Exchange Rates and Purchasing Power Parity in Imperfectly Competitive Markets

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Abstract

In this paper we make use of some recent results in strategic market games literature in order to question the validity of the Purchasing Power Parity (PPP) theory in a frictionless N-country exchange economy where agents have market power over commodity and currency markets. We identify individual equilibrium strategies that are compatible with the failure of PPP and result to exchange rate inconsistency. We then show that equilibrium PPP deviations and inconsistencies tend to zero as the number of agents in the economy increases.

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1 Introduction

The Purchasing Power Parity (PPP) theory states that the price of any commodity should be the same in all countries, once prices are converted to a common currency. Therefore, the purchasing power of money is uniform across countries, irrespectively of the choice of the price index we use to measure it. PPP is actually another expression of the "Law of One Price" (LOP) in an international economy with multiple currencies and alike LOP, it draws its foundation from the unrestricted ability of agents to arbitrage prices.

Empirical literature, see Rogoff [11], suggests that PPP deviations are large in the shortrun and die out extremely slowly in the long run. The violation of PPP has been attributed to imperfect international price arbitrage which is caused by non-tradeability of goods or services and market frictions like tariffs, taxes, transportation and transaction costs.

Trade barriers result to costly participation of agents in certain international commodity markets and cause market segmentation. When markets are segmented, imperfect competition is relevant to PPP failure: oligopolistic firms may discriminate prices for the same commodity across different international markets, the so-called "pricing-to-market" by Krugman [8], see also Goldberg and Knetter [4] for a survey. When markets are integrated, in the absence of any market frictions, perfect price arbitrage will force common currency price equalization across countries to marginal cost plus a common markup, which might be positive or zero depending on the nature of competition. In other words, market integration together with perfect price arbitrage imply absence of price discrimination, irrespectively of markets being competitive or not. In that case, imperfect competition is believed to be irrelevant to PPP failure. The main purpose of this paper is to show that this statement is not always true and imperfect competition alone may be responsible for PPP failure, even in completely frictionless markets with perfect price arbitrage.

We model imperfect competition in a general equilibrium setup using a variant of the Shapley and Shubik [12] market game model[1] with fiat money, as in Postlewaite and Schmeidler [10], allowing however for multiple currencies. An advantage of strategic market game models over competitive models is that they provide an explicit price formation process. In competitive frictionless economies, due to the law of one price, one market is as good as many in terms of allocation of resources, and thus the market structure is irrelevant to set of competitive equilibria. Recent work by Koutsougeras [5] has established that this is not true when competition is imperfect: by increasing the number of markets for a commodity new equilibria arise that violate the law of one price and render the market structure relevant to the set of Nash equilibrium allocations. This type of analysis fits perfectly to an international trade context where multiple markets for the same commodity exist in different countries. Moreover, in the model we will present, besides the commodity markets, the exchange rate

[1] See Giraud [3] and the special issue on strategic market games of the Journal of Mathematical Economics 39, 2003.

is determined under imperfect competition giving rise to additional strategic considerations.

The failure of PPP in our model is attributed solely to the fact that agents' activities have non-negligible effects on market clearing prices of commodities and currencies. Agents may manipulate commodity prices and exchange rates in their favor by engaging in speculative trade, that is by buying for example a commodity in the national market and selling it abroad and at the same time depreciate the domestic currency by buying more foreign currency. This type of strategy may be observed at equilibrium in our model, simply because equilibrium pricing allows for PPP failure. However such strategy is compatible with the notion of equilibrium because it cannot generate infinite profit: agents observe common currency price disparities of a good at equilibrium but are unable to take advantage of them, even infinitesimally, due to the adverse price effects of their actions on commodity prices and currencies to the value of their whole trades, that render any contemplated budget feasible change of their actions at equilibrium unfavorable in terms of utility. So equilibrium in our model is characterised by PPP failure and bounded international arbitrage. Such characteristics would never arise in a competitive model for if any deviation from PPP existed at equilibrium, agents would have incentive to exploit it by trading infinite amounts without affecting commodity prices and exchange rates, a contradiction given the notion of equilibrium.

It may be argued that the international market is the largest conceivable market in terms of participants in the commodity and currency markets, therefore, assuming absence of market frictions, the effect of agents' actions on prices might indeed be very small to qualify as a possible candidate for the failure of PPP. Following the asymptotic convergence approach in Koutsougeras [6], we show that deviations from PPP are large only when the number of agents in the economy is small, however such deviations vanish only at the limit where the number of agents tends to infinity. This result depends only on the number of agents in the economy and not in any particular replication of agents' characteristics, providing thus a non-cooperative foundation of PPP theory. It is also distinct from the approaches of Dubey and Shubik [2] and Mas-Colell [9], since at the limit, although PPP holds, commodity prices and exchange rates need not be competitive, if the economy is replicated in such a way that it is not atomless.

Our approach of an international economy builds on the models of the type of Shubik and Wilson [13], Postlewaite and Schmeidler [10] with N paper monies (currencies) exchangeable for goods, where each currency can be directly exchanged with all other currencies (complete markets), see Amir et al.[1],. With multiple currencies, inconsistent exchange rates may arise[2], that nevertheless tend to vanish when the number of agents increases. It is also shown that the size of PPP deviations is positively related to the number of currencies traded.

The organization of the paper is as follows: In Section 2 the construction of

[2]Inconsistent relative prices my arise at equilibrium in the model of Amir et al. [1] due to liquidity constraints.

the model is presented, in Section 3 we derive a necessary equilibrium condition that relates the prices of a commodity in different markets with the exchange rate, in section 4 we identify individual equilibrium strategies that are compatible with the failure of PPP and result to exchange rate inconsistency, in section 5 we study the asymptotic behavior of PPP deviations and we conclude in Section 6.

2 The Model

Suppose that the world economy consists of a set of N countries that are engaged in the trade of commodities with each other. Each country has its own currency, thus there are N currencies indexed by $n \in N$. Let I be the set of individuals of the world economy[3], indexed by $i \in I$. Let L be the set of commodities available in each country, $l \in L$. All sets are finite.

The trade of commodities is realized via trading posts. In each country $n \in N$ there are L trading posts, one trading post for each commodity $l \in L$. Therefore a commodity l is traded in N countries or different trading posts, making the total number of commodity markets in the world economy NL. The pair l, n denotes commodity l traded in country n.

The trade of currencies is also realized via trading posts, the foreign exchange markets. It is assumed that each country accepts payments for the transaction of commodities only in its own currency, thus the trading post for l, n accepts and makes payments only in currency n. Currency is necessary for commodity transactions to realize. Individuals must pay their purchases of commodities in the currency of the country of origin. Their sales of commodities are paid in the currency of the country of destination[4]. Each currency can be pairwise traded against $N-1$ currencies so we have $N(N-1)/2$ trading posts for currencies. The location of a trading post for currency is not essential since we assume that placing a bid or an offer is costless.

The net supply of currencies is zero for the overall economy. Individuals may obtain a certain currency by selling commodities or other currencies.

2.1 The Commodity Markets

It is assumed that commodity markets in all countries are freely accessible by all individuals and there are no transaction or transportation costs. The formation of prices in commodity markets is realized according to the standard trading post mechanism.

In order to purchase commodity $l \in L$ from country $n \in N$,an individual $i \in I$ must send a bid in country's n currency units. Let $b^i_{l,n}$ be the bid sent to

[3] The location or nationality of individuals is unimportant since all goods are assumed perfectly tradeable.

[4] For example an agent must pay in US dollars when he purchases commodities from the US and receives dollars when he sells commodities to the US.

the l, n trading post. In order to sell a commodity an individual makes an offer $q^i_{l,n}$ of commodity $l \in L$, a physical quantity, to trading post l, n. The

The price of commodity l in country n is given according to the following rule:

$$p_{l,n} = \frac{\sum_{i \in I} b^i_{l,n}}{\sum_{i \in I} q^i_{l,n}} \tag{1}$$

where the numerator is the aggregate bids and the denominator the aggregate offers sent to the l, n trading post.

In exchange to his bid $b^i_{l,n}$, an individual receives $b^i_{l,n}/p_{l,n}$ units of good l from trading post l, n. In exchange to his offer $q^i_{l,n}$, he receives $q^i_{l,n} p_{l,n}$ units of n currency.

2.2 The Currency Markets

Let $b^i_{k,n}$ be the bid for n currency denominated in k currency, $k \in N, k \neq n$. It represents also an offer for sale of k currency for the purchase of n currency. Similarly $b^i_{n,k}$ is the bid for k currency or the offer of n currency. Given $(b^i_{k,n}, b^i_{n,k})_{i \in I}$, the k/n exchange rate is

$$t^n_k = \frac{\sum_{i \in I} b^i_{k,n}}{\sum_{i \in I} b^i_{n,k}} \equiv \frac{1}{t^k_n}.$$

In exchange to his bid $b^i_{k,n}$ ($b^i_{n,k}$), an individual receives $b^i_{k,n} t^k_n$ (resp. $b^i_{n,k} t^n_k$) units of country's n (resp. k) currency.

2.3 Agents

Let $e^i = (e^i_1, ..., e^i_l, ..., e^i_L) \in \Re^L_{++}$, be the endowment of individual i. A consumption bundle is $x = (x_1, ..., x_l, ..., x_L) \in \Re^L_+$ and $x_l \in \Re_+$ is consumption of good l. Preferences are represented by a utility function over consumption bundles $u^i(x)$, $u^i : \Re^L_+ \to \Re$. We assume that the utility function is twice continuously differentiable, strictly concave and that the indifference curves passing through the endowment do not intersect the axes. The overall economy is defined as $\mathcal{E} = \{(\Re^L_+, u^i, e^i) : i \in I\}$.

Commodity deliveries must be made in physical commodities so that they cannot exceed endowments,

$$\sum_{n \in N} q^i_{l,n} \leq e^i_l \text{ for each } l \in L. \tag{2}$$

On the other hand, there is no restriction on the bids an individual may make either on the commodity or the currency markets, provided he does not go bankrupt, i.e. he does not violate his budget constraints.

A strategy (b^i, q^i) for agent i consists of $N(N-1)$ currency bids, LN commodity bids and LN commodity offers,

$$(b^i, q^i) = \left((b^i_{k,n})_{k,n \in N, k \neq n}, (b^i_{l,n}, q^i_{l,n})_{l \in L, n \in N}\right)$$

An agent's strategy set is given by

$$\Omega^i = \left\{ (b^i, q^i) \in \Re_+^{2LN+N(N-1)} : \sum_{n \in N} q^i_{l,n} \le e^i_l \text{ for each } l \in L \right\}.$$

Given a strategy profile $\{(b^i, q^i) \in \Omega^i : i \in I\}$, we use capital letters to denote aggregate bids and offers for currency or commodities. Let $B_{k,n} = \sum_i b^i_{k,n}$ be the aggregate bid for currency $n \in N$ in units of $k \in N$ currency, $B_{l,n} = \sum_i b^i_{l,n}$ the aggregate bid for commodity $l \in L$ in country $n \in N$ and $Q_{l,n} = \sum_i q^i_{l,n}$ the aggregate offer for commodity l in country's n trading post.

Each individual faces N budget constraints, one for each currency,

$$\sum_{k \in N, k \neq n} b^i_{n,k} + \sum_{l \in L} b^i_{l,n} \le \sum_{k \in N, k \neq n} b^i_{k,n} t^k_n + \sum_{l \in L} q^i_{l,n} p_{l,n}, \qquad \forall n = 1, ...N. \tag{3}$$

where the right hand side in (3) is the money receipts in n currency from the sale of $k \neq n$, $k \in N$ currencies and the sale of commodities to country n. These receipts must provide agent i with enough n currency in order to finance his bids for k currencies and commodities in country n. The agent is bankrupt if he bids more currency than that he collects, for any $n \in N$.

Individual's i consumption of commodity l is given by

$$x^i_l = \begin{cases} e^i_l + \sum_{n \in N} \left(\frac{b^i_{l,n}}{p_{l,n}} - q^i_{l,n} \right) & \text{if (3) is satisfied} \\ e^i_l - \sum_{n \in N} q^i_{l,n} & \text{otherwise.} \end{cases}$$

The problem of i is to find a strategy $(b^i, q^i) \in \Omega^i$, given the strategies of all other players $(B^{-i}, Q^{-i}) = ((B^{-i}_{l,n}, Q^{-i}_{l,n})_{l \in L, n \in N}, (B^{-i}_{k,n})_{k,n \in N, k \neq n})$, so that his utility is maximized,

$$\max_{b^i, q^i \in \Omega^i} u^i \left(x^i(b^i, q^i); (B^{-i}, Q^{-i}) \right) \tag{4}$$

$$: \sum_{k \in N, k \neq n} b^i_{n,k} + \sum_{l \in L} b^i_{l,n} \le \sum_{k \in N, k \neq n} b^i_{k,n} t^k_n + \sum_{l \in L} q^i_{l,n} p_{l,n}, \qquad \forall n = 1, ..., N \tag{5}$$

Individuals choose how much currency or commodity to bid and offer to the respective markets, taking as given the strategies of all the players in the overall economy.

The market game of this economy Γ, consists of a set of players I, their strategy sets Ω^i, the outcomes x^i, and the payoffs $u^i(x^i)$. A Nash equilibrium (NE) for Γ is a profile $\left\{(b^i, q^i) \in \Omega^i : i \in I\right\}$ such that $(b^i, q^i) \in \arg\max u(x^i)$ and (3) is satisfied with equality for every $n \in N, i \in I$.

3 Equilibrium Pricing

The Lagrangean of i's maximization problem is

$$L^i = u^i \left(x^i(b^i, q^i); (B^{-i}, Q^{-i}) \right) + \sum_{n \in N} \lambda^i_n \left(\sum_{k \in N, k \neq n} \left(b^i_{k,n} t^k_n - b^i_{n,k} \right) + \sum_{l \in L} \left(q^i_{l,n} p_{l,n} - b^i_{l,n} \right) \right).$$

The first order necessary conditions reduce to the following equations

$$(p_{l,n})^2 = \frac{1}{\lambda_n^i} \frac{\partial u}{\partial x_l^i} \frac{B_{l,n}^{-i}}{Q_{l,n}^{-i}}, \forall l \in L, n \in N \tag{6}$$

and

$$(t_n^k)^2 = \frac{\lambda_k^i}{\lambda_n^i} \frac{B_{n,k}^{-i}}{B_{k,n}^{-i}}, \forall k, n \in N, n \neq k. \tag{7}$$

Combining (6) and (7) we obtain the following equation which relates the price of the same commodity in any two countries $n, k \in N$.

$$\left(\frac{p_{l,n}}{p_{l,k}}\right)^2 = (t_n^k)^2 \frac{B_{k,n}^{-i}}{B_{n,k}^{-i}} \frac{B_{l,n}^{-i}}{Q_{l,n}^{-i}} \frac{Q_{l,k}^{-i}}{B_{l,k}^{-i}}, \forall l \in L, \forall k, n \in N, \forall i \in I. \tag{8}$$

The above equation is the analogue of the standard competitive PPP equation, in an imperfectly competitive economy. It relates the price of the same commodity in two countries with the currency exchange rate. In competitive economies, the PPP theory suggests that income should have the same purchasing power in every country (in this model, equal bids denominated in common currency should buy exactly the same quantity of the good irrespectively of the country) or equivalently that a good should sell at the same price in every country, once prices are denominated in a common currency, that is

Definition 1 ***(absolute PPP)*** *The PPP holds when*

$$p_{l,n} = t_n^k p_{l,k}, \forall l \in L, \forall n, k \in N \tag{9}$$

Exchange rate in frictionless competitive economies are consistent.

Definition 2 *Exchange rates are consistent when for any triple of currencies $n, m, k \in N$*

$$t_m^n t_k^m t_n^k = 1 \tag{10}$$

For the standard PPP theory to hold in this imperfectly competitive economy, it is required by (8) that at a Nash equilibrium

$$\frac{B_{k,n}^{-i}}{B_{n,k}^{-i}} \frac{B_{l,n}^{-i}}{Q_{l,n}^{-i}} \frac{Q_{l,k}^{-i}}{B_{l,k}^{-i}} = 1, \forall l \in L, \forall n, k \in N, \forall i \in I. \tag{11}$$

It is also evident from (7) that at a Nash equilibrium the exchange rates are consistent if and only if

$$\frac{B_{n,k}^{-i}}{B_{k,n}^{-i}} \frac{B_{k,m}^{-i}}{B_{m,k}^{-i}} \frac{B_{m,n}^{-i}}{B_{n,m}^{-i}} = 1, \forall n, m, k \in N, i \in I. \tag{12}$$

However it is not guaranteed that equations (11) and (12) will hold for all Nash equilibria of the game and consequently the standard PPP equation may not be valid for some equilibria of the game or exchange rates may be inconsistent. In fact in the next section, we will identify configurations of individual equilibrium strategies that fail to satisfy (11) and (12)and consequently invalidate the standard PPP equation (9) or the exchange rate consistency condition (12). This conjecture was first suggested by Koutsougeras [5] and [6] who demonstrated the failure of the law of one price in imperfectly competitive commodities markets with multiple trading posts per commodity and further developed by Koutsougeras and Papadopoulos [7] for strategic asset markets. In this model, where prices and exchange rates are determined endogenously at equilibrium, the possible failure of the PPP theory originates not only from the imperfectly competitive organization of trades in the commodities markets but also from strategic aspects in the currency market. The relative price ratio $p_{l,n}/p_{l,k}$ of a commodity traded in the two countries may not adjust completely so as to equalize to the exchange rate and on the other hand the exchange rate may not adjust completely to the relative price ratio for some commodities. The reasons for such incomplete price-currency adjustments hinge upon individual behavior, which is not negligible in this setup.

4 Characterization of Equilibria

Proposition 3 *If at a Nash equilibrium the PPP holds, then exchange rates are consistent.*

Proof. From (9) we have $t_m^n = p_{l,m}/p_{l,n}, t_n^k = p_{l,n}/p_{l,k}, t_k^m = p_{l,k}/p_{l,m}$. Multiplying we have

$$t_m^n t_n^k t_k^m = (p_{l,m}/p_{l,n})(p_{l,n}/p_{l,k})(p_{l,k}/p_{l,m}) = 1$$

■

However the converse is not necessarily true.

4.1 Equilibria with PPP Deviations

In this section, individual strategies that invalidate PPP are identified in terms of the net trades they result to. The net trade of an agent in the currency or commodity market is defined as quantity purchased minus quantity sold.

Let $z_{l,n}^i = \frac{b_{l,n}^i}{p_{l,n}} - q_{l,n}^i$ be the net trade of individual $i \in I$ for commodity $l \in L$ in country's $n \in N$ trading post[5]. Let $z_{n,k}^i = b_{k,n}^i t_n^k - b_{n,k}^i$ be the net trade of individual $i \in I$ in country's $n \in N$ currency against currency $k \in N$. If $z_{n,k}^i > 0$, then individual i is a net buyer of currency n against currency k (i.e. a net seller of currency k against currency n). The net trade of individual $i \in I$ of currency n against all other currencies $k \in N, k \neq n$, is given by

$$z_n^i = \sum_{k \in N, k \neq n} z_{n,k}^i = \sum_{k \in N, k \neq n} b_{k,n}^i t_n^k - \sum_{k \in N, k \neq n} b_{n,k}^i$$

[5]We have cross hauling when $b_{l,J}^i q_{l,J}^i > 0$ for $i \in I$.

where the first term on the r.h.s is the purchases of currency n by selling k currencies and the second term is the sales of n currency for buying k currencies.

Proposition 4 *If at a Nash equilibrium of the game either i)* $z^i_{l,n} \geq 0$, $z^i_{l,k} \leq 0$, $z^i_{n,k} \geq 0$ *or ii)* $z^i_{l,n} \leq 0$, $z^i_{l,k} \geq 0$, $z^i_{n,k} \leq 0$ *with* $z^i_{l,n}, z^i_{l,k}, z^i_{n,k}$ *not all zero for at least one* $i \in I, l \in L$ *then the Purchasing Power Parity equation (**??**) fails, i.e.* $p_{l,n} \neq p_{l,k} t^k_n$.

Proof. Since $B^{-i}_{l,n} = (Q^{-i}_{l,n} - z^i_{l,n}) p_{l,n}$ and $B^{-i}_{k,n} = (B^{-i}_{n,k} - z^i_{n,k}) t^n_k$, so we may rewrite (8) as

$$\frac{p_{l,n}}{p_{l,k} t^k_n} = \frac{(B^{-i}_{n,k} - z^i_{n,k})}{B^{-i}_{n,k}} \frac{(Q^{-i}_{l,n} - z^i_{l,n})}{Q^{-i}_{l,n}} \frac{Q^{-i}_{l,k}}{(Q^{-i}_{l,k} - z^i_{l,k})}, \forall n, k \in N, l \in L, i \in I. \quad (13)$$

Then for type i) trades we obtain

$$\frac{p_{l,n}}{p_{l,k} t^k_n} < 1$$

and the opposite inequality for type ii) trades. ■

Corollary 5 *If at a Nash equilibrium the PPP pricing is violated,* $p_{l,n} \neq t^k_n p_{l,k}$ *for some* $l \in L$*, then every agent* $i \in I$ *is trading a non-zero quantity at least in one of the following markets: the* n/k *currency market,* $z^i_{n,k} \neq 0$*, the market for commodity* l *in country* n*,* $z^i_{l,k} \neq 0$*, the market for commodity* l *in country* k*,* $z^i_{l.k} \neq 0$.

Proof. Suppose the opposite were true, that is for at least one $i \in I$, $z^i_{n,k} = z^i_{l,n} = z^i_{l,k} = 0$. Then (13) reduces to $\frac{p_{l,n}}{p_{l,k} t^k_n} = 1$. ■

We may distinguish three types of individual strategies that proposition 13 refers to: i) trades of opposite sign in the two markets for commodity l, e.g. buy l from home country and sell l abroad, ii) trades of the same sign in one market for commodity l and the currency of the country where l is located, e.g. buy l from home country and buy home currency, iii) a trade in one market for commodity l or the currency market, e.g. either buy l from one country only or just buy currency.

According to type i) strategy an agent finds it optimal at equilibrium to do speculative trade, that is export a domestic cheap commodity for a higher price abroad or import a cheap commodity for a higher domestic price. This strategy suggests an agent must be exploiting a certain arbitrage opportunity at equilibrium. The interesting feature of this speculative behavior is that it does preserve deviations from PPP at equilibrium because it is limited by the fact that it moves prices to the right direction: bidding more for the cheap commodity increases its market price while selling in the expensive market lowers its market price. Additionally, by bidding for the currency of the country where the commodity is cheaper will lower the selling price of the commodity even more. PPP failure is sustained at equilibrium because agents realize that these price effects would alter the whole value of their trades, thus resulting either to violation of the budget constraints or the individual optimality condition (**??**).

4.2 Equilibria with Inconsistent Exchange Rates

Proposition 6 *If at a Nash equilibrium for any triple of currencies $k, n, m \in N$, $z^i_{n,k} \geq 0$, $z^i_{k,m} \geq 0$, $z^i_{m,n} \geq 0$ or $z^i_{n,k} \leq 0$, $z^i_{k,m} \leq 0$, $z^i_{m,n} \leq 0$ with $z^i_{n,k}, z^i_{k,m}, z^i_{m,n}$ not all zero for at least one $i \in I$, then the exchange rates are inconsistent.*

Proof. From (13) we have for the pairs of currencies $(n,k), (k,m), (m,n)$

$$\begin{aligned}
\frac{p_{l,n}}{p_{l,k}t^k_n} &= \frac{(B^{-i}_{n,k} - z^i_{n,k})}{B^{-i}_{n,k}} \frac{(Q^{-i}_{l,n} - z^i_{l,n})}{Q^{-i}_{l,n}} \frac{Q^{-i}_{l,k}}{(Q^{-i}_{l,k} - z^i_{l,k})}, \\
\frac{p_{l,k}}{p_{l,m}t^m_k} &= \frac{(B^{-i}_{k,m} - z^i_{k,m})}{B^{-i}_{k,m}} \frac{(Q^{-i}_{l,k} - z^i_{l,k})}{Q^{-i}_{l,k}} \frac{Q^{-i}_{l,m}}{(Q^{-i}_{l,m} - z^i_{l,m})}, \\
\frac{p_{l,m}}{p_{l,n}t^n_m} &= \frac{(B^{-i}_{m,n} - z^i_{m,n})}{B^{-i}_{m,n}} \frac{(Q^{-i}_{l,m} - z^i_{l,m})}{Q^{-i}_{l,m}} \frac{Q^{-i}_{l,n}}{(Q^{-i}_{l,n} - z^i_{l,n})}.
\end{aligned}$$

By multiplying the above conditions side by side we obtain

$$t^n_k t^k_m t^m_n = \frac{(B^{-i}_{n,k} - z^i_{n,k})}{B^{-i}_{n,k}} \frac{(B^{-i}_{k,m} - z^i_{k,m})}{B^{-i}_{k,m}} \frac{(B^{-i}_{m,n} - z^i_{m,n})}{B^{-i}_{m,n}} \tag{14}$$

Given the condition of the proposition, if $z^i_{n,k} \geq 0$, $z^i_{k,m} \geq 0$, $z^i_{m,n} \geq 0$, then all terms in the r.h.s. of (14) are less than or equal to one, with at least on term being strictly less than one since $z^i_{n,k}, z^i_{k,m}, z^i_{m,n}$ are not all zero and consequently

$$t^n_k t^k_m t^m_n < 1.$$

Equivalently we have

$$t^n_k t^k_m t^m_n > 1,$$

when $z^i_{n,k} \leq 0$, $z^i_{k,m} \leq 0$, $z^i_{m,n} \leq 0$. ■

Corollary 7 *If at a Nash equilibrium an agent is doing triangular arbitrage in the currency markets, then the exchange rates are inconsistent.*

Proof. It follows directly from the proposition for $z^i_{n,k}$, $z^i_{k,m}$, $z^i_{m,n}$ all strictly positive or all strictly negative. ■

5 Asymptotic behavior of PPP deviations

(This section refers to the 2-country case where $N = \{I, J\}$. In the nest section we extend to the N-country case.)

Consider a Nash equilibrium profile $(b, q) = \{(b^i, q^i) \in \Omega^i : i \in I \cup J\}$ such that for some $l \in L, p_{l,I} \neq t^J_l p_{l,J}$. Then we know from corollary 5 that every agent in the economy is trading a non-zero quantity in at least one market for commodity l or the currency market.

Suppose without loss of generality that

$$p_{l,I} < p_{l,J} t_I^J$$

Then it is true that

$$\frac{p_{l,J} t_I^J}{p_{l,I}} - 1 > 0$$

Now define

$$f^l(b,q) \equiv \frac{p_{l,J} t_I^J}{p_{l,I}} - 1$$

We can view $f^l(b,q)$ as a measure of deviation from PPP pricing. The higher $f^l(b,q)$ is, the greater the deviation and when $f^l(b,q)$ is zero, no deviation exists. The next lemma shows that such deviation is bounded above for every agent in the economy

Lemma 8 *$f^l(b,q) \leq (1+\xi^i)^3 - 1, \forall i \in I \cup J$ where $\xi^i = \max\left\{ \frac{b^i_{J,I}}{B^{-i}_{J,I}}, \frac{b^i_{l,I}}{B^{-i}_{l,I}}, \frac{q^i_{l,J}}{Q^{-i}_{l,J}} \right\}$.*

Proof. Manipulating (**??**) we obtain

$$\frac{p_{l,J} t_I^J}{p_{l,I}} = \left(\frac{B^{-i}_{I,J}}{B_{I,J}} \frac{Q^{-i}_{l,I}}{Q_{l,I}} \frac{B^{-i}_{l,J}}{B_{l,J}} \right) \left(\frac{B_{J,I}}{B^{-i}_{J,I}} \frac{B_{l,I}}{B^{-i}_{l,I}} \frac{Q_{l,J}}{Q^{-i}_{l,J}} \right)$$

The term in the left parenthesis is obviously less than or equal to 1. So we have

$$\begin{aligned}
\frac{p_{l,J} t_I^J}{p_{l,I}} &\leq \frac{B_{J,I}}{B^{-i}_{J,I}} \frac{B_{l,I}}{B^{-i}_{l,I}} \frac{Q_{l,J}}{Q^{-i}_{l,J}} \\
&\leq \frac{B^{-i}_{J,I} + b^i_{J,I}}{B^{-i}_{J,I}} \frac{B^{-i}_{l,I} + b^i_{l,I}}{B^{-i}_{l,I}} \frac{Q^{-i}_{l,J} + q^i_{l,J}}{Q^{-i}_{l,J}} \\
&\leq \left(1 + \frac{b^i_{J,I}}{B^{-i}_{J,I}}\right) \left(1 + \frac{b^i_{l,I}}{B^{-i}_{l,I}}\right) \left(1 + \frac{q^i_{l,J}}{Q^{-i}_{l,J}}\right)
\end{aligned}$$

By the definition of ξ^i we conclude that

$$\frac{p_{l,J} t_I^J}{p_{l,I}} - 1 \leq \left(1 + \xi^i\right)^3 - 1, \forall i \in I \cup J$$

■

Once we found an upper bound for $f^l(b,q)$ which is individual specific, we shall try to associate it with the number of agents in the economy. In the sequel the symbol $| \, . \, |$ denotes the cardinal number of a set.

Proposition 9 *Consider a Nash Equilibrium (b,q) of the game such that $f^l(b,q) > 0$. Then*

$$i) \quad \forall \epsilon > 0, \quad |I| \geq \frac{3\sqrt[3]{1+\epsilon}}{-1+\sqrt[3]{1+\epsilon}} \Longrightarrow f^l(b,q) \leq \epsilon$$

$$ii) \quad \text{If } |I| > 3, \text{ then } f^l(b,q) \leq \frac{9(3 - 3|I| + |I|^2)}{(|I|-3)^3}$$

Proof. Suppose that $f^l(b,q) > \epsilon$. From lemma 8 we have that $\left(1+\xi^i\right)^3 - 1 > \epsilon$, hence $\xi^i > -1 + \sqrt[3]{1+\epsilon}, \forall i \in I \cup J$. So by the definition of ξ^i we have either $\frac{b^i_{J,I}}{B^{-i}_{J,I}} > -1 + \sqrt[3]{1+\epsilon}$ or $\frac{b^i_{l,I}}{B^{-i}_{l,I}} > -1 + \sqrt[3]{1+\epsilon}$ or $\frac{q^i_{l,J}}{Q^{-i}_{l,J}} > -1 + \sqrt[3]{1+\epsilon}, \forall i \in I \cup J$. Taking $\frac{b^i_{J,I}}{B^{-i}_{J,I}} > -1 + \sqrt[3]{1+\epsilon}$ we have $\frac{b^i_{J,I}}{B_{J,I}} > (-1 + \sqrt[3]{1+\epsilon})\frac{B^{-i}_{J,I}}{B_{J,I}} = (-1+\sqrt[3]{1+\epsilon})\left(1 - \frac{b^i_{l,I}}{B_{l,I}}\right)$ or equivalently $\frac{b^i_{J,I}}{B_{J,I}} > \frac{-1+\sqrt[3]{1+\epsilon}}{\sqrt[3]{1+\epsilon}}$. By manipulating the other two inequalities involving $b^i_{l,I}/B^{-i}_{l,I}$ and $q^i_{l,J}/Q^{-i}_{l,J}$ we conclude that

$$\forall i \in I \cup J,\ \text{either}\ \frac{b^i_{J,I}}{B_{J,I}} > \frac{-1+\sqrt[3]{1+\epsilon}}{\sqrt[3]{1+\epsilon}}\ \text{or}\ \frac{b^i_{l,I}}{B^{-i}_{l,I}} > \frac{-1+\sqrt[3]{1+\epsilon}}{\sqrt[3]{1+\epsilon}}\ \text{or}\ \frac{q^i_{l,J}}{Q_{l,J}} > \frac{-1+\sqrt[3]{1+\epsilon}}{\sqrt[3]{1+\epsilon}}.$$

Define

$$\begin{aligned}
V_t &= \left\{ i \in I \cup J : \frac{b^i_{J,I}}{B_{J,I}} > \frac{-1+\sqrt[3]{1+\epsilon}}{\sqrt[3]{1+\epsilon}} \right\}, \\
V_b &= \left\{ i \in I \cup J : \frac{b^i_{l,I}}{B_{l,I}} > \frac{-1+\sqrt[3]{1+\epsilon}}{\sqrt[3]{1+\epsilon}} \right\}, \\
V_q &= \left\{ i \in I \cup J : \frac{q^i_{l,J}}{Q_{l,J}} > \frac{-1+\sqrt[3]{1+\epsilon}}{\sqrt[3]{1+\epsilon}} \right\}.
\end{aligned}$$

We have associated the above sets of individuals to commodity l which traded in the trading post of each country. Then it is true that

$$\begin{aligned}
|V_t|\frac{-1+\sqrt[3]{1+\epsilon}}{\sqrt[3]{1+\epsilon}} &< \sum_{i\in V_t}\frac{b^i_{J,I}}{B_{J,I}} \le 1 \Rightarrow |V_t| < \frac{\sqrt[3]{1+\epsilon}}{-1+\sqrt[3]{1+\epsilon}} \\
|V_b|\frac{-1+\sqrt[3]{1+\epsilon}}{\sqrt[3]{1+\epsilon}} &< \sum_{i\in V_b}\frac{b^i_{l,I}}{B_{l,I}} \le 1 \Rightarrow |V_b| < \frac{\sqrt[3]{1+\epsilon}}{-1+\sqrt[3]{1+\epsilon}} \\
|V_q|\frac{-1+\sqrt[3]{1+\epsilon}}{\sqrt[3]{1+\epsilon}} &< \sum_{i\in V_q}\frac{q^i_{l,J}}{Q_{l,J}} \le 1 \Rightarrow |V_q| < \frac{\sqrt[3]{1+\epsilon}}{-1+\sqrt[3]{1+\epsilon}}
\end{aligned}$$

and since every agent is either trading commodity l in at least one trading post or the currency market, then

$$I = V_t \cup V_b \cup V_q$$

and consequently

$$\begin{aligned}
|I| &\le |V_t| + |V_b| + |V_q| \Rightarrow \\
|I| &< \frac{3\sqrt[3]{1+\epsilon}}{-1+\sqrt[3]{1+\epsilon}} \qquad (15)
\end{aligned}$$

So if $f^l(b,q) > \epsilon$, then (15) is true. Then,

$$\text{if} \quad |I| \geq \frac{3\sqrt[3]{1+\epsilon}}{-1+\sqrt[3]{1+\epsilon}} \Longrightarrow f^l(b,q) \leq \epsilon$$

which proves part *i)* of the theorem.

Given that $|I| > 3$, we solve $|I| \geq \frac{3\sqrt[3]{1+\epsilon}}{-1+\sqrt[3]{1+\epsilon}}$ for ϵ and we conclude that

$$\text{if} \quad \frac{9(3-3\,|I|+|I|^2)}{(|I|-3)^3} \leq \epsilon \Rightarrow f^l(b,q) \leq \epsilon$$

■

Now take any sequence of market games Γ^n where $|I^n| \to \infty$, and a sequence of corresponding equilibria $(b^n, q^n) \in NE(\Gamma^n)$.

Corollary 10 $|I^n| \to \infty \Rightarrow f^l(b^n, q^n) \to 0$.

Proof. From proposition (9) we have that

$$f^l(b^n, q^n) \leq \frac{9(3-3\,|I^n|+|I^n|^2)}{(|I^n|-3)^3}$$

so as $|I^n| \to \infty$, $f^l(b^n, q^n) \to 0$. ■

5.1 Asymptotic behavior of PPP deviations and inconsistent exchange rates

If PPP deviations tend to vanish as the number of agents tends to infinity, we should expect from proposition (3) that at the limit exchange rates will be consistent[6]. Here we shall extend the 2-country asymptotic convergence result to multiple countries. The proof is similar to the 2-country case so it is omitted.

Consider a Nash equilibrium profile $(b,q) = \{(b^i, q^i) \in \Omega^i : i \in I\}$. A commodity $l \in L$ is traded in N different countries. Given its price in some country, we may have at most $N-1$ deviations from PPP. Without loss of generality suppose that $p_{l,1} = \min_{n \in N} p_{l,n}$ and define $f^l(b,q)$

$$f^l(b,q) \equiv \sup_{n \in N} \frac{p_{l,n} t_1^n}{p_{l,1}} - 1$$

Proposition 11 *Consider a Nash Equilibrium* (b,q) *of the game such that* $f^l(b,q) > 0$ *and* $N \geq 2$. *Then*

$$i) \quad \forall \epsilon > 0, \quad |I| \geq \frac{(2|N|-1)\sqrt[3]{1+\epsilon}}{-1+\sqrt[3]{1+\epsilon}} \Longrightarrow f^l(b,q) \leq \epsilon$$

$$ii) \quad \text{If} \quad |I| > 2|N|-1, \; \text{then } f^l(b,q) \leq \frac{|I|^3}{(|I|-2|N|+1)^3} - 1$$

[6] Independently from PPP deviations, one could construct a measure of exchange rate inconsistency and show that it tends to 0 as the number of agents increases.

Now take any sequence of market games Γ^s where $|I^s| \to \infty$, and a sequence of corresponding equilibria $(b^s, q^s) \in NE(\Gamma^s)$.

Corollary 12 $|I^s| \to \infty \Rightarrow f^l(b^s, q^s) \to 0$.

Notice that when $N = 2$, the proposition reduces to the 2-country case.

The size of PPP deviations is positively related to the number of currencies traded. For example, the participation of some countries in a monetary union i.e. adoption of common currency, results in less currency markets available and decreases the size of PPP deviations.

6 Conclusion

The main objective of this paper was to show that in international markets, imperfect competition *per se*, without market frictions or market segmentation, may be another cause for the failure of PPP and and may result to inconsistent exchange rates when multiple currencies are being traded. By increasing the number of agents in the international economy without necessarily replicating their characteristics, PPP deviations and exchange rate inconsistencies tend to vanish. At the limit, PPP holds and exchange rates are consistent even if the economy is not competitive. This provides a foundation of PPP theory in international competitive market models where by construction the number of agents is infinite.

References

[1] AMIR, R., S. SAHI, M. SHUBIK and S. YAO (1990), "A strategic market game with complete markets", *Journal of Economic Theory* , 51, 126-143.

[2] DUBEY, P. and M. SHUBIK (1978), "A theory of money and financial institutions. 28. The noncooperative equilibria of a closed trading economy with market supply and bidding strategies", *Journal of Economic Theory* , 17, 1-20.

[3] GIRAUD, G. (2003), "Strategic market games: an introduction", *Journal of Mathematical Economics*, 39, 355-375.

[4] GOLDBERG, P.K. and M.M. KNETTER (1997), "Goods prices and exchange rates: what have we learned", *Journal of Economic Literature*, 35, 1243-1272.

[5] KOUTSOUGERAS, L. C. (2003a), "Non-Walrasian equilibria and the law of one price", *Journal of Economic Theory,* 108, 169-175.

[6] KOUTSOUGERAS, L. C. (2003b), "Convergence to no arbitrage equilibria in market games", *Journal of Mathematical Economics*, 39, 401-420.

[7] KOUTSOUGERAS, L. C. and K.G. PAPADOPOULOS (2004), "Arbitrage and equilibrium in strategic security markets", *Economic Theory*, 23, 553-568

[8] KRUGMAN, P. R (1987), "Pricing to market when the exchange rate changes", in *Real-financial linkages among open economies*. Eds.: SVEN W. ARNDT and J. DAVID RICHARDSON. Cambridge, MA: MIT Press, 49-70.

[9] MAS-COLELL, A. (1982), "The Cournotian foundations of Walrasian equilibrium theory: an exposition of recent theory", in *Advances in Economic Theory*. Ed.: HILDENBRAND, W., Cambridge University Press, Cambridge, 183-224.

[10] POSTLEWAITE, A. and D. SCHMEIDLER (1978), "Approximate Efficiency of Non-Walrasian Nash Equilibria", *Econometrica*, 46, 127-135.

[11] ROGOFF, K. (1996), "The Purchasing Power Parity Puzzle", *Journal of Economic Literature*, 34, 647-668.

[12] SHAPLEY, L. and M. SHUBIK (1977), "Trade Using One Commodity as a Means of Payement", *Journal of Political Economy*, 85, 937-968.

[13] SHUBIK, M. and C. WILSON (1977), "The optimal bankruptcy rule in a trading economy using fiat money", *Zeitschrift fur Nationaloekonomie*, 37, 337-354.