cases}$$

In particular, $\alpha_k(r_i)$ is the expected gross payoff to firm i when it reports a signal r_i, the

[14] Recall that an allocation rule can be associated with one and only one partition $\{B1(r_i),\ldots,B6(r_i)\}$ of the signal space of buyer i's competitors.

others report truthfully and the profile of reports belong to the set $Bk(r_i)$ for $k = 1, 2, \ldots, 6$. Observe that by construction, the following equality must hold:

$$V_{iQ}(s_i, s_i) = \sum_{k=1}^{6} \alpha_k(s_i) - m_i(s_i). \tag{29}$$

Analogously, I define $\beta_k(r_i, s_i)$ for $k = 4, 5$,or 6 as follows:

$$\beta_k(r_i, s_i) = \begin{cases} E\{\pi_1(\cdot;\cdot,\cdot)\,|\hat{s}_{-i} \in Bk(r_i)\} \text{ if } \hat{s}_{-i} \in Bk(r_i), \\ 0 \text{ if } \hat{s}_{-i} \notin Bk(r_i). \end{cases}$$

It follows from the above definition that $\beta_k(r_i, s_i)$ is the expected value of the marginal gross payoff to firm i when it reports r_i, the others report truthfully and the profile of reports belong to the set $Bk(r_i)$ for $k = 1, 2, \ldots, 6$. The explicit forms of $\alpha_k(r_i)$ and $\beta_k(r_i, s_i)$ are presented in the Appendix. The expected payment of a firm in an incentive compatible direct mechanism is presented in the following proposition.

Proposition 3 *In the truth-telling equilibrium of an incentive compatible and individually rational direct mechanism in which a firm with signal* 0 *obtains a net payoff of* $V_Q(0,0)$*, the expected payment of firm* i *with signal* s_i *is given by*

$$m(s_i) = \sum_{k=1}^{6} \left[\alpha_k(s_i) - \int_0^{s_i} \beta_k(s,s)\,ds\right] - V_Q(0,0) \text{ provided } V_Q(0,0) \geq \underline{\pi}. \tag{30}$$

Proof. See the Appendix. ■

The above proposition shows that the expected payment of a firm in two mechanisms are the same whenever these mechanisms have the same partition $\{Bk(r_i)\}_{k=1}^{6}$, and the same net payoff of the firm with signal 0. Also, notice that, under the condition that a firm with signal 0 earns a payoff of $V_Q(0,0)$, the equilibrium payment function $m_i(\cdot)$ is the same for all the firms. Consequently, the subscript i from the expected payment function $m_i(s_i)$ has been dropped.

I now use Proposition 3 to determine the revenue of the seller in the truth-telling equilibrium of an incentive compatible direct mechanism. Below, I define the *industry virtual payoff* and show that the *ex ante* expected revenue of the seller is the expected value of the industry virtual payoff. The industry virtual payoff depends on the *ex post* allocation rule and hence, the seller maximizes her *ex ante* expected revenue by choosing the allocation rule that maximizes the expected industry virtual payoff.

Definition: Given the profile of signals $\left(s^3_{(1)}, s^3_{(2)}, s^3_{(3)}\right)$ and the allocation rule Q, such that

$$\left(s^3_{(1)}, s^3_{(2)}, s^3_{(3)}\right) \in Ak\left(\hat{r}|Q\right); \, k = 1, \ldots, 6,$$

the industry virtual payoff, denoted by $\lambda_{Ak(\hat{r}|Q)}\left(s^3_{(1)}, s^3_{(2)}, s^3_{(3)}\right)$, is given by

$$\begin{aligned} & \lambda_{Ak(\hat{r}|Q)}\left(s^3_{(1)}, s^3_{(2)}, s^3_{(3)}\right) \\ = & \textstyle\sum_{j=0}^{3}\{\pi\left(\cdot;\cdot,\cdot\right) - I_w\left(s^3_{(j)}\right)\dfrac{1-G\left(s^3_{(j)}\right)}{g\left(s^3_{(j)}\right)}\pi_1\left(\cdot;\cdot,\cdot\right) | \left(s^3_{(1)}, s^3_{(2)}, s^3_{(3)}\right) \in Ak\left(\hat{r}|Q\right)\}; \\ k = & 1, \ldots, 6, \end{aligned} \tag{31}$$

where $I_w\left(s^3_{(j)}\right)$ is an indicator variable that takes value 1 if the firm with type $s^3_{(j)}$ purchases a license.

As an example, suppose the profile of signals $\left(s^3_{(1)}, s^3_{(2)}, s^3_{(3)}\right)$ belongs to the set $A4\left(\hat{r}|Q\right)$ under the seller's allocation rule. Then,

$$\begin{aligned} \lambda_{A4(\hat{r}|Q)}\left(s^3_{(1)}, s^3_{(2)}, s^3_{(3)}\right) &= \pi\left(s^3_{(1)}; s^3_{(2)}, 0\right) + \pi\left(s^3_{(2)}; s^3_{(1)}, 0\right) + \pi\left(0; s^3_{(1)}, s^3_{(2)}\right) \\ &- \frac{1-G\left(s^3_{(1)}\right)}{g\left(s^3_{(1)}\right)}\pi_1\left(s^3_{(1)}; s^3_{(2)}, 0\right) - \frac{1-G\left(s^3_{(2)}\right)}{g\left(s^3_{(2)}\right)}\pi_1\left(s^3_{(2)}; s^3_{(1)}, 0\right). \end{aligned}$$

The industry virtual payoff is the gross industry payoff less the information rents of the licensees. The information rent depends on the distribution function of the signals and the signals of the licensees. It also follows from (4) that $\pi_{12}\left(\cdot;\cdot,\cdot\right) < 0$ and hence, the information rent of a licensee is non-increasing in the signal of another licensee. Below, I use (30) and (31) to determine the revenue of the seller in the truth-telling equilibrium of any incentive compatible direct mechanism.

Proposition 4 *The ex ante expected revenue of the seller in the truth-telling equilibrium of an incentive compatible direct mechanism is*

$$R_Q = \int_0^1 \int_0^{s^3_{(1)}} \int_0^{s^3_{(2)}} \lambda_{Ak(\hat{r}|Q)}\left(s^3_{(1)}, s^3_{(2)}, s^3_{(3)}\right) f^3_{123}\left(\cdot\right) ds^3_{(3)} ds^3_{(2)} ds^3_{(1)} - 3V_Q\left(0,0\right), \tag{32}$$

where

$$\left(s^3_{(1)}, s^3_{(2)}, s^3_{(3)}\right) \in Ak\left(\hat{r}|Q\right); \, k = 1, \ldots, 6.$$

Proof. See the Appendix. ■

This proposition states that the revenue of the seller in the truth-telling equilibrium of a direct mechanism is the expected industry virtual payoff, less the product of the number of firms and the payoff of the firm with signal zero. Notice that the industry virtual payoff in (32) given any profile of signals depends on the seller's allocation rule.

4.3 Optimal Allocation

The allocation rule that maximizes (32) is defined to be the optimal allocation rule and the associated mechanism is said to be the optimal mechanism. Below, I use Proposition 4 to determine the optimal mechanism when the payoff function exhibits signal-dependent externalities. It follows from (32) that in the optimal mechanism, the expected industry virtual payoff is maximized and the payoff of the firm with signal zero is minimized. In order to describe the allocation that maximizes the expected industry virtual payoff, I define

$$\lambda^* \left(s^3_{(1)}, s^3_{(2)}, s^3_{(3)}\right) = \max \left\{\lambda_{A1(\hat{r}|Q)} (\cdot) , \ldots , \lambda_{A6(\hat{r}|Q)} (\cdot)\right\} .$$

Further, one has to check that the allocation that maximizes the expected industry virtual payoff satisfies the conditions for incentive compatibility given by (22) and (23). Notice that the expression in (32) relies on (22) only and therefore, there is no guarantee that the optimal allocation derived by maximizing the expression in (32) satisfies (23). In models of sales without externalities, this problem is usually solved by assuming a regularity condition—in such cases, the regularity condition states that the virtual value is an increasing function of the signal.

In my model, I can solve the problem using an appropriate regularity condition and this is described below. I define a problem to be *regular* if the following conditions are true: *(i)* Given any profile of signals $\left(s^3_{(1)}, s^3_{(2)}, s^3_{(3)}\right)$, either $\lambda^* (\cdot) = \lambda_{A6(\hat{r}|Q)} (\cdot)$ or $\lambda^* (\cdot) = \lambda_{A4(\hat{r}|Q)} (\cdot)$; hence, in the optimal allocation, the seller either sells both the licenses to the firm with the highest signal (when $\lambda^* (\cdot) = \lambda_{A6(\hat{r}|Q)} (\cdot)$) or to the two firms with the two highest signals (when $\lambda^* (\cdot) = \lambda_{A4(\hat{r}|Q)} (\cdot)$), *(ii)* The allocation induced by $\lambda^* (\cdot)$ belongs to the NDC class. Notice that if the design problem is regular, then the allocation that maximizes the industry virtual payoff is incentive compatible, because allocations that belong to the NDC class satisfy (23).[15]

In the proposition below, it has been assumed that if one of the firms decides to stay out of the mechanism, then the seller can credibly commit to allocate a license to each of the firms who participate in the mechanism.[16] Under such a commitment, the payoff to a firm

[15] Figueroa and Skreta (2005) have a different regularity condition because they do not assume that $\pi_{12} (\cdot; \cdot, \cdot) < 0$.

[16] Such a commitment is similar in spirit to Jehiel, Moldovanu, and Stachetti ((1996), p. 820) and in Kamien, Oren, and Tauman (1992). A description of the mechanism by Kamien et al. is also available at

that decides not to participate in the mechanism is

$$\underline{\pi} = \int_0^1 \int_0^{s^2_{(1)}} \pi\left(0; s^2_{(1)}, s^2_{(2)}\right) f^2_{12}(\cdot)\, ds^2_{(2)} ds^2_{(1)}. \tag{33}$$

The value of $\underline{\pi}$ in (33) is used in the description of the optimal mechanism in Proposition 5.

Proposition 5 *Suppose that the design problem is regular, and if a firm does not participate in the mechanism, the seller can credibly commit to allocate one license each to the two other firms. Moreover, suppose that the payoff function exhibits signal-dependent externalities. Then the revenue of the seller in the truth-telling equilibrium of the optimal mechanism is given by*

$$R^*_Q = \int_0^1 \int_0^{s^3_{(1)}} \int_0^{s^3_{(2)}} \lambda^*\left(s^3_{(1)}, s^3_{(2)}, s^3_{(3)}\right) f^3_{123}(\cdot)\, ds^3_{(3)} ds^3_{(2)} ds^3_{(1)} - 3\underline{\pi} \tag{34}$$

where $\underline{\pi}$ is defined by (33).

Proof. See the Appendix. ■

The above proposition states that in a regular problem, the seller can achieve her optimal revenue by choosing the allocation rule that maximizes the industry virtual payoff for every profile of reports. It follows from the definition of the industry virtual payoff that the following three factors affect the seller's revenue when she selects one licensee (the firm with the highest signal) instead of two licensees (the firms with the two highest signals):

1. The industry gross profit may increase or decrease, and hence, the effect of this factor on the seller's revenue is ambiguous. In the examples below, the industry gross profit is always maximized by choosing two licensees.

2. If the seller selects only one licensee, then the firm with the second highest signal cannot earn any information rent and this increases the revenue of the seller. Notice that this factor will be present even in a model with no externalities; however, the magnitude of this effect depends on the externality parameter. This factor reduces the number of licensees.

3. If the seller selects only one licensee, then the information rent of the firm with the highest signal increases because $\pi_{12}(\cdot;\cdot,\cdot) < 0$. This factor therefore reduces the revenue of the seller. Notice that this factor emerges only in the presence of externalities.

The choice of the optimal number of licensees therefore depends on the relative strength of each of the above-mentioned factors which in turn depends on the nature of the product

Kamien ((1992), p. 348).

market. Below, I show the role of several channels through which the product market influences the optimal number of licensees.

4.4 Role of the product market in determining the optimal allocation

In order to show the role of several product market factors in determining the optimal number of licensees, I need to make more explicit assumptions about the product market. For the purpose of the discussion below, I consider the scenario described in Example 1 when $\tau = 4$. In the example, firms compete in quantites and a winner imposes a higher level of externalities on the others when the level of product differentiation is low.

Remark 1 *Significant externalities may cause the seller to select a fewer number of licensees in the presence of private information compared to what she would have done if firms had no private information.*

Suppose the signals follow the uniform distribution. In the table below, I present the optimal number of licensees when the profile of signals is

$$\left(s^3_{(1)}, s^3_{(2)}, s^3_{(3)}\right) = (0.9, 0.8, 0.3).$$

	Externality Parameter		
	$\mu = 0$	$\mu = 0.5$	$\mu = 1$
Signals are Publicly Observable	2	2	2
The Signal of a Firm is its Private Information	2	2	1

Table 3: The optimal number of licensees when the profile of signals is 0.9, 0.8, and 0.3, as a function of the externality parameter

Notice that the optimal number of licensees is always two when the signal of each firm is publicly observable. Furthermore, the optimal number of licensees remains at two even when the signal of each firm is its private information, as long as the externality parameter is 0 or 0.5. However, when the externality parameter is 1, the seller selects only one licensee in the optimal allocation when the signal of each firm is its private information. In this example, private information by itself does not induce the seller to select a fewer number of licensees. Instead, private information combined with the presence of significant externalities induces the seller to sell licenses to a fewer number of firms.

Remark 2 *When firms have private information, a higher level of externality leads to a decrease in the expected number of licensees.*

Consider the scenario described in the previous remark. For each possible profile of signals, I compute the optimal number of licensees and use this to compute the expected number of licensees. In the table below, I show how the expected number of licensees vary with the externality parameter (level of product differentiation in the industry). Observe that an

	Externality Parameter		
	$\mu = 0$	$\mu = 0.5$	$\mu = 1$
Expected number of licensees	1.47	1.27	1.1

Table 4: The expected number of licensees as a function of the externality parameter

increase in the externality parameter (or, a reduction in the level of product differentiation) leads to a decrease in the expected number of licensees.

In the model, I assume that the signals of firms are independenly drawn from a common distribution $G(s)$. Below, I show that the nature of the distribution function $G(s)$ is an important determinant of the optimal number of licensees.

Remark 3 *Suppose $G_A(s)$ and $G_B(s)$ are two different distributions of signals such that $G_A(s)$ first-order stochastically dominates $G_B(s)$. Then, the expected number of licensees is higher under $G_A(s)$.*

To interpret the above remark, notice that when the signals are drawn from $G_A(s)$ instead of $G_B(s)$, each firm has a greater likelihood of drawing a higher signal. In other words, when signals are drawn from $G_A(s)$ instead of $G_B(s)$, each firm has a greater likelihood of achieving a lower cost of production. Therefore, when firms are likely to be more efficient users of the technology, the expected number of licensees goes up.

For an illustration of the above remark, consider the Beta (2,1) distribution and the Uniform distribution; the Beta (2,1) distribution first-order stochastically dominates the Uniform distribution. In the table below, I present the expected number of licensees under each of these distributions.

Distribution	Externality Parameter		
	$\mu = 0$	$\mu = 0.5$	$\mu = 1$
Uniform	1.47	1.27	1.1
Beta (2,1)	1.71	1.50	1.25

Table 5: The expected number of licensees under the Uniform and the Beta(2,1) distributions

Notice that the expected number of licensees is higher under the Beta (2,1) distribution than under the Uniform distribution, for every value of the externality parameter and the question is why is this the case. It follows from the discussion above that when the seller increases the number of licensees from one to two, her revenue changes due to change in three factors- the change in industry profit and the change in information rents of the firms with the highest and second highest signals. However, in this example, when the seller increases the number of licensees from one to two, the changes in the information rents are not substantially different across the two distributions. In contrast, when the seller selects two licensees instead of one, the industry profit increases more substantially when the signals are drawn from the Beta (2,1) distribution and this explains why the expected number of licensees is higher when the signals are drawn from the Beta (2,1) distribution instead of the Uniform distribution.

5 Conclusion

This paper analyzes the seller's revenue from the sale of two identical licenses, both when the signal of each firm is publicly observable, and when the signal of each firm is its private information. It is assumed that if a firm refuses to participate in the mechanism, then the seller can credibly commit to allocate a license to its competitors. When the signal of each firm is publicly observable, the seller's optimal allocation is the one that maximizes the industry payoff. In contrast, when the signal of each firm is its private information, the seller selects the allocation rule that maximizes the industry virtual payoff. Such an allocation may or may not be different from the allocation that maximizes the industry payoff. We find that the presence of private information leads the seller to sometimes sell licenses to a smaller number of firms.

An important assumption in this article is that the private information of each firm is uni-dimensional. It follows from Jehiel and Moldovanu (2001) that an efficient allocation cannot be implemented when the signals are multi-dimensional. An interesting extension of this paper is to analyze revenue-maximizing allocations when the signals are multi-dimensional.

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Appendix

A Proof of Lemma 1

If firm i does not participate in the mechanism, then its payoff $\underline{\pi}_i^c$ depends on how the seller allocates the licenses in such an eventuality. Notice that, if firm i does not participate in the mechanism, then the seller can choose to allocate a license to each of its competitors, or a license to one of its competitors, or to not sell any license. Hence, firm i's payoff if it does not participate in the mechanism is either $\pi(0; s^2_{(1)}, s^2_{(2)})$, $\pi(0; s^2_{(1)}, 0)$, $\pi(0; s^2_{(2)}, 0)$, or $\pi(0; 0, 0)$. Next, notice that it follows from (2)–(4) that

$$\pi(0; s^2_{(1)}, s^2_{(2)}) = \min\left\{\pi(0; s^2_{(1)}, s^2_{(2)}), \pi(0; s^2_{(1)}, 0), \pi(0; s^2_{(2)}, 0), \pi(0; 0, 0)\right\}. \tag{35}$$

For example,

$$\begin{aligned} & \pi(0; s^2_{(1)}, s^2_{(2)}) - \pi(0; s^2_{(1)}, 0) \\ = & \int_0^{s^2_{(2)}} \pi_3 \left(0; s^2_{(1)}, s\right) ds \\ \leq & \ 0 \text{ because } \pi_3 \left(\cdot; \cdot, \cdot\right) < 0. \end{aligned}$$

Notice that (35) holds regardless of the seller's choice of the allocation rule a in the mechanism. It also follows from (8) that in the optimal mechanism, the payoff of every non-participating firm has to be minimized. Therefore, we obtain the result.

B Relationship between the Partition $A\left(\hat{r}|Q\right)$ and the Partition $B\left(r_i\right)$

Given a profile of reports, let a *rank-set pair* be the pair whose first element is the rank of firm i's report in the profile of reports, and whose second element is the subset of $A\left(\hat{r}|Q\right)$ that contains the profile of reports. For example, if firm i's report is the second highest among the three reports (that is, if $r_i = r^3_{(2)}$), and if the profile of reports $\left(r^3_{(1)}, r^3_{(2)}, r^3_{(3)}\right) = \left(s^2_{(1)}, r_i, s^2_{(2)}\right) \in A3\left(\hat{r}|Q\right)$, then the corresponding rank-set pair is $\left(2, A3\left(\hat{r}|Q\right)\right)$. Notice that, given a partition $A\left(\hat{r}|Q\right) = \left\{A1\left(\hat{r}|Q\right), \ldots, A6\left(\hat{r}|Q\right)\right\}$ and a profile of reports $\hat{r}$, each rank-set pair can be associated with a unique set $Bk\left(r_i\right)$; $k = 1, \ldots, 6$. However, each set $Bk\left(r_i\right)$; $k = 1, \ldots, 6$, can result from several rank-set pairs. For example, one can check that the set $B1\left(r_i\right)$ can be generated by three rank-set pairs $\left(1, A1\left(\hat{r}|Q\right)\right)$, $\left(2, A6\left(\hat{r}|Q\right)\right)$ and $\left(3, A6\left(\hat{r}|Q\right)\right)$. In the following table, I associate each set $Bk\left(r_i\right)$; $k = 1, \ldots, 6$, with the rank-set pairs that generate $Bk\left(r_i\right)$.

Subsets	Corresponding Rank-Set Pairs
$B1\left(r_i\right)$	$\left(1, A1\left(\hat{r}|Q\right)\right)$ or $\left(2, A6\left(\hat{r}|Q\right)\right)$ or $\left(3, A6\left(\hat{r}|Q\right)\right)$
$B2\left(r_i\right)$	$\left(1, A2\left(\hat{r}|Q\right)\right)$ or $\left(2, A2\left(\hat{r}|Q\right)\right)$ or $\left(3, A1\left(\hat{r}|Q\right)\right)$
$B3\left(r_i\right)$	$\left(1, A3\left(\hat{r}|Q\right)\right)$ or $\left(2, A5\left(\hat{r}|Q\right)\right)$ or $\left(3, A4\left(\hat{r}|Q\right)\right)$
$B4\left(r_i\right)$	$\left(1, A4\left(\hat{r}|Q\right)\right)$ or $\left(2, A4\left(\hat{r}|Q\right)\right)$ or $\left(3, A5\left(\hat{r}|Q\right)\right)$
$B5\left(r_i\right)$	$\left(1, A5\left(\hat{r}|Q\right)\right)$ or $\left(2, A3\left(\hat{r}|Q\right)\right)$ or $\left(3, A3\left(\hat{r}|Q\right)\right)$
$B6\left(r_i\right)$	$\left(1, A6\left(\hat{r}|Q\right)\right)$ or $\left(2, A1\left(\hat{r}|Q\right)\right)$ or $\left(3, A2\left(\hat{r}|Q\right)\right)$

Table 6: Derivation of the partition $B(r_i)$ from the partition A

B.1 The probability of the realization of the set $Bk(r_i)$; $k = 1, 2, \dots, 6$

Let the indicator function $I_{Bk}(r_i, \hat{s}_{-i})$ take the value 1 if the profile of reports is an element of $Bk(r_i)$, and let $I_{Bk}(r_i, \hat{s}_{-i})$ be 0 otherwise. Formally,

$$I_{Bk}(r_i, \hat{s}_{-i}) = \begin{cases} 1 & \text{if} \quad \hat{s}_{-i} \in Bk(r_i); \quad k = 1, 2, \dots, 6, \\ 0 & \text{otherwise.} \end{cases}$$

By definition, $\phi_{Bk}(r_i)$ is the probability that the ordered profile of signals of firm i's competitors belong to the set $Bk(r_i)$ when firm i reports its signal as r_i, the rivals report truthfully, and the seller commits to the allocation rule Q. The function $\phi_{Bk}(r_i)$ is related to the indicator function $I_{Bk}(r_i, \hat{s}_{-i})$ as follows:

$$\begin{aligned} \phi_{Bk}(r_i) &\equiv P\{\hat{s}_{-i} \in Bk(r_i)\}; \quad k = 1, \dots, 6 \\ &= \int_{\hat{s}_{-i}} I_{Bk}(r_i, \hat{s}_{-i}) f_{12}^2(\cdot)\, d\hat{s}_{-i}. \end{aligned}$$

C Definitions of $\alpha_k(s_i)$ and $\beta_k(r_i, s_i)$

The functions $\alpha_k(s_i)$; $k = 1, \dots, 6$, are defined as follows:

$$\alpha_1(r_i) = \int_{\hat{s}_{-i} \in B1(r_i)} \pi\left(0; s_{(1)}^2, 0\right) f_{12}^2(\cdot)\, d\hat{s}_{-i},$$

$$\alpha_2(r_i) = \int_{\hat{s}_{-i} \in B2(r_i)} \pi\left(0; s_{(2)}^2, 0\right) f_{12}^2(\cdot)\, d\hat{s}_{-i},$$

$$\alpha_3(r_i) = \int_{\hat{s}_{-i} \in B3(r_i)} \pi\left(0; s_{(1)}^2, s_{(2)}^2\right) f_{12}^2(\cdot)\, d\hat{s}_{-i},$$

$$\alpha_4(r_i) = \int_{\hat{s}_{-i} \in B4(r_i)} \pi\left(s_i; s_{(1)}^2, 0\right) f_{12}^2(\cdot)\, d\hat{s}_{-i},$$

$$\alpha_5(r_i) = \int_{\hat{s}_{-i} \in B5(r_i)} \pi\left(s_i; s_{(2)}^2, 0\right) f_{12}^2(\cdot)\, d\hat{s}_{-i},$$

$$\alpha_6(r_i) = \int_{\hat{s}_{-i} \in B6(r_i)} \pi(s_i; 0, 0)\, f_{12}^2(\cdot)\, d\hat{s}_{-i}.$$

Analogously, the functions $\beta_k(r_i, s_i)$; $k = 1, \dots, 6$, are defined as follows:

$$\beta_1(r_i, s_i) = \beta_2(r_i, s_i) = \beta_3(r_i, s_i) = 0,$$

$$\beta_4(r_i, s_i) = \int_{\hat{s}_{-i} \in B4(r_i)} \pi_1\left(s_i; s^2_{(1)}, 0\right) f^2_{12}(\cdot)\, d\hat{s}_{-i} ,$$

$$\beta_5(r_i, s_i) = \int_{\hat{s}_{-i} \in B5(r_i)} \pi_1\left(s_i; s^2_{(2)}, 0\right) f^2_{12}(\cdot)\, d\hat{s}_{-i},$$

$$\beta_6(r_i, s_i) = \int_{\hat{s}_{-i} \in B6(r_i)} \pi_1(s_i; 0, 0)\, f^2_{12}(\cdot)\, d\hat{s}_{-i}.$$

D Proof of Proposition 3

From (22), it follows that:

$$V_{iQ}(s_i, s_i) = V_{iQ}(0,0) + \int_0^{s_i} \sum_{k=1}^{6} \beta_k(s,s)\, ds. \tag{36}$$

Combining (29) and (36), I obtain:

$$m_i(s_i) = \sum_{k=1}^{6} \left[\alpha_k(s_i) - \int_0^{s_i} \beta_k(s,s)\, ds\right] - V_{iQ}(0,0).$$

E Proof of Proposition 4

Substituting (30) in (21), I obtain the following relation:

$$\begin{aligned} R_Q &= 3\int_0^1 \sum_{k=1}^{6} \left[\alpha_k(s_i) - \int_0^{s_i} \beta_k(s,s)\, ds\right] g(s_i)\, ds_i - 3V_Q(0,0) \\ &= 3\sum_{k=1}^{6} \left[\int_0^1 \alpha_k(s_i)\, g(s_i)\, ds_i - \int_0^1 \int_0^{s_i} \beta_k(s,s)\, g(s_i)\, ds ds_i\right] - 3V_Q(0,0). \end{aligned} \tag{37}$$

Furthermore, I integrate by parts the second expression in the right hand side of (37), and obtain the following expression for the seller's revenue:

$$\begin{aligned} R_Q &= 3\sum_{k=1}^{6} \int_0^1 \gamma_k(s_i)\, g(s_i)\, ds_i \; - 3V_Q(0,0) \\ \text{where } \gamma_k(s) &\equiv \alpha_k(s) - \beta_k(s,s)\, \frac{1-G(s)}{g(s)}; \quad k = 1, 2, \ldots, 6. \end{aligned} \tag{38}$$

I now simplify the expression in (38). For the sake of exposition, I only present the simplification for the case in which the seller commits to sell both the licenses to the firm with the highest signal for all possible profile of reports; one can use the same technique for any arbitrary allocation rule. Under such a commitment,

$$\left(s^3_{(1)}, s^3_{(2)}, s^3_{(3)}\right) \in A6\left(\hat{r}|Q\right) \text{ for all possible values of } \left(s^3_{(1)}, s^3_{(2)}, s^3_{(3)}\right).$$

Pick any arbitrary $s_i \in [0,1]$. Notice that if firm i with signal s_i has the highest signal among the three firms, then the set $B6\left(s_i\right)$ is realized. Moreover, if $s_i = s^3_{(2)}$ or $s^3_{(3)}$, then the set $B1\left(s_i\right)$ is realized. From (38), it follows that

$$R_Q = 3\sum_{k=1}^{6}\int_0^1 \gamma_k\left(s_i\right) g\left(s_i\right) ds_i \ - 3V_Q\left(0,0\right). \tag{39}$$

Therefore, I first evaluate the expression

$$\sum_{k=1}^{6}\gamma_k\left(s_i\right) = \gamma_6\left(s_i\right) + \gamma_1\left(s_i\right) + \gamma_1\left(s_i\right). \tag{40}$$

By expanding the expression in the right hand side of (39), I obtain the following:

$$\begin{aligned}\sum_{k=1}^{6}\gamma_k\left(s_i\right) \ &= \ \int_0^{s_i}\int_0^{s^2_{(1)}}\left\{\pi\left(s_i;0.0\right) - \frac{1-G\left(s_i\right)}{g\left(s_i\right)}\frac{\partial\pi\left(s_i;0.0\right)}{\partial s_i}\right\} f^2_{12}\left(\cdot\right) ds^2_{(2)} ds^2_{(1)} \\ &+ \int_{s_i}^1\int_0^{s_i}\pi\left(0;s^2_{(1)},0\right) f^2_{12}\left(\cdot\right) ds^2_{(2)} ds^2_{(1)} \\ &+ \int_{s_i}^1\int_{s_i}^{s^2_{(1)}}\pi\left(0;s^2_{(1)},0\right) f^2_{12}\left(\cdot\right) ds^2_{(2)} ds^2_{(1)}.\end{aligned} \tag{41}$$

Notice that, in the first term on the right hand side of (41), $s_i = s^3_{(1)}$, and hence,

$$s^2_{(1)} = s^3_{(2)} \text{ and } s^2_{(2)} = s^3_{(3)}.$$

Similarly, in the second term $s_i = s^3_{(2)}$, and in the third term, $s_i = s^3_{(3)}$. Also, the following relation holds:

$$3g\left(s_i\right) f^2_{12}\left(s_{(1)}, s_{(2)}\right) = f^3_{123}\left(s_i, s_{(1)}, s_{(2)}\right).$$

Therefore, I can re-write the first term (on the right hand side) of (41) as follows:

$$3\int_0^1 \gamma_6(s_i)\, g(s_i)\, ds_i$$

$$= 3\int_0^1 \int_0^{s_i} \int_0^{s^2_{(1)}} \left\{\pi(s_i; 0.0) - \frac{1-G(s_i)}{g(s_i)}\frac{\partial\pi(s_i; 0.0)}{\partial s_i}\right\} g(s_i)\, f^2_{12}(\cdot)\, ds^2_{(2)} ds^2_{(1)} ds_i$$

$$= 3\int_0^1 \int_0^{s^3_{(1)}} \int_0^{s^3_{(2)}} \left\{\pi\left(s^3_{(1)}; 0.0\right) - \frac{1-G\left(s^3_{(1)}\right)}{g\left(s^3_{(1)}\right)}\frac{\partial\pi\left(s^3_{(1)}; 0.0\right)}{\partial s^3_{(1)}}\right\} g\left(s^3_{(1)}\right) f^2_{12}(\cdot)\, ds^3_{(3)} ds^3_{(2)} ds^3_{(1)}$$

$$= \int_0^1 \int_0^{s^3_{(1)}} \int_0^{s^3_{(2)}} \left\{\pi\left(s^3_{(1)}; 0.0\right) - \frac{1-G\left(s^3_{(1)}\right)}{g\left(s^3_{(1)}\right)}\frac{\partial\pi\left(s^3_{(1)}; 0.0\right)}{\partial s^3_{(1)}}\right\} f^3_{123}(\cdot)\, ds^3_{(3)} ds^3_{(2)} ds^3_{(1)}.$$

Similarly, I can expand the other terms to obtain the seller's revenue as follows:

$$\begin{aligned} R_Q &= \int_0^1 \int_0^{s^3_{(1)}} \int_0^{s^3_{(2)}} \left\{\pi\left(s^3_{(1)}; 0, 0\right) + 2\pi\left(0; s^3_{(1)}, 0\right) - \frac{1-G\left(s^3_{(1)}\right)}{g\left(s^3_{(1)}\right)}\frac{\partial\pi\left(s^3_{(1)}; 0.0\right)}{\partial s^3_{(1)}}\right\} \\ &\quad \times f^3_{123}(\cdot)\, ds^3_{(3)} ds^3_{(2)} ds^3_{(1)} - 3V_Q(0,0) \\ &= \int_0^1 \int_0^{s^3_{(1)}} \int_0^{s^3_{(2)}} \lambda_{Ak(\hat{r}|Q)}\left(s^3_{(1)}, s^3_{(2)}, s^3_{(3)}\right) f^3_{123}(\cdot)\, ds^3_{(3)} ds^3_{(2)} ds^3_{(1)} - 3V_Q(0,0)\,. \end{aligned}$$

Analogously, the revenue of the seller under any arbitrary allocation rule can be determined.

F Proof of Proposition 5

It follows from inspection of (32) that in the optimal mechanism, the payoff of a firm with signal 0, given by $V_Q(0,0)$, has to be minimized, subject to

$$V_Q(0,0) \geq \underline{\pi}.$$

Hence, in the optimal mechanism, I must have

$$V_Q(0,0) = \underline{\pi}$$

where $\underline{\pi}$ is defined in (33). Moreover, by construction, the maximum value of

$$\int_0^1 \int_0^{s^3_{(1)}} \int_0^{s^3_{(2)}} \lambda_{Ak(\hat{r}|Q)}\left(s^3_{(1)}, s^3_{(2)}, s^3_{(3)}\right) f^3_{123}(\cdot)\, ds^3_{(3)} ds^3_{(2)} ds^3_{(1)} \tag{42}$$

is given by

$$\int_0^1 \int_0^{s^3_{(1)}} \int_0^{s^3_{(2)}} \lambda^* \left(s^3_{(1)}, s^3_{(2)}, s^3_{(3)}\right) f^3_{123} \left(\cdot\right) ds^3_{(3)} ds^3_{(2)} ds^3_{(1)}. \tag{43}$$

Notice that maximizing the expression in (42) and minimizing $V_Q(0,0)$ are independent of each other. Hence, I have the result.

EXTENDED APPENDIX NOT MEANT FOR PUBLICATION

G Proof of Proposition 2

I denote the payoff of firm i in the truth-telling equilibrium of the direct mechanism as follows:

$$\Psi_{iQ}(s_i) = V_{iQ}(s_i, s_i).$$

First, I prove the necessity part. Note that incentive compatibility implies that

$$\Psi_{iQ}(s_i) \geq V_{iQ}(r_i, s_i) \text{ for all } r_i,\ s_i \in [0,1]. \tag{44}$$

Moreover, I can re-write $V_{iQ}(r_i, s_i)$ as follows:

$$V_{iQ}(r_i, s_i) = \Psi_{iQ}(r_i) + \sum_{k=1}^{2} \Phi_k(r_i) [\Pi(r_i, s_i|k) - \Pi(r_i, r_i|k)] \tag{45}$$

$$= \Psi_{iQ}(r_i) + \int_{r_i}^{s_i} \sum_{k=1}^{2} \Phi_k(r_i) \Pi_2(r_i, s|k)\, ds. \tag{46}$$

Therefore, from (44) and (46), I find that incentive compatibility implies the following condition:

$$\Psi_{iQ}(s_i) - \Psi_{iQ}(r_i) \geq \int_{r_i}^{s_i} \sum_{k=1}^{2} \Phi_k(r_i) \Pi_2(r_i, s|k)\, ds, \tag{47}$$

and, by interchanging the variables, I find that,

$$\Psi_{iQ}(r_i) - \Psi_{iQ}(s_i) \geq \int_{r_i}^{s_i} \sum_{k=1}^{2} \Phi_k(s_i) \Pi_2(s_i, s|k)\, ds. \tag{48}$$

Combining (47) and (48), I obtain the following inequality:

$$\int_{r_i}^{s_i} \sum_{k=1}^{2} \Phi_k(s_i) \Pi_2(s_i, s|k)\, ds \geq \Psi_{iQ}(s_i) - \Psi_{iQ}(r_i) \geq \int_{r_i}^{s_i} \sum_{k=1}^{2} \Phi_k(r_i) \Pi_2(r_i, s|k)\, ds. \tag{49}$$

Notice that, the above inequality implies (23). Next, I divide all the terms in (49) and let $r_i \to s_i$ to obtain the result that

$$\Psi'_{iQ}(s_i) = \sum_{k=1}^{2} \Phi_k(s_i) \Pi_2(s_i, s_i|k) \tag{50}$$

and hence,

$$\Psi_{iQ}(s_i) = \Psi_{iQ}(0) + \int_0^{s_i} \sum_{k=1}^{2} \Phi_k(s) \Pi_2(s, s|k)\, ds, \tag{51}$$

which is (22).

Next, I prove the sufficiency part. Suppose, (22) and (23) are satisfied, but the mechanism is not incentive compatible. Then, there exists s_i and r_i such that

$$V_{iQ}(r_i, s_i) > \Psi_{iQ}(s_i), \tag{52}$$

and substituting (46) in (52), I obtain that,

$$\int_{r_i}^{s_i} \sum_{k=1}^{2} \Phi_k(r_i) \Pi_2(r_i, s|k)\, ds > \Psi_{iQ}(s_i) - \Psi_{iQ}(r_i). \tag{53}$$

Furthermore, using (22) in (53), I obtain

$$\int_{r_i}^{s_i} \sum_{k=1}^{2} \Phi_k(r_i) \Pi_2(r_i, s|k)\, ds > \int_{r_i}^{s_i} \sum_{k=1}^{2} \Phi_k(s) \Pi_2(s, s|k)\, ds. \tag{54}$$

Notice that (54) contradicts (23).

H Proof of Corollary 1

Pick any two values of s_i, say s_i' and s_i''. Without loss of generality, let $s_i' < s_i''$. Using (23), I obtain that

$$\Phi_1(s_i') \Pi_2(s_i', s_i'|1) + \Phi_2(s_i') \Pi_2(s_i', s_i'|2) \leq \Phi_1(s_i'') \Pi_2(s_i'', s_i'|1) + \Phi_2(s_i'') \Pi_2(s_i'', s_i'|2). \tag{55}$$

Further, because the payoffs are convex in the signal s, therefore,

$$\Pi_2(s_i'', s_i'|1) \leq \Pi_2(s_i'', s_i''|1) \tag{56}$$

and

$$\Pi_2(s_i'', s_i'|2) \leq \Pi_2(s_i'', s_i''|2). \tag{57}$$

Combining (55), (56) and (57), I obtain the result.

I Proof of the claim that the allocation in which the firm with the second highest report purchases both the licenses is not incentive compatible

Notice that for the allocation in which the firm with the second highest report purchases both the licenses,

$$\begin{aligned} & \Phi_1(r_i)\,\Pi_2(r_i, s_i|1) + \Phi_2(r_i)\,\Pi_2(r_i, s_i|2) \\ = \; & \pi_1(s_i; 0, 0) \int_{r_i}^{1} \int_{0}^{r_i} f_{12}^2(\cdot)\, ds_{(2)}^2 ds_{(1)}^2. \end{aligned}$$

Therefore,

$$\begin{aligned} & \frac{\partial}{\partial r_i} \{\Phi_1(r_i)\,\Pi_2(r_i, s_i|1) + \Phi_2(r_i)\,\Pi_2(r_i, s_i|2)\} \\ = \; & \pi_1(s_i; 0, 0) \\ & \times \left[-\int_0^{r_i} f_{12}^2\left(r_i, s_{(2)}^2\right) ds_{(2)}^2 + \int_{r_i}^{1} f_{12}^2\left(s_{(1)}^2, r_i\right) ds_{(1)}^2 \right] \end{aligned} \tag{58}$$

Notice that the expression in (58) is negative for $r_i = 1$. Moreover, this expression is also continuous in r_i. Hence, the expression in (58) is negative in the neighborhood of 1, and violates the incentive compatibility condition (23).