CHARACTERIZING CONSISTENCY WITH MONOMIALS (PRELIMINARY VERSION)

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Abstract. This paper shows that beliefs are consistent iff they can be constructed from monomials which converge to the strategies. This characterization seems relatively tractable and is derived from the definition of consistency by means of linear algebra alone. The paper also applies its monomial characterization to repair a nontrivial fallacy in the proofs of Kreps and Wilson's insightful theorems.

1. Introduction

Imagine that you have twin daughters and that you've just put them to bed for the night. You've arranged things so that it is in their best interest to fall asleep and so that it is in your best interest to follow through on your dire threats if either of them is foolish enough to start giggling again. So, you settle into your easy chair with a good book and a nice cup of tea.

Then, in contrast to all your careful reasoning, you hear noise. What would you think? In particular, is the bigger of the two giggling, is the smaller of the two giggling, or are the two of them giggling simultaneously? (Your twins have identical voices even though one of them has grown slightly faster than the other.)

I thank Ian King, Andy McLennan, Val Lambson, and participants at the 2006 Australasian Economic Theory Workshop at UNSW. I also thank the University of Auckland Economics Department for its hospitality during my sabbatical. (This paper subsumes half of Streufert (2006a) and all of Streufert (2006b).)

This question appears to be irrelevant because there is absolutely no chance that either of the girls will actually giggle. Yet, what you would believe in this zero-probability event effects whom you would punish, which in turn effects the incentives facing your daughters. Thus your belief at this zero-probability event is part of what makes it zero-probability in the first place. For example, one of your daughters might go to sleep simply because she knows you will think that she alone is at fault if either or both of them giggles.

Since ordinary probability theory inevitably assigns zero probability to each of the three possibilities, it would be perfectly reasonable for you to arbitrarily believe that any one of the three possibilities has occurred, or more generally, for you to arbitrarily assign a nontrivial probability distribution over the three possibilities.

Alternatively, you might reason that both girls giggling is infinitely less likely than either of the two giggling alone simply because the coincidence of two zero-probability events seems infinitely less likely than either zero-probability event alone. This more sophisticated reasoning is incorporated into the concept of consistent beliefs, which was introduced by Kreps and Wilson (1982) (henceforth KW). Their path-breaking definition states that strategies and beliefs are consistent with one another iff they are the limit of a sequence of positive-valued strategies and beliefs which satisfy Bayes Rule (Bayes Rule is part of ordinary probability theory and works well when all probabilities are positive).

This paper's only theorem characterizes KW's concept of consistency in terms of monomials. Broadly speaking, the theorem suggests that it is useful to represent a probability with a "monomial" of the form cn^e in which c is a positive real number and e is a nonpositive integer. Monomials are vaguely like complex numbers in the sense that they extend the set of real numbers. In particular, a monomial cn^e is a real number when $e = 0$ just like a complex number $a+bi$ is a real number when $b = 0$. Accordingly, monomials are useful for situations like twins going to sleep something like complex numbers are useful for equations like $x^2 = -4$.

To be a little more precise, Theorem 2.1 shows that beliefs and strategies are consistent with one another iff monomials can be assigned to actions in such a way that (a) the strategy at each information set is the limit of the monomials assigned to the actions at that information set, and (b) the belief at each information set is found by calculating the product of the monomials along the paths leading to each of the nodes in the information set. This characterization seems more tractable than the definition of consistency, and surprisingly, it can be derived from the definition by linear algebra alone.

Section 3 applies this monomial characterization to repair a nontrivial fallacy in KW. In particular, KW not only provides the pathbreaking concept of consistency, but also provides three insightful theorems which derive the geometry of the set of sequential equilibrium assessments, the finiteness of the set of sequential equilibrium outcomes, and the perfection of strict sequential equilibria. These derivations depend upon Lemmas A1 and A2 in the KW appendix. This paper's Section 3 notes that the KW proofs of these lemmas contain a nontrivial fallacy. It then repairs the proofs by applying the monomial characterization.

Section 4 briefly discusses the paper's mathematical foundation. The heart of the paper is to derive an additive representation for a certain binary relation among the nodes (and this juncture happens to coincide with the fallacy mentioned a moment ago). This additive representation is constructed through linear algebra by mimicking Scott (1964) and Krantz, Luce, Suppes, and Tversky (1971) (this literature is largely unfamiliar to economists).

2. Characterization

2.1. Basic Definitions

This section recapitulates some notation and terminology from KW while discussing an example which will be used throughout the paper. This example corresponds to the story of twins going to sleep if one imagines that the bigger twin chooses to G (giggle) or S (sleep), that

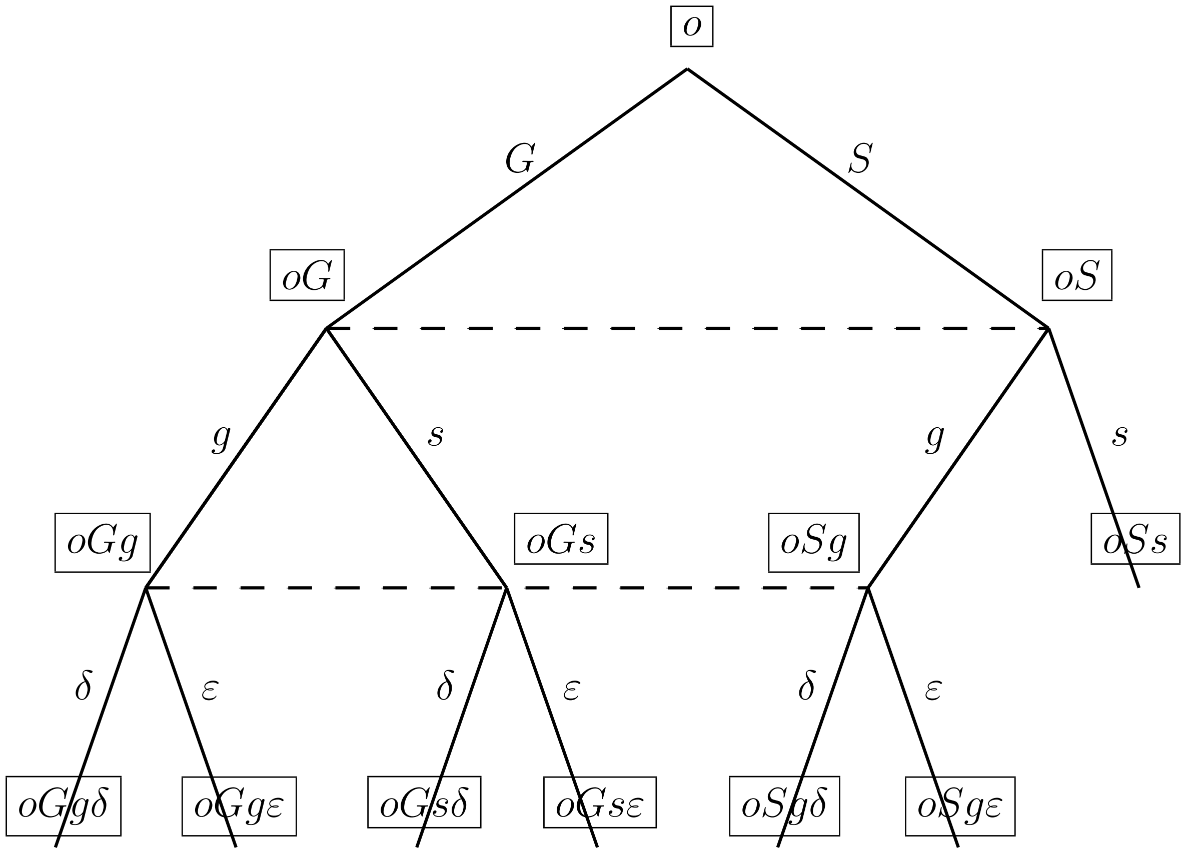

FIGURE 2.1

the smaller twin chooses to g (giggle) or s (sleep), and that you as the parent choose between the punishments δ and ε.

This paragraph and Figure 2.1 define a game form $[T, \prec, A, \alpha, H, \rho]$. The set T of nodes t contains the set $X = \{o, oG, oS, oGg, oGs, oSg\}$ of decision nodes x, which in turn contains the set $W = \{o\}$ of initial nodes. The set W is given the trivial distribution $\rho = (\rho(o)) = (1)$, and the set X is partitioned into the information sets $h \in H = \{\{o\}, \{oG, oS\}, \{oGg, oGs, oSg\}\}$. Let $H(x)$ denote the information set h which contains x. Finally, let $A = \{G, S, g, s, \delta, \varepsilon\}$ be the set of actions a, let $A(h)$ be the set of actions available from information set h, and let $\alpha(t)$ be the last action taken to reach a non-initial node t.

A strategy profile is a function $\pi{:}A{\rightarrow}[0,1]$ such that $(\forall h)$ $\Sigma_{a \in A(h)} \pi(a) = 1$. A belief system is a function $\mu{:}X{\rightarrow}[0,1]$ such that $(\forall h)$ $\Sigma_{x \in h} \mu(x) = 1$. An assessment is a strategy-belief pair (π, μ). As on KW page

872, let Ψ^0 consist of those positive-valued assessments (π, μ) for which

$$(\forall x)\ \mu(x) = \frac{\rho \circ p_{\ell(x)}(x) \cdot \Pi_{k=0}^{\ell(x)-1} \pi \circ \alpha \circ p_k(x)}{\Sigma_{x' \in H(x)}\, \rho \circ p_{\ell(x')}(x') \cdot \Pi_{k=0}^{\ell(x')-1} \pi \circ \alpha \circ p_k(x')}\ ,$$

where $p_k(x)$ is the kth predecessor of node x, and $\ell(x)$ is the number of its predecessors (in other words, let Ψ^0 consist of those positive-valued assessments that obey Bayes Rule). Then, an assessment is said to be *consistent* if it is the limit of a sequence $\{\pi_n, \mu_n\}_n$ in Ψ^0. For instance, in the example, the (π, μ) defined in the second lines of

(1)

a	G	S	g	s	δ	ε
$\pi_n(a)$	$\frac{n^{-2}}{n^{-2}+1}$	$\frac{1}{n^{-2}+1}$	$\frac{n^{-1}}{n^{-1}+1}$	$\frac{1}{n^{-1}+1}$	$\frac{1}{2}$	$\frac{1}{2}$
$\pi(a)$	0	1	0	1	$\frac{1}{2}$	$\frac{1}{2}$

and

x	o	oG	oS	oGg	oGs	oSg
$\mu_n(x)$	1	$\frac{n^{-2}}{n^{-2}+1}$	$\frac{1}{n^{-2}+1}$	$\frac{n^{-3}}{n^{-3}+n^{-2}+n^{-1}}$	$\frac{n^{-2}}{n^{-3}+n^{-2}+n^{-1}}$	$\frac{n^{-1}}{n^{-3}+n^{-2}+n^{-1}}$
$\mu(x)$	1	0	1	0	0	1

is consistent because the second line in each table is the limit of its first line, and because the (π_n, μ_n) defined in the first lines of the tables is within Ψ^0 for any value of n.

2.2. Theorem

Theorem 2.1 characterizes consistency by means of two functions defined over the set A of actions. The function e assigns an nonpositive integer "exponent" to each action, and the function c assigns a positive real "coefficient" to each action. This is simpler than the KW definition because two functions of A are simpler than a sequence of functions of A.

Theorem 2.1. *In any game form $[T, \prec, A, \alpha, \rho, H]$, an assessment (μ, π) is consistent iff there exists $c{:}A \to (0, \infty)$ and $e{:}A \to \mathbb{Z}_-$ such that*

$$(\forall a)\ \pi(a) = lim_{n\to\infty}\ c(a) n^{e(a)}\ \ \textit{and}$$

$$(\forall x)\ \mu(x) = lim_{n\to\infty}\ \frac{\rho \circ p_{\ell(x)}(x) \cdot \Pi_{k=0}^{\ell(x)-1} c \circ \alpha \circ p_k(x) n^{e \circ \alpha \circ p_k(x)}}{\Sigma_{x' \in H(x)}\ \rho \circ p_{\ell(x')}(x') \cdot \Pi_{k=0}^{\ell(x')-1} c \circ \alpha \circ p_k(x') n^{e \circ \alpha \circ p_k(x')}}$$

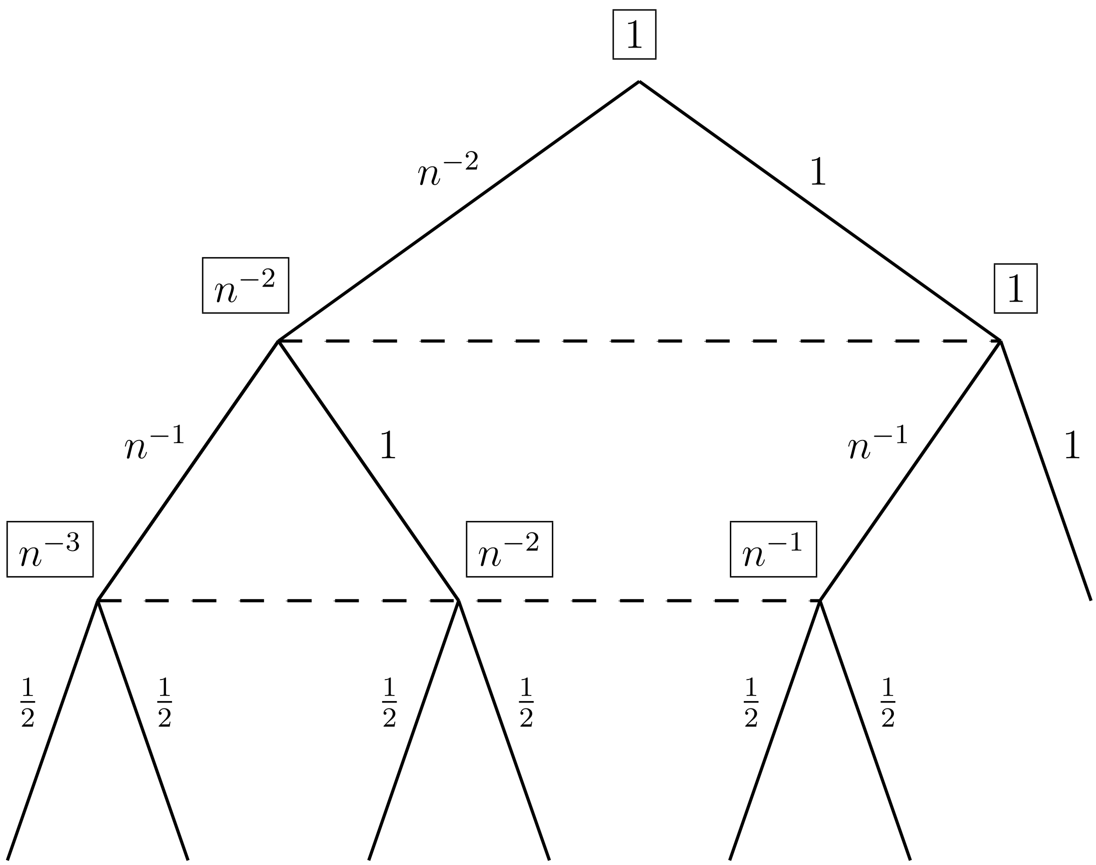

Figure 2.2

($\mathbb{Z}_-$ is the set of nonpositive integers).

Proof. Section 2.3 □

The functions c and e can be together regarded as a single function which assigns a monomial $c(a)n^{e(a)}$ to each action a. For instance, the first line in the following table defines a monomial at each action in the example

(2a)

a	G	S	g	s	δ	ε
$c(a)n^{e(a)}$	n^{-2}	1	n^{-1}	1	$\frac{1}{2}$	$\frac{1}{2}$
$\pi(a)$	0	1	0	1	$\frac{1}{2}$	$\frac{1}{2}$

The second line is then the strategy derived via the limit in Theorem 2.1's first equation.

The theorem's second equation asks one to calculate a product at each node. Fortunately, this product is just the product of the monomials along the path leading to the node. For instance, in Figure 2.2, the unboxed monomial at each action is taken from the first line of (2a) and the boxed monomial at each node is the product of the unboxed monomials above it. These boxed monomials appear in the first line of

(2b)

x	o	oG	oS	oGg	oGs	oSg
$\rho \circ p_{\ell(x)}(x) \cdot \Pi_{k=0}^{\ell(x)-1} c \circ \alpha \circ p_k(x) n^{e \circ \alpha \circ p_k(x)}$	1	n^{-2}	1	n^{-3}	n^{-2}	n^{-1}
$\mu(x)$	1	0	1	0	0	1

The second line is then the belief derived via the limit in the theorem's second equation.

By Theorem 2.1, the assessment (π, μ) defined in (2a) and (2b) is consistent. This is rather uninteresting because (2a) and (2b) are very similar to (1). In fact, it is always the case that monomials defined by c and e determine a special kind of strategy sequence $\{\pi_n\}_n$ by means of

$$(3) \qquad (\forall h)(\forall a \in A(h)) \; \pi_n(a) = \frac{c(a) n^{e(a)}}{\Sigma_{a' \in A(h)} c(a') n^{e(a')}} \; .$$

However, the converse provided by Theorem 2.1 is valuable. It shows that *any* consistent assessment can be supported with this special kind of sequence.

The following corollary is equivalent to Theorem 2.1. In both the theorem and the corollary, the first equation uses the exponents to determine the support of the strategy at each h and then uses the coefficients to determine the probabilities over that support. Similarly, the second equation uses the exponents to determine the support of the belief at each h and then uses the coefficients to determine the probabilities over that support. The corollary's formulation makes these observations more apparent.

Corollary 2.2. *In any game form* $[T,\prec,A,\alpha,\rho,H]$, *an assessment* (μ,π) *is consistent iff there exists* $c{:}A\rightarrow(0,\infty)$ *and* $e{:}A\rightarrow\mathbb{Z}_-$ *such that*

$$(\forall a)\ \pi(a) = \left(\begin{array}{cc} c(a) & \textit{if } e(a)=0 \\ 0 & \textit{if } e(a)<0 \end{array}\right) \quad \textit{and}$$

$$(\forall x)\ \mu(x) = \left(\begin{array}{cc} \dfrac{\rho\circ p_{\ell(x)}(x)\cdot\Pi_{k=0}^{\ell(x)-1}c\circ\alpha\circ p_k(x)}{\Sigma_{x'\in H^e(x)}\,\rho\circ p_{\ell(x')}(x')\cdot\Pi_{k=0}^{\ell(x)-1}c\circ\alpha\circ p_k(x')} & \textit{if } x{\in}H^e(x) \\ 0 & \textit{if } x{\notin}H^e(x) \end{array}\right)$$

where $H^e(x) = \mathit{argmax}\{\, \Sigma_{k=0}^{\ell(x')-1} e\circ\alpha\circ p_k(x') \,|\, x'{\in}H(x)\,\}$.

Proof. Section 2.3. □

Section 3 shows that Theorem 2.1 implies KW Lemmas A1 and A2 and thereby repairs a fallacy in the KW proofs. Theorem 3.1 of Perea y Monsuwe, Jansen, and Peters (1997) is weaker than the KW lemmas since it derives the analog of real but not necessarily integer exponents.

Finally, Streufert (2006a) relates Theorem 2.1's monomial characterization to an underlying concept of producthood, for relative probability, which is defined in the spirit of Kohlberg and Reny (1997).

2.3. Proof

A moment's inspection reveals that Theorem 2.1 and Corollary 2.2 are equivalent.

It is almost obvious that the monomial characterization is sufficient for consistency. In particular, if (π,μ) admits c and e which satisfy the theorem's equations, then the sequence $\{\pi_n\}_n$ defined by (3) generates a sequence $\{(\pi_n,\mu_n)\}_n$ in Ψ^0 which converges to (π,μ) (full details appear in Streufert (2006a, Section 4.5)).

It remains to show that the monomial characterization is necessary for consistency. Accordingly, suppose that (π,μ) is consistent. The remainder of this subsection will derive the corollary's version of the monomial characterization. In particular, the first part of the argument will derive exponents e, the second part will derive coefficients c, and the third part will show that c and e together satisfy the corollary's equations. (Every number in the argument is finite, and the first two parts of this argument are logically independent even though they share some notation.)

Deriving Exponents e. Define the binary relation $\preceq$ as the union of

$$\begin{aligned} \prec = &\{\ (t,p(t)) \mid \pi{\circ}\alpha(t){=}0\ \} \cup \\ &\{\ (x,y){\in}\textstyle\bigcup_h h^2 \mid \mu(x){=}0 \text{ and } \mu(y){>}0\ \}\ , \text{ and} \end{aligned}$$

$$\begin{aligned} \approx = &\{\ (t,p(t)) \mid \pi{\circ}\alpha(t){>}0\ \} \cup \\ &\{\ (p(t),t) \mid \pi{\circ}\alpha(t){>}0\ \} \cup \\ &\{\ (x,y){\in}\textstyle\bigcup_h h^2 \mid \mu(x){>}0 \text{ and } \mu(y){>}0\ \}\ , \end{aligned}$$

where $p(t)$ is the immediate predecessor of a non-initial node t. (Recall that KW lets t stand for a node and x stand for a decision node. Here every element of $\preceq$ is a pair (s,t) of nodes and every element of $\preceq\cap\bigcup_h h^2$ is a pair (x,y) of decision nodes.)

Consistency endows $\preceq$ with a special property: There cannot be a pair from $\prec$ in an indexed set $\{(s_j,t_j)\}_{j=1}^m$ of pairs from $\preceq$ whenever the set $\{(s_j,t_j)\}_{j=1}^m$ obeys

$$\Sigma_{j=1}^m 1^{s_j} = \Sigma_{j=1}^m 1^{t_j}\ , \tag{4}$$

where for any node t the row vector $1^t \in \{0,1\}^A$ is defined by

$$(1^t)_a = \begin{pmatrix} 1 & \text{if } (\exists k{\in}\{0,1,...\,\ell(t){-}1\})\ a = \alpha{\circ}p_k(t) \\ 0 & \text{otherwise} \end{pmatrix}\ .$$

To see this, take any such $\{(s_j,t_j)\}_{j=1}^m$. (4) yields that

$$(\forall n)\ \Pi_{j=1}^m \Pi_{k=0}^{\ell(s_j)-1} \pi_n{\circ}\alpha{\circ}p_k(s_j) = \Pi_{j=1}^m \Pi_{k=0}^{\ell(t_j)-1} \pi_n{\circ}\alpha{\circ}p_k(t_j)\ , \tag{5}$$

where $\{\pi_n\}_n$ is a sequence of positive-valued distributions establishing the consistency of (π,μ). (5) is equivalent to

$$(\forall n)\ \Pi_{j=1}^m P^{\pi_n}(s_j)/\rho{\circ}p_{\ell(s_j)}(s_j) = \Pi_{j=1}^m P^{\pi_n}(t_j)/\rho{\circ}p_{\ell(t_j)}(t_j)\ , \tag{6}$$

where for any node t the probability $P^{\pi_n}(t)$ is defined (as on KW page 868) by

$$P^{\pi_n}(t) = \rho{\circ}p_{\ell(t)}(t){\cdot}\Pi_{k=0}^{\ell(t)-1} \pi_n{\circ}\alpha{\circ}p_k(t)\ .$$

This (6) yields that

$$(\forall n)\ \Pi_{j=1}^{m} P^{\pi_n}(s_j)/P^{\pi_n}(t_j) = \Pi_{j=1}^{m} \rho{\circ}p_{\ell(s_j)}(s_j)/\rho{\circ}p_{\ell(t_j)}(t_j)\ ,$$

and hence (since the right-hand side is constant) that

$$\lim_n \Pi_{j=1}^{m} P^{\pi_n}(s_j)/P^{\pi_n}(t_j) \in (0,\infty)\ . \tag{7}$$

Meanwhile, consistency yields that

$$(\forall (s_j,t_j){\in}{\prec})\ \lim_n\ P^{\pi_n}(s_j)/P^{\pi_n}(t_j) = 0 \text{ and}$$
$$(\forall (s_j,t_j){\in}{\approx})\ \lim_n\ P^{\pi_n}(s_j)/P^{\pi_n}(t_j) \in (0,\infty)\ ,$$

and thus, the existence of a pair from $\prec$ would contradict (7).

The result of the previous paragraph yields that there cannot be column vectors $\beta \in \mathbb{Z}_+^{|\prec|}{\sim}\{0\}$ and $\delta \in \mathbb{Z}^{|\approx|}$ such that $\beta^T B + \delta^T D = 0$, where B and D are the matrices

$$B = [1^s{-}1^t]_{(s,t)\in\prec} \text{ and}$$
$$D = [1^s{-}1^t]_{(s,t)\in\approx}$$

whose rows correspond to pairs from $\prec$ and $\approx$. To see this, suppose that there were such β and δ. By the symmetry of $\approx$, we may define $\hat\delta \in \mathbb{Z}_+^{|\approx|}$ by

$$(\forall (s,t){\in}{\approx})\ \hat\delta_{(s,t)} = \begin{pmatrix} \delta_{(s,t)}{-}\delta_{(t,s)} & \text{if } \delta_{(s,t)}{-}\delta_{(t,s)} \geq 0 \\ 0 & \text{otherwise} \end{pmatrix}$$

so that $\delta^T D = \hat\delta^T D$, and so that consequently, we have $\beta \in \mathbb{Z}_+^{|\prec|}{\sim}\{0\}$ and $\hat\delta \in \mathbb{Z}_+^{|\approx|}$ such that $\beta^T B + \hat\delta^T D = 0$. This is equivalent to an indexed set taken from $\preceq$ which contains at least one element from $\prec$ and satisfies (4). By the previous paragraph, this is impossible.

Since the result of the previous paragraph is equivalent to (30), Proposition 4.1 shows that there is a vector $e \in \mathbb{Z}^{|A|}$ such that $Be \ll 0$ and $De = 0$. By the definitions of B and D, this is equivalent to the existence of a function $e{:}A{\to}\mathbb{Z}$ such that

$$(\forall (s,t){\in}{\prec})\ \Sigma_{j=0}^{\ell(s)-1} e{\circ}\alpha{\circ}p_j(s) < \Sigma_{j=0}^{\ell(s)-1} e{\circ}\alpha{\circ}p_j(t) \text{ and} \tag{8a}$$

$$(\forall (s,t){\in}{\approx})\ \Sigma_{j=0}^{\ell(s)-1} e{\circ}\alpha{\circ}p_j(s) = \Sigma_{j=0}^{\ell(s)-1} e{\circ}\alpha{\circ}p_j(t)\ . \tag{8b}$$

Consider any decision node x. Let h be the information set that owns it. If $\mu(x) = 0$, there must be some other node $y \in h$ such that $\mu(y) > 0$, and thus (8a) and the definition of $\prec$ yield that $x \notin H^e(x)$ (where H^e is defined as in Corollary 2.2). On the other hand, if $\mu(x) > 0$, (8) and the definitions of $\prec$ and $\approx$ yield that $x \in H^e(x)$. Hence,

$$(\forall x)\ \mu(x) > 0 \text{ iff } x \in H^e(x)\ . \tag{9}$$

When restricted to pairs (s, t) of the form $(t, p(t))$, (8) yields

$$(\forall (t,p(t)) {\in} {\prec})\ e{\circ}\alpha(t) < 0 \text{ and} \tag{10a}$$

$$(\forall (t,p(t)) {\in} {\approx})\ e{\circ}\alpha(t) = 0\ . \tag{10b}$$

Consider any action a. Let t be a node such that $a = \alpha(t)$ so that a is the action leading from $p(t)$ to t. If $\pi(a) = 0$, (10a) and the definition of $\prec$ yield $e(a) < 0$. If $\pi(a) > 0$, (10b) and the definition of $\approx$ yield that $e(a) = 0$. Hence,

$$(\forall a)\ e(a) \leq 0 \text{ and} \tag{11a}$$

$$(\forall a)\ \pi(a) > 0 \text{ iff } e(a) = 0\ . \tag{11b}$$

Deriving Coefficients c. Define the binary relation

$$\breve{\approx}\, = \{\ (x,y) {\in} \textstyle\bigcup_h h^2 \mid \mu(x) > 0 \text{ and } \mu(y) > 0\ \}\ .$$

Consistency yields that

$$(\forall (x,y) {\in} \breve{\approx})\ \lim_n P^{\pi_n}(x)/P^{\pi_n}(y) = \mu(x)/\mu(y)\ ,$$

which is equivalent to

$$(\forall (x,y) {\in} \breve{\approx})\ \lim_n \frac{\Pi_{k=0}^{\ell(x)-1} \pi_n{\circ}\alpha{\circ}p_k(x)}{\Pi_{k=0}^{\ell(y)-1} \pi_n{\circ}\alpha{\circ}p_k(y)} = \frac{\mu(x)}{\mu(y)} \frac{\rho{\circ}p_{\ell(y)}(y)}{\rho{\circ}p_{\ell(x)}(x)}\ ,$$

which is equivalent to

$$(\forall (x,y) {\in} \breve{\approx})\ \lim_n (1^x {-} 1^y)[\ln(\pi_n(a))]_{a \in A} = \ln\left(\tfrac{\mu(x)}{\mu(y)} \tfrac{\rho{\circ}p_{\ell(y)}(y)}{\rho{\circ}p_{\ell(x)}(x)}\right)\ , \tag{12}$$

where $[\ln(\pi_n(a))]_{a \in A}$ is a column vector in $\mathbb{R}^{|A|}$ so that the left-hand side is the limit of the product of a row vector with a column vector. (Every probability in (12) is positive and every logarithm is finite.)

Further, let $\check{A} = \{\ \check{a} \mid \pi(\check{a}) > 0\ \}$. Consistency yields that

$$(\forall \check{a} \in \check{A})\ \lim_n \pi_n(\check{a}) = \pi(\check{a})\ ,$$

which is equivalent to the awkward expression

$$(\forall \check{a} \in \check{A})\ \lim_n (1^{\check{a}})[\ln(\pi_n(a))]_{a \in A} = \ln(\pi(\check{a}))\ , \tag{13}$$

where $1^{\check{a}}$ is the row vector in $\{0,1\}^{|A|}$ which assumes a value of 1 at $\check{a}$ and a value of 0 elsewhere. (Every probability in (13) is positive and every logarithm is finite.)

Equations (12) and (13) can be expressed simultaneously as

$$\lim_n \begin{pmatrix} [1^x - 1^y]_{(x,y) \in \approx} \\ \text{- - - - - - - - -} \\ [1^{\check{a}}]_{\check{a} \in \check{A}} \end{pmatrix} \Big[\ln(\pi_n(a))\Big]_{a \in A} = \begin{pmatrix} [\ln(\frac{\mu(x)}{\mu(y)} \frac{\rho \circ p_{\ell(y)}(y)}{\rho \circ p_{\ell(x)}(x)})]_{(x,y) \in \approx} \\ \text{- - - - - - - - - - - - - - - -} \\ [\ln(\pi(\check{a}))]_{\check{a} \in \check{A}} \end{pmatrix}$$

and thus the column vector

$$\check{b} = \begin{pmatrix} [\ln(\frac{\mu(x)}{\mu(y)} \frac{\rho \circ p_{\ell(y)}(y)}{\rho \circ p_{\ell(x)}(x)})]_{(x,y) \in \approx} \\ \text{- - - - - - - - - - - - - - - -} \\ [\ln(\pi(\check{a}))]_{\check{a} \in \check{A}} \end{pmatrix}$$

is in the closure of the column space of the matrix

$$\check{D} = \begin{pmatrix} [1^x - 1^y]_{(x,y) \in \approx} \\ \text{- - - - - - - - -} \\ [1^{\check{a}}]_{\check{a} \in \check{A}} \end{pmatrix} .$$

Consequently, since the column space of any matrix is closed, $\check{b}$ must be in the column space of $\check{D}$. Hence, there is some $c{:}A \to \mathbb{R}_{++}$ such that

$$\begin{pmatrix} [1^x - 1^y]_{(x,y) \in \approx} \\ \text{- - - - - - - - -} \\ [1^{\check{a}}]_{\check{a} \in \check{A}} \end{pmatrix} \Big[\ln(c(a))\Big]_{a \in A} = \begin{pmatrix} [\ln(\frac{\mu(x)}{\mu(y)} \frac{\rho \circ p_{\ell(y)}(y)}{\rho \circ p_{\ell(x)}(x)})]_{(x,y) \in \approx} \\ \text{- - - - - - - - - - - - - - - -} \\ [\ln(\pi(\check{a}))]_{\check{a} \in \check{A}} \end{pmatrix} . \tag{14}$$

Because c is positive-valued (and every probability is positive), this vector equality is equivalent to the combination of

$$(\forall (x,y) \in \approx)\ \frac{\rho \circ p_{\ell(x)}(x) \cdot \Pi_{k=0}^{\ell(x)-1} c \circ \alpha \circ p_k(x)}{\rho \circ p_{\ell(y)}(y) \cdot \Pi_{k=0}^{\ell(y)-1} c \circ \alpha \circ p_k(y)} = \frac{\mu(x)}{\mu(y)}$$
$$\text{and } (\forall \check{a} \in \check{A})\ c(\check{a}) = \pi(\check{a})\ .$$

Hence, the definition of $\breve{\approx}$ yields

$$(15)\qquad \begin{gathered}(\forall(x,y){\in}\textstyle\bigcup_h h^2)\ \mu(x)>0 \text{ and } \mu(y)>0 \text{ implies}\\ \dfrac{\rho{\circ}p_{\ell(x)}(x){\cdot}\Pi_{k=0}^{\ell(x)-1}c{\circ}\alpha{\circ}p_k(x)}{\rho{\circ}p_{\ell(y)}(y){\cdot}\Pi_{k=0}^{\ell(y)-1}c{\circ}\alpha{\circ}p_k(y)} = \dfrac{\mu(x)}{\mu(y)}\ ,\end{gathered}$$

and the definition of $\breve{A}$ yields

$$(16)\qquad \begin{gathered}(\forall a)\ \pi(a)\ >\ 0 \text{ implies}\\ c(a)=\pi(a)\ .\end{gathered}$$

Conclusion. We can now derive the nonpositivity of e from (11a), the corollary's first equation from (11b) and (16), and the corollary's second equation from (9) and (15).

3. Application to KW

3.1. Some Definitions

This subsection recapitulates two less familiar definitions from KW. Both concern subsets of $A{\cup}X$. Note that KW page 880 calls *any* subset of $A{\cup}X$ a *basis*.

As on KW page 880, a basis b is *consistent* if the set

$$(17)\qquad \begin{gathered}\Psi_b = \{\text{ consistent } (\pi,\mu)\ |\\ (\forall a)\ a{\in}b \text{ iff } \pi(a){>}0 \ \text{ and }\ (\forall x)\ x{\in}b \text{ iff } \mu(x){>}0\ \}\end{gathered}$$

is nonempty. For instance, in the example, the basis

$$(18)\qquad b=\{S,s,\delta,\varepsilon,o,oS,oSg\}$$

is consistent because the (π,μ) defined in (1) belongs to Ψ_b.

As on KW Page 887, a basis b is *labelled* by a nonnegative-integer-valued function $K{:}A{\to}\mathbb{Z}_+$ if

$$(19a)\qquad (\forall h)(\exists a{\in}A(h))\ K(a)=0$$

$$(19b)\qquad (\forall a)\ a\in b \text{ iff } K(a)=0$$

$$(19c)\qquad (\forall x)\ x\in b \text{ iff } x\in \operatorname{argmin}\{J_K(x')|x'{\in}H(x)\}\ ,$$

where $J_K{:}X{\to}\mathbb{Z}_+$ is defined by

$$(20)\qquad J_K(x)=\Sigma_{k=0}^{\ell(x)-1}K{\circ}\alpha{\circ}p_k(x)\ .$$

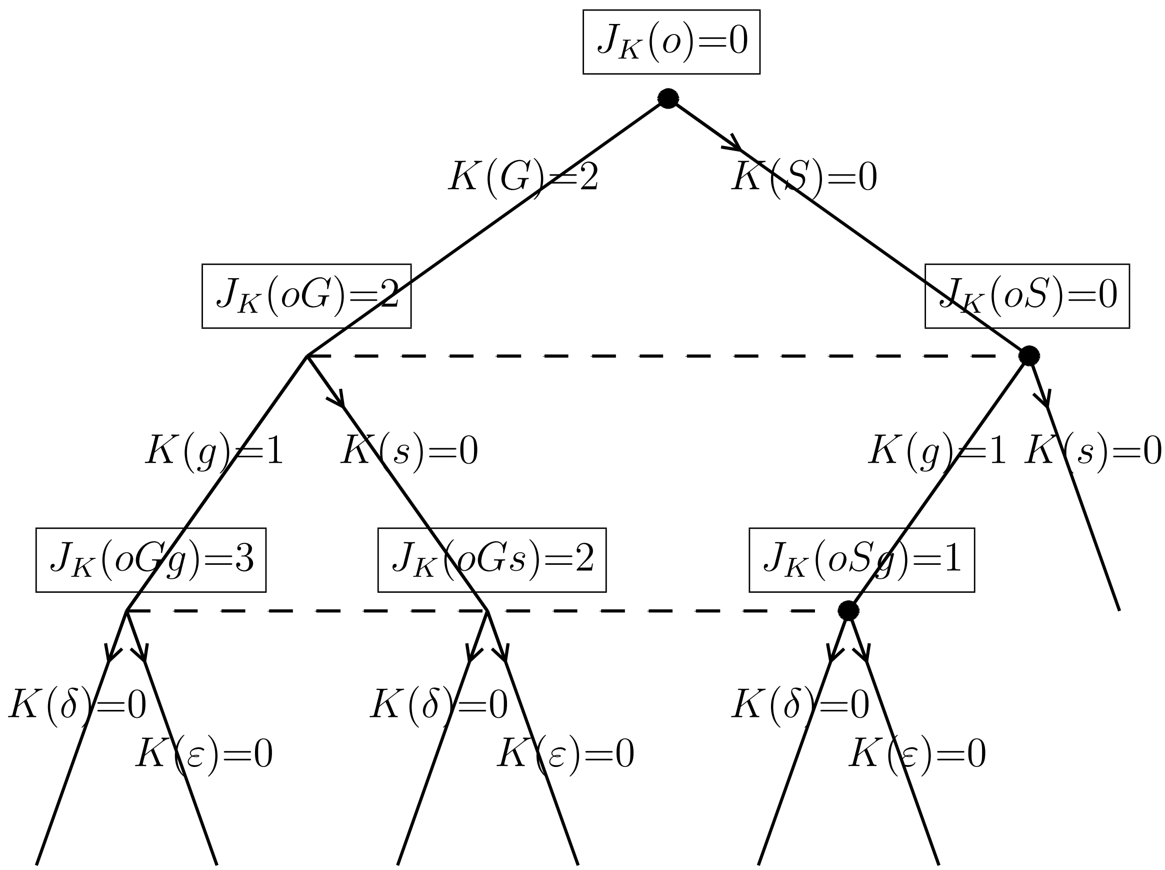

FIGURE 3.1

For instance, in the example, the b defined in (18) is labelled by the K defined in Figure 3.1. To see this, first note that the figure calculates J_K, then note that the figure also depicts b with arrows for actions and dots for nodes, and finally, inspect each of the three conditions in (19).

3.2. A FALLACY

On KW page 888, Lemma A2's proof draws upon Lemma A1. On KW page 887, Lemma A1 appears as follows.

"Lemma A1: The basis b is consistent (Ψ_b is nonempty) if and only if a b labelling exists."

In other words, Lemma A1 states that a basis is consistent iff it can be labelled. (The consistency of b is synonymous with the nonemptiness of Ψ_b.)

Lemma A1's proof appears on KW page 887. I find the argument unconvincing. In particular, its second paragraph does not show how to label an arbitrary consistent basis. The remainder of this section examines this paragraph sentence-by-sentence (the fallacy occurs in the very last sentence).

"Now suppose that b is a consistent basis."

In the example, the b defined at (18) is consistent. The proof should tell us how to label this basis b with a function K.

"Since Ψ_b is nonempty, there exists a sequence $\{(\mu_n, \pi_n)\}_n \subseteq \Psi^0$ with the limit (μ, π) belonging to Ψ_b."

In the example, the sequence $\{(\mu_n, \pi_n)\}_n \subseteq \Psi^0$ defined in the first lines of (1) has the limit (μ, π) defined in the second lines of (1) and this (μ, π) is an element of Ψ_b for the b defined at (18).

"Let M denote the finite set of all first degree, single term multinomials with coefficient one in the symbols $a \in A$."

Thus M consists of 1, each action, each pair of actions, each triple of actions, and so forth. In the example, elements of M include 1, S, and Sg, and accordingly, the elements of M will be useful in describing the paths that reach nodes (1 describes the empty path taken to the initial node). (The set M also happens to contain many other multinomials like $GSgs$ which do not correspond to paths that reach nodes, but these extra multinomials don't impose much of a burden.)

"For $m \in M$, let m_n represent m evaluated with $a = \pi_n(a)$."

In the example, if $m = Sg$, then the number $m_n = (Sg)_n$ is Sg evaluated with $S = \pi_n(S)$ and $g = \pi_n(g)$, which reduces to $\pi_n(S)\pi_n(g)$, which by the definition (1) of $\{\pi_n\}_n$ is $\frac{1}{n^{-2}+1}\frac{n^{-1}}{n^{-1}+1}$.

"Without loss of generality, we can assume that for every pair m and m' from M, the sequence m_n/m'_n converges either to zero, to infinity, or to some strictly positive number. (This is wlog because we can look along a subsequence of $\{(\mu_n, \pi_n)\}_n$ for which it is true.)"

In the example, consider $m = Sg$ and $m' = Gg$. Recall from the last step that $(Sg)_n$ is $\frac{1}{n^{-2}+1}\frac{n^{-1}}{n^{-1}+1}$. Similarly, $(Gg)_n$ is $\frac{n^{-2}}{n^{-2}+1}\frac{n^{-1}}{n^{-1}+1}$. Thus

$J(m')$	m'							
0	1							
0	S							
1	Sg					•	•	
2	G				•	•	•	
2	Gs				•	•	•	
4	Gg		•	•	•	•	•	
		Gg	Gs	G	Sg	S	1	m
		4	2	2	1	0	0	$J(m)$

Table 3.1

$(Sg)_n/(Gg)_n$ is $1/n^{-2} = n^2$, which happens to converge to infinity. In fact, all such ratio sequences in the example converge either to zero, to infinity, or to some strictly positive number (in other words, the subsequence argument is unnecessary in the example).

"Define $m \dot{<} m'$ if $\lim_n m_n/m'_n = \infty$; then $\dot{<}$ is an asymmetric and negatively transitive binary relation on M."

In the example, $Sg \dot{<} Gg$ because $\lim_n (Sg)_n/(Gg)_n = \infty$ by the last step. Many similar calculations reveal that the restriction of $\dot{<}$ to $\{1, G, S, Gg, Gs, Sg\}^2$ is the set of pairs (m, m') that receive a dot • in Table 3.1 ($Sg \dot{<} Gg$ appears as the dot with Sg on the horizontal axis and Gg on the vertical axis). (Also, elements of M outside of $\{1, G, S, Gg, Gs, Sg\}$ are excluded because they do not correspond to paths to decision nodes.)

"Since M is finite there exists an integer valued function J on M with $m \dot{<} m'$ if and only if $J(m) < J(m')$. We can pick J so that $J(m) = 0$ for the $\dot{<}$−least m—then $J(m) \geq 0$ for all m."

In the example, such a J appears in the last row (and first column) of Table 3.1. The same J also appears in the boxes of Figure 3.2. (To be precise, we are only concerning ourselves with the restriction of J to $\{1, G, S, Gg, Gs, Sg\}$.)

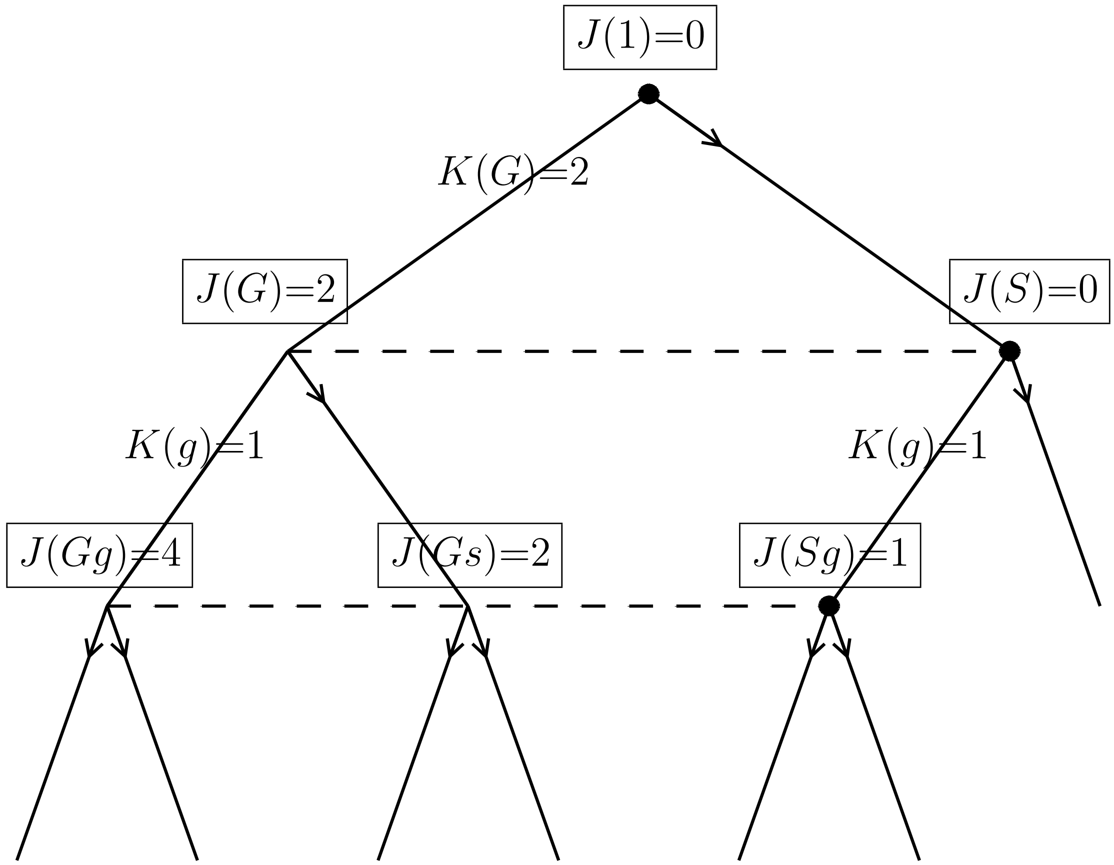

FIGURE 3.2

"For each $x \in X$ there is an associated $m^x \in M$, namely $m^x = \Pi_{\ell=0}^{\ell(x)-1}\alpha(p_\ell(x))$." (For $x \in W$, $m^x = 1$.)

Each m^x is the list of actions leading to x. In the example, $m^{oSg} = Sg$ and $\{m^x|x{\in}X\} = \{1, G, S, Gg, Gs, Sg\}$.

"Now for each a pick an arbitrary $x \in H(a)$ such that $J(m^x)$ is minimal over $x \in H(a)$ and define

$$K(a) = J(m^x{\cdot}a) - J(m^x) \text{ .''} \tag{21}$$

(First an insignificant remark: I take $H(a)$ to be the information set from which the action a can be chosen. In other words, I take $H(a)$ to equal $A^{-1}(a)$, as defined on KW Page 867.)

Consider the action $a = g$ in the example. It can be chosen from the information set $H(g) = \{oG, oS\}$, and from the bottom row in Table 3.1 or from the boxes in Figure 3.2, we have that $J(G) = 2$ and

$J(S) = 0$. Hence $x = oS$ is, in the above words from KW evaluated at $a = g$, "an arbitrary $x \in H(g)$ such that $J(m^x)$ is minimal over $x \in H(g)$" (in fact, it is the only such x). Hence (21) sets

$$K(g) = J(m^{oS}{\cdot}g) - J(m^{oS}) = J(Sg) - J(S) = 1\ . \tag{22}$$

Similarly consider the action $a = G$ in the example. It can be chosen from the information set $H(G) = \{o\}$. Hence $x = o$ is (trivially) "an arbitrary $x \in H(G)$ such that $J(m^x)$ is minimal over $x \in H(G)$." Thus (21) sets

$$K(G) = J(m^o{\cdot}G) - J(m^o) = J(G) - J(1) = 2\ . \tag{23}$$

"We leave to the reader the relatively easy tasks of proving that $K(a)$ is well-defined (i.e., the choice of a $J(m^x)$-minimal $x \in H(a)$ is irrelevant) and that K so defined is a b labelling (with, of course, $J_K(x) = J(m^x)$)."

The equation $J_K(x) = J(m^x)$ cannot be derived. Consider the example. There $J_K(oGg) = K(G)+K(g) = 2+1 = 3$ by (20), (22), and (23). Yet $J(m^{oGg}) = J(Gg) = 4$ by the definition of J in the last row of Table 3.1.

The difficulty lies in the choice of the function J. I deliberately chose $J(Gg) = 4$. Had I alternatively chosen $J(Gg) = 3$, there would have been no problem with this example at the last stage of the proof.

However, making a judicious choice of J is a nontrivial problem. Not any representation of the binary relation $\dot{<}$ will do. Rather, it has to be additive across actions. Finding that additive representation lies at the heart of Theorem 2.1's proof (see Section 4.2 for further discussion).

3.3. Proofs of KW Lemmas A1 and A2

We begin with three simple lemmas which will be used repeatedly. To get oriented, note that any two of (24), (26), and (27) imply the third.

Lemma 3.1. *$(\pi, \mu) \in \Psi_b$ iff (π, μ) is consistent,*

$$(\forall a)\ a \in b \text{ iff } \pi(x) > 0\ ,\ \textit{and} \tag{24a}$$

$$(\forall x)\ x \in b \text{ iff } \mu(x) > 0\ . \tag{24b}$$

Proof. This follows from the definition (17) of Ψ_b. $\square$

LEMMA 3.2. *b can be labelled iff there exists $e{:}A{\to}\mathbb{Z}_-$ such that*

$$(\forall h)(\exists a{\in}A(h))\ e(a) = 0\ , \tag{25}$$

$$(\forall a)\ a \in b \text{ iff } e(a) = 0\ , \text{ and} \tag{26a}$$

$$(\forall x)\ x \in b \text{ iff } x \in H^e(x)\ . \tag{26b}$$

Proof. (19) is equivalent to the combination of (25) and (26) after J_K has been substituted out, Corollary 2.2's H^e has been substituted in, and K and $-e$ have been identified. $\square$

LEMMA 3.3. *If (π, μ) and (c, e) satisfy Corollary 2.2's equations, then*

$$(\forall a)\ \pi(a) > 0 \text{ iff } e(a) = 0 \text{ and} \tag{27a}$$

$$(\forall x)\ \mu(x) > 0 \text{ iff } x \in H^e(x)\ . \tag{27b}$$

Proof. Obvious. $\square$

PROPOSITION 3.4 (KW Lemma A1). *A basis is consistent iff it can be labelled.*

Proof. Suppose b is consistent. Then $\Psi_b \neq \varnothing$, and thus by Lemma 3.1 there is a consistent assessment (π, μ) satisfying (24). Because (π, μ) is consistent, Corollary 2.2 yields (c, e) satisfying its equations. By Lemma 3.3, the corollary's equations yield (27), which together with (24) yields (26). The corollary's first equation and the well-definition of π yield (25). Hence, by Lemma 3.2, b can be labelled.

Conversely, suppose that b can be labelled. Then by Lemma 3.2 there exists some e which satisfies (25) and (26). Define c by

$$c(a) = 1/|\{\ a'{\in}A(H(a))\ |\ e(a'){=}0\ \}|\ .$$

Because of (25) and the normalization in the definition of c, we can construct π and μ to satisfy Corollary 2.2's equations. Then by the corollary itself, (π, μ) is consistent. Further, by Lemma 3.3, the corollary's equations yield (27), which together with (26) yields (24). Hence, by Lemma 3.1, $(\pi, \mu) \in \Psi_b$. Hence b is consistent. $\square$

As on KW page 888, define

$$\Xi_b = \{\ c{:}A{\to}(0,\infty) \mid (\forall h)\ \Sigma_{a\in b\cap A(h)}c(a) = 1\ \}\ ,$$

let π^b map $c \in \Xi_b$ to

$$\pi^b(c)(a) = \begin{pmatrix} c(a) & \text{if } a \in b \\ 0 & \text{if } a \notin b \end{pmatrix}\ , \tag{28a}$$

and let μ^b map $c \in \Xi_b$ to

$$\mu^b(c)(x) = \left(\begin{array}{cc} \dfrac{\rho{\circ}p_{\ell(x)}(x){\cdot}\Pi_{k=0}^{\ell(x)-1}c{\circ}\alpha{\circ}p_k(x)}{\Sigma_{x'\in b\cap H(x)}\rho{\circ}p_{\ell(x')}{\cdot}\Pi_{k=0}^{\ell(x')-1}c{\circ}\alpha{\circ}p_k(x')} & \text{if } x \in b \\ 0 & \text{if } x \notin b \end{array} \right) \tag{28b}$$

(the KW symbol ξ has been replaced by c, the KW multinomials m^x have been substituted out, and the restriction $(\forall w)\ \rho(w) = 1/|W|$ arbitrarily imposed at the start of KW Section A.1 has been relaxed).

Proposition 3.5 (KW Lemma A2). *For any consistent b, Ψ_b is the image of Ξ_b under the mapping (π^b, μ^b).*

Proof. Take any assessment (π, μ) in Ψ_b. By Lemma 3.1, we have (24). Further, since (π, μ) is consistent, Corollary 2.2 yields the existence of (c, e) which satisfy its equations. By Lemma 3.3, the corollary's equations yields (27), which together with (24) yields (26). We now assemble three facts. [a] $c \in \Xi_b$ by the corollary's first equation, the well-definition of π, and (26a). [b] $\pi = \pi^b(c)$ by the corollary's first equation, definition (28a), and (26a). [c] $\mu = \mu^b(c)$ by the corollary's second equation, definition (28b), (26b), and by the fact that $(\forall x)\ b{\cap}H(x) = H^e(x)$ by (26b). By these three facts, (π, μ) is in the image of Ξ_b under (π^b, μ^b).

Conversely, take any consistent b and any $c \in \Xi_b$. By Proposition 3.4 and Lemma 3.2, there is some e which satisfies (26). Then, $\pi^b(c)$ satisfies Corollary 2.2's first equation by definition (28a) and (26a). Further, (26b) yields $(\forall x)\ b{\cap}H(x) = H^e(x)$, and thus, $\mu^b(c)$ satisfies the corollary's second equation by definition (28b) and (26b). Since $(\pi^b(c), \mu^b(c))$ satisfies the corollary's equations by the last two sentences, the corollary itself yields that $(\pi^b(c), \mu^b(c))$ is consistent.

Therefore, since definition (28) yields that $(\pi^b(c), \mu^b(c))$ satisfies (24), Lemma 3.1 yields $(\pi^b(c), \mu^b(c)) \in \Psi_b$. □

4. Mathematical Foundation

4.1. Linear Algebra

This paper uses nothing more than linear algebra. In particular, the most advanced results used are the following proposition (which is used to derive e at (8)) and the closedness of any matrix's column space (which is used to derive c at (14)).

The following proposition states that there is a solution to the system of linear inequalities and equalities in (29) precisely when the rows used to define those inequalities and equalities are "independent" in the sense of (30). The result appears in Krantz, Luce, Suppes, and Tversky (1971), and their proof, in turn, depends only on high-school-level results for systems of linear equalities of the form $Ax = b$.

Proposition 4.1. *For any matrices $B \in \mathbb{Q}^{bk}$ and $D \in \mathbb{Q}^{dk}$, the following are equivalent.*

$$(\exists x {\in} \mathbb{Z}^k)\ Bx \ll 0 \text{ and } Dx = 0. \tag{29}$$

$$\text{Not } (\exists \beta {\in} \mathbb{Z}^b_+ {\sim} \{0\})(\exists \delta {\in} \mathbb{Z}^d)\ \beta^T B + \delta^T D = 0. \tag{30}$$

($\mathbb{Q}$ denotes the set of rationals, $\mathbb{Z}$ denotes the set of integers, and $Bx \ll 0$ means that every element of the vector Bx is negative.)

Proof. Krantz, Luce, Suppes, and Tversky (1971, Theorem 2.7 on page 62 together with the first six sentences on page 63) after replacing their m' with b, their m'' with d, their $[\alpha_i]_{i=1}^{m'}$ with $-B$, their $[\beta_i]_{i=1}^{m''}$ with D, their λ with β, and their μ with δ. □

4.2. Additive Representation

As stated a moment ago, Proposition 4.1 is used to derive the existence of exponents e which satisfy (8). Equation (8) states that e provides an additive representation of the binary relation $\preceq$. This is the heart of the proof (and it corresponds to the fallacy noted in Section 3.2: the converse $\dot{>}$ of the KW relation $\dot{<}$ is an extension of $\prec$).

Krantz, Luce, Suppes, and Tversky (1971, Sections 2.3 and 9.2) explain that Proposition 4.1 is fundamental to a literature which is largely unfamiliar to economists. As economists, we are accustomed to deriving additive representations over continuous domains, from assumptions of separability, by means of topological results due to Debreu (1960), Gorman (1968), and their successors. Meanwhile, there is another literature which derives additive representations over discrete domains, from cancellation laws, by means of algebraic results due to Scott (1964) and his successors. The insights of this alternative literature can be applied here because consistency implies the sentence containing (4), and this sentence essentially specifies cancellation laws.

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