Optimal Mechanisms for an Auction Mediator*

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Abstract

We consider a multi-period auction with a seller who has a single object for sale, a large population of potential buyers, and a mediator of the trade. The seller and every buyer have independent private values of the object. The mediator designs an auction mechanism which maximizes her revenue subject to certain constraints for the traders. In each period the seller auctions the object to a set of buyers drawn at random from the population. The seller can re-auction the object (infinitely many times) if it is not sold in previous interactions. We characterize the class of mediator-optimal auction mechanisms. One of such mechanisms is a Vickrey auction with a reserve price where the seller pays to the mediator a fixed percentage from the closing price.

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1 Introduction

This paper considers the model of *Internet-style* trade which can be described as follows. There are a mediator, a large population of buyers, and a seller who has a single object for sale. We assume that the seller cannot deal directly with buyers, instead, the trade must be mediated. At the initial period 0, the mediator establishes a trade procedure (conventionally called "auction mechanism") through which she is allowed to collect some part of the trade surplus.[1] The seller observes the auction mechanism and decides either to consume the object, or to put it for sale. If the object is consumed, the games ends. If the object is put for sale at period $t \geq 1$, a set of n buyers is drawn randomly from the buyers' population and the auction takes place. There are two important features in our model. First, whenever the seller fails to sell the object, he is allowed to offer it for (re-)sale again, as many times as he wants. Secondly, in every trade the seller faces a different set of bidders drawn from a large population.

We characterize the class of mediator-optimal mechanisms, where the mediator commits to a mechanism in advance and is not allowed to change it during the game. Moreover, we demonstrate how to implement an optimal mechanism. It turns out that the *closing-fee Internet auction* is one of such mechanisms. In the closing-fee Internet auction, the seller (repeatedly) sells the object via a Vickrey auction. In every auction he selects a reserve price and, if the object is sold, pays to the mediator a *closing fee* (a percentage of the closing price). The fee is selected by the mediator in advance, it is commonly known and fixed through the entire trade process.

There are important implications of our results. The mediated trade can

[1] By an auction mechanism we understand a game with incomplete information set up by the mediator and played by the traders in which a desirable outcome occurs as a Bayesian Nash equilibrium.

be viewed as a principal-agent model, where the mediator (the principal) designs an incentive mechanism for the seller (the agent) such that behavior of the seller maximizes the mediator's payoff. The standard principal-optimal solution of the principal-agent model is that the principal collects a fixed fee from the agent, and after that the agent maximizes his payoff. If there is uncertainty of the agent's type, then the principal prefers to use a more sophisticated mechanism which makes the agent to report her type truthfully and which discriminates between the agents of different types. In contrast, in our model the mediator-optimal solution is to collect a percentage of the seller's payoff rather than a fixed fee, and, even though there is uncertainty of the seller's type, the only discrimination in effect is that the sellers are divided into two groups with respect to their use values: Those who are willing to auction the object and those who are not.

Surprisingly, the existence of an auction mediator, an independent player whose influence on strategic behavior of traders is essential, is not illuminated in the literature. The up-to-date research concentrates on mechanisms which achieve ex-post efficiency or which maximize the seller's revenue (for overview see, e.g., Krishna 2002, Chapter 5). In contrast, we focus on the question of optimal mechanisms for the mediator. This question has a profound relevance to the problem of maximizing profit by giant commercial trade-mediating institutions which run internet auctions (e.g., eBay, Yahoo, Amazon).

Our two main assumptions are consistent with the real-life observations. Indeed, a seller has the re-sale option in real life and this option has essential impact on players' strategic behavior, as noted, for example, by Fudenberg at al. (1985), Milgrom (1987), Gupta and Lebrun (1999), Haile (2000, 2003). Our second assumption: the seller faces a different set of bidders drawn from a large population in each period - is new and crucial for our analysis. We think that it is a realistic assumption for Internet auctions. In contrast, the existing literature on auctions with resale assumes that there is the same set of bidders in all auctions, which implies that the optimal reserve price declines due to Bayesian updating of the bidders' private values distribution after every auction (see Fudenberg at al., 1985; McAfee and Vincent, 1997).

The first related work that we are aware of is Myerson and Satterthwaite (1983) who analyze a bilateral trade mediated by a "broker", assuming that the traders have independent private values for the traded good. In particular, Myerson and Satterthwaite describe a direct revelation[2] mechanism for the broker which maximizes her payoff subject to individual rationality and incentive compatibility constraints for the traders. A variety of works extends Myerson and Satterthwaite (1983) to the study of two-sided markets mediated by "platforms", starting with the double auction of Wilson (1985) and including (but not limited to) Rochet and Tirole (2003), Hagiu (2004), Reisinger (2004), and Armstrong (2006). Instead, the focus of this paper is the mediated interaction of one seller and many buyers. In particular, we generalize Myerson and Satterthwaite's (1983) model in one of our model extensions (see Section 5.3).

The paper is organized as follows. The model is described in Section 2. In Section 3 we characterize the mediator-optimal mechanisms. Section 4 describes a simple implementation of the optimal mechanism which has applications for many Internet auctions. In Section 5 we present some extensions of our model. Section 6 discusses assumptions of the model. The Appendix contains omitted proofs.

[2] A mechanism is *direct* if the traders are asked to report their "types", i.e., their private values. Further, it is *revelation* (or *truthful*) if it is a Nash equilibrium for the traders to reveal their values truthfully.

2 The Model

We consider a model of repeated auctions where a seller can re-auction the object in the next period, if the object is not sold in the current period. In contrast to the standard problem where the seller auctions the object one and only one time, in our model the seller may decide not to sell the object at all, or to re-auction it (infinitely) many times, until it is sold. The model is designed to capture Internet-style auctions.

Let player 0 be the seller and let $\mathcal{N}$ be a large homogeneous population of bidders. The seller has one object for sale. Let $v_0 \in [0,1]$ be the *use value* of the seller and v_i be the *use value* of bidder $i \in \mathcal{N}$. Assume that all use values are independent, furthermore, bidders' use values are identically distributed on the interval $[0,1]$ according to the distribution function F_b, and the seller's use value is distributed on the same interval according to the distribution function F_s. Let f_b and f_s be the corresponding density functions. We assume that f_b and f_s are strictly positive and continuous on $[0,1]$.

At period $t = 0$, a mediator chooses a trade mechanism which will be used thereafter in the game.

At period $t = 1, 2, \ldots$ a random sample of n buyers is selected from population $\mathcal{N}$. The seller either consumes the object (and the game ends) or puts it for sale (and the game proceeds to period $t+1$). At period $t+1$, the object is allocated and the payments are transferred according to the selected mechanism. If the object is sold to one of the buyers, the game ends. Otherwise, a new random sample of n buyers is selected from population $\mathcal{N}$ and the seller either consumes the object (and the game ends) or puts it for sale and so on.

We make the following assumptions.

Assumption 1. In every period a new sample $N = \{1, \ldots, n\}$ of buyers is drawn. Every buyer plays only once and has no information about past plays.

Assumption 2. The buyers are anonymous, that is, a buyer's strategy depends on her type (use value), but not on her name.

Assumption 3. There is a discount factor δ, $0 < \delta \leq 1$, common for all players. We assume that if the seller decides to auction the object at period t, an outcome of the auction is determined in the next period $t+1$. Thus, payoffs of all players obtained at period $t+1$ are discounted by δ relative to period t.

Assumption 4. The mediator chooses a mechanism only once at period $t = 0$. The mechanism depends only on the current-period reports, i.e., it is independent of time and the history of play.

Assumption 5. The mediator, the seller and all buyers are risk neutral.

We consider the class of direct mechanisms. In a direct mechanism the seller and each buyer simultaneously and confidentially report their use values to the mediator, and the mediator then determines who gets the object and how much each buyer must pay as some functions of the vector of reported use values. Formally, a direct mechanism is a pair $(\mathbf{p}, \mathbf{x})$ where[3] $\mathbf{p} : [0,1]^{n+1} \to \Delta^{n+1}$ describes probabilities of various outcomes and $\mathbf{x} : [0,1]^{n+1} \to \mathbb{R}^{n+1}$ describes payments of the traders as functions of their *reported* use values. Namely, given the vector of reports at period t, $\mathbf{w}^t = (w_0^t, w_1^t, \ldots, w_n^t)$, $p_i(\mathbf{w}^t)$ is the probability that bidder i gets the object, $i = 1, \ldots, n$, $p_0(\mathbf{w}^t) = 1 - \sum_{i=1}^n p_i(\mathbf{w}^t)$ is the probability that the

[3] Δ^{n+1} denotes the unit simplex in $(n+1)$-dimensional space.

seller retains the object; $x_i(\mathbf{w}^t)$ is a payment of bidder $i = 1, \ldots, n$ to the mediator, and $x_0(\mathbf{w}^t)$ is a payment of the mediator to the seller. Note that for every $i = 1, \ldots, n$, x_i is allowed to be non-zero even if buyer i does not receive the object.

Let $h^t = (\mathbf{w}^1, \ldots, \mathbf{w}^t)$ be the history of play up to time t. A symmetric bidding strategy of a bidder, $\omega : [0,1] \to [0,1]$, is her bid as a function of her actual use value.[4] A seller's strategy $\left(\left(\alpha^1, q^1\right), ... \left(\alpha^t, q^t\right), ...\right)$ specifies the probability, α^{t+1}, that the seller auctions the object at period $t+1$ and his bid, q^{t+1}, as a function of history h^t and his use value v_0. Denote $q^t(v_0) = q(h^{t-1}, v_0)$, and $\alpha^t(v_0) = \alpha(h^{t-1}, v_0)$.

A seller's strategy profile $(q^t, \alpha^t)_{t=1}^{\infty}$ is *stationary* if $q^t = q^1$ and $\alpha^t = \alpha^1$ for all $t = 1, 2, \ldots$. Since the mechanism $(\mathbf{p}, \mathbf{x})$ does not vary with t, there exists a stationary seller's strategy (q^*, α^*) and a bidders' strategy ω^* which constitute a subgame perfect equilibrium.

Lemma 1 (Revelation Principle) *Given a mechanism* $(\mathbf{p}, \mathbf{x})$ *and a stationary equilibrium* $(\omega^*, (q^*, \alpha^*))$ *of the correspondent game, there exists a direct revelation mechanism* $(\mathbf{p}', \mathbf{x}')$ *which has a payoff-equivalent stationary equilibrium* (ω', q', α^*) *such that* $\omega'(v_i) = v_i$ *and* $q'(v_0) = v_0$*.*

Proof. For every $\mathbf{v} \in \mathbf{V}$ define $\mathbf{p}'(v_0, v_1, \ldots, v_n) := \mathbf{p}(q(v_0), \omega(v_1), \ldots, \omega(v_n))$ and define $\mathbf{x}'(v_0, v_1, \ldots, v_n) := \mathbf{x}(q(v_0), \omega(v_1), \ldots, \omega(v_n))$. **End of proof.**

Without loss of generality we assume that $(\mathbf{p}, \mathbf{x})$ is a direct revelation mechanism. Fix $(\mathbf{p}, \mathbf{x})$ and consider period t. Let $N = \{1, \ldots, n\}$ be the set of bidders drawn at random from the population $\mathcal{N}$ at period t. Let $\mathbf{v}^t = (v_0, v_1^t, \ldots, v_n^t)$ be the vector of use values of the seller and buyers at period t (the seller's use value does not vary with time). Denote by $\mathbf{f}$ the

[4] By Assumption 2 we consider only symmetric bidding strategies. By Assumption 1, a bidding strategy cannot depend on the history of play.

joint density of $\mathbf{v}^t$; by $\mathbf{v}^t_{-i}$ and $\mathbf{f}_{-i}$ the vector of use values and its joint densities without i's coordinate, $i = 0, \ldots, n$. Let $\mathbf{V} = [0,1]^{n+1}$ be the space of use value vectors and let $\mathbf{V}_{-i}$ be the space value of use vectors without i's coordinate.

For every $i = 0, 1, \ldots, n$ denote by $\bar{p}_i(w)$ the probability of i to obtain (retain for $i = 0$) the object, conditional on i's value v_i,

$$\bar{p}_i(v_i) = \int_{\mathbf{V}_{-i}} p_i(v_i, \mathbf{v}_{-i})\mathbf{f}_{-i}(\mathbf{v}_{-i})d\mathbf{v}_{-i}.$$

Also, for every $i = 0, 1, \ldots, n$ denote by $\bar{x}_i(v_i)$ the expected payment of buyer i to the mediator (from the mediator to the seller for $i = 0$) conditional on i's value v_i,

$$\bar{x}_i(v_i) = \int_{\mathbf{V}_{-i}} x_i(v_i, \mathbf{v}_{-i})\mathbf{f}_{-i}(\mathbf{v}_{-i})d\mathbf{v}_{-i}.$$

Then, the expected utility of bidder $i = 1, \ldots, n$ is defined by

$$U_i^t(v_i^t) = \delta(v_i^t \bar{p}_i(v_i^t) - \bar{x}_i(v_i^t)). \tag{1}$$

The discount factor δ appears here because of our assumption that if an auction starts at period t, the players are "locked in" until period $t+1$, when the auction outcome is realized. Thus, payoffs of the bidders are discounted by one period.

The expected seller's gain from the auction is defined by

$$U_0^t(v_0) = -v_0 + \delta\left[\bar{x}_0(v_0) + \bar{p}_0(v_0)(v_0 + \alpha^{t+1}(v_0)U_0^{t+1}(v_0))\right]. \tag{2}$$

That is, at period t the seller gives up the object of value v_0 (the first term of the right-hand side of (2)) and at period $t+1$ he obtains the discounted payoff, the sum of the expected payment $\bar{x}_0(v_0)$ and, if the object is not sold, the value of the object v_0 and the next-period expected gain from the auction $\alpha(v_0)U_0^{t+1}(v_0)$.

Finally, the expected utility of the mediator is defined by

$$U_M^t = \delta \int_{\mathbf{V}} \left(\sum_{i=1}^{n} x_i(\mathbf{v}) - x_0(\mathbf{v}) + p_0(\mathbf{v}) \alpha^{t+1}(v_0) U_M^{t+1} \right) \mathbf{f}(\mathbf{v}) d\mathbf{v}. \tag{3}$$

A direct revelation mechanism is *feasible* if it satisfies the following constraints:

(a) *Individual rationality.* For each period $t = 1, 2, \ldots$ and each buyer $i = 1, \ldots, n$, and each $v_i \in [0, 1]$

$$U_i^t(v_i) \geq 0, \tag{4}$$

and for each $v_0 \in [0, 1]$

$$\alpha^t(v_0) U_0^t(v_0) \geq 0. \tag{5}$$

The constraint (5) means that the seller expects to obtain a non-negative gain whenever he assigns a positive probability on auctioning the object, $\alpha^t(v_0) > 0$.

(b) *Incentive compatibility.* For each trader $i = 0, 1, \ldots, n$, each period $t = 1, 2, ...$, and each use value $v_i, w_i \in [0, 1]$

$$U_i^t(v_i) \geq U_i^t(w_i | v_i), \tag{6}$$

where $U_i^t(w_i | v_i)$ is the expected utility of trader $i = 0, 1, \ldots, n$ if she reports w_i when her true use value is v_i, that is, for each $i = 1, \ldots, n$

$$U_i^t(w_i | v_i) = \delta(v_i \bar{p}_i(w_i) - \bar{x}_i(w_i)) \tag{7}$$

and

$$U_0^t(w_0 | v_0) = -v_0 + \delta \left[\bar{x}_0(w_0) + \bar{p}_0(w_0)(v_0 + \alpha^{t+1}(v_0) U_0^{t+1}(v_0)) \right].$$

Note that by Assumption 2 the next-period expected payoff $U_0^{t+1}(v_0)$ does not depend on the current report w_0.

3 Mediator-optimal Mechanisms

3.1 Seller's Decision to Auction the Object

Let CV^* be the discounted continuation value of the seller who always auctions the object,

$$CV^* = \max_{w_0} \delta \left(\bar{x}_0(w_0) + \bar{p}_0(w_0) CV^* \right). \tag{8}$$

Note that CV^* is independent from v_0, because the object is never consumed.

Lemma 2 *In equilibrium, the seller's decision to auction the object at period t depends on the discounted continuation value, CV^*, in the following way*

$$\alpha^t(v_0) = \begin{cases} 0, & \text{if } v_0 > CV^*, \\ 1, & \text{if } v_0 < CV^*, \end{cases} \tag{9}$$

and for each $v_0 < CV^*$ and each $t = 1, 2, \ldots$

$$U_0^t(v_0) = CV^* - v_0 > 0. \tag{10}$$

Proof. By stationarity, $U_0^t(v_0) = U_0^{t+1}(v_0)$ for all t. Clearly, if $U_0^t(v_0) > 0$, then $\alpha^t(v_0) = 1$ in equilibrium. By (8), $U_0^t(v_0) > 0$ if and only if $CV^* - v_0 > 0$, thus we obtain (10) and $\alpha^t(v_0) = 1$ if $v_0 < CV^*$. Similarly, we obtain $\alpha^t(v_0) = 0$ if $v_0 > CV^*$. **End of proof.**

Note that $CV^* = v_0$ is a zero probability event, thus without any effect on the result we can assume $\alpha^t(v_0) = 1$ for this case.

The game ends in the first period, if the sellers' use value is higher than the discounted continuation value from the auction, $v_0 > CV^*$. In the next subsection we analyze the situation where $v_0 \leq CV^*$, when the seller auctions the object until it is sold.

3.2 Analysis of a Stage Game

Fix the seller's realized use value $v_0 \leq CV^*$ and period t. Denote by U_M^*, the expected payoff of the mediator at any period t, conditional on $v_0 \leq CV^*$. Then

$$U_M^* = \delta \int\limits_{\mathbf{V}_{-0}} \left(\sum_{i=1}^{n} x_i(\mathbf{v}) - x_0(\mathbf{v}) + p_0(\mathbf{v}) U_M^* \right) \mathbf{f}(\mathbf{v}_{-0}) d\mathbf{v}_{-0}. \tag{11}$$

Note that $U_M^t = U_M^{t+1} = U_M^*$.

We now characterize the set of all feasible mechanisms as a function of U_M^* and CV^*. Denote by $C_b(v_i)$ the *virtual value* of bidder i, $i = 1, \ldots, n$,

$$C_b(v_i) = v_i - \frac{1 - F_b(v_i)}{f_b(v_i)}. \tag{12}$$

The difference $v_i - C_b(v_i)$ is the *information rent* of bidder i (see discussion by Krishna 2002, Section 5.2.3). We assume that function $C_b(\cdot)$ is strictly increasing. This condition is known as the *Myerson's regularity condition* (Myerson, 1981). Let

$$Q(\mathbf{p}, z) = \delta \left(z + \int_{\mathbf{V}_{-0}} \left[\sum_{i=1}^{n} \left(C_b(v_i) - z \right) p_i(\mathbf{v}) \right] \mathbf{f}_{-0}(\mathbf{v}_{-0}) d\mathbf{v}_{-0} \right).$$

We have the following result.

Theorem 1 *Suppose that the seller's realized use value v_0 satisfies $v_0 \leq CV^*$. Then at any period $t \geq 1$, every feasible mechanism $(\mathbf{p}, \mathbf{x})$ satisfies*

$$U_M^* + CV^* + \sum_{i=1}^{n} U_i^t(0) = Q(\mathbf{p}, U_M^* + CV^*), \tag{13}$$

and

$$U_M^* + CV^* \leq Q(\mathbf{p}, U_M^* + CV^*). \tag{14}$$

Proof. See the Appendix.

Theorem 1 characterizes the mediator-optimal mechanisms for a given CV^*. Denote by Z^* the joint expected gain of the seller and the mediator, $Z^* = U_M^* + CV^*$. Clearly, in order to maximize U_M^* for a given CV^*, it suffices to maximize Z^*.

Let

$$Q^*(z) = \delta \left(z + \int_{\mathbf{V}_{-0}} \max \left\{ 0, \max_{i=1,\ldots,n} C_b(v_i) - z \right\} \mathbf{f}_{-0}(\mathbf{v}_{-0}) d\mathbf{v}_{-0} \right).$$

Note that

$$Q^*(z) = \max_{\mathbf{p}':\mathbf{V}\to\Delta^{n+1}} Q(\mathbf{p}', z).$$

Lemma 3 *The equation $z = Q^*(z)$ has a unique solution on $[0,1]$.*

Proof. Let

$$T(z) = \int_{\mathbf{V}_{-0}} \max \left\{ 0, \max_{i=1,\ldots,n} C_b(v_i) - z \right\} \mathbf{f}_{-0}(\mathbf{v}_{-0}) d\mathbf{v}_{-0}.$$

Then we can rewrite $z = Q^*(z)$ as follows,

$$(1-\delta)z = \delta T(z). \tag{15}$$

We have $T(0) > 0$ and $T(1) = 0$ (because $C_b(\cdot) \leq 1$). Note that function $(1-\delta)z$ strictly increases and function $\delta T(z)$ (weakly) decreases on the interval $[0,1]$. Hence, there is a unique solution of the equation (15). **End of Proof.**

Corollary 1 *Let CV^* be given and suppose that $v_0 \leq CV^*$. A feasible mechanism $(\mathbf{p}, \mathbf{x})$ is mediator-optimal w.r.t. CV^* if and only if*

(i) *Z^* is a unique solution of $z = Q^*(z)$,*

(ii) $\mathbf{p} \in \operatorname*{argmax}_{\mathbf{p}':\mathbf{V}\to\Delta^{n+1}} Q(\mathbf{p}', Z^*)$, *and*

(iii) $\mathbf{x}$ *is selected to make* $U_0(v_0) = CV^* - v_0$ *and* $U_i^t(0) = 0$ *for each buyer* $i = 1, \ldots n$ *and any period* $t = 1, 2, \ldots$. *Namely,*

$$\bar{x}_0(v_0) = CV^* \left(\frac{1}{\delta} - \bar{p}_0(v_0) \right)$$

and

$$\bar{x}_i(v_i^t) = v_i^t \bar{p}_i(v_i^t) - \int_0^{v_i^t} \bar{p}_i(z) dz, \ i = 1, \ldots, n.$$

Proof. By Theorem 1, $Z^* = U_M^* + CV^*$ is maximized if

$$Z^* = \max_{\mathbf{p}' : \mathbf{V} \to \Delta^{n+1}} Q(\mathbf{p}', Z^*) \equiv Q^*(Z^*). \tag{16}$$

By Lemma 3, there exists a unique solution of (16). Since by Lemma 2, $U_0(v_0) + v_0 = CV^*$ whenever $v_0 \leq CV^*$, parts (ii) and (iii) follow from Theorem 1. **End of Proof.**

Note that by (ii) every mediator-optimal mechanism allocates the object according to the same rule $\mathbf{p}$ which grants the object to the bidder with the highest use value v_i, if $C_b(v_i) > Z^*$. If $C_b(v_i) < Z^*$, the object is not sold in the current period and the seller will re-auction it in the following period. Hence, Corollary 1 implies that any mediator-optimal mechanism is equivalent to the Vickrey auction with the reserve price $r^* = C_b^{-1}(Z^*)$, where $C_b^{-1}(\cdot)$ denotes the inverse function of $C_b(\cdot)$. Also note that the seller *always* receives the same expected return CV^* from the auction for any use value $v_0 \leq CV^*$.

3.3 Expected Payoff of the Mediator

In the previous section we described the mediator-optimal mechanism as a function of the continuation value CV^*. We shall now select CV^* which maximize the (unconditional) expected payoff of the mediator, U_M, and then derive the desired optimal mechanism $(\mathbf{p}, \mathbf{x})$.

Since for $v_0 > CV^*$ the seller does not auction the object, and thus the mediator receives zero, we have

$$U_M = \int_0^{CV^*} U_M^* f_s(v_0) dv_0 = U_M^* F_s(CV^*).$$

The mediator's expected unconditional payoff U_M is equal to the product of the mediator's expected gain conditional on $v_0 \leq CV^*$, U_M^*, and the probability that $v_0 \leq CV^*$, $F_s(CV^*)$. We have $U_M^* = Z^* - CV^*$, where Z^*, the highest joint gain of the mediator and the seller**,** is independent from the mechanism $(\mathbf{p}, \mathbf{x})$. Hence, CV^* must be a solution of the following optimization problem,

$$\max_{CV^* \in [0,1]} (Z^* - CV^*) F_s(CV^*). \tag{17}$$

That is, the expected revenue of the mediator U_M^* conditional on the event that the auction occurs will balance two opposite forces: The higher the (conditional) mediator revenue, $U_M^* = Z^* - CV^*$, the lower the probability that the seller is willing to auction the object, $F_s(CV^*)$.

From (17) and Corollary 1 we obtain the following theorem.

Theorem 2 *A feasible mechanism* $(\mathbf{p}, \mathbf{x})$ *is mediator-optimal if and only if*

(i) *The expected joint gain of the mediator (conditional on* $v_0 \leq CV^*$*) and the seller is a unique solution of equation* $Z^* = Q^*(Z^*)$*,*

(ii) *The expected payoff of the mediator is given by*

$$U_M = \max_{CV^* \in [0,1]} (Z^* - CV^*) F_s(CV^*), \tag{18}$$

(iii) *Mechanism* $(\mathbf{p}, \mathbf{x})$ *satisfies conditions (ii) – (iii) of Corollary 1 with respect to* CV^* *as in (18).*

4 Implementation

In this section we demonstrate that a mediator-optimal mechanism is implementable by a repeated Vickrey auction with a reserve price, where the mediator collects her payoff via a simple fee scheme.

Consider the following mechanism, the *Closing-fee Internet auction.* In every period, the mediator runs a Vickrey auction with a reserve price. The seller submits a reserve price, r, and every bidder submits a bid equal to her true use value. The winning bidder (if any) pays the greater of the second highest bid and the reserve price. If the object is sold, the mediator collects a *closing fee* (a percentage from the closing price),

$$\mu = \frac{U_M^*}{U_M^* + CV^*} \in [0,1]. \tag{19}$$

where $U_M^* = Z^* - CV^*$, Z^* and CV^* are defined in Theorem 2.

Theorem 3 *The Closing-fee Internet auction is optimal for the mediator.*

Proof. It is sufficient to show that $\mu = \frac{U_M^*}{U_M^*+CV^*}$ implies that the allocation rule $\mathbf{p}$ satisfies Condition (i) of Corollary 1. In a Vickrey auction, the optimal reserve price is

$$r^* = C_b^{-1}\left(\frac{1}{1-\mu}CV^*\right),$$

see, for example Krishna (2002, Section 5.2.2). Hence

$$r^* = C_b^{-1}\left(U_M^* + CV^*\right),$$

and the object is retained by the seller if $r^* > v_i$ for all $i = 1, \ldots, n$, or, equivalently, $C_b(r^*) = U_M^* + CV^* > C_b(v_i)$. **End of Proof.**

Theorem 3 describes the mediator-optimal Internet auction with just one fee. However, most of the real-life Internet auctions have two fees: A closing fee, μ, and a *listing fee*, $c \in \mathbb{R}$, a fixed fee collected at the beginning of every auction. In Matros and Zapechelnyuk (2006), we obtain the following result.

Proposition 1 *Consider the class of Internet auctions with a pair of fees, (c, μ). There exists a unique pair of fees (c^*, μ^*) which maximizes the mediator's expected payoff in this class. The fees are*

$$c^* = 0 \text{ and } \mu^* = \frac{U_M^*}{U_M^* + CV^*} \in [0, 1].$$

Theorem 3 and Proposition 1 demonstrate that any mechanism with a non-zero listing fee is not mediator-optimal. That is, the mediator does not benefit from asking a positive up-front fee in every auction, or from subsidizing the seller (i.e., when the listing fee is negative). It is interesting to note that many Internet auctions charge negligible or zero listing fees, and, for instance, eBay decreased the listing fee considerably in 2005. See Matros and Zapechelnyuk (2006) for more examples.

5 Extensions of the model

We discuss several extensions of the model in this section.

5.1 Collusion of the Seller and the Mediator

Suppose that the seller and the mediator collude.[5] Then the mediator knows the seller's use value and maximizes the joint payoff, MS^*. Similar to Lemma 2, the mediator auctions the object, if $v_0 \leq MS^*$ and consumes it, if $v_0 > MS^*$. The maximization problem (17) becomes

$$\max_{MS^* \in [0,1]} MS^* \tag{20}$$

subject to

$$MS^* = Q^*(MS^*). \tag{21}$$

[5] Equivalently, suppose that there is no mediator, and the seller himself is a mechanism designer.

From Lemma 3, we obtain

$$Z^* = Q^*(Z^*) = MS^*.$$

Therefore, the mediator's continuation value is Z^*. Since $Z^* > CV^*$, the auction occurs with higher probability than in the case of the mediator and a seller being independent. Hence, the expected joint gain of an independent seller and the mediator is less than the expected joint gain of them colluding.

5.2 Maintenance Fees

Suppose that the mediator has to pay a fixed *maintenance fee* $\eta \geq 0$ in order to run an auction each period. Again, the seller auctions the object, if $v_0 \leq CV^\eta$ and consumes it, if $v_0 > CV^\eta$. Then the expected payoff of the mediator at any period t, conditional on $v_0 \leq CV^\eta$ is

$$U_M^\eta = \delta \int\limits_{\mathbf{V}_{-0}} \left(\sum_{i=1}^{n} x_i(\mathbf{v}) - x_0(\mathbf{v}) + p_0(\mathbf{v}) U_M^\eta \right) \mathbf{f}(\mathbf{v}_{-0}) d\mathbf{v}_{-0} - \eta. \tag{22}$$

We have the following result.

Theorem 4 *Suppose that the seller's realized use value $v_0 \leq CV^\eta$. Then at any period $t \geq 1$, every feasible mechanism $(\mathbf{p}, \mathbf{x})$ satisfies*

$$U_M^\eta + CV^\eta + \sum_{i=1}^{n} U_i^t(0) + \eta = Q(\mathbf{p}, U_M^\eta + CV^\eta). \tag{23}$$

The proof is analogous to the proof of Theorem 1 and omitted here.

The mediator-optimal mechanisms w.r.t CV^η, if $v_0 \leq CV^\eta$, is similar to those one described in Corollary 1.

Corollary 2 *Let CV^η be given and suppose that $v_0 \leq CV^\eta$. Let $0 \leq \eta \leq Q^*(0)$. A feasible mechanism $(\mathbf{p}, \mathbf{x})$ is mediator-optimal w.r.t. CV^η if*

(i) *Z^η is a unique solution of equation $Z^\eta + \eta = Q^*(Z^\eta)$, and*

(ii) $\mathbf{p} \in \underset{\mathbf{p}':\mathbf{V}\to\Delta^{n+1}}{\operatorname{argmax}} Q(\mathbf{p}', Z^{\eta})$.

Again, the joint seller-mediator gain Z^{η} is maximized first, then this gain is divided between the seller and the mediator. Following the proof of Lemma 3, we obtain that Z^{η} is the unique solution of the following equation

$$(1-\delta)z + \eta = \delta T(z). \tag{24}$$

By assumption, $\eta \leq \delta T(0) = Q^*(0)$. It is straightforward to see that the total gain in this case, Z^{η}, is a decreasing function of the fee η, because Z^{η} is a unique intersection of the increasing and decreasing functions in the equation (24).

Hence, CV^{η} must be a solution of the following optimization problem

$$\max_{CV^{\eta}\in[0,1]} (Z^{\eta} - CV^{\eta}) F_s(CV^{\eta}).$$

Thus, we obtain the following result.

Theorem 5 *A feasible mechanism* $(\mathbf{p}, \mathbf{x})$ *with maintenance fee* η*,* $0 \leq \eta \leq Q^*(0)$*, is mediator-optimal if and only if*

(i) *The expected joint gain of the mediator and the seller is a unique solution of equation* $Z^{\eta} + \eta = Q^*(Z^{\eta})$*,*

(ii) *The expected payoff of the mediator is given by*

$$U_M = \max_{CV^{\eta}\in[0,1]} (Z^{\eta} - CV^{\eta}) F_s(CV^{\eta}), \tag{25}$$

(iii) *Mechanism* $(\mathbf{p}, \mathbf{x})$ *satisfies conditions (ii) – (iii) of Corollary 2 w.r.t.* CV^{η} *as in (25).*

5.3 The One-Period Model

Let us consider a special case where the seller is constrained to auction the object at most one time. This one-period model is a direct extension of Myerson and Satterthwaite's (1983) bilateral (one seller and one buyer) trade, mediated by a "broker", to the n-buyer problem.

Suppose that for any $v_0 \in [0,1]$ the seller auctions the object in the first period ($\alpha^1(v_0) = 1$) and never re-auctions it ($\alpha^t(v_0) = 0$ for all $t = 2, 3, \ldots$). For convenience, in all notations of this section we omit the affix referring to period 1. We normalize payoffs by selecting $\delta = 1$.

In a direct revelation mechanism, given v_i, the expected utility of bidder $i = 1, \ldots, n$ is defined by

$$U_i(v_i) = v_i \bar{p}_i(v_i) - \bar{x}_i(v_i), \tag{26}$$

the expected *gain* from trade for the seller is defined by

$$U_0(v_0) = -v_0 + \bar{x}_0(v_0) + v_0 \bar{p}_0(v_0), \tag{27}$$

and the expected utility of the mediator is defined by

$$U_M = \int\limits_{\mathbf{V}} \left(\sum_{i=1}^{n} x_i(\mathbf{v}) - x_0(\mathbf{v}) \right) \mathbf{f}(\mathbf{v}) d\mathbf{v}. \tag{28}$$

Denote by $C_b(v_i)$ the *virtual value* of bidder i, $i = 1, \ldots, n$, and denote by $C_s(v_0)$ the *virtual value* of the seller. Namely, C_b is defined above in (12) and C_s is given for every $v_0 \in [0,1]$ by

$$C_s(v_0) = v_0 + \frac{F_s(v_0)}{f_s(v_0)}. \tag{29}$$

We assume that $C_b(\cdot)$ is strictly increasing (the Myerson's regularity condition) and, in addition, assume that $C_s(\cdot)$ is strictly increasing. Define

$$W(\mathbf{p}) = \int\limits_{\mathbf{V}} \left[\sum_{i=1}^{n} \left(C_b(v_i) - C_s(v_0) \right) p_i(\mathbf{v}) \right] \mathbf{f}(\mathbf{v}) d\mathbf{v}.$$

The following theorem and corollaries are a straightforward generalization of Myerson and Satterthwaite's (1983). We omit the proofs.

Theorem 6 *Every feasible mechanism* $(\mathbf{p}, \mathbf{x})$ *satisfies*

$$U_M + \sum_{i=1}^{n} U_i(0) + U_0(1) = W(\mathbf{p}), \tag{30}$$

and

$$U_M \leq W(\mathbf{p}). \tag{31}$$

In particular, Theorem 6 demonstrates that the expected payoff of the mediator depends only on the rule of the object allocation, $\mathbf{p}$, and on the payoffs of players with the extreme private use values. This yields the following result of revenue equivalence.

Corollary 3 (Revenue Equivalence) *Let* $(\mathbf{p}, \mathbf{x})$ *and* $(\mathbf{p}', \mathbf{x}')$ *be two feasible revelation mechanisms. Suppose that* $\mathbf{p} = \mathbf{p}'$ *and the expected payoffs of traders with extreme use values,* $U_0(1)$ *and* $U_i(0) = 0$ *for all* $i = 1, \ldots, n$, *are the same in both mechanisms. Then the mediator's expected payoffs are the same in the two mechanisms.*

It follows from Theorem 6 that a mediator-optimal mechanism maximizes $W(\mathbf{p})$, that is, the allocation rule $\mathbf{p}$ grants the object to the trader with the highest virtual value. Thus, we have the following corollary (see also Myerson, 1981; and Myerson and Satterthwaite, 1983).

Corollary 4 *Every feasible mechanism* $(\mathbf{p}, \mathbf{x})$ *which is optimal for the mediator is equivalent*[6] *to a Vickrey auction with the seller's reserve price defined for every* $v_0 \in [0, 1]$ *by*

$$r^*(v_0) = C_b^{-1}\left(C_s(v_0)\right).$$

[6] I.e., the object allocation rule and the *expected* payoffs of the players are the same.

6 Discussion

We conclude with a brief discussion of some model assumptions.

1. It is crucial for our results that in every period the seller faces the same trade environment and, thus, he has the same expected payoff. How the trade environment is modelled is unimportant. Consequently, our results can be applied to a considerably wider class of problems. For instance, the number of bidders drawn in every period may be random, as long as it is identically distributed across periods.

2. The assumption that the mechanism is fixed and stationary is essential for our results. The real life supports this assumption: in all Internet auctions the rules and fees are fixed.

3. In our model a winning bidder is not allowed to re-auction the object. Adding this possibility for a winning bidder would not make any effect on the mediator-optimal mechanism, since the winning bidder will face the same stationary environment in the next period. This contrasts our results to Zheng (2002), who assumes that a fixed, finite set of bidders is involved in trade, thus, the initial seller and a winning bidder face different trade environments.

4. We assume that the auction mediator is a monopolist. It is interesting, however, to consider the situation with several competing mediators, and relate the results to the study of two-sided markets mediated by "platforms" (e.g, Rochet and Tirole, 2003). We are investigating that now and will report our results elsewhere.

Appendix

Proof of Theorem 1

We make use of the following two Lemmata (which are modified results of Myerson, 1981).

Lemma 4 *Let* $(\mathbf{p}, \mathbf{x})$ *be a feasible mechanism. Then, for every* $i = 1, \ldots, n$, $\bar{p}_i$ *are increasing, and for each* $v_i \in [0, 1]$

$$U_i^t(v_i^t) = U_i^t(0) + \delta \int_0^{v_i^t} \bar{p}_i(z)dz. \tag{32}$$

Proof. Using (1) we can rewrite (4) as

$$U_i^t(v_i^t) \geq U_i^t(w_i^t|\mathbf{v}_i^t) + \delta(v_i^t - w_i^t)\bar{p}_i(w_i^t),$$

for all $v_i^t, w_i^t \in [0, 1]$ and $i = 1, \ldots, n$. Then, using (4) twice (once with the roles of v_i^t and w_i^t switched), we obtain

$$\delta(v_i^t - w_i^t)\bar{p}_i(w_i^t) \leq U_i^t(v_i^t) - U_i^t(w_i^t|v_i^t) \leq \delta(v_i^t - w_i^t)\bar{p}_i(v_i^t).$$

It follows for $w_i^t = v_i^t - \varepsilon$ and arbitrary $\varepsilon > 0$ that

$$\delta\bar{p}_i(v_i^t - \varepsilon) \leq \frac{U_i^t(v_i^t) - U_i^t(w_i^t)}{\varepsilon} \leq \delta\bar{p}_i(v_i^t).$$

Thus, $\bar{p}_i$ is increasing and Riemann integrable, so, for all $v_i^t \in [0, 1]$

$$\delta \int_0^{v_i^t} \bar{p}_i(z)dz = U_i^t(v_i^t) - U_i^t(0),$$

which yields (32). **End of Proof.**

Lemma 5 *Let* $(\mathbf{p}, \mathbf{x})$ *be a feasible mechanism. Then for every* $i = 1, \ldots, n$ *and every* $v_i \in [0, 1]$

$$\int_0^1 x_i(\mathbf{v})f_b(v_i)dv_i = \int_0^1 C_b(v_i)p_i(\mathbf{v})f_b(v_i)dv_i - \frac{1}{\delta}U_i^t(0). \tag{33}$$

Proof. Using (1) and Lemma 4 we obtain

$$\bar{x}_i(v_i^t) = v_i^t \bar{p}_i(v_i^t) - \int_0^{v_i^t} \bar{p}_i(z)dz - \frac{1}{\delta}U_i^t(0),\ i = 1, \ldots, n$$

and

$$x_i(\mathbf{v}^t) = v_i^t p_i(\mathbf{v}^t) - \int_0^1 \left(\int_0^{v_i^t} p_i(z, \mathbf{v}_{-i})dz \right) f_b(v_i^t)dv_i^t - \frac{1}{\delta}U_i^t(0).$$

Note that

$$\int_0^1 \left(\int_0^{v_i^t} p_i(z, \mathbf{v}_{-i})dz \right) f_b(v_i^t)dv_i^t =$$

$$\int_0^1 \left(\int_z^1 f_b(v_i^t)dv_i^t \right) p_i(z, \mathbf{v}_{-i})dz = \int_0^1 (1 - F_b(z))\, p_i(z, \mathbf{v}_{-i})dz.$$

Therefore

$$\int_0^1 \left(\int_{\mathbf{V}_{-i}} x_i(v_i, \mathbf{v}_{-i})\mathbf{f}_{-i}(\mathbf{v}_{-i})d\mathbf{v}_{-i} \right) f_b(v_i)dv_i =$$

$$\int_0^1 \left(v_i \int_{\mathbf{V}_{-i}} p_i(v_i, \mathbf{v}_{-i})\mathbf{f}_{-i}(\mathbf{v}_{-i})d\mathbf{v}_{-i} \right) f_b(v_i)dv_i - \int_0^1 (1 - F_b(v_i))\, p_i(v_i, \mathbf{v}_{-i})dv_i - \frac{1}{\delta}U_i^t(0) =$$

$$\int_0^1 \left(v_i - \frac{1 - F_b(v_i)}{f_b(v_i)} \right) p_i(\mathbf{v}) f_b(v_i)dv_i - \frac{1}{\delta}U_i^t(0). \qquad (34)$$

Since $C_b(z) = z - \frac{1-F_b(z)}{f_b(z)}$, (34) immediately yields (33). **End of Proof.**

Proof of Theorem 1.

Using Lemma 5, we obtain from (11)

$$U_M^* = \delta \int\limits_{\mathbf{V}_{-0}} \left(\sum_{i=1}^n x_i(\mathbf{v}) - x_0(\mathbf{v}) + p_0(\mathbf{v})U_M^* \right) \mathbf{f}(\mathbf{v}_{-0})d\mathbf{v}_{-0} =$$

$$\delta \int\limits_{\mathbf{V}_{-0}} \left(\sum_{i=1}^n C_b(v_i)p_i(\mathbf{v}) - x_0(\mathbf{v}) + p_0(\mathbf{v})U_M^* \right) \mathbf{f}(\mathbf{v}_{-0})d\mathbf{v}_{-0} - \sum_{i=1}^n U_i^t(0). \quad (35)$$

From (8), since $(\mathbf{p}, \mathbf{x})$ is a direct revelation mechanism, we have

$$CV^* = \delta \left(\bar{x}_0(v_0) + \bar{p}_0(v_0)CV^* \right).$$

Thus

$$\bar{x}_0(v_0) = CV^* \left(\frac{1}{\delta} - \bar{p}_0(v_0) \right).$$

We rewrite (35) as follows,

$$U_M^* = \delta \int\limits_{\mathbf{V}_{-0}} \left(\sum_{i=1}^{n} C_b(v_i) p_i(\mathbf{v}) + (U_M^* + CV^*)\, p_0(\mathbf{v}) - \frac{CV^*}{\delta} \right) \mathbf{f}(\mathbf{v}_{-0}) d\mathbf{v}_{-0} - \sum_{i=1}^{n} U_i^t(0).$$

Since $p_0(\mathbf{v}) = 1 - \sum_{i=1}^{n} p_i(\mathbf{v})$, we obtain

$$U_M^* = -CV^* - \sum_{i=1}^{n} U_i^t(0) +$$

$$\delta \left[(U_M^* + CV^*) + \int\limits_{\mathbf{V}_{-0}} \left(\sum_{i=1}^{n} \left[C_b(v_i) - (U_M^* + CV^*) \right] p_i(\mathbf{v}) \right) \mathbf{f}(\mathbf{v}_{-0}) d\mathbf{v}_{-0} \right] =$$

$$Q(\mathbf{p}, U_M^* + CV^*) - CV^* - \sum_{i=1}^{n} U_i^t(0),$$

which immediately yields (13). Since by the IR constraint $U_i(0) \geq 0$ for all $i = 1, \ldots, n$, we obtain (14). **End of Proof.**

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