Multi-sender cheap talk with restricted state space*

Attila Ambrus† and Satoru Takahashi‡

This Version: February 2006

Abstract

This paper analyzes multi-sender cheap talk in multidimensional environments. Battaglini (2002) shows that if the state space is a multidimensional Euclidean space, then generically there exists a fully revealing equilibrium. We show that if the state space is restricted, either because the policy space is restricted or the set of rationalizable policies of the receiver is not the whole space, then Battaglini's equilibrium construction is in general not valid. We provide a necessary and sufficient condition for the existence of fully revealing equilibrium for any state space. For compact state spaces, we show that in the limit as the magnitudes of biases go to infinity, the existence of such equilibrium depends on whether the biases are of similar directions, where the similarity relation between biases depends on the shape of the state space. Our results imply that similar qualitative conclusions hold for the existence of fully revealing equilibrium for one-dimensional and multidimensional state spaces. We investigate the issue of how much information can be revealed in equilibrium if full revelation is not possible, and we address the question of robustness of equilibria.

*We would like to thank Drew Fudenberg for comments and suggestions, seminar participants at University of Pennsylvania, UC Berkeley, and the Harvard-MIT theory seminar for comments, and Niels Joaquin for valuable research assistance.

†Department of Economics, Harvard University, Cambridge, MA 02138, ambrus@fas.harvard.edu

‡Department of Economics, Harvard University, Cambridge, MA 02138, stakahas@fas.harvard.edu

1 Introduction

Sender-receiver games with cheap talk have been used extensively in both economics and political science to analyze situations in which an uninformed decision-maker acquires advice from an informed expert whose preferences do not fully coincide with those of the decision-maker. The seminal paper of Crawford and Sobel (1982) has been extended in many directions. In particular, Gilligan and Krehbiel (1989), Austen Smith (1993), and Krishna and Morgan (2001a, 2001b) investigate the case when the decision-maker can seek advice from multiple experts. However, Battaglini (2002) challenged the validity of the above models by showing that the qualitative conclusions of these models are different if one departs from the simplifying assumption that information is one-dimensional. He argues that in a multidimensional environment, no matter how large the biases of the experts are, generically there exists an equilibrium in which all information is revealed to the decision-maker. The construction provided is simple and intuitive: each sender only conveys information in directions along which her interest coincides with that of the receiver (directions that are orthogonal to the bias of the expert). Generically these directions of common interest span the whole state space; therefore, by combining the information obtained from the experts, the decision-maker can perfectly identify the state of the world. This is in sharp contrast with the conclusions obtained from unidimensional models in which full information revelation is not possible if experts have biases in different directions and the biases are large enough. Since in most relevant situations information is multidimensional, Battaglini concludes that using one-dimensional models to analyze sender-receiver games with multiple senders is problematic.

In this paper, we revisit the analysis of multidimensional cheap talk and claim that a careful comparison between unidimensional and multidimensional models reveals more similarities in qualitative predictions than what is suggested by the previous analysis. Our starting point is that Battaglini's equilibrium construction in a d-dimensional model (where $d \geq 2$) assumes that the state space is $\mathbb{R}^d$. The experts report coordinates along certain dimensions, and these dimensions span the whole space. The receiver is then supposed to choose the policy that belongs to the point identified by the above coordinates. This is always possible if the state space is the whole Euclidean space, but not necessarily so otherwise. Consider, for example, a situation in which a policymaker needs to allocate a fixed budget to "education," "military spending," and "healthcare," and this decision depends on factors that are unknown to her. Suppose she can ask for advice from two perfectly informed experts, a left-wing analyst and a right-wing analyst. Assume that the left-wing analyst has a bias towards spending more on education, while the right-wing analyst has a bias towards spending more on the military; both of them are unbiased with respect to healthcare. The situation can be depicted as in Figure 1. The state space in this example is represented by triangle ABC. B corresponds to a state in which it is optimal for the policymaker to spend the whole budget on the military; C

corresponds to a state in which it is optimal to spend all money on education; while A corresponds to a state in which it is optimal to spend no money on either education or military.[1] The left-wing analyst's bias is orthogonal to AB, in the direction of C. The right-wing analyst's bias is orthogonal to AC, in the direction of B.

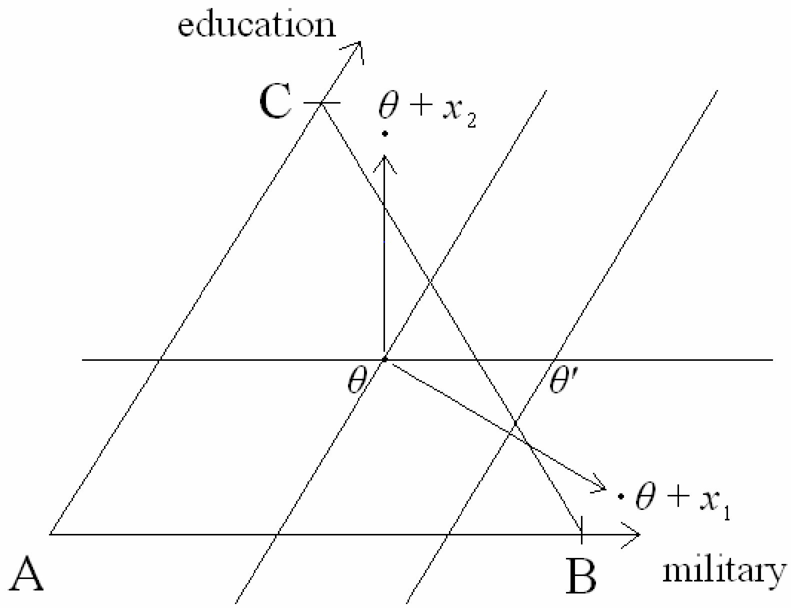

Figure 1

Battaglini's solution in this case is asking the left-wing analyst to report along a line parallel to AC, which in effect means asking how much money should be spent on the military. Similarly, the construction calls for the right-wing analyst to report along a line parallel to AB. But note that in this example, it is not true that any pair of such reports identifies a point in the state space. Consider state θ in Figure 1. If the left-wing analyst sends a truthful report, then the right-wing analyst can send many reports that are incompatible with the previous message in the sense that the only point compatible with the message pair is outside the state space (like θ' in the figure). Intuitively, these incompatible messages call for a combined expenditure on military and education that exceeds the budget. These type of incompatible reports of course

[1] The example would remain valid if there were some positive lower bounds on each type of spending. In that case, the vertices of the triangle represent scenarios in which all the remaining part of the budget above the given minimum expenditures goes to one area.

never arise if the experts indeed play according to the candidate equilibrium. Nevertheless, it is important to specify what action the policymaker takes after receiving a message like that, in order to make sure that both of the experts have the incentive to tell the truth. This raises the question whether Battaglini's construction can be extended by specifying actions after incompatible reports in such a way that it is always in the interest of both experts to tell the truth, and if not, then whether there exists any fully revealing equilibrium.

We address this question and, more generally, the issue of how much information can be transmitted in a multi-sender cheap talk model if the state space is not necessarily the whole space. We devote highlighted attention to compact state spaces in multidimension in order to be able to make a fair comparison between qualitative features of equilibria in multidimensional models and the results from the earlier literature on unidimensional models. In those models, the state space is assumed to be a compact interval. To the same extent, we argue that the case of an unbounded state space in a multidimensional model should be compared to the case of an unbounded state space in a unidimensional model.

There are some conceptual questions that need to be clarified before the analysis: namely, what it means that the state space is bounded in a cheap talk model, and whether it is a sensible assumption. We take the position of the existing literature, that a state is associated with a given optimal policy of the receiver. Therefore, the state space in the model can be bounded for two reasons. Either the policy space itself is a bounded set, as in the above example of allocating a fixed budget, or the set of rationalizable policies of the receiver is bounded. An example for the latter is the same budget allocation problem with a budget of flexible size, with the assumption that under no circumstance would the policymaker choose to set the budget outside a bounded region.[2]

Our first main result provides a necessary and sufficient condition for the existence of fully revealing equilibrium for any state space. The condition is used to establish that a fully revealing equilibrium always exists for unbounded state spaces and that, for bounded state spaces, it is always possible if the biases are small enough. It can also be used to investigate the possibility of full revelation for various pairs of biases in concrete examples.

We provide a particularly convenient characterization for the existence of fully revealing equilibrium for compact state spaces in the limit as the magnitudes of biases go to infinity. We show that the existence of such equilibrium depends on whether the senders have similar biases. Similarity of biases is defined relative to the shape of the state space: two biases are similar if the

[2] If the feasible policy space is the whole space but the set of rationalizable policies is bounded, then Battaglini's construction gives a Bayesian equilibrium. But this equilibrium is not perfect: after certain message pairs, the receiver is supposed to choose a strategy that cannot be a best response to any belief.

intersection of the minimal supporting hyperplane to the state spaces that are orthogonal to the biases contains a point of the state space. The intuition is that this point can be used to punish players if they send contradicting messages to the receiver. If the state space has a smooth boundary, then directions are similar if and only if they are exactly the same. The similarity relation between directions of biases is always reflexive and symmetric. For regular state spaces, it is also transitive.

Our results hold for any dimensions, including one. In one dimension, there are only two types of biases, the same direction and opposite directions. Biases of the former type are always similar, and biases of the latter type are never similar. Just as for multidimensional state spaces, biases with similar directions imply that full revelation is always possible in equilibrium, while non-similar directions imply that if biases are small enough, then full revelation is possible; otherwise, it is not.

We also address the question of how much information can be revealed in equilibrium if full revelation is not possible. First, we show that in this case, information loss is bounded away from zero in equilibrium. Second, we establish that if the state space satisfies some regularity conditions, then information revelation in the most informative equilibrium is also bounded away from zero, for any direction and any magnitude of the biases. This is in contrast with the case of only one sender, where Crawford and Sobel (1982) show that in a unidimensional state space no information can be transmitted if the bias of the sender is large enough, and Levy and Razin (2005) show that in a multidimensional state space there is an open set of environments in which the most informative equilibrium approaches the noninformative equilibrium as the size of bias goes to infinity. In the case of exactly opposite biases, we show that in the limit as biases go to infinity, all policy outcomes implemented in equilibrium have to be on a hyperplane that goes through the expected value of the state space (according to the prior distribution) and is orthogonal to the direction of biases.

The type of equilibria we construct in case full revelation is possible can be such that for some out-of-equilibrium message pairs, the policymaker implements a policy that is far away from states that are compatible with any of the messages sent. This raises the question whether these out-of-equilibrium beliefs are reasonable and whether there are fully revealing equilibria with reasonable out-of-equilibrium beliefs. We show that imposing a continuity property on beliefs of the receiver can further reduce the possibility of full revelation in equilibrium, and for some state spaces quite drastically. For example, if the state space is a two-dimensional set with a smooth boundary, and biases are not in exactly the same direction, then there does not exist a fully revealing continuous equilibrium, no matter how small the biases are. We motivate the continuity property we impose several ways. For example, we show that if we restrict attention to strategies that satisfy some regularity conditions, then consistency of beliefs implies the above property.

We conclude the paper by discussing some extensions of the model.

2 The model

The model we consider has the same structure as that of Battaglini (2002), with the exception that we consider state spaces that may be proper subsets of a Euclidean space. There are two senders and one receiver. The senders, labeled 1 and 2, both perfectly observe the state of the world $\theta \in \Theta$. Θ is referred to as the state space, which is a closed subset of $\mathbb{R}^d$. The prior distribution of θ is given by F. After observing θ, the senders send messages $m_1 \in M_1$ and $m_2 \in M_2$ to the receiver. The receiver observes these messages and chooses a policy $y \in Y \subseteq \mathbb{R}^d$ that affects the utility of all players. We assume that the policy space Y includes $\mathrm{co}(\Theta)$.

For any $x = (x^1, \ldots, x^d)$, $y = (y^1, \ldots, y^d) \in \mathbb{R}^d$, $x \cdot y = \sum_{j=1}^d x^j y^j$ denotes the inner product, and $|x| = \sqrt{x \cdot x}$ denotes the Euclidean norm.

For state θ and policy y, the receiver's utility is $-|y - \theta|^2$, while sender i's utility is $-|y - \theta - x_i|^2$. $x_i \in \mathbb{R}^d$ is called sender i's bias. At state θ, the optimal policy of the receiver is θ, while the set of optimal policies of sender i are the points in Y that are the closest to $\theta + x_i$ according to the Euclidean distance (which is exactly policy $\theta + x_i$ if the latter is included in the policy space). Note that the magnitude of a sender's bias does not just change his optimal policies; it also changes his preferences over the whole policy space. Intuitively, as the magnitude of bias increases, the indifference manifolds (curves when $d = 2$) of sender i at any state get closer and closer to hyperplanes (lines) that are orthogonal to x_i. We note that this formulation can be generalized without affecting the main results of the paper. In particular, the quadratic loss functions can be changed to any smooth quasiconcave utility function, and some of the results can be extended to state-dependent biases as well.[3]

Let $s_i \colon \Theta \to M_i$ denote a generic strategy of sender i in the above game, and let $y \colon M_1 \times M_2 \to Y$ denote a generic strategy of the receiver. Furthermore, let $f(m_1, m_2)$ denote the receiver's probabilistic belief of θ given messages m_1, m_2. Strategies s_1, s_2, y constitute a perfect Bayesian equilibrium if there exists a belief function f such that (i) s_i is optimal given s_{-i} and y for each $i \in \{1, 2\}$; (ii) $y(m_1, m_2)$ is optimal given $f(m_1, m_2)$ for each $(m_1, m_2) \in M_1 \times M_2$; and (iii) $f(m_1, m_2)$ is consistent with Bayes' rule for equilibrium message pairs given s_1, s_2, and F. Beliefs f like above are said to support the perfect Bayesian equilibrium (s_1, s_2, y). From now on we, refer to perfect Bayesian equilibrium simply as *equilibrium*. Note that the receiver's quadratic utility function implies

[3] See the related discussion in Section 6.

that condition (ii) above is equivalent to requiring that $y(m_1, m_2)$ be equal to the expectation of θ under $f(m_1, m_2)$. Let $\mu(m_1, m_2)$ denote this expectation.

3 Existence of fully revealing equilibrium

In this section, we investigate the existence of a fully revealing equilibrium. First, we establish a necessary and sufficient condition for existence that applies to any state space and any pair of biases. Although several useful insights are obtained from this result and it provides a tool for further analysis, the characterization is somewhat indirect, in the sense that it does not make transparent how existence of full revelation in equilibrium is related to the shape of the state space and the direction of biases. We provide a more direct necessary and sufficient condition for large enough biases.

3.1 General biases

Similarly to the well-known revelation principle in mechanism design, we do not lose generality by concentrating on truthful equilibria when investigating the existence of fully revealing equilibria. This makes our task much easier.

An equilibrium (s_1, s_2, y) is *fully revealing* if $s_1(\theta) = s_1(\theta')$ and $s_2(\theta) = s_2(\theta')$ imply $\theta = \theta'$. In this case, by Bayes' rule, $f(s_1(\theta), s_2(\theta))$ is the point mass on θ. An equilibrium (s_1, s_2, y) is *truthful* if $M_1 = M_2 = \Theta$ and $s_1(\theta) = s_2(\theta) = \theta$ for every $\theta \in \Theta$. A truthful equilibrium is fully revealing.

Lemma 1 (Battaglini (2002, Lemma 1)) *For any fully revealing equilibrium, there exists a truthful equilibrium which is outcome-equivalent to the fully revealing equilibrium.*

Proof: For any fully revealing equilibrium (s_1, s_2, y), consider a strategy profile $(\tilde{s}_1, \tilde{s}_2, \tilde{y})$ such that each sender reports the state θ truthfully and the receiver has a belief $\tilde{f}(\theta_1, \theta_2) = f(s_1(\theta_1), s_2(\theta_2))$ and takes an action $\tilde{y}(\theta_1, \theta_2) = \mu(s_1(\theta_1), s_2(\theta_2))$. Then, similarly to the standard argument about the revelation principle, each sender's strategy is optimal. Also, for each message pair (θ_1, θ_2), the sender's action is optimal because $\tilde{y}(\theta_1, \theta_2)$ is the expectation of θ under belief $\tilde{f}(\theta_1, \theta_2)$. Also, Bayes' rule is satisfied whenever it can apply because belief $\tilde{f}(\theta, \theta) = f(s_1(\theta), s_2(\theta))$ is a point mass on θ when two reports coincide. QED

In cheap talk games, sequential rationality is a weak requirement. In particular, in truthful equilibria, after incompatible reports $\theta \neq \theta'$, belief $f(\theta, \theta')$ can be an arbitrary distribution on Θ. The only restriction is that no sender has a strict incentive not to report the true state, to change the beliefs of the receiver, given that the other sender reports the truth.

Let $B(x,r) = \{y \in \mathbb{R}^d \mid |y - x| < r\}$ be the open ball with center x and radius r. For each sender i, $B(\theta + x_i, |x_i|)$ is the set of policies that are preferred to θ by sender i at state θ.

Proposition 2 *Belief f supports a truthful equilibrium if and only if, for every θ, $\theta' \in \Theta$,*

$$f(\theta,\theta) \text{ is a point mass on } \theta, \tag{1}$$

$$\mu(\theta,\theta') \notin B(\theta' + x_1, |x_1|), \tag{2}$$

$$\mu(\theta,\theta') \notin B(\theta + x_2, |x_2|). \tag{3}$$

Proof: (1) comes from Bayes' rule. (2) is the condition for sender 1 not to strictly prefer reporting θ to reporting truthfully when the true state is θ'. (3) is similar to (2). QED

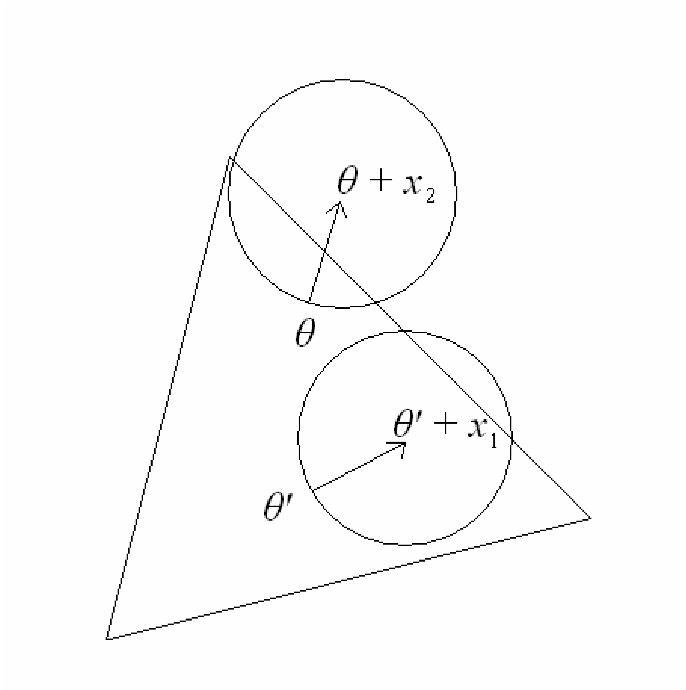

Figure 2

Figure 2 illustrates this graphically: in order to keep incentives for truthtelling both at state θ and θ', it is necessary that the policy chosen after message pair

(θ, θ') be a point that is both outside $B(\theta' + x_1, |x_1|)$ (otherwise, sender 1 would find it profitable to pretend that the state is θ in case the true state is θ') and $B(\theta + x_2, |x_2|)$ (otherwise, sender 2 would find it profitable to pretend that the state is θ' in case the true state is θ).

The above conditions give necessary and sufficient conditions for the existence of fully revealing equilibrium, stated in the next proposition. The result is a generalization of Proposition 1 of Battaglini (2002) to general state spaces in any dimension.

Proposition 3 *There exists a fully revealing equilibrium if and only if* $B(\theta' + x_1, |x_1|) \cup B(\theta + x_2, |x_2|) \not\supseteq \mathrm{co}(\Theta)$ *for all* θ, $\theta' \in \Theta$.

Proof: By Lemma 1 and Proposition 2, a fully revealing equilibrium exists if and only if there exists $\mu(\theta, \theta')$ satisfying (1)–(3). Since $\mu(\theta, \theta')$ is in the convex hull of Θ, if $B(\theta' + x_1, |x_1|) \cup B(\theta + x_2, |x_2|) \supseteq \mathrm{co}(\Theta)$ for some θ, $\theta' \in \Theta$ then (2)–(3) cannot hold simultaneously for any $\mu(\theta, \theta')$. Otherwise, for every $\theta \neq \theta' \in \Theta$, let $\mu(\theta, \theta')$ be an arbitrary element of $\mathrm{co}(\Theta)/(B(\theta' + x_1, |x_1|) \cup B(\theta + x_2, |x_2|))$. QED

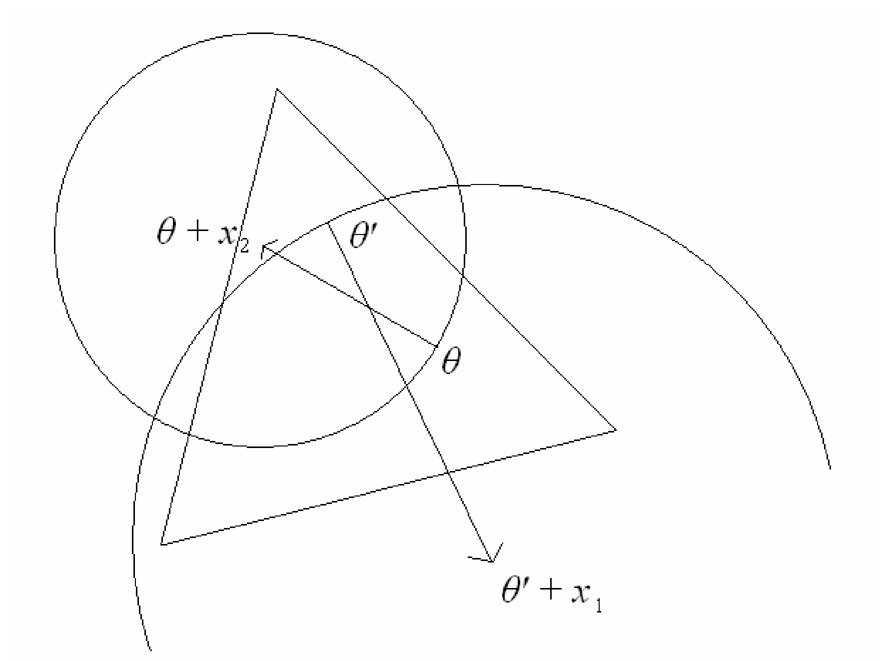

Figure 3

There cannot be a fully revealing equilibrium whenever there exists a pair (θ, θ') of states such that the open balls $B(\theta' + x_1, |x_1|)$ and $B(\theta + x_2, |x_2|)$ cover the convex hull of the state space. Figure 3 depicts a pair like that. Note that the existence of fully revealing equilibrium depends only on the shape of the state space Θ and the biases x_1, x_2, not on the prior distribution F.

Proposition 3 can be used to derive the following general results on the existence of fully revealing equilibrium.

Definition: x_1 and x_2 are *in the same direction* if $x_1 = \alpha x_2$ for some $\alpha \geq 0$ or $x_2 = 0$.

Proposition 4 *If x_1 and x_2 are in the same direction, then there exists a fully revealing equilibrium.*

Proof: Let f be the following point belief:

$$\mu(\theta, \theta') = \begin{cases} \theta \text{ if } x_2 \cdot \theta > x_2 \cdot \theta' \\ \theta' \text{ if } x_2 \cdot \theta \leq x_2 \cdot \theta' \end{cases}$$

Then f supports a fully revealing equilibrium. QED

Another general consequence of Proposition 3 is that the existence of fully revealing equilibrium depends monotonically on the magnitudes of biases.

Proposition 5 *Fix Θ. If there exists no fully revealing equilibrium for biases $x_1, x_2 \in \mathbb{R}^d$, then there exists no fully revealing equilibrium for biases $(t_1 x_1, t_2 x_2)$ for any t_1, $t_2 \geq 1$.*

Proof: $B(\theta' + t_1 x_1, |t_1 x_1|) \cup B(\theta + t_2 x_2, |t_2 x_2|) \supseteq B(\theta' + x_1, |x_1|) \cup B(\theta + x_2, |x_2|)$. Hence the claim follows from Proposition 3. QED

Finally, it can be shown that there is a fully revealing equilibrium if the biases are small enough relative to the size of the state space.

Let $\mathrm{diam}(\Theta) = \sup_{\theta, \theta' \in \Theta} |\theta - \theta'|$.

Proposition 6 *If $|x_1| + |x_2| \leq \mathrm{diam}(\Theta)/2$, then there exists a fully revealing equilibrium.*

Proof: Choose θ, $\theta' \in \Theta$ such that $|\theta - \theta'| \geq 2(|x_1| + |x_2|)$. Then no two open balls with radii $|x_1|$ and $|x_2|$ can cover the line segment between θ and θ'. QED

Corollary 7 *If Θ is unbounded, then there exists a fully revealing equilibrium.*

Proof: This claim follows from Proposition 6 since $\mathrm{diam}(\Theta) = \infty$. QED

Corollary 8 *Fix Θ. There exists $\varepsilon > 0$ such that if $\max(|x_1|, |x_2|) \leq \varepsilon$, then there exists a fully revealing equilibrium.*

Proof: If Θ consists of one point, then the claim is trivial. Otherwise, this claim follows from Proposition 6 if we set $\varepsilon = \mathrm{diam}(\Theta)/4 > 0$. QED

3.2 Examples

Our primary goal is to characterize conditions for full information revelation for large biases. Before providing the general result, it is useful to look at some concrete examples to develop intuition on how the possibility of full revelation depends on the shape of the state space and the directions and magnitudes of biases.

We analyze closed balls and hypercubes. In the next subsection, closed balls will be generalized to compact spaces with smooth boundaries and hypercubes to compact spaces with kinks.

Let D^d be the d-dimensional unit closed ball $\{\theta \in \mathbb{R}^d \mid |\theta| \leq 1\}$.

Proposition 9 *Suppose $\Theta = D^d$ with $d \geq 2$. There exists a fully revealing equilibrium if and only if x_1 and x_2 are in the same direction or $\max(|x_1|, |x_2|) \leq 1$.*

Proof: If part: By Proposition 4, we can assume that $\max(|x_1|, |x_2|) \leq 1$. For any given (θ, θ'), since $d \geq 2$, there exists a unit vector v perpendicular to $\theta' + x_1$. Let $w = -v$. We have v, $w \in D^d$. Since $|x_1| \leq 1$, (2) is satisfied both by $\mu(\theta, \theta') = v$ and by $\mu(\theta, \theta') = w$. Since $|v - w| = 2$ and $|x_2| \leq 1$, either v or w satisfies (3).

Only-if part: Suppose that x_1 and x_2 are in different directions and that $\max(|x_1|, |x_2|) > 1$. Without loss of generality, we can assume $|x_1| > 1$. By rotating the state space, we also have $x_1 = (-a, 0, \ldots, 0)$ with $a > 1$ without loss of generality. Substituting $\theta' = e := (1, 0, \ldots, 0)$ into (2), we have $|\mu(\theta, e) - (e + x_1)| \geq a$. By the triangle inequality, $\mu(\theta, e) \in D^d$, and $|e + x_1| = a - 1$, we have

$$a \leq |\mu(\theta, e) - (e + x_1)| \leq |\mu(\theta, e)| + |e + x_1| \leq 1 + (a - 1) = a.$$

Therefore, all the inequalities above hold with equality. Because $|\mu(\theta, e)| = 1$, and $\mu(\theta, e)$ and $-(e + x_1)$ are in the same direction, we have $\mu(\theta, e) = e$.

However, this violates (3) when θ is chosen appropriately. Again, without loss of generality, we have $x_2 = (b, c, 0, \ldots, 0)$ with $c \neq 0$, or $b > 0$ and $c = 0$.

For $c > 0$, we choose $\theta = (\sqrt{1-\varepsilon^2}, -\varepsilon, 0, \ldots, 0)$ for small $\varepsilon > 0$. For $c < 0$, we choose $\theta = (\sqrt{1-\varepsilon^2}, \varepsilon, 0, \ldots, 0)$ for small $\varepsilon > 0$. For $b > 0$ and $c = 0$, we choose $\theta = (1-\varepsilon, 0, \ldots, 0)$ for small $\varepsilon > 0$. In each case, we have $e \in B(\theta + x_2, |x_2|)$, which violates (3). QED

Therefore, when Θ is a closed ball, as long as x_1 and x_2 are in different directions, whether a fully revealing equilibrium exists or not is determined by how large biases are. If the biases are small enough, then we can construct a fully revealing equilibrium. If at least one of the biases is large enough, though, then there is no such equilibrium.

Consider next $[0,1]^d$, the unit hypercube in d dimensions. We say that x_1 and x_2 are *in the same orthant* if $x_1^j x_2^j \geq 0$ for every $j \in \{1, \ldots, d\}$.

Proposition 10 *Suppose* $\Theta = [0,1]^d$.

1. *If x_1 and x_2 are in the same orthant, then there exists a fully revealing equilibrium.*
2. *If x_1 and x_2 are in different orthants and* $\max_{i \in \{1,2\}} \min_{j \in \{1,\ldots,d\}} |x_i^j| > 1/2$*, then there does not exist a fully revealing equilibrium.*

Proof: For the first claim, without loss of generality, we can assume that $x_i^j \geq 0$ for all $i \in \{1,2\}$ and $j \in \{1, ..., d\}$. Let $\mu(\theta, \theta') = (0, \ldots, 0)$ for any $\theta \neq \theta'$. Then (1)–(3) are satisfied.

For the second claim, without loss of generality, we can assume that $x_1^j > 1/2$ $\forall\ j \in \{1, ..., d\}$, and $x_2^1 < 0$. Then, when $\theta' = (0, \ldots, 0)$ in (2), we have $\mu(\theta, (0, \ldots, 0)) = (0, \ldots, 0)$ for any $\theta \in [0,1]^d$. However, this violates (3) when $\theta = (\varepsilon, \ldots, 0)$ for $0 < \varepsilon < \min(-2x_2^1, 1)$. QED

The second part of the proposition establishes that if one of the biases x_i is large enough such that there is a state θ such that $B(\theta + x_i, |x_i|)$ covers the whole hypercube with the exception of θ, then no matter how small the other bias x_{-i} is, as long as it is in a different orthant, there is a state θ' such that $B(\theta' + x_{-i}, |x_{-i}|)$ covers θ (see Figure 4 for illustration).

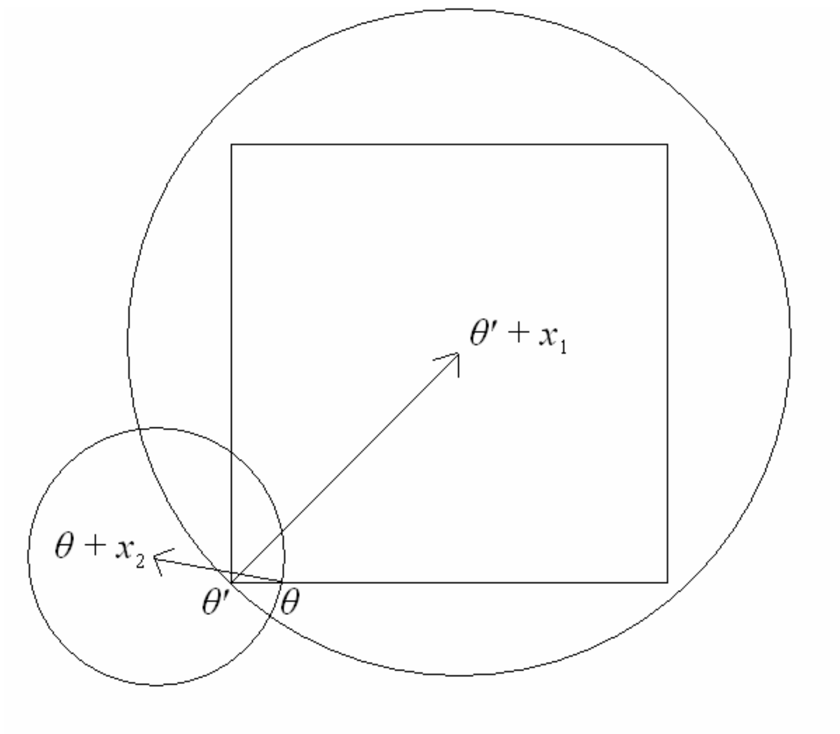

Figure 4

Combining Proposition 10 with Proposition 5 and Corollary 8 yields that if z_1 and z_2 are in different orthants, then there exists $T > 0$ such that there exists a fully revealing equilibrium for biases $(x_1, x_2) = (tz_1, tz_2)$ if $t < T$ and does not exist if $t > T$. Therefore, for biases that are in different orthants, the qualitative conditions for the existence of fully revealing equilibrium are similar to the case when the state space is a d-dimensional unit closed ball. However, for the case of biases from the same orthant, the qualitative conclusion is different. Note that the proof—that, in this case, independent of the magnitudes of biases, there always exists a fully revealing equilibrium—uses the fact that for these biases, there is a point in the state space that is minimal among points of the state space in both directions of biases. This point can serve as a punishment after any incompatible messages, which deters both senders from not revealing the true state.

3.3 Large biases

Proposition 3 establishes that the existence of fully revealing equilibrium depends on whether for any two states the state space can be covered with two open balls that are towards the direction of biases from the above states and that have radii equal to the magnitudes of biases. The next proposition shows that, for compact state spaces, the above condition for large biases is equivalent to whether the state space can be covered by the union of two open half spaces with boundaries that are orthogonal to the directions of biases.

Let S^{d-1} denote the $(d-1)$-dimensional unit sphere $\{x \in \mathbb{R}^d \mid |x| = 1\}$. S^{d-1} represents the set of possible directions in $\mathbb{R}^d$. For any $\lambda \in S^{d-1}$ and $k \in \mathbb{R}$, let $H^\circ(\lambda, k) = \{x \in \mathbb{R}^d \mid \lambda \cdot x > k\}$. $H^\circ(\lambda, \lambda \cdot x)$ is the open half space orthogonal to λ whose boundary goes through x.

Proposition 11 *Fix a compact state space Θ and the directions of biases z_1, $z_2 \in S^{d-1}$. There exists a fully revealing equilibrium with biases $(x_1, x_2) = (t_1 z_1, t_2 z_2)$ for every t_1, $t_2 \in \mathbb{R}_+$ if and only if $H^\circ(z_1, z_1 \cdot \theta') \cup H^\circ(z_2, z_2 \cdot \theta) \not\supseteq \mathrm{co}(\Theta)$ for all θ, $\theta' \in \Theta$.*

Proof: If part: The claim follows from Proposition 3 because $H^\circ(z_1, z_1 \cdot \theta') \cup H^\circ(z_2, z_2 \cdot \theta) \supseteq B(\theta' + t_1 z_1, t_1) \cup B(\theta + t_2 z_2, t_2)$ for every t_1, $t_2 \in \mathbb{R}_+$.

Only-if part: Suppose that $H^\circ(z_1, z_1 \cdot \theta') \cup H^\circ(z_2, z_2 \cdot \theta) \supseteq \mathrm{co}(\Theta)$ for some θ, $\theta' \in \Theta$. Then, since $\mathrm{co}(\Theta)$ is compact, there exists $\varepsilon > 0$ such that $H^\circ(z_1, z_1 \cdot \theta' + \varepsilon) \cup H^\circ(z_2, z_2 \cdot \theta + \varepsilon) \supseteq \mathrm{co}(\Theta)$. Since $\mathrm{co}(\Theta)$ is bounded, we have $B(\theta' + t_1 z_1, t_1) \cap \mathrm{co}(\Theta) \supseteq H^\circ(z_1, z_1 \cdot \theta' + \varepsilon) \cap \mathrm{co}(\Theta)$ and $B(\theta + t_2 z_2, t_2) \cap \mathrm{co}(\Theta) \supseteq H^\circ(z_2, z_2 \cdot \theta + \varepsilon) \cap \mathrm{co}(\Theta)$ for sufficiently large $t_1$and t_2. Hence the claim follows from Proposition 3. QED

The intuition for the result is fairly simple: as biases go to infinity, with respect to a bounded state space, the balls in Proposition 2 converge to the half spaces in Proposition 11. This result makes it possible to provide a convenient necessary and sufficient condition for the existence of fully revealing equilibrium for large enough biases, for any pair of directions of biases.

Consider a compact state space Θ. For any $\lambda \in S^{d-1}$, define $k^*(\lambda, \Theta) = \min_{\theta \in \Theta} \lambda \cdot \theta$ and let $H^*(\lambda, \Theta) = \{x \in \mathbb{R}^d \mid \lambda \cdot x \geq k^*(\lambda, \Theta)\}$. Note that the compactness of Θ implies that $k^*(\lambda, \Theta)$ and therefore $H^*(\lambda, \Theta)$ are well-defined. $H^*(\lambda, \Theta)$ is the minimal half space that is orthogonal to λ and contains Θ. Let $h^*(\lambda, \Theta)$ denote the boundary of $H^*(\lambda, \Theta)$: $h^*(\lambda, \Theta) = \{x \in \mathbb{R}^d \mid \lambda \cdot x = k^*(\lambda, \Theta)\}$ is the supporting hyperplane to Θ in the direction of λ.

For every $\theta \in \Theta$, let $N_\Theta(\theta) = \{\lambda \in \mathbb{R}^d \mid \lambda \cdot (\theta' - \theta) \leq 0\ \forall \theta' \in \Theta\}$. $\partial_\theta(\Theta)$ is the set of normal cones to Θ at point θ. Then z_1 and z_2 are *similar with respect to* Θ if there exists $\theta \in \Theta$ such that $-z_1$, $-z_2 \in N_\Theta(\theta)$.

Proposition 12 *Fix a compact state space Θ and the directions of biases z_1, $z_2 \in S^{d-1}$. The following conditions are equivalent:*

1. *There exists a fully revealing equilibrium with biases $(x_1, x_2) = (t_1 z_1, t_2 z_2)$ for every t_1, $t_2 \in \mathbb{R}_+$.*

2. $h^*(z_1, \Theta) \cap h^*(z_2, \Theta) \cap \Theta \neq \emptyset$.

3. $h^*(z_1, \Theta) \cap h^*(z_2, \Theta) \cap \mathrm{co}(\Theta) \neq \emptyset$.

4. z_1 *and* z_2 *are similar with respect to* Θ.

Proof: $1 \Rightarrow 2$: If not, then we have

$$H^*(z_1, \Theta) \cap H^*(z_2, \Theta)/(h^*(z_1, \Theta) \cap h^*(z_2, \Theta)) \supseteq \Theta.$$

Since the left-hand side of this formula is a convex subset of $H^\circ(z_1, k^*(z_1, \Theta)) \cup H^\circ(z_2, k^*(z_2, \Theta))$, we have

$$H^\circ(z_1, k^*(z_1, \Theta)) \cup H^\circ(z_2, k^*(z_2, \Theta)) \supseteq \mathrm{co}(\Theta),$$

which contradicts Proposition 11.

$2 \Rightarrow 3$: Trivial.

$3 \Rightarrow 1$: Pick any $\tilde{\theta} \in h^*(z_1, \Theta) \cap h^*(z_2, \Theta) \cap \mathrm{co}(\Theta)$. Then the claim follows from Proposition 11 because $\tilde{\theta} \notin H^\circ(z_i, z_i \cdot \theta)$ for any $i \in \{1, 2\}$ and any $\theta \in \Theta$.

$2 \Rightarrow 4$: There exists $\theta \in h^*(z_1, \Theta) \cap h^*(z_2, \Theta) \cap \Theta$. Then we have $-z_1$, $-z_2 \in N_\Theta(\theta)$.

$4 \Rightarrow 2$: Suppose that $-z_1, -z_2 \in N_\Theta(\theta)$ for some $\theta \in \Theta$. Then $\theta \in h^*(z_i, \Theta)$ for both $i \in \{1, 2\}$. QED

This proposition makes it easy to check whether for an arbitrary pair of bias directions full revelation is possible in the limit. If the intersection of the supporting hyperplanes to the state space in the given directions contains a point of the state space, then the answer is no; otherwise, it is yes (see Figure 5 below). This intersection is a lower dimensional hyperplane, and if it contains a point of the state space and $z_1 \neq z_2$, then that point has to be a kink of the state space. For example, in two dimensions, if $z_1 \neq z_2$ and $h^*(z_1, \Theta) \cap h^*(z_2, \Theta) \cap \Theta \neq \emptyset$, then $h^*(z_1, \Theta) \cap h^*(z_2, \Theta) \cap \Theta$ is a single point, which is such that there are supporting hyperplanes to Θ both in the direction of z_1 and in the direction of z_2. For a concrete example, recall the example of the d-dimensional cube with edges parallel to the axis from the previous subsection and consider $d = 2$. We saw that full revelation in equilibrium is possible even in the limit if biases go to infinity if and only if the directions of biases are in the same quadrant. Note that for each of these direction pairs, there is a vertex of the square such that there are two lines orthogonal to the biases that are tangential to the square and go through the vertex.

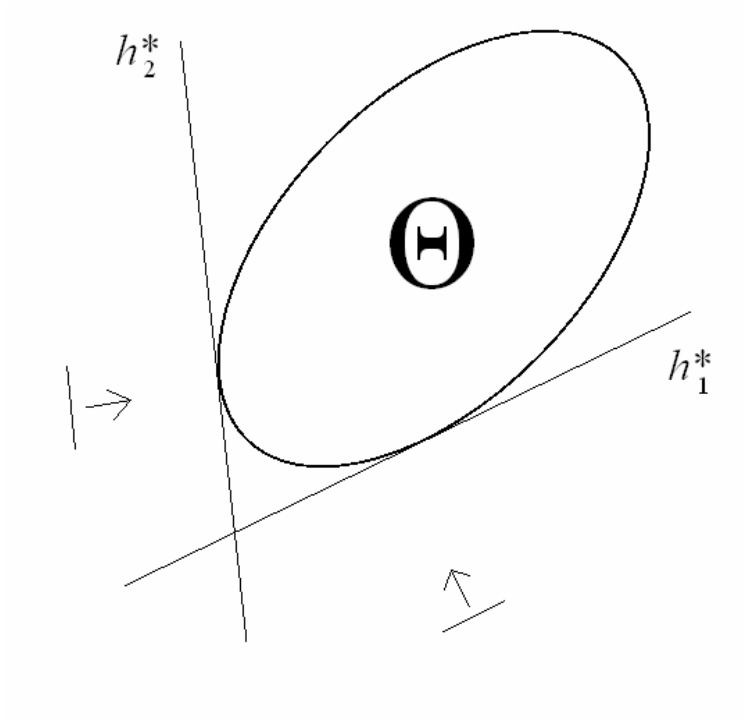

Figure 5

An immediate consequence of Proposition 12 is that for opposite biases ($z_1 = -z_2$), full revelation is possible in the limit if and only if Θ is included in a lower dimensional hyperspace that is orthogonal to the direction of biases. To see this, note that in any other case, $h^*(z_1, \Theta) \cap h^*(z_2, \Theta) = \emptyset$; therefore, $h^*(z_1, \Theta) \cap h^*(z_2, \Theta) \cap \Theta = \emptyset$.

Another conclusion we can derive from the proposition is that if Θ is a compact set with a smooth boundary, then full revelation in equilibrium is possible in the limit as biases go to infinity if and only if the biases are in the same direction. This generalizes our result from the previous subsection concerning the d-dimensional disk.

A compact Θ has a *smooth boundary* if λ, $\lambda' \in N_\Theta(\theta) \cap S^{d-1}$ implies $\lambda = \lambda'$ for any $\theta \in \Theta$.

Corollary 13 *Fix a compact state space Θ with a smooth boundary and the directions of biases z_1, $z_2 \in S^{d-1}$. There exists a fully revealing equilibrium with biases $(x_1, x_2) = (t_1 z_1, t_2 z_2)$ for every $t_1, t_2 \in \mathbb{R}_+$ if and only if $z_1 = z_2$.*

Proof: It follows from Proposition 12. QED

We also can show from Proposition 12 that we can assume the state space to be convex without loss of generality when we discuss the possibility of full revelation for large biases. This follows because the third condition in Proposition 12 depends only on $\text{co}(\Theta)$.

Note that the similarity relation used in Proposition 12 is reflexive and symmetric, but not necessary transitive. It can be shown that if Θ is *regular* in the sense that $\lambda \in N_\Theta(\theta)$ and $\lambda \in N_\Theta(\theta')$ for some $\lambda \in S^{d-1}$ implies $\theta = \theta'$, then the relation is also transitive.

We conclude the section by comparing the possibility of full revelation of information in one dimension and in more than one dimension. In each case, the same general result applies: if the state space is compact, then for biases in similar directions, full revelation of information is possible for any magnitudes of biases; for biases that are not in similar directions, the magnitudes of biases matter: full revelation of information is possible for small biases, but not possible for large enough biases. There is a potential difference between one dimension and many dimensions concerning whether nonsimilar directions are generic or not. In one dimension, there are only two types of direction pairs: the same direction and opposite directions. The former directions are always similar while the latter directions are always nonsimilar as long as the state space is not a singleton; therefore, neither of them is generic. In more than one dimension, the similarity relation depends on the shape of the state space. For state spaces with smooth boundaries, nonsimilar directions are generic, while for any other state space, neither similar nor nonsimilar direction pairs are generic. In any case, for a two-sender cheap talk model with a compact state space, one can get the same qualitative conclusions with respect to the possibility of fully revealing equilibrium if using a one-dimensional model (which is typically much easier to analyze) and if using a multidimensional model. The only caveat is that if one considers the one-dimensional model as a simplification of a more realistic multidimensional model, and similar biases are unlikely in that multidimensional model, then the one-dimensional analysis should put more emphasis on the case of opposite biases than on the case of like biases.

4 Partial information revelation

In this section, we examine how much information can be transmitted in equilibrium if full revelation is not possible. First, we show that in this case, information transmission is bounded away from efficiency, in the sense that there is an open set of states such that the implemented policy at these states is bounded away from the optimal policy of the receiver.

Proposition 14 *There exists no fully revealing equilibrium if and only if there exist $\varepsilon > 0$ and open sets U and U' satisfying $U \cap \Theta \neq \emptyset$ and $U' \cap \Theta \neq \emptyset$ such that, for any equilibrium (s_1, s_2, μ), either $|\mu(s_1(\theta), s_2(\theta)) - \theta| > \varepsilon$ for all $\theta \in U$ or $|\mu(s_1(\theta'), s_2(\theta')) - \theta'| > \varepsilon$ for all $\theta' \in U'$.*

Proof: The if part is trivial. For the only if part, suppose that there exists no fully revealing equilibrium. Then Θ is bounded, and there exist $\widetilde{\theta}, \widetilde{\theta}' \in \Theta$ such that

$$B(\widetilde{\theta}' + x_1, |x_1|) \cup B(\widetilde{\theta} + x_2, |x_2|) \supseteq \mathrm{co}(\Theta)$$

Since the left-hand side is the union of two open balls and the right-hand side is the compact set, the above inclusion holds for other two nearby balls. Namely, there exist $\varepsilon > 0$ and neighborhoods U of $\widetilde{\theta}$ and U' of $\widetilde{\theta}'$ such that

$$B(\theta' + x_1, |x_1| - \varepsilon) \cup B(\theta + x_2, |x_2| - \varepsilon) \supseteq \mathrm{co}(\Theta)$$

for any $\theta \in U$ and $\theta' \in U'$.

For any equilibrium (s_1, s_2, μ) and any $\theta \in U$, $\theta' \in U'$, we must have either $|\mu(s_1(\theta), s_2(\theta)) - \theta| > \varepsilon$ or $\left|\mu(s_1(\theta'), s_2(\theta')) - \theta'\right| > \varepsilon$ because otherwise we have $B(\theta' + x_1, \left|\theta' + x_1 - \mu(s_1(\theta'), s_2(\theta'))\right|) \cup B(\theta + x_2, |\theta + x_2 - \mu(s_1(\theta), s_2(\theta))|) \supseteq \mathrm{co}(\Theta)$, where the first ball is the set of policies sender 1 strictly prefers to $\mu(s_1(\theta'), s_2(\theta'))$ at state θ', and the second ball is the set of policies sender 2 strictly prefers to $\mu(s_1(\theta), s_2(\theta))$ at state θ. Therefore, similarly to the proof of Proposition 3, no matter what $\mu(s_1(\theta), s_2(\theta'))$ is, either sender 1 wants to report θ at state θ' or sender 2 wants to report θ' at state θ, which contradicts the equilibrium condition.

Therefore, if $|\mu(s_1(\theta), s_2(\theta)) - \theta| \leq \varepsilon$ for some $\theta \in U$, then $\left|\mu(s_1(\theta'), s_2(\theta')) - \theta'\right| > \varepsilon$ for all $\theta' \in U'$. Otherwise, $|\mu(s_1(\theta), s_2(\theta)) - \theta| \leq \varepsilon \ \forall\ \theta \in U$. QED

The proof establishes that if there is no fully revealing equilibrium, then there exist two open balls and a positive constant such that if in an equilibrium the implemented policy for at least one state in one ball is closer than epsilon to the state itself, then at every state in the other ball, the difference between the implemented policy and the state is at least as much as this constant. Note that the balls are defined independently of the equilibrium at hand; hence the above property applies to all equilibria. This is worth pointing out because typically there are many different types of equilibria, and it is hard to find nontrivial properties that hold for every equilibrium.

Next we establish that if the prior distribution is continuous and the expected value of the state space is an interior point of the state space (which holds, for example, if Θ is convex and full dimensional), then information transmission in the most informative equilibrium is bounded away from zero.[4] First, we

[4] Note that the claim is about the most informative equilibrium. As is well-known in the literature, there is always a babbling equilibrium in which no information is transmitted.

show the result for the case of non-opposite biases, where a stronger claim can be established: that there exists an open set of states such that there is an equilibrium that is fully revealing with respect to this set, independently of the magnitudes of biases.

Let $\widehat{\theta} = E(\theta)$, where E denotes the expectation with respect to the prior distribution F.

Proposition 15 *Assume F is continuous. If $\widehat{\theta}$ is an interior point of Θ and $z_1, z_2 \in S^{d-1}$ are not opposite directions, then there is an open set $C \subseteq \Theta$ such that for any t_1, $t_2 \in \mathbb{R}_+$, there is an equilibrium with biases $(x_1, x_2) = (t_1 z_1, t_2 z_2)$ such that $y(s_1(\theta), s_2(\theta)) = \theta$ for all $\theta \in C$.*

Proof: Let $\bar{t} > 0$ be such that for any t_1, $t_2 \leq \bar{t}$, full revelation is possible with biases $(x_1, x_2) = (t_1 z_1, t_2 z_2)$. Since z_1, z_2 are not orthogonal, $B(\widehat{\theta} + \bar{t} z_1, \bar{t}) \cap B(\widehat{\theta} + \bar{t} z_2, \bar{t}) \cap \Theta$ has at least one interior point $\widetilde{\theta}$. Then, by the continuity of F, there exist an open neighborhood C of $\widetilde{\theta}$ and an open neighborhood C' of $\widehat{\theta}$ such that for every t_1, $t_2 \geq \bar{t}$, $\theta \in B(\theta' + t_1 z_1, t_1) \cap B(\theta' + t_2 z_2, t_2)$ for all $\theta \in C$ and $\theta' \in C'$ and $E(\theta | \theta \notin C) \in C'$. Then the following is an equilibrium for biases $(x_1, x_2) = (t_1 z_1, t_2 z_2)$ with message spaces $M_1 = M_2 = C \cup \{m^{-C}\}$: $s_1(\theta) = s_2(\theta) = \theta$ for $\theta \in C$ and $s_1(\theta) = s_2(\theta) = m^{-C}$ otherwise, while $y(\theta, \theta) = \theta$ for $\theta \in C$ and $y(\theta', \theta'') = E(\theta | \theta \notin C)$ if either $\theta' = m^{-C}$, $\theta'' = m^{-C}$, or $\theta' \neq \theta''$. QED

Consider now the case of opposite biases. Let $\widehat{H}_i = \{y \in \mathbb{R}^d : x_i \cdot (y - \widehat{\theta}) = 0\}$ for $i = 1, 2$. $\widehat{H}_1$ and $\widehat{H}_2$ are the hyperplanes going through the expected value of the state space (according to the prior distribution) that are orthogonal to the biases. For the case of opposite biases, $\widehat{H}_1 = \widehat{H}_2 \equiv \widehat{H}$.

Proposition 16 *Assume $d \geq 2$ and F is continuous. Let $z_1, z_2 \in S^{d-1}$ be such that $z_1 = -z_2$. If $\widehat{\theta}$ is an interior point of Θ, then there exists an open neighborhood C of $\widehat{\theta}$ such that for any t_1, $t_2 \in \mathbb{R}_+$, there is an equilibrium with biases $(x_1, x_2) = (t_1 z_1, t_2 z_2)$ such that $y(s_1(\theta), s_2(\theta)) = \mathrm{proj}_{\widehat{H}} \theta$ for all $\theta \in C$.*

Proof: The assumptions above imply that there exists $C \in \Theta$ such that C is an open neighborhood of $\widehat{\theta}$ and $E(\theta | \theta \in C, \mathrm{proj}_{\widehat{H}} \theta = \overline{\theta}) = \overline{\theta}$ for all $\overline{\theta} \in \mathrm{proj}_{\widehat{H}} C$. Then the following is an equilibrium for biases $(x_1, x_2) = (t_1 z_1, t_2 z_2)$ with message spaces $M_1 = M_2 = (\mathrm{proj}_{\widehat{H}} C) \cup \{m^{-C}\}$: $s_1(\theta) = s_2(\theta) = \mathrm{proj}_{\widehat{H}} C$ for $\theta \in C$ and $s_1(\theta) = s_2(\theta) = m^{-C}$ otherwise, while $y(\theta, \theta) = \theta$ for $\theta \in \widehat{H} \cap C'$ and $y(\theta', \theta'') = E(\theta | \theta \notin C)$ if either $\theta' = m^{-C}$, $\theta'' = m^{-C}$, or $\theta' \neq \theta''$. QED

The above implies that even in the limit as the magnitudes of biases go to infinity, information transmission is bounded away from zero. This is in contrast to the one-sender case. Crawford and Sobel (1982) show that there is no informative equilibrium for large enough biases if the state space is a compact interval. In multidimensional environments, Levy and Razin (2005) provide a condition for the receiver to play at most k actions with positive probability if the magnitude of bias is sufficiently large. If this condition holds with $k = 1$, then there is no informative equilibrium for a large enough bias.[5]

It can also be shown that, for opposite biases in the limit as the magnitudes of biases go to infinity, all policies implemented in equilibrium must be contained in $\widehat{H}$.[6] This gives an upper bound on how much information can be transmitted in the limit if biases are in opposite directions. This upper bound is tight if both Θ and F are symmetric around $\widehat{H}$.

Proposition 17 *Let Θ be compact and let $z_1, z_2 \in S^{d-1}$ be such that $z_1 = -z_2$. Then for every $\varepsilon > 0$, there exists $\bar{t} \in \mathbb{R}_+$ such that for every $t_1,\ t_2 > \bar{t}$, every equilibrium of the model with biases $(t_1 z_1, t_2 z_2)$ is such that for all $\theta \in \Theta$ $\min_{x \in \widehat{H}} |y(s_1(\theta), s_2(\theta)) - x| < \varepsilon$.*

Proof: For every $y \in \mathrm{co}(\Theta)$, let $\widehat{H}(y)$ denote the hyperplane going through y that is orthogonal to the biases. For every $\varepsilon > 0$, there exists $\bar{t} \in \mathbb{R}_+$ such that for every $t_1,\ t_2 > \bar{t}$, every $y, y' \in \mathrm{co}(\Theta)$ with $z_1 \cdot y \geq z_1 \cdot y' + \varepsilon/2$, and every $\theta \in \Theta$, we have $-|y - \theta - x_1|^2 > -|y' - \theta - x_1|^2$ and $-|y - \theta - x_2|^2 < -|y' - \theta - x_2|^2$ (in words, sender 1 prefers policy y to y' in every state, while sender 2 prefers y' to y in every state). Suppose now that biases are $(t_1 z_1, t_2 z_2)$ with $t_1,\ t_2 > \bar{t}$ and that there is an equilibrium for which $z_1 \cdot y(s_1(\theta), s_2(\theta)) \geq z_1 \cdot y(s_1(\theta'), s_2(\theta')) + \varepsilon$ for some $\theta, \theta' \in \Theta$. Consider $y(s_1(\theta), s_2(\theta')))$. Then either (i) $z_1 \cdot y(s_1(\theta), s_2(\theta')) \geq z_1 \cdot y(s_1(\theta'), s_2(\theta')) + \varepsilon/2$ and hence

$$-\left|y(s_1(\theta), s_2(\theta')) - \theta' - x_1\right|^2 > -\left|y(s_1(\theta'), s_2(\theta')) - \theta' - x_1\right|^2,$$

or (ii) $z_1 \cdot y(s_1(\theta), s_2(\theta)) \geq z_1 \cdot y(s_1(\theta), s_2(\theta')) + \varepsilon/2$ and hence

$$-\left|y(s_1(\theta), s_2(\theta')) - \theta - x_2\right|^2 > -\left|y(s_1(\theta), s_2(\theta)) - \theta - x_2\right|^2,$$

contradicting that the profile is an equilibrium. Therefore, $z_1 \cdot (y(s_1(\theta), s_2(\theta)) - y(s_1(\theta'), s_2(\theta'))) < \varepsilon$ for all $\theta, \theta' \in \Theta$. Since sequential rationality implies that $E(y(s_1(\theta), s_2(\theta))) = \widehat{\theta}$, all the equilibirum policies are in the ε-neighborhood of $\widehat{H}$. QED

[5] It is not true though that informative equilibria never exist for large enough biases. Chakraborty and Harbough (2005) construct an informative equilibrium in symmetric multidimensional environments. They also show that this equilibrium construction is generically robust to small asymmetries of payoff functions and the prior distribution.

[6] Note that in one dimension, $\widehat{H}_1 = \{\widehat{\theta}\}$; therefore, in that context, this claim implies the known result that no information can be revealed in equilibrium when biases are opposite direction and the size of biases goes to infinity.

5 Robust equilibria

In cheap talk games, perfect Bayesian equilibrium (PBE) does not impose any restriction on out-of-equilibrium beliefs of the receiver. Given this great flexibility in specifying out-of-equilibrium beliefs—which is made transparent in Proposition 2—the question arises which equilibria can be supported by "plausible" beliefs. The importance of this issue in our paper is mitigated by the fact that our central results concern nonexistence of fully revealing PBE. Of course, nonexistence of fully revealing PBE implies nonexistence of any refinement of PBE that is fully revealing. However, we show how imposing a plausible refinement criterion strengthens our nonexistence results considerably for some state spaces.

An extensive investigation of robustness of PBE, and related to this investigating PBE in models with noisy state observation, is a difficult exercise for general state spaces and is beyond the scope of this paper.[7] We focus on equilibria that satisfy a particular continuity property. The property is motivated by requiring robustness to small mistakes in senders' observations, and it is satisfied by the construction provided by Battaglini (2002 and 2004) for unrestricted state spaces. We also show that a strong definition of consistency of equilibrium beliefs implies this property. We then establish that if the state space is two-dimensional and has a smooth boundary, and biases are not in exactly the same direction, then there exists no fully revealing equilibrium that satisfies the continuity property, even if the magnitudes of the biases are arbitrarily small. A similar result is shown for the case when the state space is a d-dimensional cube.

5.1 Diagonal continuity

The equilibrium construction provided in Battaglini (2002, 2004) satisfies the property that the policy implemented by the receiver is continuous in the observations made by the senders. In what follows, we investigate a requirement that is weaker than this, in that it only requires continuity at points where the observations of senders are the same.

Definition: An equilibrium (s_1, s_2, y) is called *continuous on the diagonal* if $\lim_{n\to\infty} y(s_1(\theta_1^n), s_2(\theta_2^n)) = y(s_1(\theta), s_2(\theta))$ for any sequence $\{(\theta_1^n, \theta_2^n)\}_{n\in\mathbb{N}}$ of pairs of states such that $\lim_{n\to\infty} \theta_1^n = \lim_{n\to\infty} \theta_2^n = \theta$.

Our motivation for investigating equilibria that satisfy this property comes from multiple sources. One is that we are interested in whether in restricted state spaces there exist fully revealing equilibria that can be obtained by some continuous transformation of the Battaglini construction.

[7] Battaglini (2002) investigates one definition of robustness with restricted state spaces in one dimension and unrestricted state spaces in multiple dimensions; Battaglini (2004) considers a different definition of robustness with unrestricted multidimensional state spaces.

Second, this property is equivalent to robustness to *all* small misspecification of the model. More precisely, suppose that signals that two senders receive are slightly different from the true state in reality, although all players (incorrectly) believe that both senders know the true state, they believe that other players believe that both senders know the true state, and so on. In such a situation, if the equilibrium is continuous on the diagonal, then the *ex post* loss for the receiver that arises from false beliefs is small for any realization of the true state when both senders receive signals close enough to the true state.

Third, as the next proposition shows, diagonal continuity is necessary for nonexistence of incompatible reports. The latter is a convenient property in settings where it is unclear how to specify out-of-equilibrium beliefs.[8]

Proposition 18 *For compact Θ, a fully revealing equilibrium (s_1, s_2, y) is continuous on the diagonal if*

1. *for each sender i, M_i is Hausdorff and $s_i \colon \Theta \to M_i$ is continuous, and*
2. *for each $(m_1, m_2) \in s_1(\Theta) \times s_2(\Theta)$, there exists a state $\theta \in \Theta$ such that $(s_1(\theta), s_2(\theta)) = (m_1, m_2)$.*

Proof: Consider function g on Θ that maps θ to $(s_1(\theta), s_2(\theta))$. By the assumptions, g is a continuous function onto $s_1(\Theta) \times s_2(\Theta)$. g is also one-to-one because (s_1, s_2, y) is fully revealing. Since g is a continuous bijection from compact space Θ to Hausdorff space $M_1 \times M_2$, the inverse $\mu(m_1, m_2)$ is a continuous function of $(m_1, m_2) \in s_1(\Theta) \times s_2(\Theta)$.[9] Since s_1 and s_2 are continuous, $\mu(s_1(\theta_1), s_2(\theta_2))$ is also continuous in (θ_1, θ_2). QED

The last motivation comes from consistency of beliefs, i.e., that beliefs should be limits of beliefs obtained from noisy models as the noise in senders' observations goes to zero. In Appendix A, we show that if we restrict attention to equilibria in which players' strategies satisfy some regularity conditions, then every PBE that satisfies consistency of beliefs has to satisfy diagonal continuity. The regularity conditions we impose are fairly strong, but they are needed to ensure that the conditional beliefs of the receiver in "nearby" noisy models (which are invoked in the definition of consistent beliefs) are well-defined by Bayes' rule.

[8] For example, Battaglini's (2002) equilibrium does not have incompatible reports if the state space is a whole Euclidean space.

[9] See Royden (1988), Proposition 5 of Chapter 9.

5.2 Nonexistence of fully revealing equilibria that are continuous on the diagonal

Below we show that requiring diagonal continuity can drastically reduce the possibility of full revelation in equilibrium. First we consider two-dimensional smooth compact sets. Recall Proposition 6, according to which for any pair of bias directions, if biases are small enough (positive), then there always exists a fully revealing equilibrium. As opposed to this, the next proposition shows that unless biases are exactly in the same direction, no matter how small they are, there does not exist a continuous fully revealing equilibrium.

Proposition 19 *In a two-dimensional smooth compact set Θ, if (x_1, x_2) are not in the same direction, then there does not exist a fully revealing equilibrium that satisfies diagonal continuity.*

Proof: Since Θ is a two-dimensional smooth set and (x_1, x_2) is not in the same direction, there exists $\theta \in \Theta$ such that θ is separated from other points in $\Theta \setminus (B(\theta - x_1, |x_1|) \cup B(\theta - x_2, |x_2|))$. Since $y(\theta, \theta')$ is continuous with respect to θ' at $\theta' = \theta$, when we change θ' slightly, $y(\theta, \theta')$ has to move continuously. However, we can change θ' appropriately so that we can cover by $B(\theta' - x_1, |x_1|) \cup B(\theta - x_2, |x_2|)$ the region close enough to θ. This leads to a contradiction. QED

Figure 6 illustrates the argument used in the proof: if biases are not in the same direction, then there are states θ and θ' arbitrarily close to each other (close to the boundary of the state space) such that the balls $B(\theta' - x_1, |x_1|)$ and $B(\theta - x_2, |x_2|)$ cover an open set that includes both θ and θ'. This means that in order to induce truthtelling in equilibrium, the policy implemented by the receiver after receiving messages corresponding to θ and θ' has to be "far away" from both θ and θ'. This implies that the equilibrium does not satisfy diagonal continuity in these points.

A similar nonexistence result holds for models in which the state space is a d-dimensional cube (note the difference to the result in Proposition 10).

Proposition 20 *Suppose that $\Theta = [0, W]^d$ with $W > 0$. There exists no continuous fully revealing equilibrium if $x_1^j > 0$ for all $j \in \{1, \ldots, d\}$ and $x_2^k < 0$ for some $k \in \{1, \ldots, d\}$.*

Proof: When $\theta = \theta' = (0, \ldots, 0)$, $(0, \ldots, 0)$ is separated from other points in $\Theta \setminus (B(-x_1, |x_1|) \cup B(-x_2, |x_2|))$. Then, similarly to the proof of Proposition 19, we can change θ from $(0 \ldots, 0)$ toward the positive direction of the k-th coordinate so that we can cover by $B(-x_1, |x_1|) \cup B(\theta - x_2, |x_2|)$ the region close enough to $(0, \ldots, 0)$. This leads to a contradiction. QED

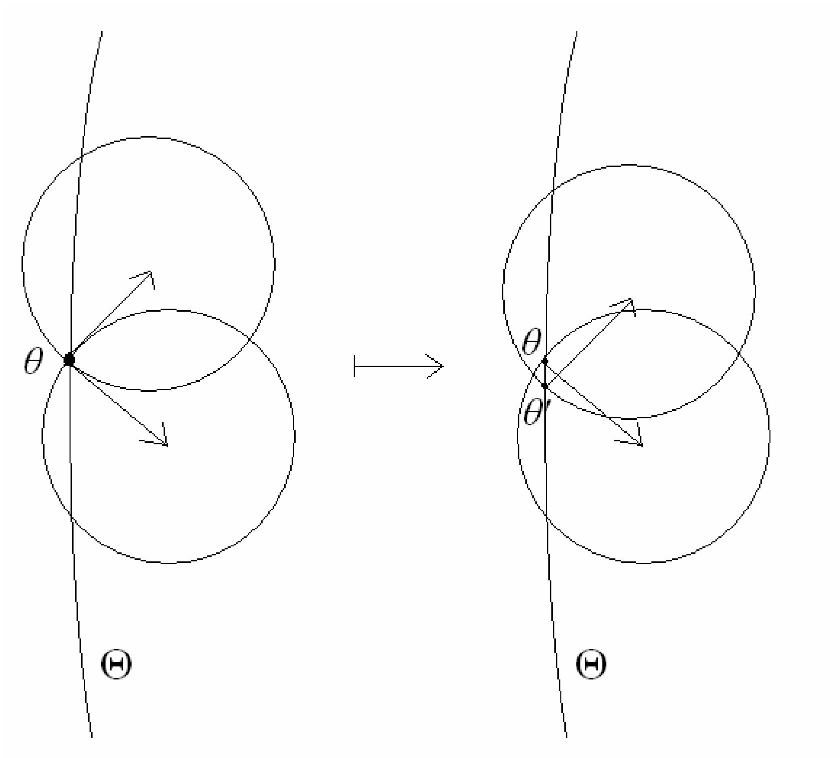

Figure 6: nonexistence of continuous truthtelling equilibrium

6 Discussion and extensions

6.1 More general preferences

The results for finite biases (Subsection 3.1) generalize to any preference specification for the senders. In particular, the quadratic loss function does not play any role in these results. Moreover, the senders' preferences (biases) can be state dependent. It can also be shown that the existence of fully revealing equilibrium is robust to introducing a small amount of state dependence into the senders' preferences. Formally, if there exists a fully revealing equilibrium in a state-independent bias model in which sender i's utility function is given by $-|y-\theta-x_i|^2$, then there exists $\varepsilon > 0$ such that there exists a fully revealing equilibrium in a state-dependent bias model in which sender i's utility function is given by $-|y-\theta-x_i(\theta)|^2$ with $|x_i(\theta)-x_i| < \varepsilon$ for all θ. The intuition behind this result is that if two open balls cover the state space as in Proposition 2, then if the balls are changed slightly, they still cover the state space. We do not formalize this claim here since the arguments are straightforward. We note that Battaglini's equilibrium construction cannot be extended to all models with state-dependent sender preferences obtained as above (if bias vectors are state dependent, then so are the directions orthogonal to the bias, which can ruin the incentives of the senders to report truthfully), even if the state space is the whole Euclidean space.

The results for large biases in Subsection 3.3 can be generalized as well, for cases when the indifference manifolds of the senders are smooth. The latter requirement is needed to make sure that indifference manifolds over a compact state space approach hyperplanes. These limit hyperplanes are not necessarily orthogonal to the bias vectors if preferences are not quadratic, but depend on the shape of the indifference curves. The assumption of state-independent biases can again be relaxed.

6.2 Long cheap talk

It is well-known that multiple rounds of costless signaling ("long cheap talk") may expand the set of equilibrium payoffs. In particular, in a one-sender model with two-stage conversation, Krishna and Morgan (2004) show that there almost always exists an equilibrium outcome that dominates all the equilibrium outcomes in the model with a single round of cheap talk. In our setting, first we show that additional rounds of communication (including allowing the receiver to send messages, as in Krishna and Morgan (2004)) do not affect the possibility of full revelation in *pure strategy* equilibrium, although they may affect partially revealing equilibria when full revelation is impossible.

Proposition 21 *If there exists no fully revealing equilibrium under a single round of cheap talk, then there also exists no fully revealing pure strategy equilibrium under multiple rounds of cheap talk.*

Proof: Suppose that there exists a fully revealing equilibrium under multiple rounds of cheap talk. Let s_0 be the receiver's strategy in the cheap-talk stage, and $s_i(\theta)$ be sender i's strategy at state θ. Each of the strategies is a sequence of messages contingent on all players' past messages. Given the sequence of messages generated by $(s_0, s_1(\theta), s_2(\theta'))$, the receiver forms the belief $\mu(\theta, \theta')$ about the state of the world. For every θ, $\theta' \in \Theta$, we have to keep sender 1 from mimicking sender 1 of type θ and playing $s_1(\theta)$ when the real state is θ'. Similarly, we have to keep sender 2 from playing $s_2(\theta')$ when the real state is θ. Thus $\mu(\theta, \theta')$ satisfies (1)–(3) as in the game with a single round of cheap talk, which contradicts the assumption. QED

In a fully revealing equilibrium without generality, it can be assumed that both the receiver and the senders use pure strategies (any mixing over actions has to be irrelevant in terms of the final outcome that is chosen). However, in a game with multiple rounds of cheap talk, off the equilibrium path players can mix in a payoff relevant manner. This means that deviatons by the senders can lead to stochastic outcomes, which can affect the existence of fully revealing equilibrium. Below we provide a necessary condition for the existence of fully revealing equilibrium in a game with multiple rounds of cheap talk. Then we show that for large biases, having multiple rounds of cheap talk (as opposed to just one) does not affect the existence of fully revealing equilibrium.

Let $D = \text{diam}(\Theta)/2$, where $\text{diam}(\Theta) = \sup_{\theta,\theta' \in \Theta} |\theta - \theta'|$. For $i = 1, 2$ let $r_i = \sqrt{\max(0, |x_i|^2 - D^2))}$.

Proposition 22 *In any game with multiple rounds of cheap talk, there exists no fully revealing equilibrium if there exist* $\theta, \theta' \in \Theta$ *such that* $B(\theta' + x_1, r_1) \cup B(\theta + x_2, r_2) \supseteq \text{co}(\Theta)$.

Proof: In a fully revealing equilibrium, for any pair of states θ and θ', it has to be true that player 1 at θ' cannot gain by deviating to playing what her strategy would prescribe at state θ, and at θ cannot gain by deviating to playing what her strategy would prescribe at state θ'. Fix any strategy profile which satisfies that at every state the policy outcome is equal to the state. Let $y(\theta, \theta')$ denote the probability distribution of policy outcomes resulting from sender 1 playing the continuation strategy that the above profile prescribes for her after observing θ and from sender 2 playing the continuation strategy that the above profile prescribes for her after observing θ'. Then since the above profile is an equilibrium, we have $-E(y(\theta, \theta') - \theta' - x_1)^2 \le -|x_1|^2$. Note that $-E(y(\theta, \theta') - \theta' - x_1)^2 = -(Ey(\theta, \theta') - \theta' - x_1)^2 - E|y(\theta, \theta') - Ey(\theta, \theta')|^2$. Since $y(\theta, \theta')$ is a distribution over $\text{co}(\Theta)$, $E|y(\theta, \theta') - Ey(\theta, \theta')|^2 \le (\text{diam}(\Theta)/2)^2 = D^2$. This means that a necessary condition for the above profile to be an equilibrium is $(Ey(\theta, \theta') - \theta' - |x_1|)^2 > |x_1|^2 - D^2$. A symmetric argument establishes that another necessary condition is $(Ey(\theta, \theta') - \theta - |x_2|)^2 \ge |x_2|^2 - D^2$. Combining the two conditions yields $Ey(\theta, \theta') \notin B(\theta' + x_1, r_1) \cup B(\theta + x_2, r_2)$. Therefore, $B(\theta' + x_1, r_1) \cup B(\theta + x_2, r_2) \supseteq \text{co}(\Theta)$ for some $\theta, \theta' \in \Theta$ implies that there does not exist a fully revealing equilibrium. QED

The above result is similar in spirit to Proposition 3: if a sender pretends to have observed a different state than the true state, then the resulting probability distribution over outcomes should yield a lower expected utility for her than revealing the true state. For quadratic utilities, the above expected utility only depends on the expectation and the variance of the resulting distribution. The variance of the distribution is bounded by a constant that depends on the diameter of the state space. This can be used to show that the expected value of the distribution has to be in the two open balls in the statement, $B(\theta' + x_1, r_1)$ and $B(\theta + x_2, r_2)$ (if player 1 played as if she observed θ and player 2 played as if she observed θ'). Note that these balls have the same centers as the ones in Proposition 3, but smaller. As opposed to Proposition 3, which provides a necessary and sufficient condition, the above claim only provides a necessary condition for the existence of fully revealing equilibrium in games with multiple rounds of cheap talk. The condition can be tightened for specific state spaces, for example, by using the fact that, for different expected values, typically the maximal variance of a distribution with the given expected value is different (if the state space is bounded).

We conclude this subsection by showing that in a bounded state space, for any fixed direction pair of biases, in the limit as the magnitude of biases go to infinity there exists fully revealing equilibrium in a game with arbitrary rounds of communication if and only if there exists one in a game with only one round of communication. This means that the results of Subsection 3.3 on large enough biases hold for games with arbitrary rounds of communication. The key insight is that the open balls in Proposition 22 converge to the ones in Proposition 3.

Proposition 23 *Fix a compact state space Θ and directions of biases $z_1, z_2 \in S^{d-1}$. If there exists $t^* \in \mathbb{R}_+$ such that for every $t_1, t_2 > t^*$ and bias pair $(x_1, x_2) = (t_1 z_1, t_2 z_2)$ there exists no fully revealing equilibrium in a game with one round of cheap talk, then there exists $t^{**} \in \mathbb{R}_+$ such that for every $t_1, t_2 > t^{**}$ and bias pair $(x_1, x_2) = (t_1 z_1, t_2 z_2)$ there exists no fully revealing equilibrium in a game with arbitrary rounds of cheap talk.*

Proof: Let $r_i(t_i) = \sqrt{\max(0, |x_i|^2 - D^2))}$ for $i = 1, 2$. Note that θ' is not on the boundary of $B(\theta' + tz_1, r_1(t_1))$, but the difference between θ' and $B(\theta' + tz_1, r_1(t_1))$ is $|t_1| - r_1(t_1) = |t_1| - \sqrt{t_1^2 - D^2} = \frac{D^2}{t_1 + \sqrt{t_1^2 - D^2}}$ for large enough t_1, which goes to 0 as $t_1 \to \infty$. A symmetric argument shows that $|t_2| - r_2(t_2) \to 0$ as $t_2 \to \infty$. Given this, the same arguments as in Proposition 11 establish that for any $\theta, \theta' \in \Theta$, we have $B(\theta' + tz_1, r_1(t_1)) \cup B(\theta + tz_2, r_2(t_2)) \not\supseteq \text{co}(\Theta)$ for all $t_1, t_2 \in \mathbb{R}_+$ if and only if $H^\circ(z_1, z_1 \cdot \theta') \cup H^\circ(z_2, z_2 \cdot \theta) \not\supseteq \text{co}(\Theta)$. The claim then follows from Propositions 11 and 22. QED.

6.3 Commitment

In the model discussed so far, we assumed that the policymaker cannot commit to a policy function, i.e., after every message pair, she has to play a best response to her updated belief. The case in which the policymaker can commit to a policy function is a mechanism-design problem, which is referred to as delegation in the cheap talk literature. Typically the equilibria of a cheap talk game with and without commitment are very different from each other, and the payoff of the receiver is strictly higher if she is able to commit to a policy function than if such commitment is not possible. However, if the set of feasible policies is equal to the set of policies rationalizable by some belief on the state space, then there is no difference between the mechanism-design problem and the cheap-talk model in terms of full revelation. Namely, if $Y = \text{co}(\Theta)$, then the receiver gets the best feasible payoff (zero) in the mechanism-design problem if and only if there exists a fully revealing equilibrium in the cheap-talk model without commitment. This is because the best feasible payoff in the mechanism-design problem can only be achieved if the mechanism implements policy θ in every state θ. Then by the revelation principle, there has to be a mechanism achieving the same outcome in which the senders both report state, and the mechanism selects policy θ after a message pair (θ, θ) for every $\theta \in \Theta$, while it selects some $y \in Y$ after incompatible messages. Note that if $Y = \mathrm{co}(\Theta)$, then the above are exactly the conditions for a truthful equilibrium in the case of no commitment, which by Lemma 1 are also the conditions for a fully revealing equilibrium.

6.4 Imperfectly observed state

The assumption in our paper, and in most of the literature in cheap talk, that the experts observe the state perfectly is not realistic in many settings. One of our motivations for examining equilibria that satisfy diagonal continuity in Section 4 is that senders might make small mistakes in observing the state. A more direct way of addressing the issue is explicitly analyzing models in which the senders' observations are noisy. Battaglini (2004) shows that the construction in Battaglini (2002), which is fully revealing in the limit as noise goes to zero, remains an equilibrium for a multidimensional Euclidean space even if observations are noisy. However, the proof of this claim relies on various special assumptions. In particular, the state is assumed to be distributed ex ante according to an improper uniform prior. We argue that the uniform prior is very special in a multi-sender cheap talk model, in the sense that it is the only prior distribution which is compatible with the Battaglini construction to be an equilibrium. It is not clear to what set of priors and how the results of Battaglini [04] can be extended. If extreme states are very unlikely—a reasonable assumption in most cases when the policy space is unbounded—then no matter how small (positive) the noise in observations is, after some message pairs the receiver thinks that it is much more likely that the noise term of one of the senders was particularly large than that a very unlikely state occurred. This can corrupt the incentives of the senders to report their observations truthfully in the Battaglini construction. Consider again the budget allocation game depicted in Figure 1, but now assume that the state space is the whole Euclidean space (the budget is of flexible size). If states outside triangle ABC are unlikely enough, then in a noisy model, the receiver will still choose a policy in or close to triangle ABC after any message pair. Therefore, thin tails of the distribution in a noisy model can invalidate Battaglini's equilibrium construction for the same reason that support restrictions do in a model where observations are perfect. The assumption of improper uniform prior avoids this problem because the tails of the distribution are extremely thick. The problem of analyzing noisy cheap talk models with general prior distributions is a difficult but important direction of future research.

7 Conclusion

This paper argues that in a cheap talk model with multiple senders, the amount of information that can be transmitted in equilibrium depends not on the dimensionality of the state space but on finer details of the model specification. These details include the shape of the boundary of the state space and how similar preferences of the senders are, where similarity is defined with respect to the state space. It is worth pointing out that the properties of the state space and sender preferences cannot be investigated independently, once we allow for general (state-dependent) preferences. For example, an open bounded state space with state-independent preferences can be transformed into an unbounded state space with state-dependent preferences in a way that the resulting games are strategically equivalent. The conclusion that qualitative predictions from a cheap talk model depend on the parameters of the model (state space, senders' preferences in different states) in a relatively complicated way suggests that in applications of cheap talk models, it is important to derive these parameters from more basic underlying objective functions.

8 Appendix A: Consistency of beliefs and diagonal continuity

In this Appendix, we show that if we restrict attention to strategies that satisfy some regularity conditions, then every equilibrium in which the receivers' beliefs are consistent satisfies diagonal continuity (as defined in 5.1).

Consider a PBE (s_1, s_2, y) and conditional beliefs of the sender $f()$ that support this equilibrium. For any $m_1 \in M_1$ and $m_2 \in M_2$, let $\mu(m_1, m_2)$ denote the expectation of θ according to $f(m_1, m_2)$.

In order to check for consistency of the above beliefs, we need to define models in which the observations of senders are noisy. We will consider a sequence of noisy models indexed by $k = 1, 2, ...$ In the noisy model indexed by k, senders 1 and 2 observe signals $\theta_1 \in \Theta$ and $\theta_2 \in \Theta$, respectively. For each true state $\theta \in \Theta$, the joint density function of signals (θ_1, θ_2) conditional on θ is given by $g^k(\theta_1, \theta_2|\theta)$. We assume that noise disappears in the limit: $g^k(\theta_1, \theta_2|\theta)$ converges in probability to (θ, θ) as $k \to \infty$.

An example for the above construction, which is similar in spirit to the one proposed in Battaglini (2004), is when

$$\theta_i = \theta + \varepsilon_k u_i,$$

where (u_1, u_2) is a truncated standard normal distribution on $\mathbb{R}^{2d}$ and $\varepsilon_k \to 0$ as $k \to \infty$. Truncation is needed to assure that θ_i belongs to Θ for sure.[10]

Fixing the senders' strategies in the sequence of noisy models to be $s_i(\theta_i)$, let $f^k(m_1, m_2)$ denote the posterior belief of the receiver in the model indexed by k, given two reports (m_1, m_2). Let $\mu^k(m_1, m_2)$ be the expectation of θ with respect to $f^k(m_1, m_2)$.

Definition: We say f is *consistent* if $f^k(m_1, m_2)$ weakly converges to $f(m_1, m_2)$ uniformly over $(m_1, m_2) \in s_1(\Theta) \times s_2(\Theta)$, i.e., for any $\varepsilon > 0$ and any continuous and bounded function b on Θ, there exists K such that we have

$$\left| \int b(\theta) f^k(m_1, m_2)(d\theta) - \int b(\theta) f(m_1, m_2)(d\theta) \right| < \varepsilon,$$

for any $(m_1, m_2) \in s_1(\Theta) \times s_2(\Theta)$ and any $k > K$.[11]

If Θ is bounded, then this definition implies that $\mu^k(m_1, m_2)$ uniformly converges to $\mu(m_1, m_2)$ as $k \to \infty$.

[10] On the other hand, noise structures like the one in Section 3 of Battaglini (2002) are not compatible with our framework because they do not admit a density function $g^k(\theta_1, \theta_2|\theta)$.

[11] We note that the requirement of uniform convergence is strong. We do not know whether consistency implies diagonal continuity if we use pointwise convergence.

To show our main result concerning consistent beliefs in the limit model, first we establish a result that applies to beliefs in the noisy models defined above. We show that $\mu^k(m_1, m_2)$ is continuous in (m_1, m_2) for any k. Intuitively speaking, in a noisy model, even if the receiver gets two pairs of messages that are a little different from each other, she does not drastically change her belief about the true state, for the difference between the message pairs does not necessarily mean a drastic difference in the true state, but means a small change in the noise contained in the senders' signals. Once we establish the continuity of μ^k, we show that continuity is inherited to μ in the limit model without noise, which implies diagonal continuity when reporting functions s_i are continuous.

In order to use Bayes' rule for continuous random variables, we impose several restrictions on senders' reporting functions. For each i, the message space M_i is a subset of a Euclidean space $\mathbb{R}^{n_i}$, and each inverse image of message m_i with respect to s_i, $s_i^{-1}(m_i) = \{\theta_i \in \Theta | s_i(\theta_i) = m_i\}$, is parametrized by $t_i \in T_i \subseteq \mathbb{R}^{d-n_i}$. That is to say, there exists a continuously differentiable bijection

$$h_i \colon M_i \times T_i \to \Theta$$

such that $m_i = s_i(\theta_i)$ if and only if $\theta_i = h_i(m_i, t_i)$ for some $t_i \in T_i$.[12]

Given (h_1, h_2), the density function of (m_1, m_2) with respect to the Lebesgue measure on $M_1 \times M_2$ conditional on the true state θ is

$$\int_{T_2} \int_{T_1} g^k(h_1(m_1, t_1), h_2(m_2, t_2) | \theta) |J_1(m_1, t_1) J_2(m_2, t_2)| dt_1 dt_2,$$

where $J_i(m_i, t_i)$ is the Jacobian of h_i at (m_i, t_i):

$$J_i(m_i, t_i) = \det \frac{\partial h_i(m_i, t_i)}{\partial(m_i, t_i)}.$$

Proposition 24 *Suppose*

1. *Θ, T_1, and T_2 are compact;*
2. *$g^k(\theta_1, \theta_2 | \theta)$ is continuous in (θ_1, θ_2), $g^k(\theta_1, \theta_2 | \theta) > 0$, and bounded;*
3. *for each sender i, $J_i(m_i, t_i)$ is continuous in m_i, $J_i(m_i, t_i) \neq 0$, and $J_i(m_i, t_i)$ is bounded.*

Then the expectation $\mu^k(m_1, m_2)$ of θ conditional on (m_1, m_2) in the k-th noisy model is continuous in (m_1, m_2).

[12] For example, in Battaglini's (2002) equilibrium construction, h_i is the identity function on $\mathbb{R}^d$; M_1 and M_2 are subspaces of $\mathbb{R}^d$ that form a coordinate system: every point in $\theta \in \mathbb{R}^d$ is uniquely expressed by a linear combination of $m_1 \in M_1$ and $m_2 \in M_2$; M_i contains sender j's bias direction; and $T_i = M_j$. Such a coordinate system exists if $d \geq 2$ and two senders' biases are not parallel.

Proof: $\mu^k(m_1, m_2)$ is given by

$$\frac{E\left[\theta \int_{T_1} \int_{T_2} g^k(h_1(m_1,t_1), h_2(m_2,t_2)|\theta) |J_1(m_1,t_1) J_2(m_2,t_2)| dt_1 dt_2\right]}{E\left[\int_{T_2} \int_{T_1} g^k(h_1(m_1,t_1), h_2(m_2,t_2)|\theta) |J_1(m_1,t_1) J_2(m_2,t_2)| dt_1 dt_2\right]}.$$

The denominator is nonzero. Also, by the Lebesgue Convergence Theorem, both the numerator and the denominator are continuous in (m_1, m_2).[13] Therefore, $\mu^k(m_1, m_2)$ is continuous with respect to (m_1, m_2). QED

Proposition 25 *Let (s_1, s_2, y) be an equilibrium in the limit game. In addition to the assumptions in Proposition 24, suppose that $m_i = s_i(\theta_i)$ is continuous in θ_i for each $i = 1$, 2. Then every equilibrium that is supported by a consistent belief is continuous on the diagonal.*

Proof: By Proposition 24, $\mu^k(m_1, m_2)$ is continuous in (m_1, m_2). Since $\mu^k(m_1, m_2)$ converges to $\mu(m_1, m_2)$ uniformly over (m_1, m_2), $\mu(m_1, m_2)$ is also continuous in (m_1, m_2), and hence $\mu(s_1(\theta_1), s_2(\theta_2))$ is continuous in (θ_1, θ_2). QED

[13] See Royden (1988), Theorem 16 of Chapter 4.

9 References

Austen-Smith, D. [1990a]: "Information transmission in debate," American Journal of Political Science, 43 (1), 124-52

Austen-Smith, D. [1990b]: "Credible debate equilibria," Social Choice and Welfare, 7, 75-93

Battaglini, M. [2002] "Multiple referrals and multidimensional cheap talk," Econometrica 70 (4), 1379-140

Battaglini, M. [2004]: "Policy advice with imperfectly informed experts," Advances in Theoretical Economics 4(1), Article 1

Chakraborty, A. and R. Harbaugh [2005]: "Comparative Cheap Talk," Journal of Economic Theory, forthcoming

Crawford, V. and J. Sobel [1982]: "Strategic information transmission," Econometrica 50 (6), 1431-51

Farrell, J. and R. Gibbons [1989]: "Cheap talk with two audiences," American Economic Review 79 (5), 1214-1223

Gilligan, T. W. and K. Krehbiel [1989]: "Asymmetric information and legislative rules with a heterogeneous committee," American Journal of Political Science 33 (2), 459-90

Green, J. R. and N. Stokey [1980]: "A Two-Person Game of Information Transmission," Harvard Institute of Economic Research Discussion Paper No. 751

Levy, G. and R. Razin [2005]: "On the limits of communication in multidimensional cheap talk," mimeo LSE

Morgan, J. and V. Krishna [2001]: "A model of expertise," Quarterly Journal of Economics 116, 747-75

Morgan, J. and V. Krishna [2001]: "Asymmetric information and legislative rules," American Political Science Review 95 (2), 435-52

Morgan, J. and V. Krishna [2004]: "The art of conversation," Journal of Economic Theory 117, 147-79

Royden, H. L. [1988]: "Real analysis," third edition, Prentice Hall, Upper Saddle River, NJ

Wolinsky, A. [2002]: "Eliciting information from multiple experts," GAEB 41, 141-160