A characterization of Bird's rule

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Abstract

We propose a new test for cost allocation rules in minimum cost spanning tree games. The Shapley value, as well as some recent rules proposed in the literature, i.e. the equal remaining obligation rule (Feltkamp, Tijs and Muto (1994)) or the Dutta-Kar rule (Dutta and Kar, 2004), are vulnerable to merging maneuvers by users. Bird's rule, on the other hand, passes this test. We give a characterization of Bird's rule based on this property.

Key words: merging, Bird's rule, tree invariance, irreducible core.

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1 Introduction

Consider the issue of providing a service to a group of villages, where each village needs to be connected directly or through other villages to a source. Constructing a service link between villages is costly, so the first challenging question is to find the cheapest network that spans all the villages, i.e. a minimum cost spanning tree. Chezh mathematician Borůvka (1927) proposed an algorithm to achieve this goal.[2] Later, Jarnik (1930) formulated another algorithm which, in the literature in economics, came to be attributed to Prim (1957) and Dijkstra (1959). Kruskal (1956) also found a similar algorithm.

There is a wide literature in operations research concerning the various computational properties of these algorithms. Most of the recent research on this topic in economics, however, concerns itself with a different question: given a minimum cost spanning tree situation, how should we allocate the costs of the network to the villages?

In this paper, we consider *merging* maneuvers by groups of users. Let N denote the set of users, and consider a group of users $S \subset N$. A merging maneuver by S occurs when the users in S choose a user $i^* \in S$ to represent them in the minimum cost spanning tree game, while constructing a private network among themselves, the costs of which are borne only by S. If the cost allocation of the representative user i^* in the merged game plus the cost of this private network is less than the cost allocation of S before merging, we say that this merging maneuver is beneficial for S.

Claus and Kleitman (1973) introduced the problem of allocating costs on a minimum cost spanning tree, whereas Bird (1976) treated it with game theoretic methods. Bird proposed a specific cost allocation for the problem, closely related to the Jarnik-Prim-Dijkstra algorithm, which is now known as *Bird's rule*. Bird also showed that the proposed allocation lies in the core of a cooperative game derived from the minimum cost spanning tree problem. This game is defined as follows: the players are the villages, and the worth of a coalition is the minimal cost of connecting this coalition to the source, using only the links between the members of this coalition. One approach is to look at the *Shapley value* of this game. However, the Shapley value will frequently be an allocation outside the core.

A problem with Bird's rule is that it does not satisfy *cost monotonicity*: an increase in the cost of one of the edges of a user might decrease her cost allocation. Based on this observation, Dutta and Kar (2004) proposed a new rule that satisfies cost monotonicity, and lies in the core of the stand alone cost game. They give characterizations of this rule and Bird's rule, and argue that the main difference between these rules is in their consistency properties.

We criticize the *Dutta-Kar rule* and the Shapley value for minimum cost spanning tree games based on the fact that they are vulnerable to merging

[2] Borůvka was asked to solve this problem in relation to the issue of providing an electricity network over locations in Bohemia, Borůvka (1926b).

by coalitions. Bird's rule, on the other hand, is invulnerable to merging. We provide a characterization of Bird's rule based on this property.

The idea of the *irreducible core*, originally developed by Bird (1976), receives a lot of attention in the literature. See, for example, Feltkamp, Tijs and Muto (1994) and Bergantiños and Vidal-Puga (2004). In light of the results of Feltkamp, Tijs and Muto (1994), our main result implies that none of the rules that are selections from the irreducible core pass the merge-proofness test.

2 Preliminaries

Let $\mathcal{N}$ denote the finite set of potential users. $0 \notin \mathcal{N}$ will be given the special role of *source* or *root*. Given $N \subset \mathcal{N}$, we let $N_0 = N \cup \{0\}$. A *network* on $S \subset \mathcal{N} \cup \{0\}$ is a graph with node set S. A link in this graph is of the form (ij) for some $i, j \in S$. We denote a generic network on S by g_S. A graph g_S is said to be connected over S if, for all $i, j \in S$, i and j are connected in graph g_S, i.e. there is a path from i to j. We denote by Γ_S the set of connected graphs on S.

Given $N \in \mathcal{N}$, a cost matrix $c = [c_{ij}]_{i,j \in N_0}$ represents the cost of direct connection between users i and j. We assume that $c_{ij} > 0$ if $i \neq j$ and $c_{ii} = 0$ for all $i \in N_0$. The set of all cost matrices for N_0 is denoted by C_N. We let $\mathcal{C} = \cup_{N \subset \mathcal{N}} C_N$.

We assume that all users have to be connected to the network, so we want to construct a connected graph on N_0. Of particular importance are efficient networks, i.e. connected graphs with minimal cost. Such a graph cannot have any cycles, otherwise we could remove an edge from this cycle and still have a (cheaper) connected graph. Hence, the efficient network must be a tree. A minimum cost spanning tree (m.c.s.t. from now on) over N_0, denoted by $g[N_0, c]$, satisfies $g[N_0, c] = \arg\min_{g \in \Gamma_{N_0}} \sum_{(ij) \in g} c_{ij}$. Also, given $S \subset N_0$, $g[S, c_S]$ denotes the m.c.s.t. on users S, where c_S is the restriction of c to S. For all $S \subset N_0$, we define $m(S, c_S) = \min_{g \in \Gamma_S} \sum_{(ij) \in g} c_{ij} = \sum_{(ij) \in g[N, c_S]} c_{ij}$.

Given $N \subset \mathcal{N}$ and $c \in C_N$, we call (N, c) a m.c.s.t. problem. Note that there might be several m.c.s.t. associated with a m.c.s.t. problem. The familiar issue of actually constructing the set of m.c.s.t of a m.c.s.t problem will be dealt with in the next section. Assuming that we constructed an efficient network, we are interested in sharing the cost of the network among the users. This is achieved by a cost allocation rule.

Definition 1 *A cost allocation rule is a family of functions $\{\phi(N, \cdot)\}_{N \subset \mathcal{N}}$ such that, for all $N \subset \mathcal{N}$, $\phi(N, \cdot) : C_N \longrightarrow \mathbb{R}_+^N$ satisfies $\sum_{i \in N} \phi_i(N, c) = m(N_0, c)$ for all $c \in C_N$.*

3 Jarnik-Prim-Dijkstra Algorithm and Related Solutions

Given $N \subset \mathcal{N}$, let $\sigma : N \to \{1, 2, ..., |N|\}$ denote a priority ordering of the users in N, i.e. i is before j in the priority ordering σ if and only if $\sigma(i) < \sigma(j)$. We let $\Phi(N)$ denote the set of priority orderings of N. The Jarnik-Prim-Dijkstra algorithm (from now on J-P-D algorithm) reaches an m.c.s.t in the following manner.

Step 0 $S^0 = \{0\}$ and $g^0 = \emptyset$.

Step 1 Take the arc $\left(0i^1\right)$ such that $c_{0i^1} = \min_{i \in N} \{c_{0i}\}$. If there are several arcs satisfying this condition, choose $\left(0i^1\right)$ where $i^1 = \arg\min_{i \in N} \sigma(i)$. Let $S^1 = \left\{0, i^1\right\}$ and $g^1 = \left\{\left(0i^1\right)\right\}$.

Step $p+1$ Take an arc $\left(ji^{p+1}\right)$ such that $c_{ji^{p+1}} = \min_{k \in S^p, l \in N \setminus S^p} \{c_{kl}\}$. If there are several arcs satisfying this condition, choose $\left(0i^{p+1}\right)$ where $i^{p+1} = \arg\min_{i \in N \setminus S^p} \sigma(i)$. Let $S^{p+1} = S^p \cup \left\{i^{p+1}\right\}$ and $g^{p+1} = g^p \cup \left\{\left(ji^{p+1}\right)\right\}$.

This algorithm terminates after $|N|$ steps. Note that different priority orderings may lead to different minimum cost spanning trees. We write $g^\sigma[N_0, c]$ to denote the m.c.s.t resulting from the priority ordering σ. The total cost of $g^\sigma[N_0, c]$ is the same for all $\sigma \in \Phi(N)$ (see Prim (1956)). It follows that when we have a unique m.c.s.t, $g^\sigma[N_0, c]$ is independent of σ.

Bird (1976) studied cost allocation on a minimum cost spanning tree and proposed what is now referred to as *Bird's rule*, a rule that is based on the J-P-D algorithm. To introduce this rule, we need some more notation, which follows the notation in Dutta and Kar (2004).

Given a tree g, the (unique) path from i to j is a set $U^{(N,c)}(i,j,g) = \{i_1, i_2, ..., i_K\}$, where $(i_{k-1}i_k) \in g$ for all $k = 1, 2, .., K$ and $i_1, i_2, ..., i_K$ are all distinct users with $i_1 = i$ and $i_K = j$. The predecessor set of user i in g is defined as $P^{(N,c)}(i,g) = \left\{k \in N_0 | k \neq i, k \in U^{(N,c)}(0,i,g)\right\}$. The immediate predecessor of user i, denoted by $\alpha^{(N,c)}(i,g)$ is the user that comes right before i : $\left(\alpha^{(N,c)}(i,g), i\right) \in g$ and $\alpha^{(N,c)}(i,g) \in P^{(N,c)}(i,g)$. The successors of user i are those users that come after i : $S^{(N,c)}(i,g) = \left\{k \in N | i \in P^{(N,c)}(k,g)\right\}$. The immediate successors of user i is denoted by $\beta^{(N,c)}(i,g) = \left\{k \in N | \alpha^{(N,c)}(k,g) = i\right\}$. When there is no room for confusion, we will drop the superscript (N,c) and the argument of g.

Given a priority ordering σ, Bird's allocation on the m.c.s.t $g^\sigma[N,c]$ is given by

$$B_i^\sigma(N,c) = c_{\alpha(i, g^\sigma[N,c])i} \ \forall i \in N.$$

When there are several m.c.s.t, Bird takes averages over the priority orderings according to which ties are broken in the J-P-D algorithm.

$$B(N,c)=\sum_{\sigma\in\Phi(N)}\frac{1}{|N|!}B^{\sigma}(N,c).$$

When there is no room for confusion, we write B_i instead of $B_i(N,c)$. Also, we use $B_i^{\sigma}(N,c)$ to denote $c_{\alpha(i,g^{\sigma}[N_0,c])i}$ for notational ease.

A merging maneuver by a group of users $S\subset N$ happens in the following manner: The users in S merge into $i^*\in S$ and the new problem becomes (N^*,c^*) with $N^*=N\backslash S\cup\{i^*\}$, $c^*_{ij}=c_{ij}$ for $\{i,j\}\subset N^*\backslash i^*$ and $c^*_{i^*j}=\min_{i\in S}c_{ij}$ for all $j\in N^*\backslash i^*$.

When a merging maneuver occurs, the users in S construct a private network among themselves, and then they connect to the network through the merged user i^*. Accordingly, when S merges, they have to pay for the cost share of i^* in the merged problem (N^*,c^*), and they pay for the cost of $g[S,c_S]$, which we defined to be $m(S,c_S)$. We are now ready to give the following definition.

Definition 2 *A cost allocation rule ϕ is vulnerable to merging if there exist $N\subset\mathcal{N}$,$c\in C_N, S\subset N$, and $i^*\in S$ such that*

$$\sum_{i\in S}\phi_i(N,c)>\phi_{i^*}(N^*,c^*)+m(S,c_S).\tag{1}$$

We say that ϕ is merge-proof if it is not vulnerable to merging.

The lemma below shows that merge-proofness is a very strong property.

Lemma 1 *On $\mathcal{C}$, there is no merge-proof cost allocation rule.*

Proof. Take a cost allocation rule ϕ and a m.c.s.t. problem (N,c) such that $N=\{1,2,3\}$, $c_{0i}=4$ for $i=1,2,3$ and $c_{ij}=1$ for all $i,j\in N, i\neq j$. Now, the cost of a m.c.s.t is $m(N,c)=6$. Let $\phi=\phi(N,c)$, and suppose that ϕ is merge-proof.

Suppose $\phi_1\leq 2$. Let (N^2,c^2) and (N^3,c^3) denote the problems where the coalition $S=\{2,3\}$ merges into 2 and 3, respectively. Let $x=\phi(N^2,c^2)$ and $y=\phi(N^3,c^3)$. Note that in (N,c), the cost allocation to S, $\phi_1+\phi_2$, is at least 4. The cost of the private network on S is 1. Therefore, for ϕ to be merge-proof, we must have $x_2\geq 3$ and $x_1\leq 2$. A similar argument yields $y_3\geq 3$ and $y_1\leq 2$. Now let i^* denote the user in S who has the higher cost allocation in the problem (N,c) : $\phi_{i^*}=\max\{\phi_2,\phi_3\}\geq\frac{6-\phi_1}{2}$. If the $\{1,i^*\}$ coalition merges into 1, they pay the cost of their private network, which is 1, and the cost allocation of user 1 in problem $(N^{S\backslash i^*},c^{S\backslash i^*})$, which is not bigger than 2. This move is beneficial for $\{1,i^*\}$ as their cost allocation in the original problem (N,c) is at least $\frac{6+\phi_1}{2}$. Therefore, we must have $\phi_1>2$, a contradiction. ■

This lemma implies that, if we are to insist on merge-proofness, we have to restrict the domain of cost matrices. In this paper, we use the following domain restriction:

Definition 3 *For all $N \subset \mathcal{N}$, $C_N^1 = \{c \in C_N | c$ induces a unique m.c.s.t. on $N_0\}$. We write $\mathcal{C}^1 = \cup_{N \subset \mathcal{N}} C_N^1$.*

We assume that any group of users have free access to the source. In other words, they can always break away from the grand coalition and construct their own network to connect to the source, using only their own links. Hence, we look at the stand alone cost of a coalition and explore core stability. Accordingly, the cost of coalition $S \subset N$ is defined to be the cost of a m.c.s.t. on $S \cup \{0\}$, i.e. $m(S_0, c_{S_0})$. We say that a cost allocation $z \in \mathbb{R}^N$ is in the core of the m.c.s.t problem (N, c) if, $\sum_{i \in N} z_i = m(N_0, c_{N_0})$ and for all $S \subset N$, we have $\sum_{i \in S} z_i \leq m(S_0, c_{S_0})$. We write $Core(N, c)$ to denote the set of cost allocations that are in the core of (N, c).

Definition 4 *A cost allocation rule ϕ satisfies core selection (CS) if, for all $N \subset \mathcal{N}, c \in C_N$, $\phi(N, c) \in Core(N, c)$.*

The following property for cost allocation rules suggests that we only use the information from the minimum cost spanning tree while allocating costs.

Definition 5 *A cost allocation rule ϕ satisfies tree invariance (TI) if, for all $N \subset \mathcal{N}$ and $c, \bar{c} \in C_N^1$ such that c and $\bar{c}$ have the same m.c.s.t*

$$\phi_i(N, c) = \phi_i(N, \bar{c}) \text{ for all } i \in N.$$

We are now ready to state our main result.

Theorem 1 *On $\mathcal{C}^1$, the cost allocation rule ϕ satisfies MP, CS, and TI if and only if $\phi = B$.*

Proof. We already know B satisfies CS (Bird, 1976). Also, Bird's rule is defined using information only from the m.c.s.t. of a m.c.s.t. problem, so that it also satisfies TI. We now show that B satisfies MP. To this purpose, fix $N \subset \mathcal{N}$ and $c \in C_N^1$..

Take any $S \subset N$. Suppose S merges into $i^* \in S$. The new problem becomes (N^*, c^*) as defined above. Let $B^* = B(N^*, c^*)$. Choose $\sigma^* \in \Phi(N^*)$ such that $B_{i^*}^{\sigma^*}(N^*, c^*) = B_{i^*}^{\sigma^*} = \min_{\sigma \in \Phi(N^*)} B_{i^*}^{\sigma}(N^*, c^*)$. Let $i^1 \in S$ be such that $c_{(\alpha(i^*) i^1)} = c_{(\alpha(i^*) i^*)}$. Now consider an ordering σ of N such that, for all i with $\sigma^*(i) < \sigma^*(i^*)$, $\sigma(i) = \sigma^*(i)$ and $\sigma(i^1) = \sigma^*(i^*)$. Let $B = B^{\sigma}(N, c) = B(N, c)$.

Now let τ and τ^* denote the orderings according to which the users in N and N^* are added to the networks $g^{\sigma}[N_0, c]$ and $g^{\sigma^*}[N_0^*, c^*]$ according to the J-P-D algorithm, respectively. Note that none of the comparisons made by the J-P-D algorithm are affected by the merging maneuver, until the step at which the merged user i^* is added to the network. This implies that, for all users with $\tau^*(i) < \tau^*(i^*)$, we have $\tau(i) = \tau^*(i)$ and $\tau(i^1) = \tau^*(i^*)$. Hence, $B_{i^1} = B_{i^*}^{\sigma^*}$. It follows from this argument that in graph $g^{\sigma}[N_0, c]$, none of the users in $S \backslash i^1$ lie on the path from 0 to i^1.

We showed that $B_{i^1} = B_{i^*}^{\sigma^*} \leq B_{i^*}^*$. A sufficient condition for the merging move not to be beneficial is

$$\sum_{i \in S \setminus i^1} B_i \leq m(S, c_S) \tag{2}$$

Now let $g^1 = \{(\alpha(i)\, i)\}_{i \in N \setminus S \cup i^1}$ and consider the graph $g = g^1 \cup g[S, c_S]$. Note that g^1 is a subgraph of $g[N_0, c]$ and is therefore a forest. Let $G_1, G_2, ..., G_K$ be the set of disjoint trees in $g^1 : G_k \cap G_l = \emptyset$ for all $k, l = 1, 2, ..., K, k \neq l$, and $\cup_{k=1}^K G_k = g^1$. Without loss of generality, assume that $i^1 \in G_1$. This implies that $0 \in G_1$ as well, since we argued above that $\nexists j \in S$ such that $j \in U(0, i^1, g[N_0, c])$. Now, as all users in $S \setminus i^1$ are connected to i^1 through the network $g[S, c_S]$, they are also connected to 0 in g. By construction, for all $k = 2, 3, ..., K$ there exists a user j_k in tree G_k such that $\alpha(j_k) \in S \setminus i^1$. This in turn implies that all users in $\cup_{k=2}^K G_k$ are connected to 0 in graph g.

Hence, g is a connected graph with $n+1$ vertices and n edges. This implies that g must be a tree, and the cost of g must be no less than the cost of the m.c.s.t. on N_0, i.e.

$$\sum_{i \in N \setminus S \cup i^1} B_i + m(S, c_S) \leq \sum_{i \in N \setminus S \cup i^1} B_i + \sum_{i \in S \setminus i^1} B_i.$$

This immediately gives us the desired inequality (2), establishing that B is merge-proof.

To show the only if part, take a cost allocation rule ϕ that satisfies MP, CS, and TI. We use an induction argument on $|N|$ to show that $\phi = B$. It is clear that ϕ coincides with B on problems with $|N| = 1$.

Induction Argument 1: Assume that ϕ coincides with B on problems with $|N| \leq K$, where $1 \leq K$.

Take a m.c.s.t. problem (N, c) with $|N| = K+1$, and let $\phi = \phi(N, c)$. We first show that for all users $i \in N$ such that $S(i) = \emptyset$ and $\alpha(i) = 0$, we have $\phi_i = B_i$. This is because CS implies $\phi_i \leq B_i$ and $\sum_{j \in N \setminus i} \phi_i \leq \sum_{j \in N \setminus i} B_i$. Recalling $\sum_{j \in N} \phi_i = \sum_{j \in N} B_i$, it follows that $\phi_i = B_i$.

Next, we will use an induction argument on the number of users in $S(i)$ for a non-leaf user i to show that $\phi = B$. First note that for any user $i \in N$, CS implies $\sum_{j \in i \cup S(i)} \phi_j \geq \sum_{j \in i \cup S(i)} B_j$, since $m((N \setminus S(i))_0) = \sum_{j \in N \setminus S} B_j$.

Now consider a non-leaf user i. When the coalition $i \cup S(i)$ merges into $\{i\}$, the new problem is one with less than K players and hence by the Induction Argument 1, the merged user pays B_i in the new problem. Now, as the cost of the spanning tree on $i \cup S(i)$ is $\sum_{j \in S(i)} B_j$, MP implies that we must have $\sum_{j \in i \cup S(i)} \phi_j \leq \sum_{j \in i \cup S(i)} B_j$ as well, establishing the following observation.

Observation 1: $\sum_{j \in i \cup S(i)} \phi_j = \sum_{j \in i \cup S(i)} B_j$ for all non-leaf users $i \in N$.

Now consider a non-leaf user i such that $|S(i)| = 1$. Let $S(i) = \{i_1\}$.

Let $a = B_i$ and $b = B_{i_1}$. Now, Observation 1 implies $\phi_i + \phi_{i_1} = a + b$.

As i_1 is a leaf, $S(i_1) = \emptyset$ and CS implies that $\phi_{i_1} = b + x$ for some $x \geq 0$. Suppose $x > 0$. Take a user $i_2 \notin N$, and consider the problem $(\bar{N}, \bar{c})$, where

$\bar{N} = N \cup \{i_2\} \setminus \{i_1\}, \bar{c}_{ij} = c_{ij}$ for all $\{i,j\} \subset \bar{N} \setminus \{i_2\}$, $\bar{c}_{ji_2} = c_{ji_1}$ for all $j \in \bar{N} \setminus \{i\}$, and $\bar{c}_{ii_2} = d < b$. Check that $\bar{c} \in C^1_N$, and that $\beta^{(\bar{N},\bar{c})}(i) = \{i_2\}$. Let $\bar{\phi} = \phi(\bar{N}, \bar{c})$. CS implies $\bar{\phi}_{i_2} = d + y$ for some $y \geq 0$, and CS and MP imply $\bar{\phi}_i + \bar{\phi}_{i_2} = a + d$.

Next, consider the problem $\left(\tilde{N}, \tilde{c}\right)$, where $\tilde{N} = N \cup \{i_2\}, \tilde{c}_{ij} = c_{ij}$ for all $\{i,j\} \subset \tilde{N} \setminus \{i_2\}$, $\tilde{c}_{ji_2} = \bar{c}_{ji_2}$ for all $j \in \tilde{N} \setminus \{i_1\}$ and $c_{i_1 i_2} = b + \varepsilon$ for some $0 < \varepsilon < x$. Check that $\beta^{(\tilde{N},\tilde{c})}(i) = \{i_1, i_2\}$ and $\{i_1, i_2\}$ are leaves in graph $\mathrm{g}\left[\tilde{N}_0, \tilde{c}\right]$.

Let $\tilde{\phi} = \phi\left(\tilde{N}, \tilde{C}\right)$ and note that Observation 1 implies $\tilde{\phi}_i + \tilde{\phi}_{i_1} + \tilde{\phi}_{i_2} = a + b + d$. Now, suppose $\tilde{\phi}_{i_1} < b + x$, which implies $\tilde{\phi}_i + \tilde{\phi}_{i_2} > a - x + d$. Consider a merging move by users $\{i, i_2\}$ into $\{i\}$.The m.c.s.t. corresponding to the resulting problem is the same as that of the original problem (N, c) and so by TI, user $\{i\}$ pays $a - x$ in the merged problem. The $\{i, i_2\}$ coalition pays this plus the cost of their private network, $c_{ii_2} = d$. Hence, $\{i, i_2\}$ coalition pays $a - x + d$ when they merge, a contradiction to MP. Hence, $\tilde{\phi}_{i_1} \geq b + x$.

By a similar argument to the one above, we can also show $\tilde{\phi}_{i_2} \geq d + y$. Now, suppose $\{i_1, i_2\}$ coalition merges into $\{i_2\}$. Again, the m.c.s.t. corresponding to the resulting problem is the same as that of the problem $\left(\bar{N}, \bar{c}\right)$ and so by TI, user $\{i_2\}$ pays $d + y$ in the merged problem. The cost of the private network $(i_1 i_2)$ is $b + \varepsilon$. Hence, the coalition $\{i_1, i_2\}$ when they merge pays $d + y + b + \varepsilon < d + y + b + x \leq \tilde{\phi}_{i_1} + \tilde{\phi}_{i_2}$. This is a contradiction to MP, establishing that $\phi_{i_1} = b$, and since $\phi_i + \phi_{i_1} = a + b$, $\phi_i = a$.

Induction Argument 2: Fix a positive integer $M \leq K - 1$ and assume that, for all users $j \in N$ such that $S(j)$ contains at most M users, $\phi_k = B_k$ for all $k \in j \cup S(j)$.

Consider a user $i \in N$ such that $S(i)$ contains $M + 1$ users. We consider three cases, and make use of the following observation, which follows from the construction of a m.c.s.t. using the J-P-D algorithm.

Observation 2: Given (N, c), take $i_1, i_2 \in N$ such that $(i_1 i_2) \notin g[N_0, c]$. Consider the problem $(N, \bar{c})$ where $\bar{c}_{ij} = c_{ij}$ for $\{i,j\} \neq \{i_1, i_2\}$ and $\bar{c}_{i_1 i_2} > \max\{B_{i_1}, B_{i_2}\}$. We have $g[N_0, c] = g[N_0, \bar{c}]$.

Case 1. Everyone in $\beta(i)$ is a non-leaf user. Let $\beta(i) = \{i_1, i_2, \cdots, i_m\}$. Now, by Induction Argument 2, for all $l = 1, 2, ..., m$, we have $\phi_j = B_j$ for all $j \in i_l \cup S(i_l)$. Therefore,

$$\sum_{j \in S(i)} \phi_j = \sum_{l=1}^{m} \sum_{j \in i_l \cup S(i_l)} \phi_j = \sum_{l=1}^{m} \sum_{j \in i_l \cup S(i_l)} B_j = \sum_{j \in S(i)} B_j$$

Since $\sum_{j \in i \cup S(i)} \phi_j = \sum_{j \in i \cup S(i)} B_j$ by Observation 1, we have $\phi_i = B_i$.

Case 2. Everyone in $\beta(i)$ is a leaf. Let $\beta(i) = \{i_1, i_2, ..., i_m\}, m \geq 2$.

Let $a = B_i$ and assume, without loss of generality, that $B_{i_1} \leq B_{i_2} \leq \cdots \leq B_{i_m}$. For each $i_k \in \beta(i)$, CS implies $\phi_{i_k} = B_{i_k} + x_k$ for some $x_k \geq 0$. Now, suppose there exists $k \in \{1, 2, ..., m\}$ such that $x_k > 0$. Take any $i_l \neq i_k$

and consider the problem $(N,\bar{c})$ where $\bar{c}_{ij}=c_{ij}$ for all $\{i,j\}\neq\{i_k,i_l\}$, and $\bar{c}_{i_k i_l}=\max\{B_{i_k},B_{i_l}\}+\varepsilon$ for some $0<\varepsilon<x_k$. By Observation 2, the m.c.s.t corresponding to the problem $(N,\bar{c})$ is the same as that of (N,c). This implies that $\phi(N,\bar{c})=\phi(N,c)=\phi$. When the coalition $\{i_k,i_l\}$ merges into $i_{\arg\min\{k,l\}}$, they have to construct the egde $(i_k i_l)$ which costs $\max\{B_{i_k},B_{i_l}\}+\varepsilon$. The merged user pays his Bird allocation $\arg\min\{B_{i_k},B_{i_l}\}$ by Induction Argument 1. Now MP implies $\min\{B_{i_k},B_{i_l}\}+\max\{B_{i_k},B_{i_l}\}+\varepsilon=B_{i_k}+B_{i_l}+\varepsilon\geq\phi_1+\phi_k=B_{i_k}+B_{i_l}+x_k$, a contradiction. This, together with Observation 1, implies that $\phi_i=B_i$.

Case 3. $\beta(i)=\{i_1,i_2,\cdots,i_m,j_1,j_2,\cdots,j_n\}$**, where everyone in** $\{i_1,i_2,\cdots,i_m\}$ **is a non-leaf user and everyone in** $\{j_1,j_2,\cdots,j_n\}$ **is a leaf.** For all $l=1,2,...m$, $\beta(i_l)$ contains at most M non-leaf users, and hence by Induction Argument 2, $\phi_j=B_j$ for all $j\in i_l\cup S(i_l)$. For every $l=1,2,...,n$, CS implies $\phi_{j_l}=B_{j_l}+x_l$ for some $x_l\geq 0$. Now suppose there exist $j_l\in\{j_1,j_2,\cdots,j_n\}$ such that $\phi_{j_l}=B_{j_l}+x_l$ for some $x_l>0$. Next. consider the problem $(N,\bar{c})$ where $\bar{c}_{ij}=c_{ij}$ for all $\{i,j\}\neq\{i_1,j_l\}$ and $\bar{c}_{i_1 j_l}=\max\{B_{i_1},B_{j_l}\}+\varepsilon$ for some $0<\varepsilon<x_l$. We know that the m.c.s.t corresponding to $(N,\bar{c})$ is the same as that of (N,c) by Observation1, and so by TI, $\phi(N,\bar{c})=\phi$. Consider a merging maneuver by $S=\{i_1,j_l\}$ into $i^*=\arg\min_{j\in S}B_j$. The new problem has at most K users and so by Induction Argument 1, we have $\phi_{i^*}(N,\bar{c})=B_{i^*}$. This means that in problem $(N,\bar{c})$, the cost share to coalition S, $B_{i_1}+B_{j_l}+x_l$, is more than what they would have paid had they merged, $\min\{B_{i_1},B_{j_l}\}+\max\{B_{i_1},B_{j_l}\}+\varepsilon=B_{i_1}+B_{j_l}+\varepsilon$. This is a contradiction to MP, and therefore we must have $x_1=x_2=\cdots=x_n=0$. In other words, $\phi_j=B_j$ for all $j\in S(i)$. This, together with Observation 1, implies that $\phi_i=B_i$.

This ends the induction argument, and establishes that $\phi=B$. ■

4 Discussion

In order to understand the implications of our theorem, we should first introduce the notion of the *irreducible core* of a m.c.s.t. problem, first discussed in Bird (1976). We define the *canonical cost matrix* of a given a m.c.s.t problem $(N,c), c\in C_N^1$, denoted by $c^{can}(N,c)$. The matrix $c^{can}(N,c)$ is such that, $g[N_0,c]$ is a m.c.s.t in problem $(N,c^{can}(N,c))$ and, for all cost matrices $c'\in C_N$ with $c'_{ij}\leq c^*(N,c)_{ij}$ for all $i,j\in N_0$ and $c'\neq c^*(N,c)$, $g[N_0,c]$ is not a m.c.s.t in problem (N_0,c').[3] For example, consider the problem $(N,\bar{c})$ where $N=\{1,2,3\}$, $\bar{c}_{01}=4,\bar{c}_{12}=\bar{c}_{13}=1$, and $\bar{c}_{02}=\bar{c}_{03}=\bar{c}_{23}=10$. The irreducible cost matrix $c^{can}(N,\bar{c})$ is given by the cost matrix c in Example 1. Bergantiños and Vidal-Puga (2004) prove that it is unique.

A cost allocation rule ϕ is a selection from the irreducible core if, for all $N\subset\mathcal{N}, c\in C_N$ we have $\phi(N,c)\in Core(N,c^{can}(N,c))$. There is a wide class of cost allocation rules that are selections from the irreducible core. The cost sharing game $(N,c^{can}(N,c))$ is submodular (Bird, 1976), and the extreme

[3] This definition of the canonical matrix appears in Bergantiños and Vidal-Puga (2004). Bird (1976) calls $c^{can}(N,c)$ the *minimal network.*

points of the irreducible core are the Bird allocations induced by all the different priority orderings of N. As a result, the *Shapley value* of $(N, c^{can}(N, c))$ is equal to Bird's rule applied to the problem $(N, c^{can}(N, c))$. Feltkamp, Tijs and Muto (1994) call this rule the *Equal Remaining Obligations (ERO) rule* , and Branzei, Moretti, Norde and Tijs (2003) call it the *p-value*. See these papers and Bergantiños and Vidal-Puga (2004) for alternative characterizations of this rule.

The class of cost allocation rules constructed in Feltkamp, Tijs and Muto (1994) are selections from the irreducible core and satisfy TI on $\mathcal{C}^1$. Among these rules are Bird's rule (Bird, 1976), Dutta-Kar rule (Dutta and Kar, 2004) and the Equal Remaining Obligations rule. As it is mentioned in that paper, $Core(N, c^{can}(N, c)) \subset Core(N, c)$. Therefore, if the cost allocation ϕ is an irreducible core selection, it satisfies CS. Also, since these rules use only information from the m.c.s.t, they satisfy TI. Our main result implies that, on domain $\mathcal{C}^1$, Bird's rule is the only merge-proof one among these rules.

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