Subgame perfect equilibria for stochastic games

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Abstract

For an n-person stochastic game with Borel state space S and compact metric action sets $A_1, A_2, \ldots, A_n$, sufficient conditions are given for the existence of subgame perfect equilibria. One result is that such equilibria exist if the law of motion $q(\cdot|s,a)$ is, for fixed s, continuous in $a = (a_1, a_2, \ldots, a_n)$ for the total variation norm and the payoff functions $f_1, f_2, \ldots, f_n$ are bounded, Borel measurable functions of the sequence of states $(s_1, s_2, \ldots) \in S^{\mathbb{N}}$ and, in addition, are continuous when $S^{\mathbb{N}}$ is given the product of discrete topologies on S.

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1 Introduction.

The stochastic games we study have n players $1, \ldots, n$. The state space S is a Borel subset of a Polish space. Each player i has a compact metric action set A_i. The set $\Delta(A_i)$ of probability measures defined on the Borel subsets of A_i is equipped with its usual weak topology and, hence, $\Delta(A_i)$ is also compact metrizable. Let $A = A_1 \times A_2 \times \cdots \times A_n$ have its product topology and it too is compact metrizable. The law of motion q is a conditional probability distribution on S given $S \times A$ with the interpretation that, if the players choose actions $a = (a_1, a_2, \ldots, a_n) \in A$ at state $s \in S$, then $q(\cdot|s,a)$ is the conditional distribution of the next state. We always assume that $q(B|s,a)$ is Borel measurable jointly in (s,a) for B a Borel subset of S, and we will make further assumptions below.

The game begins at some initial state s_1. The players choose actions $a^1 = (a_1^1, a_2^1, \ldots, a_n^1)$ and the next state s_2 has distribution $q(\cdot|s_1, a^1)$. This process is iterated to generate a random *history* or *play*

$$h = (s_1, (a^1, s_2), (a^2, s_3), \ldots) \in H = S \times (A \times S)^{\mathbb{N}}.$$

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Here $a^k = (a_1^k, a_2^k, \ldots, a_n^k)$ is, for each k, the n-tuple of actions chosen by the players at stage k. Each player i has a bounded, Borel measurable payoff function $f_i : H \mapsto \mathbb{R}$ and receives $f_i(h)$ as payoff at history h. Let $f = (f_1, f_2, \ldots, f_n)$ be the n-tuple of payoff functions.

Denote by S^* the disjoint union of the sets $S, S \times (A \times S), S \times (A \times S)^2, \ldots$; that is,

$$S^* = \bigcup_{k \geq 0} [S \times (A \times S)^k].$$

The elements of S^* are called *partial histories.*

A *strategy* σ_i for player i assigns to each partial history $p = (s_1, (a^1, s_2), \ldots, (a^{k-1}, s_k))$ in S^* the conditional distribution $\sigma_i(p) \in \Delta(A_i)$ for a_i^k given p. Formally, a strategy for player i is a Borel function from S^* into $\Delta(A_i)$. It is assumed that the players choose their actions independently. So the conditional distribution of a^k given p is the product measure

$$\sigma(p) = \sigma_1(p) \times \sigma_2(p) \cdots \times \sigma_n(p) \tag{1.1}$$

on A.

An n-tuple $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_n)$ consisting of a strategy for each player is called a *profile.* A profile σ together with an initial state s_1 determines the distribution $P_\sigma = P_{s_1,\sigma}$ of the history h. Note that, by an abuse of notation, we write $\sigma(p)$ for the n-tuple $(\sigma_1(p), \sigma_2(p), \ldots, \sigma_n(p))$ as well as for the product measure in (1.1). The meaning will always be clear from the context.

The stochastic game $\Gamma(f, s_1)$ begins at state s_1. The players select strategies to form a profile σ and each player i receives the expected payoff

$$E_\sigma f_i = \int f_i(h)\, P_\sigma(dh).$$

We write $E_\sigma f$ for the vector of expected payoffs $(E_\sigma f_1, E_\sigma f_2, \ldots, E_\sigma f_n)$.

To define a subgame of $\Gamma(f, s_1)$ let $p = (s_1, (a^1, s_2), \ldots, (a^{k-1}, s_k))$ be a partial history. The length of p, written $lh(p)$, is defined to be $k - 1$. (Thus $lh(s) = 0$ for $s \in S$.) Denote by $l(p)$ the last state s_k of p. The *subgame* $\Gamma(f, p)$ is the stochastic game with initial state $s_k = l(p)$ and payoff functions $f_i p$ defined for histories

$$h' = (s_k, (b^1, t_2), (b^2, t_3), \ldots)$$

to be

$$(f_i p)(h') = f_i(s_1, (a^1, s_2), \ldots, (a^{k-1}, s_k), (b^1, t_2), (b^2, t_3), \ldots).$$

Thus $f_i p$ is the section of f_i at p. We use $\Gamma(f, \cdot)$ to denote the collection of all the games $\Gamma(f, p)$, $p \in S^*$. (Notice that the game $\Gamma(f, s)$ is itself the subgame $\Gamma(f, p)$ for which $p = s$.)

For a strategy σ_i and a partial history p, the *conditional strategy given* p is written $\sigma_i[p]$. If $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_n)$ is a profile of strategies, the *conditional profile* $\sigma[p]$ is just the profile of conditional strategies $(\sigma_1[p], \sigma_2[p], \ldots, \sigma_n[p])$.

Now we can define a *subgame perfect equilibrium* (or SPE for short) for $\Gamma(f, \cdot)$ as being a profile σ such that, for all $p \in S^*$, the conditional profile $\sigma[p]$ is a Nash equilibrium for the subgame $\Gamma(f, p)$.

To prove the existence of an SPE, further conditions on the game are necessary. (Without additional assumptions there need not exist an SPE even for one-person games.) To state one of the conditions, let $\Delta(S)$ be the set of probability measures defined on the Borel subsets of the state space S.

Condition 1. Variation norm continuity. *For every fixed $s \in S$, the law of motion $q(\cdot|s,a)$ is continuous in a for the total variation norm on $\Delta(S)$.*

The next condition is inspired by Dubins and Savage (1976), who study (finitely additive) probability measures on an infinite product of spaces each of which is given the discrete topology.

Definition 1. *Suppose that each of the nonempty sets $X_1, X_2, \ldots$ has the discrete topology and the product $Y = X_1 \times X_2 \times \cdots$ has the product topology. Then a continuous function g defined on Y is called* DS-continuous.

Now we can state our first theorem.

Theorem 1.1. *Assume condition 1. Assume further that the payoff functions $f_1, f_2, \ldots, f_n$ depend only on the sequence of states and are DS-continuous from H into $\mathbb{R}$. Then $\Gamma(f,\cdot)$ has a subgame perfect equilibrium.*

Recall that we always assume that the payoff f is bounded and Borel measurable for the product of the original Polish topologies. Of course, the assumption of DS-continuity is (typically) weaker than an assumption of continuity for the product of the original topologies.

An example in Harris et al (1995) shows this theorem does not hold in general for payoff functions that depend on actions as well as states. However, the corresponding result is true when the action sets are finite.

Theorem 1.2. *Assume that the action sets $A_1, A_2, \ldots, A_n$ are finite and that the payoff functions $f_1, f_2, \ldots, f_n$ are DS-continuous from H into $\mathbb{R}$. Then $\Gamma(f,\cdot)$ has a subgame perfect equilibrium.*

Fudenberg and Levine (1983) established the special case of Theorem 1.2 in which the state space is finite. See also section 6.3 in Harris et al (1995).

Our last result is for games with additive payoffs.

Condition 2. Additive rewards. *Assume that, for $i = 1, 2, \ldots, n$,*

$$f_i(h) = \sum_{k=1}^{\infty} r_{ik}(s_k, a^k)$$

for all $h = (s_1, (a^1, s_2), (a^2, s_3), \ldots) \in H$, where the functions $r_{ik} : S \times A \mapsto \mathbb{R}$ are uniformly bounded, Borel in s for fixed a, and continuous in a for fixed s. Assume also that the convergence of the partial sums $\sum_{k=1}^{m} r_{ik}(s_k, a^k)$ is uniform on H as $m \to \infty$.

Theorem 1.3. *Under conditions 1 and 2, $\Gamma(f,\cdot)$ has a subgame perfect equilibrium.*

Theorem 1.3 implies the existence of an SPE for discounted games in which the daily reward function of each player is bounded, Borel in s, and continuous in a.

Antecedents. The arguments in this paper owe a great deal to the earlier work of Amir (1991), Mertens and Parthasarathy (1991, 2003) and Solan (1998). Indeed, the method we use in the next section is abstracted from Solan's proof that SPE's exist for discounted stochastic games. A crucial tool for us, as it was for Solan, is the "measurable "measurable" choice theorem" of Mertens (2003). We also rely on concepts introduced in the gambling theory of Dubins and Savage (1976).

The existence of stationary equilibria for discounted stochastic games has been proved under various assumptions. See, for example, Nowak (2003b), and Parthasarathy and Sinha (1989). Of course, stationary equilibria are, in particular, subgame perfect.

Since subgame perfect equilibria do not always exist, it is natural to look for subgame perfect correlated equilibria. Their existence has been established in different contexts by Harris et al (1995), Nowak and Raghavan (1992), Nowak (2003a), and Solan and Vieille (2002) among others.

Outline. In the next section we prove an abstract existence result to the effect that an SPE exists for $\Gamma(f, \cdot)$ when the payoff f is DS-continuous and can be uniformly approximated by a sufficiently nice function g such that $\Gamma(g, \cdot)$ has an SPE. This abstract result leads to the proofs of Theorems 1.1-1.3 in the last three sections.

2 An adaptation of a proof of E. Solan

E. Solan (1998) gave a nice proof of the existence of SPE's for discounted stochastic games. In this section we adapt his methods to prove a technical result which will be the key to our proofs of Theorems 1.1-1.3.

We make the following assumptions throughout this section.

Assumption 1. *The payoff function $f = (f_1, f_2, \dots, f_n)$ is Borel and DS-continuous from H into $[-R, R]^n$ where R is a fixed positive real number.*

Assumption 2. *There exist Borel functions $g^m : H \mapsto [-R, R]^n$, $m \in \mathbb{N}$ with the following properties:*

(i) $||g^m - f|| \to 0$ as $m \to \infty$, where, for a function $\phi : H \mapsto [-R, R]^n$, $||\phi|| = \sup_h |\phi(h)|$ and $|\cdot|$ is the usual norm on Euclidean n-space.

(ii) for each m, there is a subgame perfect equilibrium σ^m in the game $\Gamma(g^m, \cdot)$ with corresponding equilibrium payoff $V^m : S^* \mapsto [-R, R]^n$, that is, for each $p \in S^*$,

$$V^m(p) = E_{\sigma^m[p]}(g^m p). \tag{2.1}$$

Furthermore, for each $p \in S^*$, $\sigma^m(p)$ is an equilibrium profile in the one-move game with payoff $\int V^m(pat)\, q(dt|l(p), a)$ and equilibrium payoff

$$V^m(p) = \int \int V^m(pat)\, q(dt|l(p), a) \sigma^m(p)(da), \tag{2.2}$$

and the family of functions $\{\int V^m(pat)\, q(dt|l(p), a) \,:\, m \in \mathbb{N}\}$ is equicontinuous in a.

Remark 2.1. ***Notation.***

(i) For $p \in S^*$, $a \in A$, and $t \in S$, the notation pat used above denotes the partial history that consists of the coordinates of p followed by a and t.

(ii) Recall that we use $\sigma^m(p)$ to denote the product measure $\sigma_1^m(p) \times \sigma_2^m(p) \times \cdots \sigma_n^m(p)$ in $\Delta(A)$, and also for the n-tuple $(\sigma_1^m(p), \sigma_2^m(p), \ldots, \sigma_n^m(p))$.

Here is the technical result we will need.

Theorem 2.2. *Under assumptions 1 and 2, the game $\Gamma(f, \cdot)$ has a subgame perfect equilibrium.*

The proof will use certain properties of multifunctions which we present in four lemmas.

Let X be a Borel subset of a Polish space, let Y be a Polish space, and let $\{\psi_m\}$ be a sequence of Borel functions from X into Y. The multifunction $Ls(\psi_m)$ from X to subsets of Y assigns to each $x \in X$ the set $Ls(\psi_m)(x)$ of all $y \in Y$ such that, for every open subset V of Y containing y, $\psi_m(x) \in V$ for infinitely many m. We write $Gr(Ls(\psi_m))$ for the graph of $Ls(\psi_m)$, namely the set

$$Gr(Ls(\psi_m)) = \{(x, y) \,:\, y \in Ls(\psi_m)(x)\}.$$

Lemma 2.3. *$Gr(Ls(\psi_m))$ is a Borel subset of $X \times Y$, and, for each x, $Ls(\psi_m)(x)$ is a closed subset of Y.*

Proof. Let $\{V_m\}$ be a countable base for the topology of Y. Then

$$\begin{aligned}(x, y) \notin Gr(Ls(\psi_m)) &\text{ iff } (\exists m)[y \in V_m \;\&\; (\exists k)(\forall l \geq k)(\psi_l(x) \notin V_m)] \\ &\text{ iff } (\exists m)[y \in V_m \;\&\; x \in \liminf_l \psi_l^{-1}(Y - V_m)].\end{aligned}$$

Hence,

$$(X \times Y) - Gr(Ls(\psi_m)) = \bigcup_m [\liminf_l \psi_l^{-1}(Y - V_m) \times V_m],$$

which is clearly Borel. The proof that $Ls(\psi_m)(x)$ is closed is completely straightforward. $\square$

A multifunction F from X to Y is defined to be *Borel measurable* if, for every open subset U of Y, the set $\{x \in X \,:\, F(x) \cap U \neq \emptyset\}$ is a Borel subset of X.

Lemma 2.4. *Assume that, for each x, the set $\{\psi_m(x) \,:\, m \geq 1\}$ is precompact in Y. Then (a) $Ls(\psi_m)$ is a Borel measurable multifunction with nonempty compact values, and (b) $Ls(\psi_m)$ admits a Borel measurable selector.*

Proof. Part (a) follows from the previous lemma, the precompactness assumption, and the Kunugui - Novikov theorem. (See 4F.12 in Moschovakis (1980) or 4.7.11 in Srivastava (1998).) Part (b) is a consequence of (a) and the selection theorem of Kuratowski and Ryll-Nardzewski. (See Corollary 5.2.5 in Srivastava (1998).) $\square$

The next lemma records, for ease of reference, a part of Lemma 3, page 69, in Hildenbrand (1974).

Lemma 2.5. *Let $Y = [-R, R]^n$ and let μ be a probability measure on the Borel subsets of Y. Suppose $\lim_m \int \psi_m \, d\mu$ exists. Then there is a Borel selector ψ of $Ls(\psi_m)$ such that $\int \psi \, d\mu = \lim_m \int \psi_m \, d\mu$.*

Our last lemma specializes a deep result of Mertens (2003) to a Borel setting.

Lemma 2.6. *Suppose that Y and Z are Borel subsets of Polish spaces and F is a Borel measurable multifunction on $Y \times Z$ with nonempty compact subsets of $[-R, R]^n$ as values. Let $q(\cdot|y)$ be a Borel measurable transition function from Y to Z. Define a multifunction G on Y as follows:*

$$G(y) = \{\int f(y,z)\, q(dz|y) \,:\, f \text{ is a Borel selector of} F\}.$$

Then

(i) G is a Borel measurable multifunction with nonempty compact values,

(ii) there is a Borel measurable function $g : Gr(G) \times Z \mapsto [-R, R]^n$ such that, for every $(y, x) \in Gr(G)$, $g(y, x, \cdot)$ is a selector for the multifunction $F(y, \cdot)$ and

$$x = \int g(y, x, z)\, q(dz|y).$$

Remark 2.7. *The multifunction G in Lemma 2.6 can also be described, for each $y^* \in Y$, as $G(y^*) = \{\int g(z)\, q(dz|y^*) \,:\, g$ is a Borel selector of $F(y^*, \cdot)\}$. To see this, fix a Borel selector ϕ of F and define*

$$f(y,z) = \begin{cases} g(z) & \text{if } y = y^*, \\ \phi(y,z) & \text{if } y \neq y^*. \end{cases}$$

The proof of Theorem 2.2 now proceeds in a number of steps.

Step 1. Define the multifunction D on S^* by $D(p) = Ls(V^m)(p)$. By Lemma 2.3, $Gr(D)$ is a Borel subset of $S^* \times [-R, R]^n$. In addition, for each $p \in S^*$, $\{V^m(p) : m \geq 1\}$ is precompact since $[-R, R]^n$ is compact. So, by Lemma 2.4, D is Borel measurable with nonempty compact values.

Step 2. Define $u^m : S^* \times A \mapsto [-R, R]^n$ by

$$u^m(p, a) = \int V^m(pat)\, q(dt|l(p), a).$$

By assumption 2, u^m is a Caratheodory function (Borel in p and continuous in a). Also the set $\{u^m(p, \cdot) : m \geq 1\}$ is an equicontinuous subset of the space $C = C(A, [-R, R]^n)$ of continuous functions from A to $[-R, R]^n$. Hence, by the Arzela-Ascoli theorem, this set is precompact in the topology of uniform convergence on C.

Step 3. Next define $\phi^m : S^* \mapsto [-R, R]^n \times C \times \Delta(A)$ by setting

$$\phi^m(p) = (V^m(p), u^m(p, \cdot), \sigma^m(p)).$$

Plainly, ϕ^m is Borel measurable. Also note that, for each $p \in S^*$, the set $\{\sigma^m(p) : m \geq 1\}$ is precompact in the topology of weak convergence on $\Delta(A)$. Observe that a limit point of $\{\sigma^m(p) : m \geq 1\}$ is again a product measure on A.

Step 4. Define another multifunction G on $Gr(D)$ by

$$G(p,x) = \{(v,\nu) \in C \times \Delta(A) \, : \, (x,v,\nu) \in Ls(\phi^m)(p)\}.$$

It is easy to see that $Gr(G)$ is a Borel subset of $S^* \times [-R,R]^n \times C \times \Delta(A)$. Also, for each $(p,x) \in Gr(D)$, $G(p,x)$ is the x-section of the nonempty compact set $Ls(\phi^m)(p)$, and so is itself compact. Hence, by Lemma 2.4, G is a Borel measurable multifunction on the Borel set $Gr(D)$ with nonempty compact values, and there are Borel functions v^* and ν^* on $Gr(D)$ into $C = C(A,[-R,R]^n)$ and $\Delta(A)$, respectively, such that $(v^*(p,x), \nu^*(p,x)) \in G(p,x)$ for every $(p,x) \in Gr(D)$. (We remind the reader that, as mentionned at the end of step 3, $\nu^*(p,x)$ is a product measure on A.)

Step 5. We define a third multifunction Ω on $S^* \times A$ by

$$\Omega(p,a) = \{\int g(t)\, q(dt|l(p),a) \, : \, g \text{ is a Borel selector of } D(pa\cdot)\}.$$

By Lemma 2.6 and Remark 2.7, Ω is a Borel measurable multifunction with nonempty compact values (so that $Gr(\Omega)$ is a Borel subset of $S^* \times A \times [-R,R]^n$) and there exists a Borel function $\psi : Gr(\Omega) \times S \mapsto [-R,R]^n$ such that

(i) $\psi(p,a,x,t) \in D(pat)$, for all $t \in S$, and

(ii) $\int \psi(p,a,x,t)\, q(dt|l(p),a) = x$, for every $(p,a,x) \in Gr(\Omega)$.

Step 6. Claim: If $(p,x) \in Gr(D)$, then $(\forall a)[v^*(p,x)(a) \in \Omega(p,a)]$.
To verify this claim, let $(p,x) \in Gr(D)$ and fix $a \in A$. Choose a subsequence $\{u^{m_i}(p,\cdot)\}$ of $\{u^m(p,\cdot)\}$ such that $u^{m_i}(p,\cdot)$ converges uniformly to $v^*(p,x)$. So, in particular, $u^{m_i}(p,a)$ converges to $v^*(p,x)(a)$. The last statement can be written as

$$\lim_i \int V^{m_i}(pat)\, q(dt|l(p),a) = v^*(p,x)(a). \tag{2.3}$$

So, by Lemma 2.5, there is a Borel function $g : S \mapsto [-R,R]^n$ such that g is a selector of $D(pa\cdot)$ and

$$\lim_i \int V^{m_i}(pat)\, q(dt|l(p),a) = \int g(t)\, q(dt|l(p),a). \tag{2.4}$$

It follows from (2.3) and (2.4) that $v^*(p,x)(a) \in \Omega(p,a)$.

Step 7. Let $\bar{D} = \{(p,a,x,t) \in S^* \times A \times [-R,R]^n \times S \, : \, x \in D(p)\}$ and define $\psi^* : \bar{D} \mapsto [-R,R]^n$ by setting

$$\psi^*(p,a,x,t) = \psi(p,a,v^*(p,x)(a),t),$$

where ψ is the function introduced in Step 5. Then ψ^* is well-defined by Step 6. Plainly, ψ^* is a Borel function. Note that it follows from (ii) of Step 5 that

$$v^*(p,x)(a) = \int \psi^*(p,a,x,t)\, q(dt|l(p),a) \tag{2.5}$$

for every $(p,a,x,t) \in \bar{D}$.

Step 8. Claim: For $(p, x) \in Gr(D)$, $\nu^*(p, x)$ is an equilibrium profile in the one-move game with payoff $v^*(p, x)(a)$ and the corresponding equilibrium payoff is x.

To see this, choose a subsequence $\{(V^{m_i}(p), u^{m_i}(p, \cdot), \sigma^{m_i}(p))\}$ of $\{(V^m(p), u^m(p, \cdot), \sigma^m(p))\}$ such that

$$\lim_i (V^{m_i}(p), u^{m_i}(p, \cdot), \sigma^{m_i}(p)) = (x, v^*(p, x), \nu^*(p, x)). \tag{2.6}$$

Since $\sigma^{m_i}(p)$ is an equilibrium payoff in the one-move game with payoff $u^{m_i}(p, \cdot)$ by assumption 2(ii) and $V^{m_i}(p)$ is the corresponding equilibrium payoff by virtue of (2.2), the claim follows from (2.6).

Step 9. We are now in a position to define a profile τ for the game $\Gamma(f, \cdot)$. (It will turn out that τ is an SPE.) First define a function $\pi : S^* \mapsto [-R, R]^n$ by recursion as follows: for $s \in S$, set $\pi(s) = \xi(s)$ where ξ is a fixed, but arbitrary, Borel selector for $D(s)$; and, for $p \in S^*$, $a \in A$, and $t \in S$, let

$$\pi(pat) = \psi^*(p, a, \pi(p), t).$$

Using step 5 and induction on the length of p, one proves easily that $\pi(p) \in D(p)$ and that π is well-defined. Clearly, π is a Borel function. The profile τ can now be defined on S^* by

$$\tau(p) = \nu^*(p, \pi(p)).$$

So τ is also clearly Borel.

Step 10. Let $V(p) = \int fp(h)\, P_\tau(dh)$, $p \in S^*$. (Recall that P_τ is the probability measure induced by τ on the Borel subsets of H.)

Claim: $V(p) = \pi(p)$.

We prove the claim for $p = s \in S$. (The proof for other p's is similar and is omitted.) Let $\epsilon > 0$ and fix

$$h = (s_1, (a^1, s_2), (a^2, s_3), \ldots) \in H.$$

By the DS-continuity of f (assumption 1), we can choose k so large that with

$$p = p_k(h) = (s_1, (a^1, s_2), \ldots, (a^{k-1}, s_k)),$$

we have $||f(ph') - f(h)|| < \epsilon/2$ for all $h' \in (A \times S)^{\mathbb{N}}$. (Here ph' is the history that consists of the coordinates of p followed by those of h'.) Next choose M so large that $||g^m - f|| < \epsilon/2$ for all $m \geq M$. So, if $q \supseteq p$, $m \geq M$, and $x \in D(q)$, then $||x - f(h)|| \leq \epsilon$. To see this, observe that

$$||g^m(qh') - f(h)|| \leq ||g^m(qh') - f(qh')|| + ||f(qh') - f(h)|| < \epsilon.$$

It follows from (2.1) that $||V^m(q) - f(h)|| \leq \epsilon$ for all $m \geq M$ so that $||x - f(h)|| \leq \epsilon$. Consequently, if $l \geq k$ and $x \in D(p_l(h))$, then $||x - f(h)|| \leq \epsilon$. Hence, by step 9, $||\pi(p_l(h)) - f(h)|| \leq \epsilon$, and therefore

$$\lim_{m \to \infty} \pi(p_m(h)) = f(h). \tag{2.7}$$

Denote the history generated by the probability measure P_τ with initial state s by

$$(s, Y_1, Y_2, \ldots, Y_m, \ldots).$$

(Thus $Y_m = (a^m, s_{m+1})$, $m \geq 1$.) Then the process $\pi(s), \pi(sY_1), \ldots, \pi(sY_1 \cdots Y_m), \ldots$ is a martingale. Indeed,

$$
\begin{aligned}
E_\tau[\pi(sY_1 \cdots Y_m Y_{m+1} \,|\, sY_1 \cdots Y_m = p] &= \int\int \psi^*(p, a, \pi(p), t)\, q(dt|l(p), a)\, \nu^*(p, \pi(p))(da) \\
&= \int v^*(p, \pi(p))(a)\, \nu^*(p, \pi(p))(da) \\
&= \pi(p),
\end{aligned}
$$

where the last two equalities are by (2.5) and step 8, respectively.

By (2.7), $\pi(sY_1 \cdots Y_m) \to f(sY_1 \cdots Y_m Y_{m+1} \cdots)$ so that, by the dominated convergence theorem,

$$\lim_m E_\tau[\pi(sY_1 \cdots Y_m)] = E_\tau(f) = V(s).$$

But, by the martingale property,

$$E_\tau[\pi(sY_1 \cdots Y_m)] = \pi(s)$$

for all $m \geq 1$. Hence $\pi(s) = V(s)$ and the proof of the claim is complete for $p = s$.

Step 11. The last step in the proof of Theorem 2.2 is to show that τ is an SPE for $\Gamma(f, \cdot)$. Let $\tau = (\tau_1, \tau_2, \ldots, \tau_n)$ and fix a strategy σ for player i. Let $\hat{\tau} = (\tau^{-i}, \sigma)$. Note that $E_{\tau[p]}(f_i p)$ is just the ith coordinate $V(p)_i$ of $V(p)$ as defined in step 10.

Again we denote the history generated by $P_{\hat{\tau}}$ with initial state s by $(s, Y_1, Y_2, \ldots, Y_m, \ldots)$.

Claim: The process $V(s)_i, V(sY_1)_i, \ldots, V(sY_1 Y_2 \cdots Y_m)_i, \ldots$ is a supermartingale under $P_{\hat{\tau}}$.

To verify this, let $\lambda(p)$ be the product measure

$$\lambda(p) = \tau_1(p) \times \cdots \times \sigma(p) \times \cdots \times \tau_n(p),$$

for $p \in S^*$. Then calculate:

$$
\begin{aligned}
&E_{\hat{\tau}}[V(sY_1 Y_2 \cdots Y_m Y_{m+1})_i \,|\, sY_1 Y_2 \cdots Y_m = p] \\
&\qquad = E_{\hat{\tau}}[\pi(sY_1 Y_2 \cdots Y_m Y_{m+1})_i \,|\, sY_1 Y_2 \cdots Y_m = p] \\
&\qquad = \int\int \psi^*(p, a, \pi(p), t)_i\, q(dt|l(p), a)\, \lambda(da) \\
&\qquad \leq \int\int \psi^*(p, a, \pi(p), t)_i\, q(dt|l(p), a)\, \nu^*(p, \pi(p))(da) \\
&\qquad = \pi(p)_i \\
&\qquad = V(p)_i.
\end{aligned}
$$

Here the inequality is by virtue of the fact that $\tau(p) = \nu^*(p, \pi(p))$ is an equilibrium profile in the one-move game with payoff $\int \psi^*(p, a, \pi(p), t)_i\, q(dt|l(p), a) = v^*(p, \pi(p))(a)$ by step 8. Since $V(sY_1 Y_2 \cdots Y_m) \to f$, it follows from the supermartingale property that

$$E_{\hat{\tau}}(f_i) = \lim_m E_{\hat{\tau}}[V(sY_1 Y_2 \cdots Y_m)_i] \leq V(s)_i = E_\tau(f_i).$$

This proves that τ is an equilibrium profile in the game $\Gamma(f, s)$. The proof that, for $p \in S^*$, $\tau[p]$ is an equilibrium profile in the game $\Gamma(f, p)$ is similar and is omitted.

The proof of Theorem 2.2 is now complete.

3 Finitary games and the proof of Theorem 1.1

In order to deduce Theorem 1.1 from Theorem 2.2, we first identify the class of functions that will be used to approximate the payoff f as in Assumption 2.

Let t be a Borel function from $S^{\mathbb{N}}$ to $\{0, 1, \ldots\} \cup \{\infty\}$. We say that t is a *Borel, stopping time* if, given elements $x = (s_1, s_2, \ldots)$ and $y = (r_1, r_2, \ldots)$ in $S^{\mathbb{N}}$ such that $t(x) < \infty$ and y agrees with x in the first $t(x)$ coordinates, then $t(x) = t(y)$. If, in addition, $t(x) < \infty$ for *all* $x \in S^{\mathbb{N}}$, then t is called a *stop rule*. A Borel function $g : S^{\mathbb{N}} \mapsto \mathbb{R}$ is a *Borel, finitary function* if there exists a Borel stop rule t such that $g(x) = g(y)$ whenever y agrees with x in the first $t(x)$ coordinates. In this case, the function g is said to be determined by time t. (See Dubins and Savage (1976) for a discussion of the properties of finitary functions.)

Borel finitary functions will play the role of the functions g^m in Assumption 2. First we establish that they can be used to uniformly approximate the payoff functions of Theorem 1.1. As in the previous section, R denotes a fixed positive real number.

Lemma 3.1. *Suppose that $\phi : S^{\mathbb{N}} \mapsto [-R, R]$ is a Borel, DS-continuous function. Then, for every $\epsilon > 0$, there is a Borel, finitary function $\psi : S^{\mathbb{N}} \mapsto [-R, R]$ such that $\sup\{|\phi(x) - \psi(x)| : x \in S^{\mathbb{N}}\} \leq \epsilon$.*

Proof. Let $\{a_1, a_2, \ldots, a_m\}$ be an ϵ-net in $[-R, R]$ and set $V_i = \phi^{-1}((a_i - \epsilon, a_i + \epsilon))$, $1 \leq i \leq m$. Then the sets V_i are Borel. For each i, there is, by Corollary 2.4 in Maitra et al (1990), a Borel stopping time t_i such that $[t_i < \infty] = V_i$. Set $t = \min\{t_i \,|\, 1 \leq i \leq m\}$. Clearly, t is a Borel stopping time. Moreover, since $\cup_{1 \leq i \leq m} V_i = S^{\mathbb{N}}$, t is a stop rule. For $1 \leq i \leq m$, define

$$W_i = \{x \in S^{\mathbb{N}} \,|\, t(x) = t_i(x) \,\&\, t_j(x) > t(x),\ 1 \leq j < i\}.$$

It is now easy to check that $W_i \subseteq V_i$, $W_i \cap W_j = \emptyset$ for $i \neq j$, $\cup_i W_i = S^{\mathbb{N}}$, and each W_i is Borel. Set $\psi = a_i$ on W_i, $1 \leq i \leq m$. Then ψ is obviously Borel. It is determined by time t and is therefore finitary. By construction, $|\phi(x) - \psi(x)| \leq \epsilon$ for all x. □

We need a bit of notation. Define $\Phi : H \mapsto S^{\mathbb{N}}$ by

$$\Phi((s_1, (a_1, s_2), (a_2, s_3), \ldots)) = (s_1, s_2, s_3, \ldots).$$

Recall that $S^* = \bigcup_{k \geq 0} [S \times (A \times S)^k]$ and let $\tilde{S} = \bigcup_{k \geq 1} S^k$. Next define $\Psi : S^* \mapsto \tilde{S}$ by

$$\Psi((s_1, (a_1, s_2), \ldots, (a_{k-1}, s_k)) = (s_1, s_2, \ldots . s_k).$$

Suppose that $f = (f_1, f_2, \ldots, f_n)$ is the Borel, DS-continuous payoff function of Theorem 1.1. Since the f_i are bounded, we can assume that $f : H \mapsto [-R, R]^n$. Also, since f depends only on the sequence of states, we can find $\bar{f} : S^{\mathbb{N}} \mapsto [-R, R]^n$ such that $f = \bar{f} \circ \Phi$ and $\bar{f}$ is Borel, DS-continuous on $S^{\mathbb{N}}$.

By Lemma 3.1, there is, for each $\epsilon > 0$, a Borel, finitary function $\bar{g} : S^{\mathbb{N}} \mapsto [-R, R]^n$ such that $||\bar{g} - \bar{f}|| < \epsilon$. Define g on H by $g(h) = (\bar{g} \circ \Phi)(h)$. Then g approximates f on H uniformly within ϵ. Most of the remainder of this section is devoted to the study of the game $\Gamma(g, \cdot)$, which we call a *finitary* game. At the end of the section we will deduce Theorem 1.1 from Theorem 2.2 and the properties of finitary games.

For our treatment of finitary games, we will not need the full strength of Condition 1 and we will replace it by the weaker condition below.

Condition 3. Feller continuity. *For fixed $s \in S$, the law of motion $q(\cdot|s,a)$ is Feller continuous in a; that is, for every bounded, real-valued function ϕ on S, $\int \phi(t)\, q(dt|s,a)$ is continuous in a.*

Theorem 3.2. *Assume condition 3. Let $\bar{g} : S^{\mathbb{N}} \mapsto [-R,R]^n$ be a Borel, finitary function and let $g : H \mapsto [-R,R]^n$ be the function $(\bar{g} \circ \Phi)$. Then there exist Borel functions $\bar{\sigma} : \tilde{S} \mapsto \Delta(A)$ and $\bar{V} : \tilde{S} \mapsto [-R,R]^n$ such that*

(i) $\sigma = \bar{\sigma} \circ \Psi$ is a subgame perfect equilibrium in the finitary game $\Gamma(g,\cdot)$ and the corresponding equilibrium payoff is $V = \bar{V} \circ \Psi$;

(ii) for each $p \in S^*$, the one-move game with payoff $\int V(pat)\, q(dt|l(p),a)$ has equilibrium profile $\sigma(p)$ with corresponding equilibrium payoff $V(p)$.

Proof. Let t be a Borel stop rule on $S^{\mathbb{N}}$ such that $\bar{g}$ is determined by time t. The stop rule t defines a tree on S as follows:

$$\begin{aligned}(s_1, s_2, \ldots, s_m) \in T \text{ iff } & (\exists h \in S^{\mathbb{N}})(t(s_1, s_2, \ldots, s_m h) \geq m) \\ \text{iff } & (\forall h \in S^{\mathbb{N}})(t(s_1, s_2, \ldots, s_m h) \geq m).\end{aligned}$$

Thus T is both analytic and coanalytic, hence Borel.

Furthermore, T is a Borel tree on S, which means

(i) T is a Borel subset of $\tilde{S}$, and

(ii) T is closed under initial segments.

Observe that, since t is everywhere finite, T is a Borel well-founded relation (i.e. T has no infinite branches). It now follows, courtesy of a result of Moschovakis (1980, section 4C.14), that there is a coanalytic, non-Borel subset C of a Polish space Z, a function η on C onto ω_1, the first uncountable ordinal, and a Borel function ξ on T into C such that

(a) η is a coanalytic norm on C, and

(b) $(s_1, s_2, \ldots, s_l) \in T$ & $m < l$ implies $\eta(\xi((s_1, s_2, \ldots, s_l))) < \eta(\xi((s_1, s_2, \ldots, s_m)))$.

For the definition of a coanalytic norm, see Moschovakis (1980, pages 200-201). We will use only the following two properties of coanalytic norms:

(c) For every ordinal $\alpha < \omega_1$, the set $\{z \in C : \eta(z) = \alpha\}$ is a Borel subset of Z (Moschovakis (1980, 4C.7)),

(d) If K is an analytic subset of C, then there is $\alpha^* < \omega_1$ such that $\eta(z) \leq \alpha^*$ for every $z \in K$.

Define a function $i : T \mapsto \omega_1$ by $i((s_1, s_2, \ldots, s_m)) = \eta(\xi((s_1, s_2, \ldots, s_m)))$. For every $\alpha < \omega_1$, let

$$T_\alpha = \{(s_1, s_2, \ldots, s_m) \in T : i((s_1, s_2, \ldots, s_m)) = \alpha\}.$$

Then each T_α is a Borel subset of $\tilde{S}$ by virtue of (c). Also, since $\xi(T)$ is an analytic subset of C, it follows from (d) that there is $\bar{\alpha} < \omega_1$ such that $T = \cup_{\alpha \leq \bar{\alpha}} T_\alpha$.

The following lemma will be needed. In the lemma and the sequel, we use the notation $\mathcal{B}(X)$ to denote the Borel σ-field of a topological space X.

Lemma 3.3. *Assume condition 3. Suppose Y is a Borel subset of a Polish space and let $\mathcal{D}$ be a countably-generated sub-σ-field of $\mathcal{B}(Y)$. Let $\zeta : Y \times S \mapsto [-R, R]^n$ be a $\mathcal{D} \times \mathcal{B}(S)$- measurable function. Let $G(y, s)$ be the one-move game with payoff $\int \zeta(y, t)\, q(dt|s, a)$. Then there exist $\mathcal{D}\times\mathcal{B}(S)$- measurable functions $\rho : Y \times S \mapsto \Delta(A)$ and $W : Y \times S \mapsto [-R, R]^n$ such that $\rho(y, s)$ is an equilibrium profile in the game $G(y, s)$ and $W(y, s)$ is the corresponding equilibrium payoff, i.e.*

$$W(y, s) = \int \int \zeta(y, t)\, q(dt|s, a)\rho(y, s)(da).$$

Proof. Let $F(y, s)$ be the set of all equilibrium profiles $(\mu_1, \mu_2, \ldots, \mu_n)$ for the game $G(y, s)$. It is easy to verify that $F(y, s)$ is a nonempty compact subset of the set of all profiles $\mathcal{P} = \Delta(A_1) \times \Delta(A_2) \times \cdots \times \Delta(A_n)$. Also the graph of F belongs to $\mathcal{D}\times\mathcal{B}(S)\times\mathcal{B}(\mathcal{P}))$. To see this, imitate the proof of Lemma 2.1 in Maitra and Sudderth (2003) and use the fact that the function

$$(y, s, a) \mapsto \int \zeta(y, t)\, q(dt|s, a),$$

being $\mathcal{D} \times \mathcal{B}(S)$-measurable for fixed a and continuous in a for fixed (y, s) is, in fact, $\mathcal{D} \times \mathcal{B}(S) \times \mathcal{B}(A)$-measurable. (See Theorem 3.1.30 in Srivastava (1998).) An application of Theorem 5.7.1 of Srivastava (1998) will now yield a $\mathcal{D}\times\mathcal{B}(S)$-measurable selector $\rho : Y \times S \mapsto \mathcal{P}$ for F. Finally, it is easily checked that W is $\mathcal{D} \times \mathcal{B}(S)$-measurable. □

Now let $\mathcal{C}$ be the smallest σ-field on S^* which makes Ψ measurable, i.e., $\mathcal{C} = \Psi^{-1}(\mathcal{B}(\tilde{S}))$. Fix μ_i^*, $i = 1, 2, \ldots, n$, $s^* \in S$, and let $x^* = (s^*, s^*, \ldots)$ be that point in $\tilde{S}$ all of whose coordinates are s^*. Now, for $p \in \Psi^{-1}(\tilde{S} - T)$, the function $x \mapsto \bar{g}(\Psi(p)x)$ is constant for $x \in \tilde{S}$. So we define

$$V(p) = \bar{g}(\Psi(p)x^*), \qquad \sigma(p) = \mu_1^* \times \mu_2^* \times \cdots \mu_n^*.$$

Then

(e) V and σ are defined on the set $\Psi^{-1}(\tilde{S} - T) \in \mathcal{C}$ and are measurable with respect to the restriction of $\mathcal{C}$ to $\Psi^{-1}(\tilde{S} - T)$,

and, for $p \in \Psi^{-1}(\tilde{S} - T)$,

(f) $\sigma(p)$ is an equilibrium profile in the one-move game with payoff $\int V(pat)\, q(dt|l(p), a)$, and

(g) $V(p) = \int\int V(pat)\, q(dt|l(p), a)\sigma(p)(da)$.

We will now extend the definitions of the functions V and σ to all of S^* so that they are $\mathcal{C}$-measurable and properties (f) and (g) continue to hold. Since $\Psi^{-1}(T) = \cup_{\alpha\leq\bar{\alpha}}\Psi^{-1}(T_\alpha)$, the definitions of V and σ will proceed by transfinite induction.

So suppose that $\alpha \leq \bar{\alpha}$ and V, σ have been defined for all $p \in \cup_{\beta<\alpha}\Psi^{-1}(T_\beta)$ so that (f) and (g) still hold and also that

(h) V and σ are measurable with respect to the restriction of $\mathcal{C}$ to $(\cup_{\beta<\alpha}\Psi^{-1}(T_\beta)) \cup \Psi^{-1}(\tilde{S} - T)$.

We will now define V and σ on the $\mathcal{C}$-set $\Psi^{-1}(T_\alpha)$. For each $p \in \Psi^{-1}(T_\alpha)$, note that, by property (b) above, $pat \in (\cup_{\beta<\alpha}\Psi^{-1}(T_\beta)) \cup \Psi^{-1}(\tilde{S} - T)$, so that $V(pat)$ and $\sigma(pat)$ are defined for all $a \in A$, $t \in S$. We next apply Lemma 3.3 with $Y = \Psi^{-1}(T_\alpha)$, $\zeta(p,t) = V(pa^*t)$, where a^* is a fixed element of A, and the σ-field $\mathcal{D}$ equal to the restriction of $\mathcal{C}$ to $\Psi^{-1}(T_\alpha)$. Let ρ and W be the functions whose existence is asserted in Lemma 3.3. We now define, for $p \in \Psi^{-1}(T_\alpha)$,

$$\sigma(p) = \rho(p, l(p)), \qquad V(p) = W(p, l(p)).$$

It is straightforward to check that σ and V satisfy (f)-(h). Since $\bar{\alpha}$ is countable, this completes the extension of σ and V to S^* so that (f)-(h) are satisfied.

It remains to be verified that, for all $p \in S^*$, the conditional profile $\sigma[p]$ is an equilibrium in the game $\Gamma(g,p)$ and that $V(p) = E_{\sigma[p]}(gp)$ is the corresponding equilibrium payoff. This is trivially true for $p \in \Psi^{-1}(\tilde{S} - T)$ since the function gp is constant for such p. For $p \in \Psi^{-1}(T)$, we will prove the assertion by another transfinite induction. So suppose the assertion is true for all $p \in \cup_{\beta<\alpha}\Psi^{-1}(T_\beta)$ and let $p \in \Psi^{-1}(T_\alpha)$. Then, by property (b), $pat \in (\cup_{\beta<\alpha}\Psi^{-1}(T_\beta)) \cup \Psi^{-1}(\tilde{S} - T)$. Let player i deviate by using $\tau[p]$ at p and let $\bar{\sigma} = (\sigma^{-i}, \tau)$ be the resulting profile. Recall that g_i, the ith coordinate of g, is the payoff function for player i and calculate as follows:

$$\begin{aligned}
E_{\sigma[p]}(g_i p) &= \int\int E_{\sigma[pat]}(g_i pat)\, q(dt|l(p),a)\sigma(p)(da) \\
&= \int\int V(pat)_i\, q(dt|l(p),a)\sigma(p)(da) \\
&\geq \int\int V(pat)_i\, q(dt|l(p),a)\bar{\sigma}(p)(da) \\
&= \int\int E_{\sigma[pat]}(g_i pat)\, q(dt|l(p),a)\bar{\sigma}(p)(da) \\
&\geq \int\int E_{\bar{\sigma}[pat]}(g_i pat)\, q(dt|l(p),a)\bar{\sigma}(p)(da) \\
&= E_{\bar{\sigma}[p]}(g_i p).
\end{aligned}$$

Here the second and fourth equalities and the second inequality are by virtue of the inductive hypothesis, while the first inequality holds because of (f). Finally,

$$\begin{aligned}
E_{\sigma[p]}(gp) &= \int\int E_{\sigma[pat]}(gpat)\, q(dt|l(p),a)\sigma(p)(da) \\
&= \int\int V(pat)\, q(dt|l(p),a)\sigma(p)(da) \\
&= V(p),
\end{aligned}$$

where the second equality is by virtue of the inductive hypothesis and the third is by (g).

□

We are now ready to complete the proof of Theorem 1.1. So assume Condition 1 of the Introduction and note that it implies the weaker Condition 2 of this section. The payoff function $f : H \mapsto [-R, R]^n$ is assumed to be Borel, DS-continuous, and to depend only on the sequence of states. Hence, there is a Borel, DS-continuous function $\bar{f} : S^{\mathbb{N}} \mapsto [-R, R]^n$ such that $f = \bar{f} \circ \Phi$. Use Lemma 3.1 to choose Borel, finitary functions $\bar{g}^m : S^{\mathbb{N}} \mapsto [-R, R]^n$ such that $||\bar{g}^m - \bar{f}|| \to 0$ as $m \to \infty$. Let $g^m = \bar{g}^m \circ \Phi$. Then also $||g^m - f|| \to 0$ as $m \to \infty$.

For each m, one can choose, by virtue of Theorem 3.2, Borel functions $\bar{\sigma}^m : \tilde{S} \mapsto \Delta(A)$ and $\bar{V}^m : \tilde{S} \mapsto [-R, R]^n$ such that, if $\sigma^m = \bar{\sigma}^m \circ \Psi$ and $V^m = \bar{V}^m \circ \Psi$, then

(i) σ^m is a subgame perfect equilibrium in the game $\Gamma(g^m, \cdot)$ and the corresponding equilibrium payoff is V^m;

(ii) for each $p \in S^*$, the one-move game with payoff $\int V^m(pat)\, qdt|l(p), a)$ has equilibrium profile $\sigma(p)$ with corresponding equilibrium payoff $V^m(p)$.

Finally, it follows from Condition 1 and Lemma 3.6 in Solan (1998) that, for each $p \in S^*$, the family

$$\{\int V^m(pat)\, q(dt|l(p), a) \;:\; m \in \mathbb{N}\} = \{\int \bar{V}^m(\Psi(p)t)\, q(dt|l(p), a) \;:\; m \in \mathbb{N}\}$$

is equicontinuous in a. So Assumption 2 of section 2 is verified; Assumption 1 is true by hypothesis. Theorem 1.1 now follows from Theorem 2.2.

4 Finite action sets and the proof of Theorem 1.2

Assume the hypotheses of Theorem 1.2; that is, the actions sets $A_1, A_2, \ldots, A_n$ are finite and the payoff functions $f_1, f_2, \ldots, f_n$ are bounded, Borel, and DS-continuous from H to $\mathbb{R}$. Note that the payoffs may now depend on actions as well as states.

We use the "partial history trick" to deduce Theorem 1.2 from Theorem 1.1. We take S^* to be our state space and $\bar{q}$ to be our law of motion, where

$$\bar{q}(pat|p, a) = q(t|l(p), a).$$

We will define $\bar{f}$ on the new history space $\bar{H} = S^* \times (A \times S^*)^{\mathbb{N}}$ as follows: Let $\bar{h} = ((p_1), (b^1, p_2), (b^2, p_3) \ldots) \in \bar{H}$. If there exists an $h = (s_1, (a^1, s_2), (a^2, s_3), \ldots) \in H$ such that

$$p_k = (s_1, (a^1, s_2), \ldots, (a^{k-1}, s_k)) \tag{4.1}$$

for all $k = 1, 2, \ldots$, we set $\bar{f}(\bar{h}) = f(h)$. If there is no such $h \in H$, let $m = m(\bar{h})$ be the largest integer, possibly zero, such that (4.1) holds for all $k = 1, 2, \ldots, m$ for some $h = (s_1, (a^1, s_2), (a^2, s_3), \ldots)$ and let

$$\bar{f}(\bar{h}) = \begin{cases} f(s^*, (a^*, s^*), (a^*, s^*), \ldots), & \text{if } m = 0 \\ f(s_1, (a^1, s_2), \ldots, (a^{m-1}, s_m), (a^*, s^*), (a^*, s^*), \ldots), & \text{if } m \geq 1. \end{cases}$$

where a^* and s^* are fixed elements of A and S, respectively. (Note that $m(\bar{h})$ could also be described as the largest m such that, for all $k = 1, 2, \ldots, m$, $lh(p_k) = k - 1$ and p_k extends p_{k-1}.) Then $\bar{f}$ is bounded, Borel, DS-continuous, and depends only on states. Since A is finite, $\bar{q}$ satisfies Condition 1. Hence, by Theorem 1.1, $\Gamma(\bar{f}, \cdot)$ has a subgame perfect equilibrium. It is now immediate that $\Gamma(f, \cdot)$ has a subgame perfect equilibrium.

5 Additive payoffs and the proof of Theorem 1.3

In this section we assume both Condition 1 and Condition 2 of the Introduction. We write r_k for the profile $(r_{1k}, r_{2k}, \ldots, r_{nk})$ of the player's reward functions at stage k, $k \geq 1$. Let

$$g^m(h) = \sum_{k=1}^{m} r_k(s_k, a_k)$$

for all $h = (s_1, (a^1, s_2), (a^2, s_3), \ldots) \in H$ and $m \geq 1$. For each m, the function g^m is a bounded, Borel, finitary function on H. By Condition 2, $||g^m - f|| \to 0$ as $m \to \infty$. Thus f is bounded, Borel, and DS-continuous. Assume, without loss of generality, that the range of f and the range of $g^m, m \geq 1$, is contained in $[-R, R]^n$.

First we fix m and analyze the game $\Gamma(g^m, \cdot)$. Let $W_0^m(t) = 0$ for $t \in S$, and, for $1 \leq k \leq m$, define W_k^m on S by induction as follows:

$$W_k^m(s) = \int \int [r_{m-k+1}(s,a) + W_{k-1}^m(t)]\, q(dt|s,a) \rho_k^m(s)(da),$$

where $\rho_k^m(s)$ is an equilibrium profile in the one-move game with payoff

$$r_{m-k+1}(s,a) + \int W_{k-1}^m(t)\, q(dt|s,a).$$

Condition 3, which follows from Condition 1, ensures that the functions ρ_k^m can be chosen to be Borel so that the functions W_k^m are also Borel. (See the proof of Lemma 2.1 in Maitra and Sudderth (2003).)

We now define V^m and σ^m on S^*. Let $p = (s_1, (a^1, s_2), \ldots, (a^{l-1}, s_l)) \in S^*$ and fix $\mu_i^* \in \Delta(A_i)$, $i = 1, 2, \ldots, n$.

Case 1. If $l > m$, set

$$V^m(p) = \sum_{k=1}^{m} r_k(s_k, a^k), \qquad \sigma^m(p) = \mu_1^* \times \mu_2^* \times \cdots \times \mu_n^*.$$

Case 2. If $1 \leq l \leq m$, set

$$V^m(p) = \sum_{k=1}^{l-1} r_k(s_k, a^k) + W_{m-l+1}^m(s_l), \qquad \sigma^m(p) = \rho_{m-l+1}^m(s_l).$$

Then V^m and σ^m are Borel functions on S^*. It is easy to check that, for each $p \in S^*$, $\sigma^m(p)$ is an equilibrium profile in the one-move game with payoff $\int V^m(pat)\, q(dt|l(p), a)$ and that $V^m(p)$ is the corresponding equilibrium payoff. This is obvious if $l > m$. For $1 \leq l \leq m$, the assertion follows by induction from the equality

$$\int V^m(pat)\, q(dt|l(p), a) = \sum_{k=1}^{l-1} r_k(s_k, a^k) + [r(s_l, a) + \int W_{m-l}^m(t)\, q(dt|s_l, a)] \qquad (5.1)$$

Finally, by repeating the argument in the last paragraph of the proof of Theorem 3.2, it is not hard to see that σ^m is a subgame perfect equilibrium in the game $\Gamma(g^m, \cdot)$

with corresponding equilibrium payoff V^m. (The argument is essentially a standard backward induction as in Rieder (1991) or Lemma 2.1 of Maitra and Sudderth (2003).)

Next Condition 1 and (5.1) imply, by another application of Lemma 3.6 in Solan (1998), that, for fixed $p \in S^*$, the family

$$\{ \int V^m(pat)\, q(dt|l(p), a) \;:\; m \in \mathbb{N}\}$$

is equicontinuous in a. Thus Assumptions 1 and 2 of section 2 hold. So Theorem 1.3 now follows from Theorem 2.2.

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