

TWO NOTES ON THE BLOTTO GAME

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ABSTRACT. We exhibit a new equilibrium of the classic Blotto game in which players allocate one unit of resources among three coordinates and try to defeat their opponent in two out of three. It is well known that a mixed strategy will be an equilibrium strategy if the marginal distribution on each coordinate is $U\left[0, \frac{2}{3}\right]$. All known examples of such distributions have two-dimensional support. Here we exhibit a distribution which has one-dimensional support and is simpler to describe than previous examples. The construction generalizes to give one-dimensional distributions with the same property in higher-dimensional simplexes as well.

As our second note, we give some results on the equilibrium payoffs when the game is modified so that one player has greater available resources. Our results suggest a criterion for equilibrium selection in the original symmetric game, in terms of robustness with respect to a small asymmetry in resources.

1. INTRODUCTION

Consider a game in which the two players simultaneously select vectors from $[0, 1]^N$ whose coordinates sum to 1, and are considered to have won a coordinate if they select a higher number than their opponent in that coordinate. After Laslier and Picard (2002), we call the game in which a player's payoff is the number of coordinates won minus the number lost the *plurality game*. Alternatively, the object could be simply to win a majority of coordinates. This is called the *majority game*. This game can be interpreted as a contest between

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politicians allocating advertising money among N states in a simplified electoral college system, where each state is won by the side with greater spending. The classic case $N = 3$ was first described by Borel (1921), and equilibria were first given in Borel and Ville (1938). It is often called the Colonel Blotto game, as it could be interpreted as a model of resource allocation in warfare, assuming that even a small advantage in resources allocated to a given battle is enough to win that battle completely. In the special case that $N = 3$ and budgets are equal, the majority game and plurality game coincide because a player can never win in 0 or 3 coordinates.

It is well-known that a mixed strategy given by a distribution on $[0, 1]^N$ whose marginal distribution on each coordinate is *Uniform* $[0, \frac{2}{N}]$ will be an equilibrium strategy in the plurality game. Borel and Ville (1938) found two examples of such distributions for $N = 3$, one with support on the inscribed disc in the triangular representation of the simplex and one, called the Hex equilibrium, with support on the full hexagon $\{x_1, x_2, x_3 \in [0, \frac{2}{3}] : x_1 + x_2 + x_3 = 1\}$ which is the set of best responses in both equilibria. We will exhibit an equilibrium strategy here, with the same marginal distributions, that has one-dimensional support – in particular, its support consists of two line segments. The construction generalizes to give a solution for the N -dimensional plurality game. Specifically, we give a distribution on $\{x \in [0, 1]^N : \sum x_i = 1\}$ with support on $N - 1$ parallel line segments and the desired marginals.

Also, there are no current results on the modification of the majority game in which players have different total wealth, i.e. one player picks a vector whose sum is 1 while the other picks a vector whose sum is r . We will provide bounds on the equilibrium payoffs in such a game as a function of r . We obtain particularly tight bounds when r is close to 1, thereby characterizing the marginal impact of a small advantage in available resources.

2. RECENT LITERATURE

Laslier and Picard (2002) apply equilibria of the Blotto game to analyze the redistribution of goods that results from two-party electoral competition. In

particular, they give the Lorenz curve and determine other measures of the inequality that would result from the distributions prescribed by the disc equilibrium. Kvassov (2003) analyzes Blotto-style contests with the modification that players do not necessarily use all their available resources. This approach would be justified in the many applications in which resources have an alternative use or can be saved for the next period. Our framework, on the other hand, in which resources must be spent immediately or lost, would frequently be appropriate in the context of campaign spending or warfare. Kvassov also allows for asymmetric budgets. Unlike our results on asymmetric budgets in Section 4, which focus on the majority game, he focuses on a game in which the objective is to win as many coordinates as possible (plurality game). Clearly this would be appropriate in auctions or other contexts where each coordinate won has value, while the majority perspective would usually be appropriate in an electoral context.

The majority game is also relevant in the scenario studied by Szentes and Rosenthal (2003), a simultaneous auction for three objects (chopsticks) in which the marginal value of acquiring a second object is high compared to the first. They are able to completely describe the equilibria of such auctions, which are closely related to Blotto games. The key difference is that the lower bidder for each object does not pay, whereas the usual Blotto game is similar to an all-pay auction. The all-pay condition is a sensible model when resources cannot be recovered, as in campaign spending or warfare.

3. ONE-DIMENSIONAL EQUILIBRIUM

In this section, we will exhibit a one-dimensional distribution on the N -dimensional simplex $\Delta_N = \left\{ x \in \mathbb{R}^N : x_i \geq 0, \sum_{i=1}^N x_i = 1 \right\}$ whose marginal distribution on each coordinate is *Uniform* $\left[0, \frac{2}{N}\right]$ and which is therefore an equilibrium of the plurality game. We can depict this distribution graphically in the case $N = 3$ as a uniform distribution on the two line segments pictured in Figures 1 and 2. Figure 1 depicts the simplex explicitly as a subset of \mathbb{R}^3 , while in Figure 2 we have the usual two-dimensional representation which we

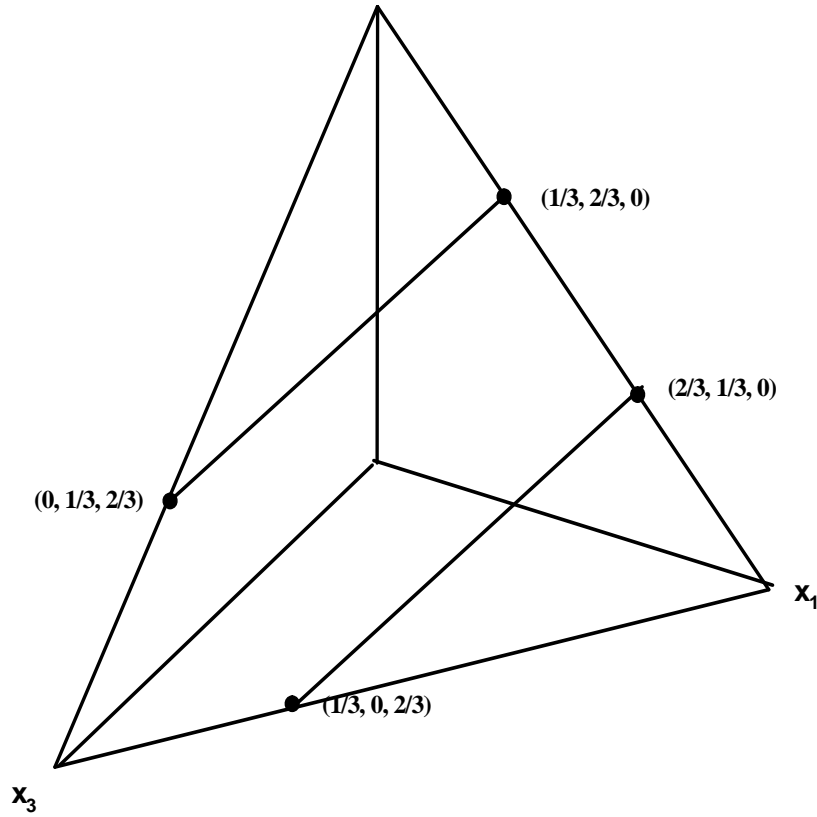


FIGURE 1

will use henceforth. This is obtained by letting the plane of the page be the plane $x_1 + x_2 + x_3 = 1$. Notice that the distribution of x_3 is uniform on each line segment individually. Also, coordinate x_1 is distributed $U\left[0, \frac{1}{3}\right]$ on the left-hand segment in Figure 2 and $U\left[\frac{1}{3}, \frac{2}{3}\right]$ on the right-hand segment, yielding the correct distribution overall. Similarly, coordinate x_2 is distributed $U\left[0, \frac{1}{3}\right]$ on the right-hand segment and $U\left[\frac{1}{3}, \frac{2}{3}\right]$ on the left-hand segment. In general, our distribution will be uniform on $N-1$ parallel line segments in the N -dimensional simplex, as described below.

Proposition 1. *Let $T = 1 + 2 + \dots + (N-1) = \frac{N(N-1)}{2}$. The uniform distribution on the $N-1$ parallel line segments whose endpoints are given by $\frac{1}{T}(k, k+1, \dots, N-2, 0, \dots, k-1, N-1)$ and $\frac{1}{T}(k+1, k+2, \dots, N-1, 1, \dots, k, 0)$*

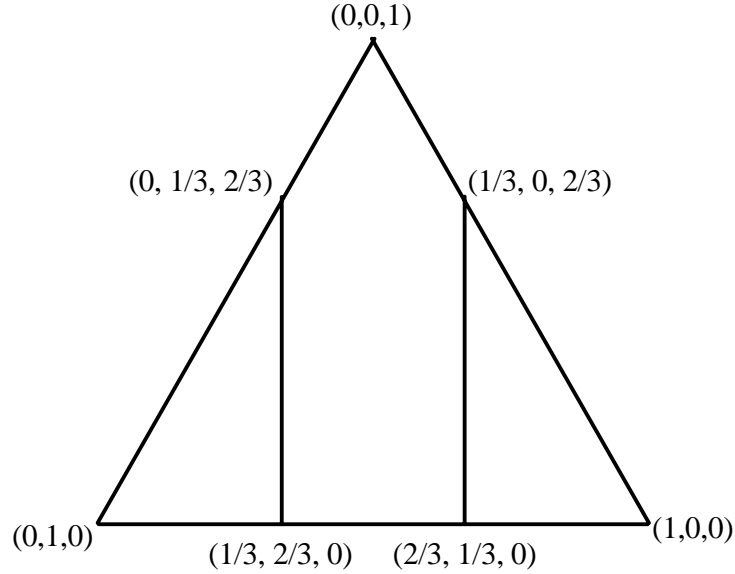


FIGURE 2

for $k = 0, 1, \dots, N - 2$ gives a distribution on Δ_N whose marginal distribution on each coordinate is Uniform $[0, \frac{2}{N}]$.

Proof. First notice that the coordinates of each endpoint sum to 1, so that each line segment is indeed contained in the simplex. Also, the distribution of the last coordinate x_N , which plays a special role, is $U [0, \frac{N-1}{T}] = U [0, \frac{2}{N}]$ on each segment. The distribution of the first coordinate is $U [\frac{k}{T}, \frac{k+1}{T}]$ on the k th segment, yielding the correct overall distribution as k runs from 0 to $N - 2$. A similar argument applies to coordinates x_2 through x_{N-1} . \square

Notice that we would get a different distribution if we relabelled the coordinates; if we take the average of the distributions formed by the possible labellings we get a distribution which, like the classic examples, is symmetric between the coordinates. In the 3-dimensional case this is a uniform distribution on the six-pointed star pictured in Figure 3.

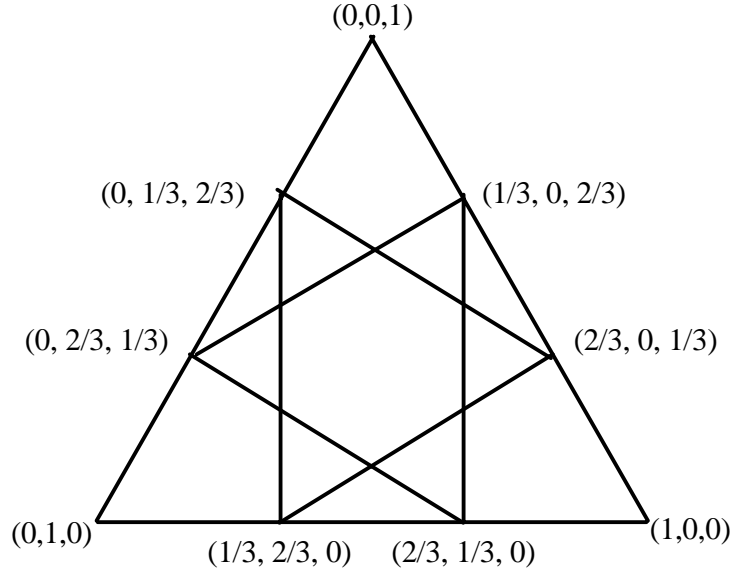


FIGURE 3

Distributive Implications and the Lorenz Curve

Laslier and Picard (2002) compute the average Lorenz curve that would result if wealth were distributed as in the disc equilibrium. If we order a division of one unit of wealth among N individuals so that $y_1 \leq y_2 \leq \dots \leq y_n$, the Lorenz curve is defined by the partial sums $c_k(y) = \sum_{i=1}^k y_i$. Given the mixed strategy defined above, a straightforward computation shows that the expected values of these partial sums are $l_k(N) = \frac{k^2}{N^2}$. In the limit, this approximates an average Lorenz curve of $c(t) = t^2$; that is, the average proportion of total wealth held by the poorest fraction t of the population is t^2 . In contrast, for the disc equilibrium Laslier and Picard find that the corresponding curve is $c_d(t) = t - \frac{1}{4} \sin \pi t$, which lies above our curve, so that there is more inequality in our equilibrium. Indeed, they find that the limit of the Gini index of inequality (defined as twice the area between the Lorenz curve and the diagonal) is $\frac{1}{\pi}$, while for us it is $\frac{1}{3}$.

4. ASYMMETRIC BUDGETS

In this section we will analyze the majority game in the case $N = 3$, with the modification that player 2 has a total budget of 1 unit but player 1 has a total budget of r . Note that in the plurality game payoffs are completely determined by the marginal distributions on each coordinate, because of the additively separable utility functions. This property does not hold in the majority game except in the special case of equal budgets. In general, this makes it more difficult to describe equilibria, but we will be able to show some results giving bounds on the equilibrium payoffs for different values of r . In particular, let $w(r)$ be the equilibrium probability of winning for player 1. This section will establish some properties of the function w .

It will be convenient to modify our tie-breaking rule and specify that the player with the larger budget wins all ties, as suggested by Kvassov (2003). This ensures that payoffs are weakly lower-semicontinuous, which along with the fact that we have a constant-sum game with compact action spaces, and discontinuities lie in a lower-dimensional space, allows us to apply a result of Dasgupta and Maskin (1986) to guarantee existence of a mixed-strategy equilibrium.

We note that in two-player zero-sum games, it is appropriate to speak of equilibrium *strategies* rather than *strategy profiles*. Indeed, we can define an equilibrium strategy as one which guarantees that the player receives at least his equilibrium (or maximin) payoff, and any pair of equilibrium strategies will be an equilibrium in the usual sense. Similarly, we can speak of an ε -equilibrium strategy as one which guarantees that the player comes within ε of his maximin payoff.

We observe that changing r to $\frac{1}{r}$ effectively interchanges the roles of the two players, so that we have the following:

Fact 1: $w(\frac{1}{r}) = 1 - w(r)$.

Because of this symmetry, we will focus on the case $r > 1$. We will now specify exactly how much of an advantage is necessary for player 1 to guarantee victory.

Proposition 2. $w(r) = 1$ if and only if $r \geq \frac{3}{2}$.

Proof. If $r \geq \frac{3}{2}$ player 1 can guarantee victory by choosing the vector $(\frac{r}{3}, \frac{r}{3}, \frac{r}{3})$; player 2 cannot win because beating player 1 in two coordinates would require more than $\frac{2r}{3} \geq 1$ unit of wealth.

Now suppose $r < \frac{3}{2}$, and let player 2 use the strategy which is uniformly distributed on the simplex. Take any action of player 1, and assume without loss of generality $x_1 \leq x_2 \leq x_3$, so that $x_1 + x_2 \leq \frac{2}{3}r < 1$. Then the region of player 2's action space in which he wins coordinates 1 and 2 is an equilateral triangle of side $1 - x_1 - x_2 \geq 1 - \frac{2}{3}r$, so for fixed r , there is a positive lower bound on his winning probability proportional to $(1 - \frac{2}{3}r)^2$. \square

We also have the following:

Proposition 3. If $r \geq \frac{5}{4}$ then $w(r) \geq \frac{2}{3}$.

Proof. Suppose player 1 uses an equal mixture among the three vectors given by $(\frac{1}{2}, \frac{1}{2}, r - 1)$ and its permutations. We claim that any vector chosen by player 2 defeats at most one of these three vectors, so player 1 is guaranteed to win at least two-thirds of the time. Suppose to the contrary, so that without loss of generality player 2 has a vector (x_1, x_2, x_3) which wins against $(\frac{1}{2}, \frac{1}{2}, r - 1)$ and $(\frac{1}{2}, r - 1, \frac{1}{2})$. In order to win against the first of these vectors we must have $x_3 > r - 1 > \frac{1}{4}$ and in order to win against the second vector we must have $x_2 > r - 1 > \frac{1}{4}$, implying that $x_1 < \frac{1}{2}$. Then, we would have to win against both vectors in both the second and third coordinate, implying x_2 and x_3 are both greater than $\frac{1}{2}$ which is impossible. \square

As our main result in this section, we will determine the marginal impact of a player having a small advantage in available resources. The general idea is that

the equilibrium strategies from the symmetric case will still approximate equilibrium strategies here. From this argument, we will get the following result, which gives tight bounds on $w(r)$ when r is close to 1. It will turn out to be important that the stronger player uses an equilibrium strategy with bounded density. The Hex equilibrium has density proportional to $\max_i |x_i - \frac{1}{3}|$, which is bounded above. Thus, the one-dimensional distribution given in the previous section, or the disc equilibrium, which has unbounded density near the boundary, would be inferior to the Hex strategy when budgets are slightly asymmetric, in the sense that they do not approximate equilibrium strategy as closely, giving the weaker player a higher maximum payoff.

Proposition 4. *There exists $A > 0$ such that for all $r > 1$, $\frac{3}{2}r - 1 - A(r - 1)^2 \leq w(r) \leq \frac{3}{2}r - 1$.*

Proof. Assume that $r > 1$ and that player 2 employs a strategy which has marginal distribution $U[0, \frac{2}{3}]$ on each coordinate. Then for any vector in player 1's choice set $\{(x_1, x_2, x_3) : x_1 + x_2 + x_3 = r\}$ his probability of winning in coordinate i is $p_i = \min(\frac{3}{2}x_i, 1)$. Then $p_1 + p_2 + p_3 \leq \frac{3}{2}r$, with equality if $x_1, x_2, x_3 \leq \frac{2}{3}$. With $r > 1$, player 1 could win in one, two, or all three coordinates. Let the probability that he wins in exactly j coordinates be q_j . We can now compute his expected number of coordinates won in two different ways, as $q_1 + 2q_2 + 3q_3 = p_1 + p_2 + p_3$. Thus, the probability that player 1 wins a majority is $q_2 + q_3 = p_1 + p_2 + p_3 - (q_1 + q_2 + q_3) - q_3 = p_1 + p_2 + p_3 - 1 - q_3 \leq \frac{3}{2}r - 1$. Therefore, $w(r) \leq \frac{3}{2}r - 1$.

Now, to get a lower bound on $w(r)$, assume player 1 employs a scaled version of the Hex strategy which has marginal distribution $U[0, \frac{2}{3}r]$ on each coordinate, and has maximum density $\bar{\rho}$. We will seek the action for player 2 that maximizes his winning probability. For each vector (y_1, y_2, y_3) in player 2's choice set, his probability of winning each coordinate i is $p_i = \min(\frac{3}{2r}y_i, 1)$. Since player 2 will always win in 0, 1, or 2 coordinates, a derivation similar to that above gives that his winning probability is $s_2 = p_1 + p_2 + p_3 - 1 + s_0$ where s_j is the probability that he wins exactly j coordinates. For a fixed action of player 2, the region in player 1's action space for which player 2 wins in 0

coordinates will be an equilateral triangle of side $r - 1$, as depicted in Figure 4. He will want to choose his action so that this triangle is in a region of maximal density¹, which is why it is important that the density of player 1's strategy be bounded. We then have that player 2's winning probability satisfies

$$\begin{aligned} s_2 &\leq p_1 + p_2 + p_3 - 1 + \bar{\rho} \frac{\sqrt{3}}{4} (r - 1)^2 \\ &\leq \frac{3}{2r} - 1 + \bar{\rho} \frac{\sqrt{3}}{4} (r - 1)^2 \end{aligned}$$

Therefore

$$w(r) \geq 1 - s_2^{\max} \geq 2 - \frac{3}{2r} - \bar{\rho} \frac{\sqrt{3}}{4} (r - 1)^2$$

Now we will use the expansion

$$\begin{aligned} \frac{1}{r} &= \frac{1}{1 - (1 - r)} = 1 + (1 - r) + (1 - r)^2 + (1 - r)^3 \dots \\ &< 1 + (1 - r) + (1 - r)^2 \end{aligned}$$

when $r > 1$. Substituting this into the inequality above, we have

$$\begin{aligned} w(r) &\geq 2 - \frac{3}{2} [1 + (1 - r) + (1 - r)^2] - \bar{\rho} \frac{\sqrt{3}}{4} (r - 1)^2 \\ &= \frac{3}{2} r - 1 - \left(\bar{\rho} \frac{\sqrt{3}}{4} + \frac{3}{2} \right) (r - 1)^2 \end{aligned}$$

as desired. □

In the course of the proof, we also showed

Corollary 1. *Fix $r > 1$ and let $\varepsilon = A(r - 1)^2$. Any equilibrium strategy of the symmetric game is an ε -equilibrium strategy for the weaker player. The Hex strategy is an ε -equilibrium strategy for the stronger player, while any other equilibrium strategy of the symmetric game with higher maximum density is not.*

Our bounds on $w(r)$, which only differ by a second-order term for r close to 1, also yield the following result.

¹It may momentarily seem obscure that he wants to maximize his probability of losing all three coordinates. The key is that his expected number of coordinates won is constant, and winning exactly one coordinate is a useless waste of resources.

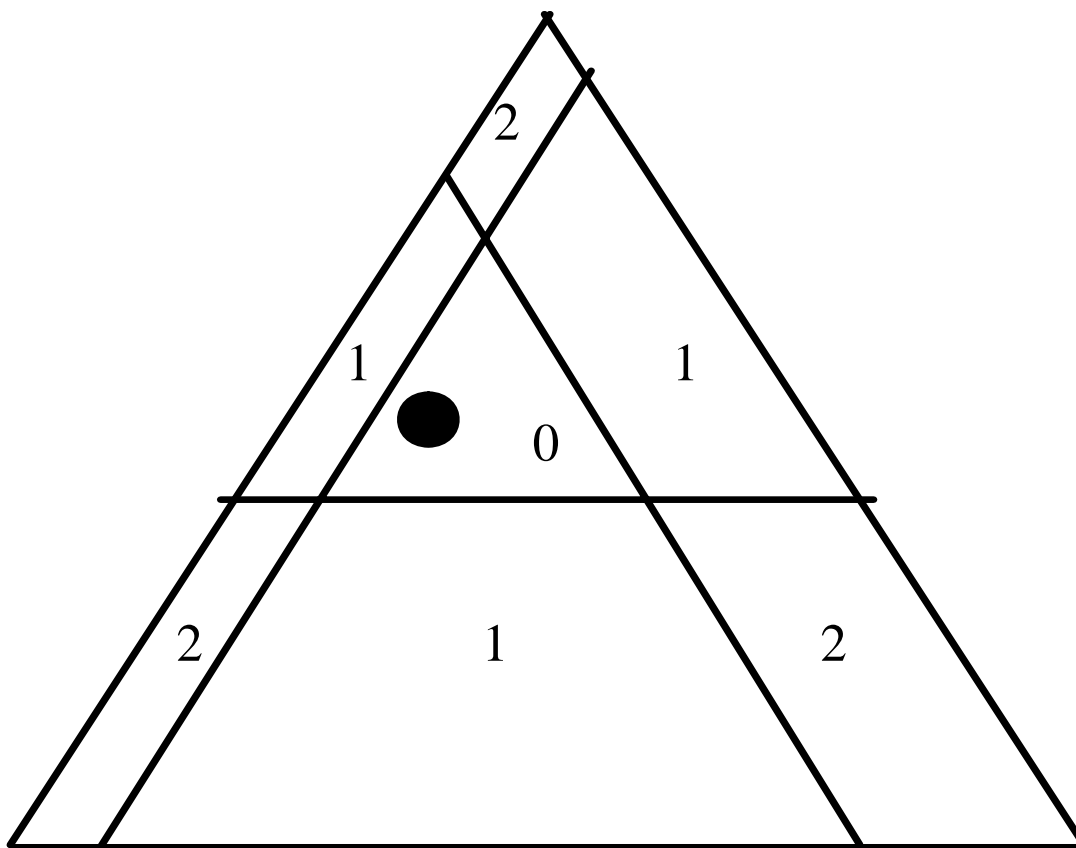


FIGURE 4. The simplex represents the action space of player 1, who is assumed to have fixed a mixed strategy. The point represents an action (y_1, y_2, y_3) for player 2. The label in each region is the number of coordinates player 2 would win if 1's action were in that region, so he is trying to maximize the total mass under player 1's strategy of the regions labeled 2. This turns out to be equivalent to maximizing the mass of the triangle labeled 0.

Corollary 2. $w'(1) = \frac{3}{2}$.

Proof. For the right derivative this is immediate. The left derivative follows from a short derivation using symmetry considerations (i.e. Fact 1). \square

5. CONCLUDING REMARKS

The proof of our final proposition suggests that an equilibrium strategy of the standard Blotto game will be more robust to small asymmetries in players' available resources when the density of the strategy is bounded above. It is relatively easy to show that the Hex strategy has the smallest maximum density of any distribution with the appropriate marginals, so that it is maximally robust from this point of view. The new distribution which we presented, of course, does not do well under this criterion, since it has one-dimensional support. It does have the aesthetic advantage of being extremely easy to describe and verify.

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