Efficiency in Coalition Games with Externalities*

Isa E. Hafalir†

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Abstract

A natural extension of superadditivity is not sufficient to imply that the grand coalition is efficient when externalities are present. We provide a condition –analogous to convexity– that is sufficient for the grand coalition to be efficient and show that this also implies that the (appropriately defined) core is nonempty. Moreover, we propose a mechanism which implements the most efficient partition for all coalition formation games and characterize the payoff division of the mechanism.

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1 Introduction

Most of the work on cooperative game theory tries to understand how coalitions behave in environments in which players can cooperate with each other. The central questions this body of literature asks are: first, which coalitions should form and second, how the gains of cooperation should be shared. Economic environments with no externalities (in which what a group of players can achieve by cooperating is independent of what other players do) are best modelled as *Characteristic Function Games* (*CFGs*), introduced by Von Neumann - Morgenstern (1947). Coalition formation games in economic environments with externalities (in which what a group of players can achieve by cooperating depends on what other coalitions form) were first modelled by Lucas and Thrall (1963) as *Partition Function Games* (*PFGs*).

As Maskin (2003) points out, the two most important concepts in cooperative game theory, the core and the Shapley value, presume that the *grand coalition* (the

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†Department of Economics, Penn State University, University Park, PA 16802, USA. E-mail: isaemin@psu.edu

coalition of all players) would form. Most of the literature following early findings in cooperative game theory, and working on CFGs or PFGs, either assumes or models the grand coalition to be formed as a result of the game. In this paper, we argue that for some economic environments, the formation of the grand coalition is not very natural if one takes efficiency—the maximization of total surplus—into account. Maskin (2003) also argues that there are games in which it is implausible to expect that the grand coalition will form. However, his argument is not related to the inefficiency of the resulting partition; rather, it is about the incentive to free-ride if the merging of coalitions exerts a positive externality on other coalitions.

Bloch (1996), Ray and Vohra (1999), and Yi (1997) model economic environments with externalities and show that their noncooperative games might result in finer partitions than the grand coalition. However, they do not take the efficiency of the resulting coalition into account; nor do they provide a characterization of the payoff division of the resulting game. Bloch (1996) assumes that the division of coalitional surplus is exogenously fixed: the proposers only choose which coalition they will offer. He shows that any core stable allocation can be attained as a stationary perfect equilibrium of the game. Ray and Vohra (1999) consider a game in which the proposers offer a coalition and a contingent payoff division. They prove that there exists a stationary equilibrium of their game and provide an algorithm to determine the resulting partition. Yi (1997) characterizes and compares stable coalition structures under some different rules of coalition formation.

Myerson (1977), Bolger (1989), Pham Do and Norde (2002), Macho-Stadler, Perez-Castrillo and Wettstein (2004) and Albizuri, Arin and Rubio (2005) give axiomatic extensions of the Shapley value for CFGs to PFGs. They all assume that the grand coalition will form, even if it is not efficient. In the first part of their paper, de Clippel and Serrano (2005) require the efficiency of the grand coalition and provide upper and lower bounds for the players' payoffs. They also characterize a value by strengthening their marginality assumption.[1] In the second part of their paper, de Clippel and Serrano (2005) consider the case in which the grand coalition does not form and characterize a payoff configuration on the basis of Myerson's (1980) principle of balanced contributions. Their result is similar to Maskin (2003): they argue that considerations of coalition formation may induce formation of finer partitions than the grand coalition, even if the grand coalition is efficient.

Maskin (2003) provides an axiomatic characterization of a generalized Shapley value and exhibits a mechanism that implements it. His mechanism has the interesting property that the grand coalition may not necessarily form. He axiomatizes the solution to the following noncooperative game: at stage k, agent k enters the room and all players with the lowest index in coalitions in the room simultaneously bid for k. Player k either accepts one of the bids or makes his own (singleton) coalition and the game moves to the $k+1st$ stage. At the end of the game, the lowest index players

[1] The value they characterize coincides with the value proposed in this paper when the game is fully cohesive (see Section 4).

in each coalition distribute the promised bids and keep the rest for themselves. If there are negative externalities, the grand coalition always form, though it may be inefficient. If there are positive externalities, this game might result in a partition finer than the grand coalition, but again this may be inefficient.

For CFGs, the assumption of superadditivity is commonly used. It says that what two coalitions can get by merging should not be less than the sum of what they get separately. This assumption implies the efficiency of the grand coalition in environments with no externalities. However, as we will show in Section 2, a straightforward extension of superadditivity does not imply efficiency of the grand coalition when externalities are present. Therefore, from an efficiency point of view, superadditive PFGs do not necessarily result in the grand coalition.

Consider a Cournot oligopoly game. Assume that when two firms merge, by cost reduction they can do better in the market (keeping other coalitions fixed), but since negative externalities are present, all other firms are worse off after they merge. When a third firm, which is worse off as a result of the merger, joins the coalition of two, superadditivity implies that their total payoff is no less than what they get separately. However, the members of the three-firm coalition do not necessarily gain over what they get in the initial stage (when firms 1 and 2 were separated) because of negative externalities. When players cannot internalize the externalities by taking decisions jointly,[2] the grand coalition is not necessarily efficient, even though the game is superadditive.

We show that while superadditivity is not sufficient, a straightforward extension of the convexity assumption in CFGs to PFGs implies that the grand coalition is efficient. In Section 3, we also show that convex PFGs have a nonempty core (for a specific definition of the core). Note that for economic environments with externalities, there can be many definitions of the core. This is because after a deviation, the payoff of the deviating group depends on what the complementary coalition does.

In Section 4, we turn to the question of a noncooperative implementation of the efficient partition in PFGs. We propose a mechanism which gives an efficient predicted partition as well as a payoff division among the players. We also provide a characterization of this value (the resulting payoff division).

2 Efficiency, Superadditivity and Convexity in PFGs

The set of players is given by $N = \{1, 2, ..., n\}$. In Characteristic Function Games (CFGs), any coalition $S \subseteq N$ generates a value $v(S)$ and this value is independent of what other agents (not in S) do. In Partition Function Games (PFGs) there can be externalities, and these are captured by taking v as a function of a coalition and a partition (which has that coalition as a member). That is, in PFGs any coalition $S \subseteq N$ generates a value $v(S; \rho)$ where ρ is a partition of N with $S \in \rho$.

[2] For instance, in this example, when three firms cannot agree to remain single.

Formally, given a partition ρ of N and a coalition $S \in \rho$, the pair $(S;\rho)$ is called an *embedded coalition* of N. The set of all embedded coalitions is denoted by $EC(N)$. A PFG is a function v that assigns to every embedded coalition $(S;\rho) \in EC(N)$, a real number $v(S;\rho)$. By convention, $\varnothing \in \rho$ and $v(\varnothing;\rho) = 0$ for all partitions ρ of N.

A PFG is said to have positive externalities if for any $C, S, T \subseteq N$, and for any partition ρ of $N - (S \cup T \cup C)$, we have

$$v(C; \{S \cup T, C\} \cup \rho) > v(C; \{S, T, C\} \cup \rho).$$

Similarly a PFG is said to have negative externalities if

$$v(C; \{S \cup T, C\} \cup \rho) < v(C; \{S, T, C\} \cup \rho).$$

In words, a game has positive (negative) externalities if merger between two coalitions makes other coalitions better (worse) off.

2.1 Superadditivity

It is well known that if a CFG is superadditive, then the grand coalition is efficient. That is, if for all $S, T \subseteq N$ with $S \cap T = \varnothing$, $v(S \cup T) \geq v(S) + v(T)$, then $v(N) \geq \sum_{S \in \rho} v(S)$ for all partitions ρ of N.

A natural extension of superadditivity to PFGs used in Maskin (2003) and several others is as follows: A PFG is *superadditive* if for any $S, T \subseteq N$ with $S \cap T = \varnothing$, and any partition ρ of $N - (S \cup T)$,

$$v(S \cup T; \{S \cup T\} \cup \rho) \geq v(S; \{S, T\} \cup \rho) + v(T; \{S, T\} \cup \rho).$$

For notational convenience, let us denote $v(S \cup T; \{S \cup T\} \cup \rho)$ by $v_\rho(S \cup T; \{S \cup T\})$ and so on. With this notation, superadditivity can be written as follows: For any $S, T \subseteq N$ with $S \cap T = \varnothing$, and any partition ρ of $N - (S \cup T)$,

$$v_\rho(S \cup T; \{S \cup T\}) \geq v_\rho(S; \{S, T\}) + v_\rho(T; \{S, T\}).$$

In PFGs, superadditivity is not enough for the efficiency of the grand coalition, as the following example shows. Because of externalities, although the merging coalitions benefit from merging, others might be worse off and the total payoff in the grand coalition might be less then the total payoff in some other partition.

Example 1 *Consider the following symmetric 3-player PFG:* $N = \{1, 2, 3\}$

$$\begin{aligned} v(\{i\}; \{\{1\}, \{2\}, \{3\}\}) &= 4 \text{ for } i = 1, 2, 3; \\ v(\{j, k\}; \{\{i\}, \{j, k\}\}) &= 9 \text{ and } v(\{i\}; \{\{i\}, \{j, k\}\}) = 1 \text{ for } \{i, j, k\} = N; \\ v(N; \{N\}) &= 11. \end{aligned}$$

This game is superadditive, but the grand coalition is not efficient since $v(N, \{N\}) = 11 < \sum_{i=1}^{i=3} v(\{i\}, \{\{1\}, \{2\}, \{3\}\}) = 12.$

In this game, the grand coalition is not efficient because there are negative externalities. It can be easily shown that if the externalities are positive and the game is superadditive, then the grand coalition is always efficient.

2.2 Convexity

A stronger assumption on value functions in CFGs is convexity, or supermodularity. Convexity implies not only that the merging of two coalitions is beneficial for them, but also that merging with bigger coalitions is more beneficial. That is, the game is convex if there are increasing returns to cooperation. A natural extension of convexity to PFGs can be given as follows:[3] A PFG is *convex* if for any $S, T \subseteq N$ and any partition ρ of $N - (S \cup T)$,

$$\begin{aligned} & v_\rho (S \cup T; \{S \cup T\}) + v_\rho (S \cap T; \{S \cap T, S - T, T - S\}) \\ \geq \;& v_\rho (S; \{S, T - S\}) + v_\rho (T; \{T, S - T\}) . \end{aligned}$$

For CFGs, this definition reduces, of course, to the usual definition which is given in Appendix A. As shown by Example 1, superadditivity by itself does not imply the efficiency of the grand coalition. We will argue below that convexity implies that any coalition can achieve at least as much as the sum of what its parts can achieve (independent of whether game has positive, negative or mixed externalities) and in particular, it implies the efficiency of the grand coalition.

Proposition 1 *If a PFG is convex, then for any coalition C, any partition ρ of $N - C$ and ρ' of C,*

$$v_\rho (C) \geq \sum_{S \in \rho'} v_\rho (S; \rho') .$$

Proof. Fix a coalition $C \subseteq N$ and a partition ρ of $N - C$. The proof will be inductive on the cardinality of the partition ρ' of C. Let us denote ρ' by $\{C_1, C_2, ..., C_k\}$ with $k \leq |C|$ (suppose $C_i \neq \varnothing$).

For notational simplicity denote $C_i \cup C_{i+1} \cup ... \cup C_j$ by $\bar{S}_{ij}$ and $\{C_i, C_{i+1}, ..., C_j\}$ by $S_{i,j}$.

Induction hypothesis: For any $3 \leq l \leq k$ and any partition ρ'' of $C_{l+1} \cup C_{l+2} \cup ... \cup C_k$,

$$v_{\rho\cup\rho''} \left(\bar{S}_{1,l}; \{\bar{S}_{1,l}\}\right) \geq v_{\rho\cup\rho''} \left(\bar{S}_{1,l-2}; \{\bar{S}_{1,l-2}, C_{l-1}, C_l\}\right) + v_{\rho\cup\rho''} (C_{l-1}; S_{1,l}) + v_{\rho\cup\rho''} (C_l; S_{1,l})$$

Induction base: For $l = 3$, in the definition of convexity take $S = \bar{S}_{1,2}$ and $T = \bar{S}_{2,3}$ (so $S \cap T = \{C_2\}$), and get

$$v_{\rho\cup\rho''} \left(\bar{S}_{1,3}; \{\bar{S}_{1,3}\}\right) + v_{\rho\cup\rho''} (\{C_2\}; S_{1,3}) \geq v_{\rho\cup\rho''} \left(\bar{S}_{1,2}; \{\bar{S}_{1,2}, C_3\}\right) + v_{\rho\cup\rho''} \left(\bar{S}_{2,3}; \{\bar{S}_{2,3}, C_1\}\right),$$

and by superadditivity applied to the right hand side of the above inequality, obtain:

$$v_{\rho\cup\rho''} \left(\bar{S}_{1,3}; \{\bar{S}_{1,3}\}\right) \geq v_{\rho\cup\rho''} (C_1; S_{1,3}) + v_{\rho\cup\rho''} (C_2; S_{1,3}) + v_{\rho\cup\rho''} (C_3; S_{1,3}) .$$

[3] For more discussion of different possible definitions of convexity for PFGs, see Appendix A. We are grateful to Geoffroy de Clippel for suggesting this analysis.

Induction proof: Assume that the induction hypothesis is true for $l = t - 1$. We need to show that it is true for $l = t$ as well.

Fix a partition ρ'' of $C_{t+1} \cup C_{t+2} \cup ... \cup C_k$. For $S = \bar{S}_{1,t-1}$ and $T = \bar{S}_{2,t}$ (so $S \cap T = \bar{S}_{2,t-1}$), from convexity, obtain:

$$\begin{aligned} & v_{\rho\cup\rho''}\left(\bar{S}_{1,t}; \{\bar{S}_{1,t}\}\right) + v_{\rho\cup\rho''}\left(\bar{S}_{2,t-1}; \{\bar{S}_{2,t-1}, C_1, C_t\}\right) \\ \geq\ & v_{\rho\cup\rho''}\left(\bar{S}_{1,t-1}; \{\bar{S}_{1,t-1}, C_t\}\right) + v_{\rho\cup\rho''}\left(\bar{S}_{2,t}; \{\bar{S}_{2,t}, C_1\}\right). \end{aligned} \tag{1}$$

Again from convexity, obtain (by $S = \bar{S}_{1,t-2}$ and $T = \bar{S}_{2,t-1}$, so $S \cap T = \bar{S}_{2,t-2}$):

$$\begin{aligned} & v_{\rho\cup\rho''}\left(\bar{S}_{1,t-1}; \{\bar{S}_{1,t-1}, C_t\}\right) + v_{\rho\cup\rho''}\left(\bar{S}_{2,t-2}; \{\bar{S}_{2,t-2}, C_1, C_{t-1}, C_t\}\right) \\ \geq\ & v_{\rho\cup\rho''}\left(\bar{S}_{1,t-2}; \{\bar{S}_{1,t-2}, C_{t-1}, C_t\}\right) + v_{\rho\cup\rho''}\left(\bar{S}_{2,t-1}; \{\bar{S}_{2,t-1}, C_1, C_t\}\right). \end{aligned} \tag{2}$$

Add up (1) and (2) and obtain:

$$\begin{aligned} & v_{\rho\cup\rho''}\left(\bar{S}_{1,t}; \{\bar{S}_{1,t}\}\right) + v_{\rho\cup\rho''}\left(\bar{S}_{2,t-2}; \{\bar{S}_{2,t-2}, C_1, C_{t-1}, C_t\}\right) \\ \geq\ & v_{\rho\cup\rho''}\left(\bar{S}_{1,t-2}; \{\bar{S}_{1,t-2}, C_{t-1}, C_t\}\right) + v_{\rho\cup\rho''}\left(\bar{S}_{2,t}; \{\bar{S}_{2,t}, C_1\}\right). \end{aligned} \tag{3}$$

Use induction hypothesis for at $l = t - 1$ to obtain:

$$\begin{aligned} & v_{\rho\cup\rho''\cup\{C_1\}}\left(\bar{S}_{2,t}; \{\bar{S}_{2,t}\}\right) \geq \\ & v_{\rho\cup\rho''\cup\{C_1\}}\left(\bar{S}_{2,t-2}; \{\bar{S}_{2,t}, C_{t-1}, C_t\}\right) + v_{\rho\cup\rho''\cup\{C_1\}}\left(C_{t-1}; S_{2,t}\right) + v_{\rho\cup\rho''\cup\{C_1\}}\left(C_t; S_{2,t}\right). \end{aligned} \tag{4}$$

Use (3), (4), and induction hypothesis to obtain:

$$v_{\rho\cup\rho''}\left(\bar{S}_{1,t}; \{\bar{S}_{1,t}\}\right) \geq v_{\rho\cup\rho''}\left(\bar{S}_{1,t-2}; \{\bar{S}_{1,t-2}, C_{t-1}, C_t\}\right) + v_{\rho\cup\rho''}\left(C_{t-1}; S_{1,t}\right) + v_{\rho\cup\rho''}\left(C_t; S_{1,t}\right),$$

which completes the induction proof.

Using induction hypothesis and the base for $l = 3$, we conclude that for any partition ρ' of C, we have

$$v_\rho(C) \geq \sum_{S\in\rho'} v_{\rho\cup\rho'}(S).$$

■

Now, we can state the immediate corollary of the above proposition, which states that convexity implies the efficiency of the grand coalition.

Corollary 1 *If a PFG is convex, then for any partition ρ of N,*

$$v(N; \{N\}) \geq \sum_{S\in\rho} v_\rho(S).$$

It should be noted that convex PFGs do not necessarily have positive externalities. Consider Example 1, with the difference that $v(N;\{N\}) = 15$ instead of 11. This game is convex, yet has negative externalities.

Let us define the PFGs with the property that any coalition can achieve at least as much as the sum of what its parts can achieve by fully cohesive[4] PFGs. Formally, a PFG is *fully cohesive* if for any coalition C, any partition ρ of $N - C$ and ρ' of C,

$$v_\rho(C) \geq \sum_{S \in \rho'} v_\rho(S; \rho').$$

In other words, a fully cohesive PFG assigns more to the subset $C \subseteq N$, than to any of its partitions, for any partition of the set $N - C$. Proposition 1 shows that convexity implies full cohesiveness. One might wonder if convexity is too strong to guarantee full cohesiveness. The example in Appendix A shows that a weaker definition of convexity (which is, for CFGs, equivalent to the convexity definition given in this paper) is not enough to guarantee full cohesiveness. Therefore, we might conclude that convexity, although it is not a necessary condition for full cohesiveness, is not a very strong one.

3 The Core

In this section, we focus on convex PFGs. Hence, in the games we consider here, the grand coalition is the most efficient partition. Therefore, any other coalition can be Pareto improved by making appropriate side-transfers.

For CFGs, a vector of payoffs $x = (x_1, x_2, ..., x_n)$ is in the core if for all $S \subset N$, we have

$$\sum_{i \in S} x_i \geq v(S).$$

A nice feature of convex CFGs is that they have a nonempty core. In CFGs, when group of agents is deciding whether or not to deviate, they do not consider what other agents would do (a coalition's value is independent of what other coalitions form). However, this is not the case in PFGs. In PFGs, one has to make assumptions about what a deviating coalition conjectures about the reaction of the others while defining the core. Hence, there can be many definitions of the core. One simple definition of the core can be given by supposing that the agents in the deviating coalition S presume that agents in $N - S$ will form singletons after the deviation.

Definition 1 *A vector of payoffs* $x = (x_1, x_2, ..., x_n)$ *is in the* core with singleton expectations, *named* s-core, *if for all* $S \subset N$, *we have*

$$\sum_{i \in S} x_i \geq v(S; \{S\} \cup [N - S]).$$

[4] This term was first defined by Currarini (2003).

where $[N-S]$ denotes the partition of $N-S$ to singletons.

Next proposition shows that any convex game has a nonempty s-core.

Proposition 2 *If a PFG is convex, then s-core is nonempty*

Proof. Define the following CFG with $\bar{v}(S) = v(S; \{S\} \cup [N-S])$. We claim that the CFG $\bar{v}$ is convex.

Take any $S, T \subset N$, with $|T-S| = |S-T| = 1$, then from definition of convexity[5] in the PFG (with $\rho = [N-(S \cup T)]$)

$$\begin{aligned} & v_\rho(S \cup T; \{S \cup T\}) + v_\rho(S \cap T; \{S \cap T, S-T, T-S\}) \\ \geq \ & v_\rho(S; \{S, T-S\}) + v_\rho(T; \{T, S-T\}), \end{aligned}$$

or in characteristic function notation,

$$\bar{v}(S \cup T) + \bar{v}(S \cap T) \geq \bar{v}(S) + \bar{v}(T). \tag{5}$$

Now, by using (5) we need to show that for all S' and T' (without the restriction $|T'-S'| = |S'-T'| = 1$), (5) is true.

We claim that (5) is true for all S' and T' with $S' \cup T' = S \cup T$.

Let $S - S' = \{s_1, .., s_k\}$ and denote $\{s_1, .., s_l\}$ by S_l for $1 \leq l \leq k$ and similarly let $T - T' = \{t_1, .., t_m\}$ and denote $\{t_1, .., t_l\}$ by T_l for $1 \leq l \leq m$.

Write down the following inequalities:

$$\begin{aligned} \bar{v}(S \cup T) + \bar{v}(S \cap T) &\geq \bar{v}(S) + \bar{v}(T); \\ \bar{v}(T) + \bar{v}((S \cap T) - T_1) &\geq \bar{v}(S \cap T) + \bar{v}(T - T_1); \\ \bar{v}(T - T_1) + \bar{v}((S \cap T) - T_2) &\geq \bar{v}((S \cap T) - T_1) + \bar{v}(T - T_2); \\ & \ \ \ldots.. \\ \bar{v}(T - T_{m-1}) + \bar{v}((S \cap T) - T_m) &\geq \bar{v}((S \cap T) - T_{m-1}) + \bar{v}(T - T_m). \end{aligned}$$

When we sum up above inequalities, we obtain:

$$\begin{aligned} \bar{v}(S \cup T) + \bar{v}((S \cap T) - T_m) &\geq \bar{v}(S) + \bar{v}(T - T_m) \\ \bar{v}(S \cup T) + \bar{v}(S \cap T') &\geq \bar{v}(S) + \bar{v}(T'). \end{aligned}$$

Then, write down the following inequalities:

$$\begin{aligned} \bar{v}(S \cup T) + \bar{v}(S \cap T') &\geq \bar{v}(S) + \bar{v}(T'); \\ \bar{v}(S) + \bar{v}((S \cap T') - S_1) &\geq \bar{v}(S \cap T') + \bar{v}(S - S_1); \\ \bar{v}(S - S_1) + \bar{v}((S \cap T') - S_2) &\geq \bar{v}((S \cap T') - S_1) + \bar{v}(S - S_2); \\ & \ \ \ldots.. \\ \bar{v}(S - S_{k-1}) + \bar{v}((S \cap T') - S_m) &\geq \bar{v}((S \cap T') - S_{m-1}) + \bar{v}(S - S_m). \end{aligned}$$

[5] Note that we are using the definition of convexity only for sets with $|T-S| = |S-T| = 1$. Therefore, this proposition would be true for weakly convex games (discussed in Appendix A) as well.

We sum up above inequalities, and we obtain:

$$\begin{aligned}\bar{v}(S\cup T)+\bar{v}((S\cap T')-S_m) &\geq \bar{v}(T')+\bar{v}(S-S_m);\\ \bar{v}(S'\cup T')+\bar{v}(S'\cap T') &\geq \bar{v}(T')+\bar{v}(S').\end{aligned}$$

This completes the proof of $\bar{v}$ convex.

A very well known result of convex CFG tells us that $x_i=\bar{v}(\{1,..,i\})-\bar{v}(\{1,..,i-1\})$ is in the core of the game (see Moulin, 1988, page 113). Hence we obtain that the s-core is nonempty for convex PFGs, since

$$x_i=v(\{1,..,i\};\{1,..,i\},\{i+1\},...,\{n\})-v(\{1,..,i-1\};\{1,..,i-1\},\{i\},...,\{n\})$$

is in the core of the PFG. ■

Note that the s-core definition is very pessimistic for PFGs with positive externalities but it is the most optimistic one for PFGs with negative externalities (which makes it very easy to block). Therefore, we immediately have the following remark.

Remark 1 *A convex PFG with negative externalities has a nonempty core (independent of agents conjectures about what will happen after the deviation.)*

Another natural core specification is given by the following definition.

Definition 2 *A vector of payoffs* $x=(x_1,x_2,...,x_n)$ *is in the* core with cautious expectations, *named* c-core, *if for all* $S\subset N$, *we have*

$$\begin{aligned}\sum_{i\in S}x_i &\geq v_{\rho^*}(S,\{S\}),\\ \textit{where } \rho^* &= \arg\min_{\rho} v_{\rho}(S,\{S\}).\end{aligned}$$

This definition of the core is analogous to the definition of α*-core* in the literature. It is easy to see that if the game has positive externalities, then $\rho^*=[N-S]$ and if the game has negative externalities, then $\rho^*=\{N-S\}$. The following corollary is an implication of Proposition 2.

Corollary 2 *A convex PFG game has a nonempty c-core.*

There can be other definitions of the core. Maskin (2003) makes the assumption that any deviating coalition S presumes the complementary coalition $N-S$.

Definition 3 *A vector of payoffs* $x=(x_1,x_2,...,x_n)$ *is in the* core with merging expectations, *named* m-core, *if for all* $S\subset N$, *we have*

$$\sum_{i\in S}x_i\geq v(S;\{S,N-S\}).$$

The following example shows that convexity does not imply a nonempty m-core.

Example 2 *Consider the following symmetric 3-player PFG:* $N = \{1, 2, 3\}$

$$\begin{aligned} v(\{i\}; \{\{1\}, \{2\}, \{3\}\}) &= 4 \textit{ for } i = 1, 2, 3; \\ v(\{j, k\}; \{\{i\}, \{j, k\}\}) &= 9 \textit{ and } v(\{i\}; \{\{i\}, \{j, k\}\}) = 6 \textit{ for } \{i, j, k\} = N; \\ v(N; \{N\}) &= 16. \end{aligned}$$

One can easily verify that above game is convex: Take $S = \{1, 2\}$ and $T = \{2, 3\}$, we have:

$$\begin{aligned} 20 &= v(N; \{N\}) + v(\{2\}; \{\{1\}, \{2\}, \{3\}\}) \\ &> v(\{1, 2\}; \{\{1, 2\}, \{3\}\}) + v(\{2, 3\}; \{\{2, 3\}, \{1\}\}) = 18. \end{aligned}$$

However, this game has an empty m-core. When a singleton deviates, he presumes that the other two will make a coalition, so he can get a payoff of 6 by deviating. This implies the grand coalition should allocate at least 18, hence the m-core is empty.

Note that the above example has positive externalities. Moreover, Maskin (2003)'s definition of the core relies on a very optimistic conjecture about the reactions of other agents when the externalities are positive.

One natural expectation of the deviating agents is that others will take this deviation as given and try to maximize their own payoff. We call this expectation as *rational expectations.*

Definition 4 *A vector of payoffs* $x = (x_1, x_2, ..., x_n)$ *is in the* core with rational expectations*, named* r-core*, if for all* $S \subset N$*, we have*

$$\begin{aligned} \sum_{i \in S} x_i &\geq v_{\rho^*}(S; \{S\}), \\ \textit{where } \rho^* &= \arg\max_{\rho} \sum_{C \in \rho} v_{\rho}(C; \{S\}). \end{aligned}$$

Although natural, convex PFGs might have an empty core with rational expectations. The game specified in Example 2 has an empty r-core. When a singleton i deviates, the best j and k achieve is obtained by forming a coalition since they can get 9 rather than a total of 8. This then implies any core allocation should allocate at least 18, which is not possible.[6]

[6] One could also look for a *consistent core* notion. Specifically, a core definition in which a deviating coalition expects that a partition the complementary coalition would form would be in the core of the reduced game. For the game in Example 2, however, the consistent core is empty. This is because when a singleton deviates, the core of the two person game contains only the grand coalition of two players.

4 A noncooperative implementation

In this section, we do not impose any restrictions on PFGs. Specifically, we do not require v to have superadditivity, convexity, or positive or negative externalities.

For PFGs, Myerson (1977), Bolger (1989), Pham Do and Norde (2002), Maskin (2003), Macho-Stadler, Perez-Castrillo and Wettstein (2004), Albizuri, Arin and Rubio (2005) and de Clippel and Serrano (2005) give extensions of Shapley value of CFGs to PFGs. Except for Maskin (2003) and de Clippel and Serrano (2005), other papers propose that the grand coalition will form. Most of these models consider the environments in which the grand coalition is the most efficient partition. Their values and implementations are either not applicable to environments in which the grand coalition is not the most efficient one or they result in inefficient partitions. Maskin (2003) proposes that for superadditive games, the grand coalition will form with negative externalities (where the grand coalition is not necessarily efficient), but might not form with positive externalities (where the grand coalition is efficient.)

Both Maskin (2003) and de Clippel and Serrano (2005) argue that inefficient outcomes may emerge in superadditive games if one introduces the considerations of coalition formation, while we argue in this paper that the "nonformation" of the grand coalition might emerge from the efficiency considerations. In an environment in which side payments are allowed, formation of inefficient partitions is implausible since a Pareto superior allocation and a payoff division would be available to the agents.

We propose a noncooperative implementation which always results in the efficient partition. Consider the following simple game.[7]

Take any ordering of the players (consider the natural ordering, the game will be the same for any other permutation σ of the players): at stage k, agent k enters the room. Suppose that before he enters, agents $1, ..., k-1$ has already formed a partition of $K-1 = \{1, ..., k-1\}$ and there are three kinds of people in the room: a boss, dependents and independents. The boss at stage $k-1$ has promised the dependents a payoff to be given at the end of the game. At stage k, agent k has all the bargaining power. k proposes a partition of $K = \{1, .., k\}$ and a payoff division for all agents in $K-1$. If everybody accepts the proposal then the proposed partition forms, k becomes the new boss and all others become dependents. If anybody rejects, the old partition and singleton coalition of k forms (k becomes an independent). At the end of the game (stage n) independents enjoy their payoff from the resulting partition. The boss gets the rest of the total payoffs, distributes the promised payoffs to the dependents and enjoys what is left. This game is played for all possible orderings of players. The *Efficient Generalized Shapley Value* ($EGSV$) of players is the average of their payoffs obtained for different orderings.

For the above game, it is not difficult to see that at every stage, the newcomers will propose an acceptable offer and become the new boss. Before showing why this

[7] We provide an alternative game and a characterization in Appendix B.

is the case, let us introduce some notation.

Let ρ^k denote a typical partition of K. Define $V\left(\rho^k\right)$ as follows:

$$V\left(\rho^k\right) = \sum_{C \in \rho^k} v\left(C; \rho^k \cup [N-K]\right),$$

where $[N-K]$ denotes the partition of $N-K$ to singletons.

Let $\overline{\rho}^k$ be defined as follows:

$$\overline{\rho}^k = \arg\max_{\rho^k} V\left(\rho^k\right).$$

That is, $\overline{\rho}^k$ is the most efficient partition given that agents $k+1, ..., n$ remain singletons. Let $\widetilde{\rho}^k$ be the proposed partition at stage k of the game. Finally, let p_i^k be the promised payoff to i at stage k.

Consider the last stage. If any of the agents rejects agent n's offer, then final partition $\rho^{n-1} \cup \{n\}$ forms (where ρ^{n-1} is equal to $\widetilde{\rho}^{n-1}$ if $n-1$'s offer was accepted at stage $n-1$), independents enjoy their resulting payoffs, dependents enjoy what has been promised, and the boss enjoys what is left. Then, for an independent agent i in $N-1$ to accept the offer, he needed to be promised at least $v\left(\{i\}; \rho^{n-1} \cup \{n\}\right)$. For a dependent agent i to accept the offer, he needed to be promised at least p_i^{n-1} and the boss needed to be promised at least $V\left(\rho^{n-1}\right)$ minus what independents and dependents get after a rejection. Hence, agent n can propose an acceptable offer at a sum of $V\left(\rho^{n-1}\right)$. If his offer got rejected, n enjoys $v\left(\{n\}; \rho^{n-1} \cup \{n\}\right)$. Whereas, by proposing the partition $\overline{\rho}^n$, he can get the payoff $V\left(\overline{\rho}^n\right) - V\left(\rho^{n-1}\right)$ which is never less than $v\left(\{n\}; \rho^{n-1} \cup \{n\}\right)$ (note that $\overline{\rho}^n$ is the most efficient partition of N). Therefore, n's (weakly) best strategy is to offer $\widetilde{\rho}^n = \overline{\rho}^n$ and promise to each agent what they get if they reject the offer. This actually proves that this game always results in the most efficient partition.

At stage $n-1$, if the proposal is rejected then partition $\rho^{n-2} \cup \{n-1\}$ forms, then (from backward induction) an independent agent i enjoys $v\left(\{i\}; \rho^{n-2} \cup \{n-1\} \cup \{n\}\right)$, dependents enjoy what has been promised, and the boss enjoys what is left from $V\left(\rho^{n-2} \cup \{n-1\}\right)$. Agent $n-1$ can then propose an acceptable offer at a sum of $V\left(\rho^{n-2}\right)$. By proposing the partition $\overline{\rho}^{n-1}$, he can get the payoff $V\left(\overline{\rho}^{n-1}\right) - V\left(\rho^{n-2}\right)$ which is never less than $v\left(\{n-1\}; \rho^{n-2} \cup \{n-1\} \cup \{n\}\right)$ (note that $\overline{\rho}^{n-1}$ is the most efficient partition of $N-1$ when n remains singleton). Therefore, $n-1$'s best strategy is to offer $\widetilde{\rho}^{n-1} = \overline{\rho}^{n-1}$ and promise each agent what they get (at the end of the game) if they reject.

Continuing in this fashion of using backward induction, we can conclude that agent k at stage k proposes the partition $\overline{\rho}^k$ and the acceptable promises of payoffs which add up to $V\left(\overline{\rho}^{k-1}\right)$. Hence, at the end of the game agent k gets a payoff of $m^k = V\left(\overline{\rho}^k\right) - V\left(\overline{\rho}^{k-1}\right)$ (for the natural ordering) and the most efficient partition is the result of the game. Then EGSV of player i, which is denoted by $\psi_i^{Sh}(v)$ is the average of these marginal contributions. That is,

$$\psi_i^{Sh}(v) = \frac{1}{n!}\sum_{\sigma} m^{\sigma(i)}.$$

Note that this value reduces to Shapley value for superadditive CFGs.

4.1 The Characterization

When we consider a fully cohesive[8] PFG, then $\widetilde{\rho}^k = \overline{\rho}^k = K$ and the payoff division of above game coincides with the value given by Pham Do and Norde (2002) and de Clippel and Serrano (2005). It is not difficult to see that for arbitrary games the value ψ^{Sh} is not additive. This is true even for (nonsuperadditive) CFGs. Consider the following example.

Example 3 *Consider the following two symmetric 3-player CFGs:* $v(\{i\}) = 2$, $v(\{i,j\}) = 1$, $v(\{1,2,3\}) = 4$ *and* $w(\{i\}) = 1$, $w(\{i,j\}) = 5$, $w(\{1,2,3\}) = 4$. *Any value which is efficient and symmetric should give the value* $(2,2,2)$ *to the players in both of the games. However, in the game* $v+w$ *this value should give* $(3,3,3)$*. Hence, the value is not additive.*

On the other hand, ψ^{Sh} is efficient-cover additive. That is, when we consider the efficient cover (or fully-cohesive cover) of two games, then EGSV is additive. More formally, let the efficient cover of the game v be defined as follows:

$$\bar{v}_\rho(S;\{S\}) = \max_{\rho'\text{: partition of } S} \sum_{C\in\rho'} v_\rho(C;\rho').$$

A value ψ is *efficient-cover additive* if $\psi_i(\bar{v}) + \psi_i(\bar{w}) = \psi_i(\bar{v}+\bar{w})$ and *fully efficient* if

$$\sum_{i\in N}\psi_i(v) = V(\overline{\rho}^n).$$

The other two axioms that will characterize ψ are null player property and symmetry. A value ψ satisfies the *null-player property* if $V^\sigma\left(\overline{\rho}^{\sigma(k)}\right) = V^\sigma\left(\overline{\rho}^{\sigma(k)-1}\right)$ for all permutations, then $\psi_k(v) = 0$. A value is *symmetric* if for players i and j, if $V^\sigma\left(\overline{\rho}^{\sigma(i)}\right) - V^\sigma\left(\overline{\rho}^{\sigma(i)-1}\right) = V^{\sigma'}\left(\overline{\rho}^{\sigma'(j)}\right) - V^{\sigma'}\left(\overline{\rho}^{\sigma'(j)-1}\right)$ where only difference between σ and σ' is the places of i and j, then $\psi_i(v) = \psi_j(v)$.

Now, we can introduce the characterization for EGSV.

Proposition 3 *A value is efficient-cover additive, fully efficient, symmetric and satisfies null-player property if and only if it is Efficient Generalized Shapley Value,* ψ^{Sh}*.*

The proof of this proposition follows from the above observations and Proposition 3 in de Clippel and Serrano (2005).

[8] The formal definition was given at the end of Section 2.

4.2 An Example

In this section, we consider an example to illustrate the difference between our results and results of de Clippel and Serrano's (2005) and Maskin's (2003) in terms of the resulting partition and payoff division. Consider the following PFG, which was considered in both de Clippel and Serrano (2005) and Maskin (2003).

Example 4 *Consider the following 3-player PFG:* $N = \{1, 2, 3\}$

$$\begin{aligned} v\left(\{i\}; \{\{1\}, \{2\}, \{3\}\}\right) &= 0 \text{ for } i = 1, 2, 3; \\ v\left(\{i\}; \{\{i\}, \{j, k\}\}\right) &= 9 \text{ for } \{i, j, k\} = N \\ v\left(\{1, 2\}; \{\{1, 2\}, \{3\}\}\right) &= 12; \\ v\left(\{1, 3\}; \{\{1, 3\}, \{2\}\}\right) &= 13; \\ v\left(\{2, 3\}; \{\{2, 3\}, \{1\}\}\right) &= 14; \\ v\left(N; \{N\}\right) &= 24. \end{aligned}$$

Note that in this game the most efficient partition is the grand coalition. However, both balanced contributions approach of de Clippel and Serrano (2005) and coalition formation game of Maskin (2003) result in the partition of a singleton and a coalition of two. More specifically, for any ordering, the first agent forms a singleton and other two joins for a coalition in de Clippel and Serrano (2005) and the one with higher index among first two agents forms a coalition with the third agent, while other agent (the one with lower index among first two agents) forms a singleton in Maskin (2003). For this game, the resulting values (the payoff division) is given by $\left(\frac{43}{6}, \frac{44}{6}, \frac{45}{6}\right)$ in de Clippel and Serrano (2005) and $\left(7, \frac{22}{3}, \frac{25}{3}\right)$ in Maskin (2003). Note that these two payoff vectors adds up to less than 24, the total attainable by the grand coalition.

Let us apply our implementation to above PFG. Consider the ordering $1, 2, 3$. In the case that agent 2 is independent when agent 3 enters the room, if either agent 1 or agent 2 rejects 3's offer, they will get 0 payoff. Therefore, agent 3 will offer the partition of $\{N\}$ and will offer payoff of 0 to both agent 1 and 2 in this case. Given this, agent 2 will offer agent 1 a payoff of 0 in the second stage (because if agent 1 rejects 2's offer, agent 2 will be an independent and agent 1 will have 0 payoff at the end of the game) and the partition of $\{1, 2\}$. Therefore, 3 will face the partition $\{1, 2\}$ in the third stage and offer payoff of 0 payoff to agent 1, payoff of 12 to agent 2 and the partition of $\{N\}$. We then can conclude that for the ordering $1, 2, 3$ the grand coalition is the resulting partition and payoff divisions are $(0, 12, 12)$. For the other orderings we can confirm that the grand coalition will form and payoff divisions are given by: $(0, 11, 13)$ if the ordering is $1, 3, 2$; $(12, 12, 0)$ if the ordering is $2, 1, 3$; $(10, 14, 0)$ if the ordering is $2, 3, 1$; $(13, 11, 0)$ if the ordering is $3, 1, 2$; and $(10, 14, 0)$ if the ordering is $3, 2, 1$. We therefore conclude that EGSV for above PFG is given by $\left(\frac{15}{2}, 8, \frac{17}{2}\right)$.

5 Conclusion

When externalities are present, the assumption of "two coalitions together can do better than what they can make separately" (superadditivity) is not enough to conclude that the grand coalition is the most efficient partition. We have identified a natural extension of convexity (supermodularity) to be a sufficient condition implying that "any number of coalitions together can do better than what they can make separately." We have also shown that convexity implies that a particular definition of the core is nonempty. As a remark, we noted that convex PFGs with negative externalities always have a nonempty core and the core with cautious expectations is also nonempty.

There have been different extensions of the Shapley value to PFGs that have been proposed, but except for Maskin (2003) and de Clippel and Serrano (2005), all implicitly or explicitly assume that the grand coalition will form. We have proposed a mechanism which always results in an *efficient* partition and provided a characterization of the resulting payoff division.

Application of our game to noncooperative setups, such as bidding rings and distribution of payoffs after the bidding in the auction theory setup, is left for future work.

A Appendix: Different convexity definitions in PFGs

For CFGs, the convexity definition "$v(S\cup T)+v(S\cap T)\geq v(S)+v(T)$ for all $S,T\subseteq N$" is equivalent to the definition "$v(S\cup T)+v(S\cap T)\geq v(S)+v(T)$ for all $S,T\subseteq N$ with $|S-T|=|T-S|\leq 1$". The proof of this argument is not difficult to obtain (see Moulin, 1988, page 112). A straightforward extension of the latter definition to PFGs can be given as: A PFG is *weakly convex* if for any $S,T\subseteq N$ with $|S-T|=|T-S|\leq 1$ and any partition ρ of $N-(S\cup T)$,

$$\begin{aligned}&v_\rho(S\cup T;\{S\cup T\})+v_\rho(S\cap T;\{S\cap T,S-T,T-S\})\\ \geq\;&v_\rho(S;\{S,T-S\})+v_\rho(T;\{T,S-T\}).\end{aligned}$$

or, if the game is superadditive and for any $S\subseteq N-\{i,j\}$ and any partition of $N-(S\cup\{i,j\})$,

$$\begin{aligned}&v_\rho(S\cup\{i,j\};\{S\cup\{i,j\}\})-v_\rho(S\cup\{i\};\{S\cup\{i\},\{j\}\})\\ \geq\;&v_\rho(S\cup\{j\};\{S\cup\{j\},\{i\}\})+v_\rho(S;\{S,\{j\},\{i\}\}).\end{aligned}$$

The following example shows that weak convexity is not equivalent to convexity. Moreover, it shows that weak convexity does not imply the efficiency of the grand coalition.

Example 5 *Consider the following symmetric 5-player PFG:* $N = \{1,2,3,4,5\} = \{i,j,k,l,m\}$

$$\begin{aligned}
v(\{i\};[N]) &= 3 \text{ for } i \in N; \\
v(\{i,j\};\{\{i,j\},\{k\},\{l\},\{m\}\}) &= 7 \text{ and } v(\{k\};\{\{i,j\},\{k\},\{l\},\{m\}\}) = 3; \\
v(\{i,j\};\{\{i,j\},\{k,l\},\{m\}\}) &= 9 \text{ and for } v(\{m\};\{\{i,j\},\{k,l\},\{m\}\}) = 8; \\
v(\{i,j,k\};\{\{i,j,k\},\{l\},\{m\}\}) &= 12 \text{ and } v(\{l\};\{\{\{i,j,k\},\{l\},\{m\}\}) = 3; \\
v(\{i,j,k\};\{\{i,j,k\},\{l,m\}\}) &= 17 \text{ and } v(\{l,m\};\{\{\{i,j,k\},\{l,m\}\}) = 6; \\
v(\{i,j,k,l\};\{\{i,j,k,l\},\{m\}\}) &= 18 \text{ and } v(\{m\};\{\{\{i,j,k,l\},\{m\}\}) = 3; \\
v(N;\{N\}) &= 25.
\end{aligned}$$

We can confirm that above game is weakly convex. First, note that the game is superadditive. Next, check the inequalities required by weak convexity. If $S = \{1\}$, weak convexity implies:

$$\begin{aligned}
5 &= v(\{1,2,3\};\{\{1,2,3\},\{4\},\{5\}\}) - v(\{1,2\};\{\{1,2\},\{3\},\{4\},\{5\}\}) \\
&\geq v(\{1,3\};\{\{1,3\},\{2\},\{4\},\{5\}\}) - v(\{1\};\{\{1\},\{2\},\{3\},\{4\},\{5\}\}) = 4,
\end{aligned}$$

and

$$\begin{aligned}
8 &= v(\{1,2,3\};\{\{1,2,3\},\{4,5\}\}) - v(\{1,2\};\{\{1,2\},\{3\},\{4,5\}\}) \\
&\geq v(\{1,3\};\{\{1,3\},\{2\},\{4,5\}\}) - v(\{1\};\{\{1\},\{2\},\{3\},\{4,5\}\}) = 6.
\end{aligned}$$

If $S = \{1,2\}$, weak convexity implies:

$$\begin{aligned}
6 &= v(\{1,2,3,4\};\{\{1,2,3,4\},\{5\}\}) - v(\{1,2,3\};\{\{1,2,3\},\{4\},\{5\}\}) \\
&\geq v(\{1,2,4\};\{\{1,2,4\},\{3\},\{5\}\}) - v(\{1,2\};\{\{1,2\},\{3\},\{4\},\{5\}\}) = 3.
\end{aligned}$$

If $S = \{1,2,3\}$, weak convexity implies:

$$\begin{aligned}
7 &= v(\{1,2,3,4,5\};\{\{1,2,3,4,5\}\}) - v(\{1,2,3,4\};\{\{1,2,3,4\},\{5\}\}) \\
&\geq v(\{1,2,3,5\};\{\{1,2,3,5\},\{4\}\}) - v(\{1,2,3\};\{\{1,2,3\},\{4\},\{5\}\}) = 6.
\end{aligned}$$

Since all inequalities are satisfied, the game is weakly convex. However, the grand coalition is not efficient, since $2v(\{1,2\};\{\{1,2\},\{3,4\},\{5\}\}) + v(\{5\};\{\{1,2\},\{3,4\},\{5\}\}) = 26 > 25 = v(N;\{N\})$. We can also confirm that above game is not convex, since for $S = \{1,2,3,4\}$ and $T = \{3,4,5\}$ we have

$$\begin{aligned}
34 &= v(S \cup T;\{S \cup T\}) + v(S \cap T;\{S \cap T, S - T, T - S\}) \\
&< v(S;\{S, T - S\}) + v(T;\{T, S - T\}) = 35.
\end{aligned}$$

B Appendix: An Alternative Implementation

The game we provide here is the same with the game in Section 4, except for the last stage: Consider the natural ordering of the players. At stage k, agent k enters the room and proposes a partition of $K = \{1, .., k\}$ and a payoff division for all agents in $K - 1$. If everybody accepts the proposal then the proposed partition forms, k becomes the new boss and all others become dependents. If anybody rejects, the old partition and singleton coalition of k forms (k becomes an independent).

At the end of the game (stage n) independents randomly form a partition among themselves with each partition forming with equal chance. The last agent, n, becomes the boss of independents and enjoy the total of payoffs from the resulting partition among independents. The boss of dependents gets the the total payoffs from partition among dependents and himself, distributes the promised payoffs to the dependents and enjoys what is left. This game is played for all possible orderings of players. The *Efficient Generalized Shapley Value2* (*EGSV2*) of players is the average of their payoffs obtained for different orderings.

For the above game, the newcomers will propose an acceptable offer and become the new boss.

Let $p(k)$ represents the number of partitions of a set with cardinality k. Let ρ^k denote a typical partition of K. Define $V_2\left(\rho^k\right)$ as follows:

$$V_2\left(\rho^k\right) = \frac{1}{p(n-k)} \sum_{\rho:\text{ partition of } N-K} \sum_{C \in \rho^k} v\left(C; \rho^k \cup \rho\right).$$

Let $\overline{\rho}^k$ be defined as follows:

$$\overline{\rho}_2^k = \arg\max_{\rho^k} V_2\left(\rho^k\right).$$

That is, $\overline{\rho}_2^k$ is the most efficient partition given that agents $k+1, ..., n$ form a random partition among themselves, all being formed with equal chance. Let $\widetilde{\rho}_2^k$ be the proposed partition at stage k of the game.

Consider the last stage. The last stage in this game is the same as the last stage in the game given in Section 4. Therefore, the weakly best option is to offer $\widetilde{\rho}_2^n = \overline{\rho}_2^n$ and promise to each agent what they get if they reject the offer.

At stage $n-1$, if the proposal is rejected then the partition $\rho^{n-2} \cup \{n-1\}$ forms and agent $n-1$ becomes an independent, then (from backward induction) an independent agent i enjoys 0 payoff, dependents enjoy what has been promised, and the boss enjoys what is left from

$$V_2\left(\rho^{n-2} \cup \{n-1\}\right) = \frac{1}{2}\left(\sum_{C \in \rho^{n-2}} v_{\rho^{n-2}}\left(C; \{n-1\} \cup \{n\}\right) + v_{\rho^{n-2}}\left(C; \{n-1, n\}\right)\right).$$

Agent $n-1$ can then propose an acceptable offer at a sum of $V_2\left(\rho^{n-2}\right)$. By proposing an offer which will not be accepted, agent $n-1$ will get 0 payoff. Therefore, $n-1$'s best strategy is to offer $\widetilde{\rho}_2^{n-1} = \overline{\rho}_2^{n-1}$ and promise each agent what they get (at the end of the game, in expected terms) if they reject.

Continuing in this fashion of using backward induction, we can conclude that agent k at stage k proposes the partition $\overline{\rho}_2^k$ and the acceptable promises of payoffs which add up to $V_2\left(\overline{\rho}_2^{k-1}\right)$. Hence, at the end of the game agent k gets a payoff of $m_2^k = V_2\left(\overline{\rho}_2^k\right) - V_2\left(\overline{\rho}_2^{k-1}\right)$ (for the natural ordering) and the most efficient partition is the result of the game. Then EGSV2 of player i, which is denoted by $\varphi_i^{Sh}(v)$ is the average of these marginal contributions. That is,

$$\varphi_i^{Sh}(v) = \frac{1}{n!}\sum_{\sigma} m_2^{\sigma(i)}.$$

When we consider a fully cohesive PFG, then $\widetilde{\rho}^k = \overline{\rho}^k = K$ and the payoff division of above game coincides with value given by Albizuri, Arin and Rubio (2005). In addition to axioms of efficient cover additivity, full efficiency and symmetry we add the following two axioms which were first given by Albizuri, Arin and Rubio (2005).

A value ψ satisfies the *oligarchy axiom* if the following holds. If there exists $S \subseteq N$ such that

$$v_\rho(C) = \begin{cases} v(N;N) & \text{if } C \supseteq S \\ 0 & \text{otherwise} \end{cases},$$

then, $\sum_{i\in S}\psi_i(v) = v(N;\{N\})$.

A value satisfies *embedded coalition anonymity* if $\psi_i(\zeta_S v) = \psi_i(v)$ for all bijections ζ_S on $\{(T;\rho) : T = S\}$ and for all $i \in N$. According to this axiom, only worths of different embedded coalitions are important, not which embedded coalitions correspond to those worths.

Now, we can introduce the characterization for EGSV2.

Proposition 4 *A value is efficient-cover additive, fully efficient, symmetric and satisfies oligarch and embedded coalition anonymity axioms if and only if it is EGSV2,* φ^{Sh}.

The proof of this proposition follows from Section 4 of this paper and Theorem 3 in Albizuri, Arin and Rubio (2005).

The difference between the payoff configurations' of the game introduced above and the game in Section 4 is that the value given by the above game can be obtained as the Shapley value of an expected CFG (with all partitions being formed with equal chance), whereas the value given by the game in Section 4 can be obtained as the Shapley value of a fictitious CFG (with $\widetilde{v}(S) = v(S;\{S\}\cup[N-S])$).

References

[1] Albizuri M., Arin J., and Rubio J. (2005), "An Axiom System for a Value For Games In Partition Function Form", International Game Theory Review, Vol. 7, No. 1, 63-72.

[2] Bloch, F. (1996). "Sequential Formation of Coalitions in Games with Externalities and Fixed Payoff Division", Games and Economic Behavior, vol. 14(1), 90-123.

[3] Bolger, E.M.(1989) "A set of axioms for a value for partition function games." International Journal of Game Theory 18: 37-44.

[4] Currarini, S. (2003) "On the Stability of Hierarchies in Games with Externalities" Working Paper, Coalition Theory Network.

[5] de Clippel, G. and Serrano, R. (2005) "Marginal Contributions and Externalities in the Value" Working Paper, Brown University.

[6] Lucas, W. and R. Thrall (1963) "n-Person Games in Partition Function Form" Naval Research Logistics Quarterly X: 281-98.

[7] Maskin, E. (2003), "Bargaining, Coalitions and Externalities" Working Paper, Institute for Advanced Study.

[8] Macho-Stadler I., Perez-Castrillo D., and Wettstein D. (2004) "Sharing the surplus: An extension of the Shapley value for environments with externalities" Working Paper, Universitat Autònoma de Barcelona.

[9] Moulin, H. (1988) "Axioms of Cooperative Decision Making" Monograph of the Econometric Society, Cambridge University Press, Cambridge.

[10] Myerson, R. (1977) "Values for games in partition function form" International Journal of Game Theory 6: 23-31.

[11] Myerson, R. (1980) "Conference Structures and Fair Allocation Rules" International Journal of Game Theory 9: 169-182.

[12] Pham Do, K. and Norde, H. (2002) "The Shapley value for partition function form games" Working Paper, Tilburg University, Center for Economic Research.

[13] Ray, D, and Vohra R. (1999). "A theory of endogenous coalition structures". Games and Economic Behavior, vol. 26, 286-336.

[14] Shapley, L. (1953), "A Value for n-Person Games," in H. Kuhn and A. Tucker (eds.), Contributions to the Theory of Games, II, Princeton: Princeton University Press, 307-317.

[15] Von Neumann, J. , Morgenstern, O. (1947) "Theory of Games and Economic Behavior". Wiley, New York.

[16] Yi, S.-S., (1997), "Stable Coalition Structures with Externalities," Games and Economic Behavior, 20: 201-237.