Common Agency with Informed Principals[1]

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Abstract

We analyze a symmetric common agency game between two privately informed principals. Principals offer contributions to a common agent who produces a public good on their behalf. Asymmetric information introduces incentive compatibility constraints which replace the requirement of *truthfulness* found in the earlier common agency literature under complete information. There exists a large class of differentiable equilibria which are ex post inefficient. Inefficiencies come from the fact that each principal wants to reduce public good production to induce the agent to reveal the types of others which have been learned from observing their contributions. This screening problem in games with voluntary contributions highlights a new source of inefficiency in public good provision which differs from the usual free-riding problem. For distributions having a linear hazard rate, closed-form equilibria are obtained. Those equilibria are interim efficient for some distributions of social weights on the different types of principals. Introducing asymmetric information on the agent's cost of producing the public good might also help to select a unique equilibrium under some circumstances.

Keywords: Common agency, informed principals, public goods, ex post and interim efficiency.

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1 Introduction

Over the past twenty years and following the seminal contributions of Wilson (1979) and Bernheim and Whinston (1986a), the common agency literature has developed an analytical framework to tackle a variety of important problems such as menu auctions,[1] public good provisions through voluntary contributions,[2] or policy formation with competing lobbying groups.[3] Given this broad range of applications, it is fair to say that common agency is by now viewed as a major piece of the toolkit of many economists, especially in the field of political economy.

The success of this model relies both on its simple underlying structure and on its clear-cut predictions. Under common agency, several principals design non-cooperatively contributions $t_i(q)$ for a common agent. The agent accepts or refuses contracts and then chooses a quantity q of public good on behalf of principals. Players' preferences are common knowledge. A priori, many equilibria of this three-stage game may be sustained thanks to the freedom in specifying contributions off the equilibrium. By imposing that contributions are *truthful*, i.e., reflect the relative preferences of the principals among alternatives, Bernheim and Whinston (1986a) significantly reduced this indeterminacy and selected efficient equilibria.[4] Since each principal's marginal preferences among alternatives are fully reflected by his truthful contribution, what this principal pays at the margin for inducing a change in the agent's decision is exactly what it is worth to him. Modulo truthfulness, common agency aggregates preferences efficiently under complete information.

This paper extends the common agency framework to an environment where principals are privately informed on their preferences. This extension is needed in a variety of circumstances. Voluntary contributions to a public good are designed by donors with an eye on how much information on their willingness to pay they convey. In political economy settings, lobbying groups have private information on the benefits they withdraw from a given policy and much of their activity consists in conveying information to a less-well informed policy-maker.

Private information on the principals' preferences introduces *incentive compatibility constraints* which replace the more ad hoc *truthfulness* requirement imposed under complete information. To understand the impact of incentive compatibility on the properties

[1] Wilson (1979) and Bernheim and Whinston (1986a).

[2] Laussel and Lebreton (1998).

[3] Grossman and Helpman (1994) and Dixit, Grossman and Helpman (1997), among many others.

[4] Multiplicity might come from the flexibility in sharing the aggregate surplus among the contributing principals and their common agent. The feasible redistributions of the aggregate surplus are described by means of simple inequalities. See Bernheim and Whinston (1986a).

of equilibria, it is useful to make a brief detour by the theory of informed principals.[5] Indeed, at a best-response to what other principals are offering, a given principal designs his own contribution not only to convey information to the common agent on his preferences but also to extract the information that this agent may have learned from observing the contributions of others. This screening role of a contribution points at the fact that *"market information"*[6] has to be learned in equilibrium. Of course, the difficulty is that "market information" is by large endogenous: this is what other principals reveal to the common agent through their offers. Standard mechanism design techniques can nevertheless still be used to compute best-responses. When choosing how much to contribute at the margin for q units of the public good, each principal behaves thus as a monopsonist in front of an agent who is privately informed on the preferences of others. Using a standard argument of the screening literature, this principal reduces his marginal contribution to reduce the rent that the agent may withdraw from privately observing the other principals' offers.

Two properties of equilibria immediately follow.

Multiplicity: When the marginal contributions of other principals do not change much with their types, the agent withdraws little rent from observing their offers. A given principal induces little downward distortions for screening purposes and the marginal contribution of that principal does not change much with his own type. The reverse happens if other principals' contributions change significantly with their own types. This generates multiple equilibria.

Inefficiency: All equilibria are ex post inefficient. Again, because each principal wants to reduce the agent's output for screening purposes, downward distortions below the first-best always occur. Principals contribute less at the margin than what the good is worth to them not because they want to hide their types to the common agent but because this limits the rent that this agent may get from learning the types of other principals.

Turning to the weaker concept of *interim efficiency*,[7] we show existence of an interim efficient equilibrium for a large class of distributions having linear hazard rates. Such an interim efficient allocation is achieved through a centralized mechanism offered by an uninformed mediator. The social weights given to the different types of principals in the welfare function that would be maximized by this mediator are then characterized. Unsurprisingly, because the agent gets a positive rent in equilibrium, he must also receive a positive weight in the social welfare function.

[5] See Maskin and Tirole (1990).

[6] See Epstein and Peters (1999) and Peters (2001). Those papers derived Revelation Principles for multiprincipal environments where principals' preferences are common knowledge. Market information captures the endogenous randomness that arises in mixed-strategy equilibria.

[7] See Holmström and Myerson (1983) and Ledyard and Palfrey (1999).

Looking then for a device to select among equilibria which would be available with less restrictive assumptions on the distributions, we introduce private information on the common agent's cost of producing the public good. Provided the distribution of this parameter has a sufficiently large support, there exists a unique equilibrium of the common agency game that can be characterized as the fixed-point of a functional operator. We give also an analytical example such that this equilibrium can be explicitly computed.

Review of the literature: The results of the earlier common agency literature under complete information have been extended in many different directions. Dixit, Grossman and Helpman (1997) introduced redistributive concerns by relaxing the quasi-linearity assumption. Laussel and Lebreton (1998) studied incomplete information on the preferences of the common agent but focused on ex ante contracting, i.e., before the agent learns the realization of his cost. In both papers efficiency is still obtained.[8]

Paralleling those papers which put aside incentives, Stole (1991), Martimort (1992, 1996), Mezzetti (1997), Biais, Martimort and Rochet (2000) and Martimort and Stole (2002, 2003) among others analyzed oligopolistic screening environments where different principals elicit information privately known by the common agent at the contracting stage. These papers stressed the impact of oligopolistic screening on the standard rent/efficiency trade-off. We focus instead on asymmetric information on the principals' side. The agent's private information vis à vis each principal is then endogenized; it is what the agent may have learned from observing the other principals' offers.

There exists a tiny literature on voluntary contributions by privately informed contributors. Menezes, Monteiro and Temini (2001) and Laussel and Palfrey (2003) both analyzed such games for a 0-1 public good. Both papers stressed the multiplicity of equilibria that arises in those environments but, as we will see, somewhat underestimate the problem by looking at a discrete decision. Menezes, Monteiro and Temini (2001) highlighted the strong ex post inefficiency of equilibria. Laussel and Palfrey (2003) focused instead on the interim efficiency of some equilibrium allocations. We show that downward distortions below the first-best exist in any equilibrium satisfying a simple monotonicity condition. We provide more comments on how the results of Laussel and Palfrey (2003) on interim efficiency and ours compare in due place.

More broadly, our paper also contributes to the general issue of understanding public good provision under asymmetric information. From Clarke (1971) and Groves (1973), it is well-known that ex post efficiency is possible under dominant strategy provided that one is ready to generate a surplus. D'Aspremont and Gerard-Varet (1979) showed that one can maintain budget balance and efficiency under Bayesian implementation. Laffont

[8] Other extensions less directly relevant for the analysis of this paper include Prat and Rustichini (2003) who studied competition among principals trying to influence multiple agents and Bergemann and Välimäki (2003) who considered dynamic issues.

and Maskin (1979), Rob (1989) and Mailath and Postlewaite (1990) stressed the role of participation constraints to generate inefficiency. The game of voluntary contributions we analyze below is an alternative to the centralized mechanisms used in this literature; something that may be attractive in the absence of a mediator to enforce such a centralized mechanism.[9] Common agency ensures voluntary participation by principals, relies on Bayesian strategies, and finally generates a positive surplus for the agent and ex post inefficiency. When a centralized mechanism is offered by an uninformed mediator, inefficiencies are due to the contributors' incentives to hide their own types to this mediator: the so-called "free-rider" problem. Under common agency, as we will see below, contributors reveal instead costlessly their types through their contract offers to the agent but want to screen this agent according to what he has learned from others: a different source of inefficiency in public good provision.

Section 2 presents the model. Section 3 characterizes the differentiable equilibria of our common agency game under asymmetric information. A special attention is given to the so-called *pointwise optimal equilibria*. Lindahl-Samuelson conditions are derived and a few properties of the equilibria, noticeably their ex post inefficiency, are discussed. Section 4 is devoted to the multiplicity problem. Section 5 reinterprets the pointwise optimal equilibria using a mechanism design approach. Section 6 discusses interim efficiency. Section 7 introduces adverse selection on the agent's cost to select a unique equilibrium allocation which is characterized as the unique fixed-point of a functional operator. Section 8 briefly concludes. Proofs are relegated to an Appendix.[10]

2 The Model

There are two risk-neutral principals P_i $(i = 1, 2)$ who derive utility from consuming a public good which is produced in non-negative quantity q.[11] This public good may be an infrastructure of variable size, a charitable activity, or it may also have a more abstract interpretation as a policy variable in a lobbying game. P_i gets a utility $V_i = \theta_i q - t_i$ from consuming q units of the good when he pays an amount t_i.

Principals are privately informed on their respective valuations θ_i. Those types are independently drawn from the same common knowledge and atomless distribution on $\Theta = [\underline{\theta}, \bar{\theta}]$ with c.d.f. $F(\cdot)$ and everywhere positive density $f = F'$. The hazard rate $\frac{1-F(\theta)}{f(\theta)}$ is a Lipschitz function. $E_\theta[\cdot]$ denotes the expectation operator with respect to the law of θ.

[9] One example one may have in mind is given by transnational public goods where no international agency is available to enforce such a mechanism.

[10] For completeness, we show there also how to construct non-differentiable equilibria.

[11] Extension to the case of more than two principals increases significantly complexity.

Contributions are collected by a risk-neutral common agent A who produces at cost $C(q)$ the public good. The function $C(\cdot)$ is twice differentiable, increasing and convex. To avoid technicalities due to corner conditions, the Inada condition $C'(0) = 0$ holds.

The game unfolds as follows. First, principals learn their preferences. Second, they offer non-cooperatively contributions $\{t_i(q, \hat{\theta}_i)\}_{\hat{\theta}_i \in \Theta}$ to the agent. Third, the agent accepts or refuses those contracts. If he refuses all contracts, the game ends. Upon acceptance of at least one offer, the agent produces an amount q. Payments are made.

We want to characterize Perfect Bayesian Equilibria (PBE) - or equilibria in short - of this game.

We will give conditions on the type distribution so that principals offer in equilibrium non-negative contributions with non-negative margin. The first condition implies that the agent is always as well-off accepting both offers. The second condition implies that the level of public good produced is greater with two principals than if only one contributes.

Benchmark: For future references, we denote by $q^*(\theta_1, \theta_2)$ the first-best level of public good. It satisfies the following standard Lindahl-Samuelson conditions:

$$C'(q^*(\theta_1, \theta_2)) = \sum_{i=1}^{2} \theta_i.$$

Note that $q^*(\cdot)$ is (strictly) monotonically increasing in each of its arguments.

3 Preliminary Results

Even though the contexts are quite different, we follow the same strategy as when computing the symmetric equilibrium of a first-price unit auction to characterize equilibria of the common agency game under asymmetric information. We first conjecture the form of such equilibrium as a menu of contributions. Within this menu, each principal chooses according to his type. By the Revelation Principle applied to that Bayesian game, there is no loss of generality in restricting to incentive compatible menus. Facing the symmetric menu of contributions $\{t(q, \hat{\theta})\}_{\hat{\theta} \in \Theta}$, principal P_i with type θ_i optimally picks $t(q, \theta_i)$.[12] We denote $p(q, \theta_i) = \frac{\partial t}{\partial q}(q, \theta_i)$ the marginal contribution of type θ_i when q units of public good

[12] It is important to distinguish incentive compatibility from the notion of "truthfulness" developed by Bernheim and Whinston (1986a). When the principals' preferences are common knowledge, truthfulness means that each principal's marginal contribution is equal to his marginal valuation, i.e., $\frac{\partial t}{\partial q}(q, \hat{\theta}_i) = \hat{\theta}_i$ for all $\hat{\theta}_i$ and all q. Under asymmetric information, incentive compatibility requires only that each principal finds optimal to pick the contribution corresponding to his type.

are produced.[13,14]

For any pair (θ_1, θ_2), let define the agent's rent when accepting both contributions as

$$U(\theta_1, \theta_2) = \max_q \sum_{i=1}^{2} t(q, \theta_i) - C(q).$$

From the fact that, in a common agency equilibrium, the agent gets more by taking both contributions than rejecting any of those or both, we must have:

$$U(\theta_1, \theta_2) \geq \max\{0, \hat{U}(\theta_1), \hat{U}(\theta_2)\} \quad \text{for all pairs} \quad (\theta_1, \theta_2) \in \Theta^2. \tag{1}$$

$\hat{U}(\theta_i) = \max_q t(q, \theta_i) - C(q)$ is the agent's rent when accepting only P_i's contribution.

At the last stage of the game, the agent chooses optimally q given that he has accepted both contributions. The interior level of public good is given by the first-order condition

$$\sum_{i=1}^{2} p(q(\theta_1, \theta_2), \theta_i) = C'(q(\theta_1, \theta_2)), \tag{2}$$

provided that the local second-order condition for the agent's problem holds, namely:

$$\sum_{i=1}^{2} \frac{\partial p}{\partial q}(q(\theta_1, \theta_2), \theta_i) - C''(q(\theta_1, \theta_2)) \leq 0. \tag{3}$$

We will first omit this last constraint and check ex post that it is satisfied at equilibrium.

Of particular relevance are non-negative contributions such that an upward shift in the principal's valuation increases the equilibrium quantity. This is of course obtained when $\frac{\partial p}{\partial \theta_i}(q, \theta_i) \geq 0$ for all (q, θ_i). Such schedules satisfy the same Spence-Mirrlees property (SMP) than the principals' preferences.

Definition 1 *: A non-negative contribution satisfies the Spence-Mirrlees Property (SMP) (resp. the Strict Spence-Mirrlees Property (SSMP)) when*

$$\frac{\partial p}{\partial \theta_i}(q, \theta_i) \geq 0 \quad (\textit{resp.} > 0) \textit{ for all} \quad (q, \theta_i).$$

Standard revealed preference arguments yield then:

[13] For technical reasons, we focus on contributions which are three times piece-wise differentiable with respect to q and $\hat{\theta}_i$. In the Appendix, we analyze also a class of non-differentiable equilibria.

[14] This specification of the contributions available to principals seems to restrict a priori their strategy. Since principals have to choose within a given menu of contributions, they are not allowed to offer more complex mechanisms. Section 5 shows that the equilibria obtained with those menus are in fact robust. They are also equilibria when principals can deviate and offer more complex mechanisms. For instance, P_i could ask the agent about the information that he learns on P_{-i} by observing his mere offer.

Lemma 1 *: In any PBE with non-negative contributions satisfying SMP:*

- $q(\theta_i, \theta_{-i})$ *is almost everywhere differentiable,*
- $\frac{\partial q}{\partial \theta_i}(\theta_i, \theta_{-i}) \geq 0$*, for all* (θ_i, θ_{-i}) *in* Θ^2*. Inequalities are strict when SSMP holds.*

Truthtelling is a Bayesian-Nash equilibrium between the principals when:

$$\theta_i = \arg\max_{\hat{\theta}_i} \Phi(\hat{\theta}_i, \theta_i) \tag{4}$$

where $\Phi(\hat{\theta}_i, \theta_i) = E_{\theta_{-i}}\left[\theta_i q(\hat{\theta}_i, \cdot) - t(q(\hat{\theta}_i, \cdot), \hat{\theta}_i)\right]$ is P_i's payoff when his type is θ_i and he picks the contribution corresponding to type $\hat{\theta}_i$ within the proposed menu.

Proposition 1 *: The first- and second-order conditions for optimality of the principal's problem (4) are respectively given by:*

$$E_{\theta_{-i}}\left[\frac{\partial q}{\partial \theta_i}(\theta_i, \cdot)\left[\theta_i + p_{-i}(q(\theta_i, \cdot), \cdot) - C'(q(\theta_i, \cdot)) - \frac{1 - F(\cdot)}{f(\cdot)}\frac{\partial p}{\partial \theta_{-i}}(q(\theta_i, \cdot), \cdot)\right]\right] = 0, \tag{5}$$

$$E_{\theta_{-i}}\left[\frac{\partial q}{\partial \theta_i}(\theta_i, \cdot)\right] \geq 0 \quad \textit{for all } \theta_i \in \Theta \textit{ and } i = 1, 2.^{15} \tag{6}$$

Pointwise-optimality: Among the equilibrium schedules satisfying conditions (5) and (6), we focus on those which are *pointwise optimal* and monotonic in the following sense:

$$\theta_i + p(q(\theta_i, \theta_{-i}), \theta_i) - C'(q(\theta_i, \theta_{-i})) = \frac{1 - F(\theta_{-i})}{f(\theta_{-i})}\frac{\partial p}{\partial \theta_i}(q(\theta_i, \theta_{-i}), \theta_{-i}) \tag{7}$$

and

$$\frac{\partial q}{\partial \theta_i}(\theta_i, \theta_{-i}) \geq 0 \quad \text{for all } (\theta_i, \theta_{-i}) \text{ in } \Theta^2,\ i = 1, 2. \tag{8}$$

Pointwise optimality will receive a clear motivation in Section 5. Unless specified otherwise, equilibria will be understood as satisfying pointwise optimality.

Condition (7) looks like the standard optimality condition for a problem involving only P_i and an agent having preferences $t + t_{-i}(q, \theta_{-i}) - C(q)$ who has also private information on θ_{-i}. The right-hand side of (7) represents then the standard adverse selection distortion. Section 5 makes again this analogy clearer.

The following lemma gives conditions which ensure that the first- and second-order conditions for the principals' and the agent's optimization problems are also sufficient.

[15] From Lemma 1 the second-order condition for the principal's problem is implied by the monotonicity properties coming from the agent's problem in a SMP equilibrium.

Lemma 2 *: Let $\{q(\theta_1, \theta_2), p(q, \theta)\}$ be a public good level and a non-negative marginal contribution satisfying SMP, (2), (7) and (8). This profile, if it exists, forms an equilibrium.*

Lindahl-Samuelson conditions: Condition (7) is also helpful in already deriving a few properties of the equilibrium schedules. To do so, consider SSMP equilibria which entail a strictly increasing output ((8) is strict over the range of $q(\cdot)$). We can thus uniquely define the inverse function $\psi(q, \theta_i)$ as $q(\theta_i, \psi(q, \theta_i)) = q$ for all θ_i and q in the range of $q(\theta_i, \cdot)$.[16] Condition (7) becomes:

$$\psi(q, \theta_i) + p(q, \theta_i) - C'(q) = \frac{1 - F(\theta_i)}{f(\theta_i)} \frac{\partial p}{\partial \theta_i}(q, \theta_i), \tag{9}$$

for all q in the range of $q(\theta_i, \cdot)$. This can be rewritten as:

$$\frac{\partial}{\partial \theta_i}[p(q, \theta_i)(1 - F(\theta_i))] = (\psi(q, \theta_i) - C'(q))f(\theta_i).$$

This is a differential equation in θ_i which can be integrated to get $p(q, \theta_i)$ as

$$p(q, \theta_i) = \frac{\varphi(q)}{1 - F(\theta_i)} + C'(q) - \frac{1}{1 - F(\theta_i)} \int_{\theta_i}^{\bar{\theta}} \psi(q, x) f(x) dx, \tag{10}$$

where $\varphi(q)$ is an integration constant.

If we impose that $p(q, \theta_i)$ is bounded around $\theta_i = \bar{\theta}$ for all q, we must have $\varphi(q) = 0$. Finally, the equilibrium schedule writes as:

$$p(q, \theta_i) = C'(q) - \frac{1}{1 - F(\theta_i)} \int_{\theta_i}^{\bar{\theta}} \psi(q, x) f(x) dx. \tag{11}$$

Taking into account that

$$p(q, \theta_i) + p(q, \psi(q, \theta_i)) = C'(q), \tag{12}$$

for all q in the range of $q(\theta_i, \cdot)$ yields

$$p(q, \psi(q, \theta_i)) = \frac{1}{1 - F(\theta_i)} \int_{\theta_i}^{\bar{\theta}} \psi(q, x) f(x) dx,$$

or using that $\psi(q, \psi(q, \theta_i)) = \theta_i$ for all θ_i

$$p(q, \theta_i) = \frac{1}{1 - F(\psi(q, \theta_i))} \int_{\psi(q, \theta_i)}^{\bar{\theta}} \psi(q, x) f(x) dx. \tag{13}$$

[16]Note that because q is in the range of $q(\theta_i, \cdot)$, $\psi(q, \theta_i)$ belongs to $[\underline{\theta}, \bar{\theta}]$.

By summing the expressions of the non-negative marginal contributions obtained from (13), we get the following Lindahl-Samuelson conditions:

$$C'(q(\theta_1,\theta_2)) = \sum_{i=1}^{2} \frac{1}{1-F(\theta_i)} \int_{\theta_i}^{\bar{\theta}} \psi(q(\theta_1,\theta_2),x)f(x)dx. \tag{14}$$

To understand (14), it is useful to come back on the definition of the equilibrium schedule given in (13). Given the equilibrium conjecture $p(\cdot)$, one may define for any type θ_i and output q, the *conjugate type* $\psi(q,\theta_i)$ which is such that the quantity q is produced when both a type and his conjugate follow the equilibrium strategy. All types corresponding to a valuation x greater than $\psi(q,\theta_i)$ are thus ready to contribute at the margin at least $p(q,\psi(q,\theta_i))$ for q units of public good in any SMP equilibrium. This is in front of those types that P_i can in fact underestimate his valuation θ_i and contribute less. By how much can he underestimate his valuation? Facing such a type x, the marginal contribution of P_i with conjugate type $\psi(q,x)$ is $p(q,\psi(q,x))$. Once q units of the good are produced with type x for P_{-i}, one can infer that P_i's marginal valuation is at least $\psi(q,x)$. (13) shows that the marginal contribution of type θ_i is an average of all such inframarginal valuations. Since x is greater than θ_i, and $\psi(q,\cdot)$ is decreasing in its second argument, that average is lower than θ_i.

Ex post inefficiency: Integrating by parts (13), we obtain:

$$p(q,\theta_i) = \theta_i + \frac{1}{1-F(\psi(q,\theta_i))} \int_{\psi(q,\theta_i)}^{\bar{\theta}} \frac{\partial\psi}{\partial\theta}(q,x)(1-F(x))dx. \tag{15}$$

From the definition of $\psi(\cdot)$, $\frac{\partial\psi}{\partial\theta}(q,x) = -\frac{\frac{\partial q}{\partial\theta_1}}{\frac{\partial q}{\partial\theta_2}}(x,\psi(q,x)) < 0$ when output is monotonically increasing. Therefore, in any symmetric SSMP equilibrium of the contribution game (if such an equilibrium exists) and with contributions having bounded derivative $\frac{\partial p}{\partial\theta}(\cdot)$ around $\bar{\theta}$, $t(q,\theta_i)$ *does not* reflect the preferences of the principal. This contrasts with Bernheim and Whinston (1986a)'s findings under complete information. Indeed, we have:

$$p(q,\theta_i) \leq \theta_i, \tag{16}$$

for all θ_i in $[\underline{\theta},\bar{\theta}]$, with a strict inequality everywhere except when $q = q(\theta_i,\bar{\theta})$.[17]

4 Equilibrium Existence, Multiplicity, Inefficiency

The qualitative properties found above do not give much information on the existence and multiplicity of equilibria, even when one restricts the analysis to SSMP equilibria

[17] Note that $p(q,\theta_i)$ can be arbitrarily extended linearly for out of equilibrium quantities $q \geq q(\theta,\bar{\theta})$ and $q \leq q(\theta,\underline{\theta})$ so that $p(q,\theta)$ is no longer given by (15) but the inequality (16) still holds.

with bounded marginal contributions. After all, the Lindahl-Samuelson conditions (14) are complex and only define $q(\cdot)$ implicitly in terms of the inverse function $\psi(q,\cdot)$.

Constructing equilibria: To get further insights on the existence and multiplicity of SSMP equilibria, it is useful to come back on the two conditions which define the marginal contribution $p(q,\cdot)$ and the conjugate type $\psi(q,\cdot)$ and to reconstruct from there an equilibrium:

$$p(q,\theta)+p(q,\psi(q,\theta)) = C'(q), \tag{17}$$

$$\theta - p(q,\theta) = \frac{1-F(\psi(q,\theta))}{f(\psi(q,\theta))}\frac{\partial p}{\partial \theta}(q,\psi(q,\theta)), \tag{18}$$

for all (q,θ), where q is in the range of the equilibrium schedule of outputs $q(\cdot)$.

Those two equations do not uniquely define an equilibrium marginal contribution. To do so, we need first to define which type $\tilde{\theta}$ is such that $q(\tilde{\theta},\tilde{\theta}) = q$ in the equilibrium under scrutiny. When both principals have type $\tilde{\theta}$, their marginal contributions are the same. For such a $\tilde{\theta}$, we must have

$$\psi(q,\tilde{\theta}) = \tilde{\theta} \text{ and } p(q,\tilde{\theta}) = \frac{C'(q)}{2}. \tag{19}$$

By definition of a conjugate type, it must also be that:

$$\psi(q,\psi(q,\theta)) = \theta, \tag{20}$$

for all θ in $[\underline{\theta},\overline{\theta}]$ and q in the range of $q(\cdot)$.

Instead of defining its output $q(\cdot)$, we may describe an equilibrium in terms of its isoquant lines $\psi(q,\theta)$. The strict monotonicity properties $\frac{\partial q}{\partial \theta_1}(\theta_1,\theta_2)>0$ and $\frac{\partial q}{\partial \theta_2}(\theta_1,\theta_2)>0$ are then satisfied whenever

$$\frac{\partial \psi}{\partial \theta}(q,\theta)<0 \quad \text{and} \quad \frac{\partial \psi}{\partial q}(q,\theta)>0 \tag{21}$$

over the whole domain of definition of $\psi(\cdot)$.

This approach in terms of isoquants is used thereafter to characterize equilibria. It illuminates the degree of freedom left in specifying the equilibrium output $Q(\theta)=q(\theta,\theta)$ along the 45 degree line. Since an isoquant is symmetric with respect to that line, it is enough to characterize the pair $\{p(q,\cdot),\psi(q,\cdot)\}$ over the interval $[\underline{\theta},\tilde{\theta}]$ and to reconstruct the equilibrium marginal contribution over the whole interval $[\underline{\theta},\bar{\theta}]$ using (17).[18]

[18] When an isoquant $\psi(q,\theta)$ is defined over $[\underline{\theta},\tilde{\theta}]$ it must be, *by definition*, that $\psi(q,\underline{\theta}) \leq \bar{\theta}$. When $\psi(q,\theta)$ is only defined over an interval $[\theta_1,\tilde{\theta}]$ for some θ_1 such that $\theta_1>\underline{\theta}$, we have in fact $\psi(q,\theta_1)=\bar{\theta}$. In other words, q does not belong to the range of the equilibrium schedule $q(\theta,\cdot)$ for $\theta<\theta_1$. We will abuse language by saying that the interval of definition of $\psi(q,\cdot)$ is still $[\underline{\theta},\tilde{\theta}]$.

Proposition 2 *: Fix any strictly increasing output schedule $Q(\theta)$ such that $Q(\theta) \leq q^*(\theta, \theta)$ (with equality only at $\bar{\theta}$). Define for any q in the range of $Q(\cdot)$ and any $\tilde{\theta}$ such that $q = Q(\tilde{\theta})$ the following system of first-order differential equations over the interval $[\underline{\theta}, \tilde{\theta}]$*

$$\frac{\partial p}{\partial \theta}(q, \theta) = \frac{\psi(q, \theta) + p(q, \theta) - C'(q)}{\frac{1-F(\theta)}{f(\theta)}}, \tag{22}$$

$$\frac{\partial \psi}{\partial \theta}(q, \theta) = -\frac{\frac{1-F(\psi(q,\theta))}{f(\psi(q,\theta))}}{\frac{1-F(\theta)}{f(\theta)}} \left(\frac{\psi(q, \theta) + p(q, \theta) - C'(q)}{\theta - p(q, \theta)} \right), \tag{23}$$

with the boundary conditions (19).

This system admits a unique solution over $[\underline{\theta}, \tilde{\theta}]$. Provided that $\frac{\partial \psi}{\partial q}(q, \theta) > 0$, the second-order conditions for the agent's and the principals' problems (3) are satisfied, and this solution defines a SSMP equilibrium pair $\{p(q, \theta), \psi(q, \theta)\}$. The conditions $\frac{\partial \psi}{\partial q}(q, \theta) > 0$ and $p(q, \theta) > 0$ are always satisfied when $\bar{\theta} - \underline{\theta}$ is small enough.

A better understanding of the multiplicity of equilibria requires a detour by the theory of informed principals, and we postpone such discussion to Section 5.

The resolution techniques and the multiplicity of equilibria found above are reminiscent of the analysis of equilibria in double-auctions made in Leininger, Linhart and Radner (1989) and Sattherwaite and Williams (1989). Those authors have developed procedures that consist in fixing the equilibrium strategies for the buyer and the seller when their valuations coincide and reconstruct numerically the bidding strategies as solutions of differential equations with lags on both sides of these critical values. Our approach is not numerical. Menezes, Monteiro and Temini (2001) and Laussel and Palfrey (2003, p. 460) use also a similar technique in their public good model with a 0-1 decision. The restriction to a 0-1 decision hides in fact the multiplicity problem that arises from the arbitrary specification of the monotonic equilibrium output on the diagonal.

Ex post inefficiency: The Lindahl-Samuelson conditions (14) derived in the case of a contribution with bounded derivative imply that equilibrium outputs are downward distorted below the first-best. This result holds in fact for all equilibria described in Proposition 2.

Corollary 1 *: An ex post efficient outcome can never be implemented at a SSMP equilibrium. Downward distortions below the first-best always occur.*

Closed-form solutions: Although generally quite complex, the derivation of some explicit solutions is feasible for a large class of distributions including, uniform and exponential among others.

Proposition 3 *: Assume that* $F(\cdot)$ *has linear hazard rate, i.e.,* $\frac{1-F(\theta)}{f(\theta)} = \beta(\bar{\theta}-\theta)+\gamma$ *where* $(\beta,\gamma) \in \mathbb{R}^2_+$ *where* $\underline{\theta} \geq \frac{1}{1+\beta}(\beta(\bar{\theta}-\underline{\theta})+\gamma)$*. Then, there exists a unique*[19] *symmetric equilibrium with a non-negative marginal contribution which is linear in* θ *and given by:*

$$p(q,\theta) = \frac{\theta}{1+\beta} + \frac{\beta}{1+\beta}C'(q) - \frac{\beta}{(1+\beta)(1+2\beta)}\bar{\theta} - \frac{\gamma}{(1+\beta)(1+2\beta)}. \tag{24}$$

It corresponds to a non-negative output:

$$C'(q(\theta_1,\theta_2)) = \frac{1+2\beta}{1+\beta}(\theta_1+\theta_2) - \frac{2\beta}{1+\beta}\bar{\theta} - \frac{2\gamma}{1+\beta}. \tag{25}$$

5 Informed Principals and Mechanism Design

In this section, we propose an alternative approach to characterize *pointwise optimal* equilibria. This approach illustrates the role of nonlinear contributions in simultaneously screening the other principal's type and signaling his own type to the agent.

Viewing the strategy of each principal as a choice within a menu $\{t(q,\hat{\theta}_i)\}_{\hat{\theta}_i\in\Theta}$ as we did so far and writing down the condition for incentive compatibility may *a priory* entail a loss of generality if we want to describe the whole set of equilibria when principals are unrestricted in the mechanisms they may offer. Indeed, a given principal might like to deviate to a more complex mechanism than a nonlinear contribution. In this section, we show that this is indeed not the case. We will describe explicitly P_i's best-response to P_{-i}'s own offer within the largest class of mechanisms available and show that it can actually be implemented as a nonlinear contribution.

For any fixed contribution offered by P_i, the design of P_{-i}'s own contribution is an informed principal problem under private values. We know from Maskin and Tirole (1990) that, under risk-neutrality, there is no loss of generality in having P_{-i} offer a contract to the agent exactly as if the latter was informed on the principal's type. Intuitively, the mechanism consisting in piling up the various contracts that would be signed by those different types if the agent was informed on P_{-i}'s preferences is incentive compatible from the principal's point of view and achieves a lower bound on the principal's payoffs. The key insight due to Maskin and Tirole (1990) is that, higher payoffs can only be achieved if the principal is risk-averse. This is obtained by pooling those contracts at the time of offering contracts and revealing the principal's type at a later communication stage only. Pooling offers relax the agent's incentive and participation constraints and improve risk-sharing among the different types of the principal. With risk-neutrality, this insurance

[19] Uniqueness is meant in terms of marginal contributions. Different equilibrium payoffs for the principals may of course be sustained by playing on the constants of integration.

motive disappears and the lowest bound on the principal's payoff is also an upper bound. Instead of offering a mechanism to the agent with a communication stage after contract's acceptance, the principal is as well off revealing his type right away by offering only one contract. For each contribution offered by P_i, P_{-i} has thus always in his best-response correspondence a separating menu of contributions.[20]

In the case of a single principal analyzed by Maskin and Tirole (1990), this equivalence between two contracting modes has no consequence. Under common agency instead, that seemingly innocuous difference in the timing of information revelation has a strategic value since it affects the way P_i contracts himself with the agent. Provided that P_{-i}'s offer reveals his type to the agent, P_i knows that he should design his contribution not only to signal his own type to the agent but also to screen P_{-i}'s type which is "endogenously" learned in equilibrium by the agent. This points at the major role that nonlinear contributions play in a common agency environment: learning over what Epstein and Peters (1999) and Peters (2001) call *market information*, i.e., the preferences of other principals.

We focus on pure strategy equilibria with separating menus which reveal all information on the principals' types to the agent through contract offers. To compute P_i's best response to any given P_{-i}'s nonlinear contribution $t_{-i}(q, \theta_{-i})$ within the largest space of possible mechanisms, we use the Revelation Principle.[21] We may thus as well restrict the analysis to direct truthful revelation mechanisms $\{t_i^D(\hat{\theta}_{-i}|\theta_i), q(\hat{\theta}_{-i}|\theta_i)\}$ where $\hat{\theta}_{-i}$ is the agent's report on θ_{-i} (that he has learned from P_{-i}'s offer). The agent's utility can then be written as:

$$\hat{U}(\hat{\theta}_{-i}, \theta_{-i}|\theta_i) = t_i^D(\hat{\theta}_{-i}|\theta_i) + t_{-i}(q(\hat{\theta}_{-i}|\theta_i), \theta_{-i}) - C(q(\hat{\theta}_{-i}|\theta_i)). \tag{26}$$

From incentive compatibility, we get:

$$U(\theta_{-i}|\theta_i) = \hat{U}(\theta_{-i}, \theta_{-i}|\theta_i) = \max_{\hat{\theta}_{-i}} \hat{U}(\hat{\theta}_{-i}, \theta_{-i}|\theta_i).$$

We assume that $t_{-i}(q, \theta_{-i})$ is twice differentiable and satisfies SSMP. Using standard techniques, we get:

- $q(\theta_{-i}|\theta_i)$ is monotonically increasing and thus almost everywhere differentiable in θ_{-i} with

$$\frac{\partial q}{\partial \theta_{-i}}(\theta_{-i}|\theta_i) \geq 0 \qquad \text{a.e.}, \tag{27}$$

[20] The reader may have recognized a feature already found in Bernheim and Whinston (1986a). To refine with truthfulness among all equilibria under complete information, they indeed first noticed that each principal has a best-response which is truthful and thus justified that focusing at equilibria in truthful schedules is meaningful. We apply the same device to justify that focusing on equilibria where principals reveal their types through a separating offer is also meaningful.

[21] See Martimort and Stole (2002, 2003) for this way of applying the Revelation Principle to compute best-responses in pure strategy equilibria of common agency games.

- $U(\theta_{-i}|\theta_i)$ is almost everywhere differentiable in θ_{-i} with

$$\frac{\partial U}{\partial \theta_{-i}}(\theta_{-i}|\theta_i) = \frac{\partial t_{-i}}{\partial \theta_{-i}}(q(\theta_{-i}|\theta_i), \theta_{-i}). \tag{28}$$

At a best-response to $t_{-i}(q, \theta_{-i})$, P_i with type θ_i must solve the following problem:

$$(P_i): \quad \max_{\{U(\cdot|\theta_i);q(\cdot|\theta_i)\}} E_{\theta_{-i}}\left[\theta_i q(\cdot|\theta_i) + t_{-i}(q(\cdot|\theta_i), \cdot) - C(q(\cdot|\theta_i)) - U(\cdot|\theta_i)\right], \tag{29}$$

subject to (27)-(28) and

$$U(\theta_{-i}|\theta_i) \geq \max\{0, \hat{U}(\theta_{-i})\}, \quad \text{for all } \theta_{-i} \in \Theta, \tag{30}$$

where (30) is the agent's ex post participation which guarantees that he makes a positive profit for all profiles (θ_i, θ_{-i}) and prefers to take both contributions that only one.

A solution to (P_i) is an allocation $\{U(\theta_{-i}|\theta_i), q(\theta_{-i}|\theta_i)\}$ or equivalently a direct revelation mechanism $\{t_i^D(\theta_{-i}|\theta_i), q(\theta_{-i}|\theta_i)\}$ (we omit the dependence on $t_{-i}(q, \theta_{-i})$) from which we can reconstruct a nonlinear contribution $t_i(q, \theta_i)$ when $q(\theta_{-i}|\theta_i)$ is invertible. Of course, since all problems (P_i) have the same constrained set whatever θ_i, the menu $\{t_i(q|\hat{\theta}_i)\}_{\hat{\theta}_i \in \Theta}$ obtained is incentive compatible from P_i's point of view.

The standard techniques for solving problems like (P_i) consists in first neglecting the second-order condition (27), second integrating by parts, and third maximizing pointwise with respect to output the virtual surplus function obtained thereby. A first difficulty is to ensure the concavity of this virtual function since it depends on the other principal's offer $t_{-i}(q, \theta_{-i})$ which is endogenous in equilibrium. A second difficulty comes from checking that the second-order condition (27) holds.[22] It turns out that those difficulties can be handled altogether when principal P_{-i} offers a contribution $t_{-i}(q, \theta_{-i})$ such that the corresponding $\psi_{-i}(q, \theta)$ function obtained from (22) as

$$\psi_{-i}(q, \theta_{-i}) = -p_{-i}(q, \theta_{-i}) + C'(q) + \frac{1 - F(\theta_{-i})}{f(\theta_{-i})} \frac{\partial p_{-i}}{\partial \theta_{-i}}(q, \theta_{-i}) \tag{31}$$

satisfies $\frac{\partial \psi_{-i}}{\partial q}(q, \theta_{-i}) > 0$ and $\frac{\partial \psi_{-i}}{\partial \theta_{-i}}(q, \theta_{-i}) < 0$.

Provided that both conditions hold, P_i's best response is well-defined and entails an optimal output which is increasing with respect to θ_{-i}. The corresponding output is then defined by the pointwise optimality condition (7).

Let us now consider $\{p(q, \theta), \psi(q, \theta)\}$ a pair defined in Proposition 2 and corresponding to a contribution $t(q, \theta)$ offered in the symmetric equilibrium of a common agency game where principals are *a priori* restricted to choose within the incentive compatible menu $\{t(q, \hat{\theta}_i)\}_{\hat{\theta}_i \in \Theta}$. From the analysis above, we get:

[22] In standard principal-agent's problems, such monotonicity results again from the monotonicity of the hazard rate plus again the concavity of the virtual surplus function and an assumption on the sign of the third derivative $\frac{\partial^3 t_{-i}}{\partial \theta^2 \partial q}(q, \theta_{-i})$.

Proposition 4 *: An equilibrium pair* $\{p(q,\theta),\psi(q,\theta)\}$ *as defined in Proposition 2 is also an equilibrium when principals can deviate to a larger space of mechanisms.*

Comparing with the more direct approach taken in Proposition 2, we observe that equilibria with separating contributions correspond to the *pointwise optimal* allocations selected in Section 4. Proposition 4 shows thus that the focus on *pointwise* optimal allocations is a rather natural requirement. It comes from the fact that each principal may as well reveal truthfully his type to the agent through his offer.

The link between the direct and the informed principal approaches becomes stronger once one notices that the condition $\frac{\partial\psi}{\partial q}(q,\theta) > 0$ that must be checked ex post on the solution of the system of differential equations (19)-(22)-(23) in the direct approach to guarantee an equilibrium corresponds in fact to the condition which guarantees concavity of the virtual surplus of each principal in the mechanism design approach.

Ex post inefficiency and multiplicity revisited: The mechanism design approach is also useful to understand inefficiencies. At a best-response, a principal wants to induce a lower production from the agent than what is ex post efficient from the point of view of the bilateral surplus of the pair he forms with the agent. This downward distortion helps indeed to reduce the information rent that the agent may get from his private knowledge of the other principal's type. This phenomenon should be contrasted with the usual "free-rider" problem for public good provision found in centralized Bayesian mechanisms à la Laffont and Maskin (1979), Rob (1989) and Mailath and Postlewaite (1990). Free-riding comes there from the contributors' incentives to underestimate their valuations. Under common agency instead, principals do not hide their own valuations to the agent but want to screen the agent about the preferences of others. This is no longer the contributors who hide information but the agent who is at the intersection of all information sets.

The informed principal approach is also useful to understand the multiplicity of equilibria. When choosing how much to contribute at the margin for q units of the public good, a given principal behaves as a monopsonist in front of an agent privately informed on the preferences of other principals. When the marginal contributions of other principals do not change much with their types, the agent withdraws little rent from observing their offers. A given principal induces little downward distortions for screening purposes and the marginal contribution of that principal does not change much with his own type. The reverse happens if other principals' contributions change significantly with their own types. This generates multiple equilibria.

6 Interim Efficiency

Under complete information, it is well known that the "truthful" equilibria of common agency game are on the Pareto-frontier of what the principals could achieve by cooperating. Under asymmetric information, one can still be interested by the normative properties of common agency equilibria provided that *interim efficiency* is used as the welfare criterion.[23] We now investigate under which circumstances an equilibrium of the common agency game under asymmetric information might be *interim efficient*.

Interim efficient allocations are obtained as solutions of a centralized mechanism design problem. An uninformed mediator offers a single mechanism to both principals, who then report their types to this mediator. This mediator maximizes a weighted sum of both the principals and the agent's utilities with the weights given to different types of the principals being possibly different. Because, we want to replicate with such centralized mechanism a symmetric common agency equilibrium, we restrict to symmetric weights which do not depend on the principal's identity.

Proposition 5 : *(Ledyard and Palfrey (1999)) A public good profile $q(\theta_1, \theta_2)$ is interim efficient if and only if there exist non-negative social weights $\alpha(\theta) \geq 0$ such that $\int_{\underline{\theta}}^{\bar{\theta}} \alpha(\theta) f(\theta) d\theta \leq 1$ and*

$$C'(q(\theta_1, \theta_2)) = \sum_{i=1}^{2} b(\theta_i) \tag{32}$$

where $b(\theta_i) = \theta_i - \frac{1-F(\theta_i)}{f(\theta_i)}(1 - \tilde{\alpha}(\theta_i))$ is increasing and $\tilde{\alpha}(\theta_i) = \frac{1}{1-F(\theta_i)} \int_{\theta_i}^{\bar{\theta}} \alpha(x) f(x) dx$.

The inequality that $\int_{\underline{\theta}}^{\bar{\theta}} \alpha(x) f(x) dx \leq 1$ captures the possibility that the common agent receives a positive weight in the social welfare function maximized by the uninformed mediator. Remember that, in a common agency equilibrium, the agent gets a non-negative ex post rent $U(\theta_1, \theta_2)$ which should be accounted for when evaluating welfare.[24]

An interim efficient allocation is necessarily such that $C'(q(\theta_1, \theta_2))$ is separable in θ_1 and θ_2; a rather strong condition. This separability has a simple origin. To compute interim efficient allocations, the valuation of each informed party is then replaced by a

[23] See Holmström and Myerson (1983) and Ledyard and Palfrey (1999).

[24] This distinguishes our notion of interim efficiency from the usual one where it is assumed that budget is always balanced ex post (see Ledyard and Palfrey (1999) for instance). Formally, the mediator maximizes now:

$$E_{\Theta \times \Theta} \left(\sum_{i=1}^{2} \alpha(\theta_i) f(\theta_i) V(\theta_i) + U(\theta_1, \theta_2) \right)$$

under the principals' incentive and participation constraints. The characterization of those interim efficient allocations follows then closely Ledyard and Palfrey (1999) and is left to the reader.

virtual valuation. This modification captures the bilateral communication between this informed party and the mediator that takes place in a centralized mechanism. Under common agency instead, allocations result from an equilibrium between mechanisms. As Section 5 makes clear, each principal signals his own type through the mere offer he makes to the agent whereas, at the same time, he screens the preferences of his rival. For each piece of information, there is too much communication compared with a centralized mechanism. This lack of coordination can benefit the agent by making him able to extract some rent from playing one principal against the other. This makes it not obvious that interim efficiency can be achieved in any equilibrium of our game.

To check for interim efficiency of some equilibrium allocations, it is useful to give an alternative expression of the Lindahl-Samuelson conditions. This expression is obtained by summing up the conditions of pointwise optimality for both principals to get first:

$$C'(q(\theta_1,\theta_2)) = \sum_{i=1}^{2}\left(\theta_i - \frac{1-F(\theta_i)}{f(\theta_i)}\frac{\partial p}{\partial \theta_i}(q(\theta_1,\theta_2),\theta_i)\right). \tag{33}$$

To obtain interim efficient equilibria of the common agency game, one might first identify simply the conditions (33) with (32) obtained at interim efficient allocations. Proceeding that way would lead to choose:

$$\frac{\partial p}{\partial \theta_i}(q,\theta_i) = 1 - \tilde{\alpha}(\theta_i). \tag{34}$$

Even though they are simple, we show in the Appendix that these identifications fail to achieve interim efficiency.

The next propositions might nevertheless be viewed as being more optimistic in our quest for interim efficiency.

Proposition 6 *: Assume that $F(\cdot)$ has linear hazard rate, i.e., $\frac{1-F(\theta)}{f(\theta)} = \beta(\bar{\theta}-\theta)+\gamma$ where $(\beta,\gamma)\in\mathbb{R}^2_+$ where $\underline{\theta} \geq \frac{1}{1+\beta}(\beta(\bar{\theta}-\underline{\theta})+\gamma)$. Then, the unique linear symmetric equilibrium defined by (24) and (25) is interim efficient. It corresponds to social weights which are constant, $\alpha(\theta)=\frac{\beta}{1+\beta}$.*

Linear hazard rate distributions are attractive because they give closed-form solutions as we already know from Proposition 3. Such an equilibrium is actually interim efficient but it corresponds to the case where the uninformed mediator gives some positive weight to the common agent in his objective function.[25] This is an important result which shows that modelers may give up the complexity of the common agency game and focus instead on a centralized mechanism provided that they stipulate a given distribution of social weights which is of course linked to the distribution of types.

[25] In the case of an exponential distribution (i.e., $\beta = 0$), principals have even zero bargaining power (see Proposition 7 below).

Proposition 7 *: Assume that* $C(q) = \frac{q^2}{2}$*, and that* $F(\theta) = 1 - exp(-\frac{\theta-\lambda}{\lambda})$ *on* $[\lambda, +\infty)$*. Then, the unique linear symmetric equilibrium corresponds to an output and a non-negative marginal contribution given by:*

$$q(\theta_1, \theta_2) = \theta_1 + \theta_2 - 2\lambda, \quad \textit{and} \quad p(q, \theta) = \theta - \lambda \geq 0. \tag{35}$$

Interim efficiency is achieved with social weights on the principals which are identically null. No other interim efficient outcome can be sustained at an equilibrium.

The important point to notice in this proposition is that there is a unique interim efficient outcome which can be implemented as a common agency equilibrium. This uniqueness contrasts with Laussel and Palfrey (2003)'s finding for a discrete 0-1 public good.[26]

7 Equilibrium Uniqueness

As shown above, one may face some difficulties in using interim efficiency as a robust criterion to select equilibrium outcomes if one does not make specific assumptions on the type distribution. Alternatively, we introduce in this section another selection device: asymmetric information on the agent's cost. Our goal is to show that, provided there is enough uncertainty on the agent's cost, the multiplicity of equilibria of the common agency game disappears and, that one may select a quite tractable solution under some weak assumptions.

Formally, suppose that the agent's marginal cost writes as $C'(q) + \varepsilon$ where the random variable ε is distributed on the interval $I = [0, \bar{\varepsilon}]$ with a cumulative distribution $G(\cdot)$ and an everywhere positive and differentiable density $g(\cdot)$.

For technical reasons, we will also make the following assumptions:

Assumption 1 *:* $\frac{d}{dx}\left(\frac{g'(x)}{g(x)}\right) \leq 0$ *for any* x*.*

Assumption 2 *:* $g'(0) \leq g(\bar{\varepsilon})g(0)$*.*[27]

[26] These authors argued that there is an open set of interim efficient outcomes which arise as common agency equilibria. They import the techniques due to Gresik (1996) in the case of double-auctions. Those techniques consist in defining a given profile of social weights for one agent (say the buyer in the double-auction case) and derive the social weights of the other (the seller) so that the trade boundary derived from equilibrium behavior fits with the trade boundary achieved at an interim efficient outcome. This leeway in specifying one family of social weights is no longer available in our context. Symmetry between the two players requires that, in the type space, the boundary between producing the public good or not in a symmetric equilibrium is replicated by an interim efficient outcome where both agents receive the same social weights. This symmetry puts lots of constraints on the possible social weights.

[27] The uniform distribution on I satisfies Assumptions 1 and 2.

Assumption 3 *: $C'(q) \geq \bar{\theta} + \theta - \Phi(\theta)$ (where $\Phi = \left(\frac{G}{g}\right)^{-1}$) for all θ and q in the equilibrium range $q(\theta, \cdot)$ and $\underline{\theta} > \frac{1}{g(\bar{\varepsilon})}$.*

To analyze equilibria under asymmetric information on the agent's cost, the most useful procedure relies on the *supply profile* due to Wilson (1993). Consider thus P_i with type θ_i who pays a marginal contribution p for q units of the public good. This principal must assess the probability that the type of P_{-i} is large enough and that the agent's cost is low enough so that those q units are indeed produced. Formally, if P_{-i} follows the symmetric equilibrium strategy $p(q, \cdot)$, the likelihood that at least q units of public good are produced is $proba\left\{p(q, \tilde{\theta}_{-i}) + p \geq C'(q) + \varepsilon\right\} = E_{\widetilde{\theta}}\left[G(p + p(q, \widetilde{\theta}) - C'(q))\right]$ where $E_{\widetilde{\theta}}(\cdot)$ is the expectation operator. From this, $p(q, \theta)$ should satisfy:

$$p(q, \theta) \in \arg\max_{p} (\theta - p) E_{\widetilde{\theta}}\left[G(p + p(q, \widetilde{\theta}) - C'(q))\right].$$

The corresponding first-order condition can be written as:

$$-E_{\widetilde{\theta}}\left[G(p(q, \theta) + p(q, \widetilde{\theta}) - C'(q))\right] + (\theta - p(q, \theta)) E_{\widetilde{\theta}}\left[g(p(q, \theta) + p(q, \widetilde{\theta}) - C'(q))\right] = 0. \tag{36}$$

Therefore, a symmetric equilibrium $p(q, \theta)$ satisfies the following functional equation:

$$p(q, \theta) = \theta - \frac{E_{\widetilde{\theta}}\left[G(p(q, \theta) + p(q, \widetilde{\theta}) - C'(q))\right]}{E_{\widetilde{\theta}}\left[g(p(q, \theta) + p(q, \widetilde{\theta}) - C'(q))\right]}. \tag{37}$$

Define now the set $B = \{p(q, \cdot) : \Theta \to \mathbb{R}^+ \text{ continuous} \mid \ p(q, \theta) \leq \theta, \text{ for all } \theta \in \Theta\}$ and consider the functional operator $T[\cdot] : B \to B$ such that:

$$T[p](q, \theta) = \theta - \frac{E_{\widetilde{\theta}}\left[G(p(q, \theta) + p(q, \widetilde{\theta}) - C'(q))\right]}{E^{\widetilde{\theta}}\left[g(p(q, \theta) + p(q, \widetilde{\theta}) - C'(q))\right]}.$$

The fixed-points of $T[\cdot]$ are thus solutions of the functional equation (37).

Proposition 8 *: Under Assumptions 1 to 3, there exists a unique fixed-point $\bar{p}(\cdot) \in B$ for the operator $T[\cdot]$. Moreover, the class of differentiability of $\bar{p}(\cdot)$ is the minimum between that of $g(\cdot)$ and $C'(\cdot)$.*

Although Proposition 8 may yield uniqueness under fairly general conditions, it may be possible to reach more precise results in specific contexts. To illustrate consider the case where $g(\cdot)$ is uniform on an interval $I = [0, \bar{\varepsilon}]$ with $\bar{\varepsilon}$ sufficiently large. The following consistency condition should hold to insure that best-responses are found at points where the probability of producing the public good is strictly positive:

$$\bar{p}(q(\theta_1,\theta_2,\varepsilon),\theta_1)+\bar{p}(q(\theta_1,\theta_2,\varepsilon),\theta_2)-C'(q(\theta_1,\theta_2,\varepsilon))\in I \text{ for all } (\theta_1,\theta_2)\in\Theta^2. \tag{38}$$

Equation (36) becomes then:

$$2\bar{p}(q,\theta)=\theta-E_{\tilde{\theta}}(\bar{p}(q,\tilde{\theta}))+C'(q).$$

From which, we derive the explicit expressions of the marginal contribution at a symmetric equilibrium and the output as:

$$\bar{p}(q,\theta)=\frac{\theta}{2}-\frac{E_{\tilde{\theta}}(\tilde{\theta})}{6}+\frac{C'(q)}{3} \text{ and } C'(q(\theta_1,\theta_2,\varepsilon))=\max\left\{\frac{3}{2}(\theta_1+\theta_2)-E_{\tilde{\theta}}(\tilde{\theta})-\varepsilon,0\right\}.$$

The consistency conditions are then satisfied when

$$3\bar{\theta}-E_{\tilde{\theta}}(\tilde{\theta})-\bar{\varepsilon}\leq C'(q)\leq 3\underline{\theta}-E_{\tilde{\theta}}(\tilde{\theta}),$$

a condition which defines the range of possible equilibrium outputs and puts a lower bound on the degree of uncertainty on cost to ensure uniqueness of the equilibrium, namely: $3(\bar{\theta}-\underline{\theta})\leq\bar{\varepsilon}$.

8 Conclusion

In this paper, we have analyzed a common agency game with privately informed principals. In checking whether the earlier lessons of the complete information literature can be extended under asymmetric information, our conclusions are contrasted.

First, asymmetric information introduces incentive compatibility constraints for the principals. Those constraints replace and put on firmer foundations the "truthfulness" requirement used in the earlier complete information literature. Far from helping in selecting a unique equilibrium allocation, private information on the principals' side introduces a new source of multiplicity for the equilibria. The contribution chosen by a given principal depends on how he expects others to change their own marginal contributions as their types change. The introduction of asymmetric information on the agent's cost function might nevertheless help to select a unique equilibrium, at least, as long as uncertainty on that cost function is sufficiently spread.

Second, ex post inefficiencies are pervasive in common agency game. The weaker criterion of interim efficiency may be satisfied by some equilibria in tractable examples characterized by a type distribution with a linear hazard rate. In which case, the distribution of social weights given to the various types of principals can be derived from the type distribution. Interestingly, the common agency equilibrium could result from a more centralized mechanism design problem where principals receive as a whole a weight less than one. Beyond that class of distribution, it is still unknown to us whether an interim efficient allocation always arises at equilibrium.

Third, the class of pointwise optimal equilibria that we stressed can easily be analyzed with standard mechanism design techniques using the theory of informed principals. Those equilibria are robust in the sense that a principal would not like to deviate to a larger space of mechanisms to improve his payoff. Using the theory of informed principals shows that inefficiencies come from the fact that principals want to screen others' types rather than hiding their own preferences contrary to the intuition underlying the "free-riding" problem. This gives a fresh look at inefficiencies in public good provision.

Our model should certainly be extended along several directions. First, other information structures would be worth being investigated, for instance, allowing correlation between the principals' types or common values aspects. Second, following Maskin and Tirole (1990), we know that risk-aversion on the principals' side forces some pooling in informed principal games. Risk-aversion on the principals' side may thus significantly change equilibrium patterns in common agency environments. Third, one could also be interested in analyzing settings with asymmetric principals. Lastly, in other institutional contexts, allocations do not result from a well-centralized mechanisms but come out of the equilibria among various decentralized mechanisms. One may think of multi-unit auctions on financial or electricity markets for instance. It would be nice to extend the approach taken in this paper to these environments. We hope to investigate some of these issues in future research.

References

Bergemann, D. and J. Välimäki, (2003), "Dynamic Common Agency", *Journal of Economic Theory*, forthcoming in *Journal of Economic Theory.*

Bernheim, D. and M. Whinston, (1986a), "Menu Auctions, Resource Allocations and Economic Influence", *Quarterly Journal of Economics*, 101: 1-31.

Bernheim, D. and M. Whinston, (1986b), "Common Agency", *Econometrica*, 54: 923-942.

Biais, B., D. Martimort and J.C. Rochet, (2000), "Competing Mechanisms in a Common Value Environment", *Econometrica*, 68: 799-837.

Clarke, E., (1971), "Multipart Pricing of Public Goods", *Public Choice*, 8: 19-33.

D'Aspremont, C. and L.A. Gerard-Varet, (1979), "Incentives and Complete Information", *Journal of Public Economics*, 11: 25-45.

Dixit, A., G. Grossman and E. Helpman, (1997), "Common Agency and Coordination", *Journal of Political Economy*, 105: 752-769.

Edlin, A. and C. Shannon, (1998), "Strict Monotonicity in Comparative Statics", *Journal of Economic Theory*, 81: 201-219.

Epstein, L. and M. Peters, (1999), "A Revelation Principle for Competing Mechanisms", *Journal of Economic Theory*, 88: 119-160.

Gresik, T., (1996), "Incentive-Efficient Equilibria of Two-Party Sealed-Bid Bargaining Games", *Journal of Economic Theory*, 68: 26-48.

Grossman, G. and E. Helpman, (1994), "Protection for Sale", *American Economic Review*, 84: 833-850.

Grossman, G. and E. Helpman, (2001), *Special Interest Politics*, MIT Press.

Groves, T., (1973), "Incentives in Teams", *Econometrica*, 41: 617-631.

Hirsch, M. and S. Smale, (1974), *Differential Equations, Dynamic Systems, and Linear Algebra*. Academic Press.

Holmström, B. and R. Myerson, (1983), "Efficient and Durable Decision Rules with Incomplete Information", *Econometrica*, 51: 1799-1819.

Laffont, J.J. and E. Maskin, (1979), "A Differentiable Approach to Expected Utility Maximizing Mechanisms", in *Aggregation and Revelation of Preferences*, ed. J.J. Laffont. North Holland.

Laussel, D. and M. Lebreton, (1998), "Efficient Private Production of Public Goods under Common Agency", *Games and Economic Behavior*, 25: 194-218.

Laussel, D. and T. Palfrey, (2003), "Efficient Equilibria in the Voluntary Contributions Mechanisms with Private Information", *Journal of Public Economic Theory*, 5: 449-478.

Ledyard, J. and T. Palfrey, (1999), "A Characterization of Interim Efficiency with Public Goods", *Econometrica*, 67: 435-448.

Leininger, W., P. Linhart and R. Radner, (1989), "Equilibria of the Sealed-Bid Mechanism for Bargaining with Incomplete Information", *Journal of Economic Theory*, 48: 63-106.

Mailath, G. and A. Postlewaite, (1990), "Asymmetric Information Bargaining Problems with Many Agents", *Review of Economic Studies*, 57: 351-368.

Martimort, D., (1992), "Multi-Principaux avec Anti-Selection", *Annales d'Economie et de Statistiques*, 28: 1-38.

Martimort, D., (1996), "Exclusive Dealing, Common Agency, and Multiprincipals Incentive Theory", *Rand Journal of Economics*, 27: 1-31.

Martimort, D. and L. Stole, (2002), "The Revelation and Delegation Principles in Common Agency Games", *Econometrica*, 70: 1659-1674.

Martimort, D. and L. Stole, (2003), "Contractual Externalities and Common Agency Equilibria", *Advances in Theoretical Economics*, Vol. 3, Issue 1, Article 4. http://www.bepress.com/bejte.

Maskin, E. and J. Tirole, (1990), "The Principal-Agent Relationship with an Informed Principal, I: The Case of Private Values", *Econometrica*, 58: 379-410.

Menezes, F., P. Monteiro and A. Temini, (2001), "Private Provision of Discrete Public Goods with Incomplete Information", *Journal of Mathematical Economics*, 35: 493-514.

Mezzetti, C., (1997), "Common Agency with Horizontally Differentiated Principals", *Rand Journal of Economics*, 28: 323-345.

Peters, M., (2001), "Common Agency and the Revelation Principle", *Econometrica*, 69: 1349-1372.

Prat, A. and A. Rustichini, (2003), "Games Played through Agents", *Econometrica*, 71: 989-1026.

Rob, R., (1989), "Pollutions Claims Settlements with Private Information", *Journal of Economic Theory*, 47: 307-333.

Sattherwaite, M. and S. Williams, (1989), "Bilateral Trade with the Sealed-Bid k-Double Auction: Existence and Efficiency", *Journal of Economic Theory*, 48: 107-133.

Stole, L., (1991), "Mechanism Design under Common Agency", mimeo University of Chicago.

Wilson, R. (1979), "Auctions of Shares", *Quarterly Journal of Economics*, 93: 675-689.

Wilson, R., (1993), *Nonlinear Pricing.* Oxford University Press.

Appendix

Proof of Lemma 1: Let us fix (θ_i, θ_{-i}) and consider $\theta_i > \theta_i'$. By definition, we have

$$t(q(\theta_i,\theta_{-i}),\theta_i) + t(q(\theta_i,\theta_{-i}),\theta_{-i}) - C(q(\theta_i,\theta_{-i})) \geq t(\tilde{q},\theta_i) + t(\tilde{q},\theta_{-i}) - C(\tilde{q}), \qquad \forall \tilde{q}.$$

Thus,

$$t(q(\theta_i,\theta_{-i}),\theta_i) - t(\tilde{q},\theta_i) \geq t(\tilde{q},\theta_{-i}) - C(\tilde{q}) - \left[t(q(\theta_i,\theta_{-i}),\theta_{-i}) - C(q(\theta_i,\theta_{-i}))\right], \qquad \forall \tilde{q}.$$

Using SMP, the l.h.s. above is lower than $t(q(\theta_i,\theta_{-i}),\theta_i') - t(\tilde{q},\theta_i')$ for all $\tilde{q} \geq q(\theta_i,\theta_{-i})$. Then, $q(\theta_i,\theta_{-i}) \geq q(\theta_i',\theta_{-i})$ and $q(\cdot)$ is almost everywhere differentiable in each of its arguments.[28] Using Theorem 2 of Edlin and Shannon (1998), inequalities are strict when SSMP holds. ∎

Proof of Proposition 1:

• **First- and Second-Order Conditions**: Integrating by parts and using (2) yields:

$$E_{\theta_{-i}}\left[t(q(\hat{\theta}_i,\cdot),\hat{\theta}_i)\right] = (F(\cdot)-1)t(q(\hat{\theta}_i,\cdot),\hat{\theta}_i)\Big|_{\underline{\theta}}^{\bar{\theta}} + E_{\theta_{-i}}\left[\frac{1-F(\cdot)}{f(\cdot)}p(q(\hat{\theta}_i,\cdot),\hat{\theta}_i)\frac{\partial q}{\partial \theta_{-i}}(\hat{\theta}_i,\cdot)\right]$$
$$= t(q(\hat{\theta}_i,\underline{\theta}),\hat{\theta}_i) + E_{\theta_{-i}}\left[\frac{1-F(\cdot)}{f(\cdot)}\left(C'(q(\hat{\theta}_i,\cdot)) - p(q(\hat{\theta}_i,\cdot),\cdot)\right)\frac{\partial q}{\partial \theta_{-i}}(\hat{\theta}_i,\cdot)\right].$$

Inserting into the maximand of (4) gives us finally:

$$\Phi(\hat{\theta}_i,\theta_i) = E_{\theta_{-i}}\left[\left(\theta_i q(\hat{\theta}_i,\cdot) - \frac{1-F(\cdot)}{f(\cdot)}\left(C'(q(\hat{\theta}_i,\cdot)) - p(q(\hat{\theta}_i,\cdot),\cdot)\right)\right)\frac{\partial q}{\partial \theta_{-i}}(\hat{\theta}_i,\cdot)\right] - t(q(\hat{\theta}_i,\underline{\theta}),\hat{\theta}_i). \tag{A1}$$

Using (A1), we get the following first-order derivative of $\Phi(\cdot)$ with respect to $\hat{\theta}_i$:

$$\frac{\partial \Phi}{\partial \hat{\theta}_i}(\hat{\theta}_i,\theta_i) = E_{\theta_{-i}}\left[\left(\theta_i - \frac{1-F(\cdot)}{f(\cdot)}\left(C''(q(\hat{\theta}_i,\cdot)) - \frac{\partial p}{\partial q}(q(\hat{\theta}_i,\cdot),\cdot)\right)\right)\frac{\partial q}{\partial \theta_i}(\hat{\theta}_i,\cdot)\frac{\partial q}{\partial \theta_{-i}}(\hat{\theta}_i,\cdot)\right]$$
$$-E_{\theta_{-i}}\left[\frac{1-F(\cdot)}{f(\cdot)}(C'(q(\hat{\theta}_i,\cdot)) - p(q(\hat{\theta}_i,\cdot),\cdot))\frac{\partial^2 q}{\partial \theta_i \partial \theta_{-i}}(\hat{\theta}_i,\cdot)\right]$$
$$-p(q(\hat{\theta}_i,\underline{\theta}),\hat{\theta}_i)\frac{\partial q}{\partial \theta_i}(\hat{\theta}_i,\underline{\theta}) - \frac{\partial t}{\partial \theta_i}(q(\hat{\theta}_i,\underline{\theta}),\hat{\theta}_i). \tag{A2}$$

Integrating by parts the second term yields

$$E_{\theta_{-i}}\left[\frac{1-F(\cdot)}{f(\cdot)}(C'(q(\hat{\theta}_i,\cdot)) - p(q(\hat{\theta}_i,\cdot),\cdot))\frac{\partial^2 q}{\partial \theta_i \partial \theta_{-i}}(\hat{\theta}_i,\cdot)\right]$$

[28] In case, there are several maximizers of $t(q,\theta_i) + t(q,\theta_{-i}) - C(q)$ the argument applies also to any selection of the correspondence.

$$= (1 - F(\cdot))(C'(q(\hat{\theta}_i, \cdot)) - p(q(\hat{\theta}_i, \cdot), \cdot))\frac{\partial q}{\partial \theta_i}(\hat{\theta}_i, \cdot)\Big|_{\underline{\theta}}^{\bar{\theta}}$$

$$-E_{\theta_{-i}}\left[\frac{1 - F(\cdot)}{f(\cdot)}\frac{\partial q}{\partial \theta_i}(\hat{\theta}_i, \cdot)\left(\frac{\partial q}{\partial \theta_{-i}}(\hat{\theta}_i, \cdot)\left(C''(q(\hat{\theta}_i, \cdot)) - \frac{\partial p}{\partial q}(q(\hat{\theta}_i, \cdot), \cdot)\right) + \frac{\partial p}{\partial \theta_i}(q(\hat{\theta}_i, \cdot), \cdot)\right)\right]$$

$$+E_{\theta_{-i}}\left[C'(q(\hat{\theta}_i, \cdot)) - p(q(\hat{\theta}_i, \cdot), \cdot)\right]$$

$$= -p(q(\hat{\theta}_i, \underline{\theta}), \hat{\theta}_i)\frac{\partial q}{\partial \theta_i}(\hat{\theta}_i, \underline{\theta})$$

$$-E_{\theta_{-i}}\left[\frac{1 - F(\cdot)}{f(\cdot)}\left\{\left(C''(q(\hat{\theta}_i, \cdot)) - \frac{\partial p}{\partial q}(q(\hat{\theta}_i, \cdot), \cdot)\right)\frac{\partial q}{\partial \theta_i}(\hat{\theta}_i, \cdot)\frac{\partial q}{\partial \theta_{-i}}(\hat{\theta}_i, \cdot) + \frac{\partial q}{\partial \theta_i}(\hat{\theta}_i, \cdot)\frac{\partial p}{\partial \theta_i}(q(\hat{\theta}_i, \cdot), \cdot)\right\}\right]$$

$$+E_{\theta_{-i}}\left[C'(q(\hat{\theta}_i, \cdot)) - p(q(\hat{\theta}_i, \cdot), \cdot)\right]. \tag{A3}$$

where the last equality comes from (2) for $\theta_1 = \hat{\theta}_i$ and $\theta_2 = \underline{\theta}$.

By Lemma 4 below, it must be that the agent's payoff with type $\hat{\theta}_i$ when $\theta_{-i} = \underline{\theta}$ is zero. This gives:

$$U(\hat{\theta}_i, \underline{\theta}) = t(q(\hat{\theta}_i, \underline{\theta}), \hat{\theta}_i) + t(q(\hat{\theta}_i, \underline{\theta}), \underline{\theta}) - C(q(\hat{\theta}_i, \underline{\theta})) = 0, \quad \text{for all } \hat{\theta}_i. \tag{A4}$$

Differentiating w.r.t. $\hat{\theta}_i$ yields:

$$\left(p(q(\hat{\theta}_i, \underline{\theta}), \hat{\theta}_i) + p(q(\hat{\theta}_i, \underline{\theta}), \underline{\theta}) - C'(q(\hat{\theta}_i, \underline{\theta}))\right)\frac{\partial q}{\partial \theta_i}(\hat{\theta}_i, \underline{\theta}) + \frac{\partial t}{\partial \theta_i}(q(\hat{\theta}_i, \underline{\theta}), \hat{\theta}_i) = 0$$

and thus using (2),

$$\frac{\partial t}{\partial \theta_i}(q(\hat{\theta}_i, \underline{\theta}), \hat{\theta}_i) = 0, \quad \text{for all } \hat{\theta}_i. \tag{A5}$$

Inserting into (A2) yields:

$$\frac{\partial \Phi}{\partial \hat{\theta}_i}(\hat{\theta}_i, \theta_i) = E_{\theta_{-i}}\left[\frac{\partial q}{\partial \theta_i}(\hat{\theta}_i, \cdot)\left(\theta_i + p(q(\hat{\theta}_i, \cdot), \cdot) - C'(q(\hat{\theta}_i, \cdot)) - \frac{1 - F(\cdot)}{f(\cdot)}\frac{\partial p}{\partial \theta_{-i}}(q(\hat{\theta}_i, \cdot), \cdot))\right)\right]. \tag{A6}$$

For $\hat{\theta}_i = \theta_i$ to be the optimal report, i.e., $\frac{\partial \Phi}{\partial \hat{\theta}_i}(\theta_i, \theta_i) = 0$, the first-order condition (5) must hold. The second-order condition for the principal's problem is

$$\frac{\partial^2 \Phi}{\partial \hat{\theta}_i^2}(\hat{\theta}_i, \theta_i)\Big|_{\hat{\theta}_i = \theta_i} \leq 0.$$

Using (A6), the Envelope Theorem and taking the total derivative of (A5) with respect to $\hat{\theta}_i$, we get

$$\frac{\partial^2 \Phi}{\partial \hat{\theta}_i^2}(\theta_i, \theta_i) = -E_{\theta_{-i}}\left[\frac{\partial q}{\partial \theta_i}(\theta_i, \cdot)\right].$$

Hence, (6) holds.

• **Non-Negative Contributions and the Agent's Participation Constraints**: We want to show that (1) always holds. We proceed with two lemmata.

Lemma 3 *: In any SMP equilibrium,* $U(\theta_1,\theta_2) \geq \hat{U}(\theta_1)$ *at any* (θ_1,θ_2) *if* $U(\underline{\theta},\theta_2) \geq \hat{U}(\underline{\theta})$ *for any* θ_2*.*

Proof: Note that

$$\frac{\partial U}{\partial \theta_1}(\theta_1,\theta_2) = \frac{\partial t}{\partial \theta_1}(q(\theta_1,\theta_2),\theta_1)$$

and

$$\frac{\partial \hat{U}}{\partial \theta_1}(\theta_1) = \frac{\partial t}{\partial \theta_1}(\hat{q}(\theta_1),\theta_1)$$

where $\hat{q}(\theta_1) = \arg\max_q t(q,\theta_1) - C(q) \leq q(\theta_1,\theta_2)$ and where the last inequality follows from the fact that marginal contributions are non-negative. Then, we have:

$$\frac{\partial U}{\partial \theta_1}(\theta_1,\theta_2) \geq \frac{\partial \hat{U}}{\partial \theta_1}(\theta_1)$$

from SMP and the lemma is proved. ■

Lemma 4 *: In any SMP symmetric equilibrium,* $U(\underline{\theta},\theta) = 0 > \hat{U}(\underline{\theta})$ *for any* θ*.*

Proof: When $p(q,\theta) \geq 0$ for all q in the range of $q(\cdot,\theta)$, one has $\hat{q}(\underline{\theta}) = q(\underline{\theta},\underline{\theta})$. Then,

$$\hat{U}(\underline{\theta}) = t(q(\underline{\theta},\underline{\theta}),\underline{\theta}) - C(q(\underline{\theta},\underline{\theta})) = -\frac{C(q(\underline{\theta},\underline{\theta}))}{2} < 0$$

in a symmetric equilibrium. ■

Putting together Lemmas 3 and 4 ensures that the relevant binding participation constraints in a SMP equilibrium are (A4). Given that (1) holds, contributions are necessarily non-negative. ■

Proof of Lemma 2:

• **Agent's Problem**: We show that the local conditions (2), the monotonicity conditions (8) and the SMP property are enough for global optimality of the agent's problem. Fix q such that $q \leq q(\theta_1,\theta_2)$ (the case $q \geq q(\theta_1,\theta_2)$ can be treated similarly). We have:

$$\sum_{i=1}^{2} t(q(\theta_1,\theta_2),\theta_i) - C(q(\theta_1,\theta_2)) - \left(\sum_{i=1}^{2} t(q,\theta_i) - C(q)\right) = \int_q^{q(\theta_1,\theta_2)} \left(\sum_{i=1}^{2} p(x,\theta_i) - C'(x)\right) dx.$$

For any $x \in [q, q(\theta_1,\theta_2)]$, define $\psi(x,\theta_1)$ from the first-order condition (2) such that $p(x,\theta_1) + p(x,\psi(x,\theta_1)) = C'(x)$. Note that (8) and the definition of ψ imply that $\psi(x,\theta_1)$ is weakly increasing in x. Then, we have:

$$\int_q^{q(\theta_1,\theta_2)} \left(\sum_{i=1}^{2} p(x,\theta_i) - C'(x)\right) dx = \int_q^{q(\theta_1,\theta_2)} (p(x,\theta_2) - p(x,\psi(x,\theta_1)))dx \geq 0$$

since $\psi(x,\theta_1) \leq \psi(q(\theta_1,\theta_2),\theta_1) = \theta_2$ and SMP holds. This shows that the local second-order condition (4) is implied by SMP and the monotonicity conditions (8).

• **Principal's Problem**: We show that the schedule satisfying (7) and (8) is not only locally incentive compatible but also globally. From (A6) and (8) we have

$$\Phi(\theta_i,\theta_i) - \Phi(\hat{\theta}_i,\theta_i) = \int_{\hat{\theta}_i}^{\theta_i} \frac{\partial \Phi}{\partial \hat{\theta}_i}(x,\theta_i)dx = \int_{\hat{\theta}_i}^{\theta_i} \left[\frac{\partial \Phi}{\partial \hat{\theta}_i}(x,\theta_i) - \frac{\partial \Phi}{\partial \hat{\theta}_i}(x,x)\right] dx$$

$$= \int_{\hat{\theta}_i}^{\theta_i} E_{\theta_{-i}} \left[\frac{\partial q}{\partial \theta_i}(x,\theta_{-i})\right] (\theta_i - x)dx.$$

By (8) this last expression is always non-negative. ■

Proof of Proposition 2: Let us fix any strictly increasing output schedule $Q(\theta)$ such that $Q(\bar{\theta}) = q^*(\bar{\theta},\bar{\theta})$. Define also $\tilde{\theta}$ such that $Q(\tilde{\theta}) = q$ for any q in the range of $Q(\cdot)$.

• **System of Differential Equations:** From (18) taken when θ is replaced by $\psi(q,\theta)$, we have:

$$\frac{1-F(\theta)}{f(\theta)}\frac{\partial p}{\partial \theta}(q,\theta) = \psi(q,\theta) - p(q,\psi(q,\theta)). \tag{A7}$$

Using (17), we get (22). Now differentiating (17) with respect to θ yields:

$$\frac{\partial p}{\partial \theta}(q,\theta) = -\frac{\partial p}{\partial \theta}(q,\psi(q,\theta))\frac{\partial \psi}{\partial \theta}(q,\theta).$$

Using (18), we get (23).

Using (17) at $\theta = \tilde{\theta}$, we obtain the initial conditions (19) of the system of differential equations. Note that this system is locally Lipschitz around $\tilde{\theta}$ for any $\tilde{\theta} < \bar{\theta}$. Hence, from Theorem 1 p. 162 in Hirsch and Smale (1974), the solution to this system is locally unique in a neighborhood on the left of $\tilde{\theta}$.[29]

The issue is that (23) fails to be Lipschitz at all pairs (θ,p) because of the denominator on the right-hand side. To solve this difficulty and prove global uniqueness of the solution, we may rewrite (22)-(23)[30] as a system of autonomous differential equations parameterized with some $x \in \mathbb{R}^+$:

$$\frac{\partial p}{\partial x}(q,x) = (\psi(q,x) + p(q,x) - C'(q))(\theta(x) - p(q,x)), \tag{A8}$$

$$\frac{\partial \psi}{\partial x}(q,x) = -\frac{1-F(\psi(q,x))}{f(\psi(q,x))}(\psi(q,x) + p(q,x) - C'(q)), \tag{A9}$$

$$\frac{d\theta}{dx}(x) = \frac{1-F(\theta(x))}{f(\theta(x))}(\theta(x) - p(q,x)), \tag{A10}$$

[29] The case $\tilde{\theta} = \bar{\theta}$ is degenerate since then (17), (18) and (A7) altogether imply that $q = q^*(\bar{\theta},\bar{\theta})$.
[30] Slightly abusing notations.

with the initial conditions

$$p(q,0) = \frac{C'(q)}{2} \quad \text{and} \quad \psi(q,0) = \theta(0) = \tilde{\theta}. \tag{A11}$$

This system is now Lipschitz everywhere when $\frac{1-F(\theta)}{f(\theta)}$ is itself Lipschitz. Hence, from Theorem 1 p. 162 and Lemma p. 171 in Hirsch and Smale (1974), its solution is globally unique on a maximal interval belonging to $\mathbb{R}^+$.

• **Monotonicity Properties $\frac{\partial \psi}{\partial \theta}(q,\theta) < 0$ and $\frac{\partial p}{\partial \theta}(q,\theta) > 0$:** Note that, for $\tilde{\theta}$, $\frac{\partial \psi}{\partial \theta}(q,\tilde{\theta}) = -1$ from (23) and thus $\frac{\partial \psi}{\partial \theta}(q,\theta) < 0$ in the neighborhood of $\tilde{\theta}$. Similarly, we get $\frac{\partial p}{\partial \theta}(q,\tilde{\theta}) = \frac{\tilde{\theta} - \frac{C'(q)}{2}}{\frac{1-F(\tilde{\theta})}{f(\tilde{\theta})}} > 0$ since $q = Q(\tilde{\theta}) < q^*(\tilde{\theta},\tilde{\theta})$.

Suppose that there exist values of θ such that $\frac{\partial \psi}{\partial \theta}(q,\theta) > 0$. Denote then by θ_1 the highest value of $\theta < \tilde{\theta}$ such that $\frac{\partial \psi}{\partial \theta}(q,\theta) = 0$. Of course $\theta_1 < \tilde{\theta}$. Then, from (22) and (23), either $\psi(q,\theta_1) = \bar{\theta}$ or $\frac{\partial p}{\partial \theta}(q,\theta_1) = 0$.

In the first case, we are done because the upper part of the isoquant is already completely defined on $[\theta_1, \tilde{\theta}]$ only. In the second case, note that we must have $\psi(q,\theta_1) + p(q,\theta_1) - C'(q) = 0$. Observe that, at such θ_1, the system of differential equations (22)-(23) is Lipschitz unless $\theta_1 = p(q,\theta_1)$. When it is Lipschitz, its solution is thus locally unique. But then the pair of constant functions $\psi(q,\theta) = \psi(q,\theta_1)$ and $p(q,\theta) = p(q,\theta_1)$ is a solution on the right of θ_1, a contradiction with the definition of θ_1 and the fact that $\frac{\partial \psi}{\partial \theta}(q,\theta) < 0$ on the right of θ_1.

Is it possible that $\theta_1 = p(q,\theta_1)$? To prove the converse, first note that (22) can be rewritten as:

$$\frac{\partial}{\partial \theta}\left[(1-F(\theta))p(q,\theta)\right] = (\psi(q,\theta) - C'(q))f(\theta).$$

Integrating yields

$$(1-F(\theta))p(q,\theta) = k + \int_{\tilde{\theta}}^{\theta} \psi(q,x)f(x)dx + C'(q)(1-F(\theta))$$

where k is a constant of integration.

But using the initial conditions (19), we get $k = -\frac{C'(q)}{2}(1-F(\tilde{\theta}))$ and thus:

$$p(q,\theta) = C'(q)\left(1 - \frac{1-F(\tilde{\theta})}{2(1-F(\theta))}\right) + \frac{1}{1-F(\theta)}\int_{\tilde{\theta}}^{\theta} \psi(q,x)f(x)dx. \tag{A12}$$

Integrating by parts on the right-hand side and manipulating, we get

$$\theta - p(q,\theta) = \theta + \psi(q,\theta) - C'(q)$$

$$+\left(\frac{C'(q)}{2}-\tilde{\theta}\right)\left(\frac{1-F(\tilde{\theta})}{1-F(\theta)}\right)-\frac{1}{1-F(\theta)}\int_{\tilde{\theta}}^{\theta}\frac{\partial\psi}{\partial\theta}(q,x)(1-F(x))dx. \tag{A13}$$

For θ_1 such that $\theta_1 = p(q,\theta_1)$, the integral on the right-hand side of (A13) is positive since $\frac{\partial\psi}{\partial x}(q,x) < 0$ on $[\theta_1, \tilde{\theta}]$ by definition of θ_1. Moreover, $2\tilde{\theta} > C'(q)$ by definition of $Q(\cdot)$. Hence, we find

$$\theta_1 + \psi(q,\theta_1) - C'(q) > 0$$

which leads (from (23)) to a contradiction with the fact that $\frac{\partial\psi}{\partial\theta}(q,\theta_1) = 0$.

- **Constructing Equilibria:** The solution $\{\psi(q,\theta), p(q,\theta)\}$ starting from the initial condition (19) allows us to construct an equilibrium if the second-order condition of the agent's problem is satisfied.

Differentiating (17) with respect to q, we get:

$$\frac{\partial p}{\partial q}(q,\theta)+\frac{\partial p}{\partial q}(q,\psi(q,\theta))-C''(q)=-\frac{\partial p}{\partial\theta}(q,\psi(q,\theta))\frac{\partial\psi}{\partial q}(q,\theta). \tag{A14}$$

Given that $\frac{\partial p}{\partial\theta}(q,\theta) > 0$, the condition $\frac{\partial\psi}{\partial q}(q,\theta) > 0$ ensures that the right-hand side of (A14) is negative, ensuring local concavity of the agent's problem. From Lemma 2, this ensures also global concavity of the agent's problem.

Note that $\theta = \psi(Q(\theta),\theta)$. Differentiating with respect to θ and taking into account that $Q(\theta)$ is increasing, we find $\frac{\partial\psi}{\partial q}(Q(\theta),\theta) > 0$. When $\bar{\theta}-\underline{\theta}$ is small enough, this positive sign holds on the whole domain and thus $\frac{\partial\psi}{\partial q}(q,\theta) > 0$.

Similarly, $p(q,\theta) > 0$ when $\bar{\theta}-\underline{\theta}$ is small enough by continuity since $p(q,\tilde{\theta}) > 0$. ■

Proof of Corollary 1: Since $\frac{\partial p}{\partial\theta}(q,\theta) > 0$ on $[\underline{\theta},\tilde{\theta}]$, we immediately obtain that $\psi(q,\theta) \geq p(q,\psi(q,\theta))$ on that interval (the inequality is strict when $\theta < \tilde{\theta} \leq \bar{\theta}$) and thus $\theta > p(q,\theta)$ on $[\tilde{\theta},\bar{\theta}]$ (except possibly when $\psi(q,\theta) = \bar{\theta}$ where this is an equality). We could as well have studied (22)-(23) over the interval $[\tilde{\theta},\bar{\theta}]$ and get similarly $\frac{\partial p}{\partial\theta}(q,\theta) > 0$ on that interval, i.e., $\psi(q,\theta) \geq p(q,\psi(q,\theta))$ on $[\tilde{\theta},\bar{\theta}]$ or $\theta \geq p(q,\theta)$ on $[\underline{\theta},\tilde{\theta}]$. ■

Proof of Proposition 3: Postulating $\frac{\partial p}{\partial\theta}(q,\theta) = k$ for some k, the equilibrium condition for principal P_1 yields:

$$\theta_1 - p(q,\theta_1) = (\beta(\bar{\theta}-\theta_2)+\gamma)k. \tag{A15}$$

Summing with a similar equation coming from principal P_2's best response yields:

$$C'(q(\theta_1,\theta_2)) = (\theta_1+\theta_2)(1+\beta k) - 2\beta k\bar{\theta} - 2\gamma k. \tag{A16}$$

Expressing $\theta_2 = \psi(q,\theta_1)$ from (A16) and inserting into (A15) yields an expression of $p(q,\theta)$ whose derivative w.r.t. θ must be equal to k. This yields $k = \frac{1}{1+\beta}$ and, finally, equations (24) and (25).

Finally, the condition $\underline{\theta} \geq \frac{1}{1+\beta}(\beta(\bar{\theta} - \underline{\theta}) + \gamma)$ ensures that marginal contributions are positive. ■

Proof of Proposition 4: For equilibria where $\frac{\partial t_{-i}}{\partial \theta_{-i}}(q, \theta_{-i}) \geq 0$, $U(\theta_{-i}|\theta_i)$ is increasing and (28) is binding at $\theta_{-i} = \underline{\theta}$ provided that the marginal contribution $\frac{\partial t_{-i}}{\partial q}(q, \theta_{-i})$ is positive. Integrating by parts, we obtain:

$$E_{\theta_{-i}}\left[U(\cdot|\theta_i)\right] = E_{\theta_{-i}}\left[\frac{1-F(\cdot)}{f(\cdot)}\frac{\partial t_{-i}}{\partial \theta_{-i}}(q(\cdot|\theta_i), \cdot)\right].$$

Inserting into (29), we have to optimize pointwise with respect to q the following expression of the virtual surplus of principal P_i:

$$S_i(q, \theta_i, \theta_{-i}) = \theta_i q + t_{-i}(q, \theta_{-i}) - C(q) - \frac{1-F(\theta_{-i})}{f(\theta_{-i})}\frac{\partial t_{-i}}{\partial \theta_{-i}}(q, \theta_{-i}).$$

This expression is concave in q when

$$\frac{\partial^2 S_i}{\partial q^2}(q, \theta_i, \theta_{-i}) = \frac{\partial p_{-i}}{\partial q}(q, \theta_{-i}) - C''(q) - \frac{1-F(\theta_{-i})}{f(\theta_{-i})}\frac{\partial^2 p_{-i}}{\partial \theta_{-i}\partial q}(q, \theta_{-i}) \leq 0.$$

Define now $\psi_{-i}(q, \theta_{-i})$ from (31). Differentiating (31) with respect to q yields that the virtual surplus is strictly concave if and only if $\frac{\partial \psi_{-i}}{\partial q}(q, \theta_{-i}) > 0$.

Optimization of the virtual surplus leads then to an output $q(\theta_{-i}|\theta_i)$ which satisfies

$$\frac{\partial S_i}{\partial q}(q(\theta_{-i}|\theta_i), \theta_i, \theta_{-i}) = 0.$$

This is the pointwise optimality condition (7) at a symmetric equilibrium. Provided that $\frac{\partial q}{\partial \theta_{-i}}(\theta_{-i}|\theta_i) \geq 0$, we have indeed the solution to (P_i).

Differentiating (7) with respect to θ_{-i}, we have:

$$\frac{\partial^2 S_i}{\partial q^2}\frac{\partial q}{\partial \theta_{-i}} = \frac{1-F(\theta_{-i})}{f(\theta_{-i})}\frac{\partial^2 p_{-i}}{\partial \theta_{-i}^2} - \left(1 - \frac{d}{d\theta_{-i}}\left(\frac{1-F(\theta_{-i})}{f(\theta_{-i})}\right)\right)\frac{\partial p_{-i}}{\partial \theta_{-i}}. \tag{A17}$$

Differentiating (31) with respect to θ_{-i} allows to simplify (A17) to get

$$\frac{\partial^2 S_i}{\partial q^2}\frac{\partial q}{\partial \theta_{-i}} = \frac{\partial \psi_{-i}}{\partial \theta_{-i}}.$$

At a symmetric equilibrium characterized in Proposition 2, given that P_{-i} makes the offer $t(q, \theta_{-i})$ defined through its non-negative margin $p(q, \theta_{-i})$ (up to a constant), P_i cannot do better than offering a direct revelation mechanism which allows to reconstruct the nonlinear schedule $t(q, \theta_i)$ itself. ■

Simple Identifications Fail to Achieve Interim Efficiency: We express first the slope of the agent's indifference curve in two different ways:

$$\frac{\frac{\partial q}{\partial \theta_1}}{\frac{\partial q}{\partial \theta_2}} = \frac{\frac{\partial p}{\partial \theta_1}(q,\theta_1)}{\frac{\partial p}{\partial \theta_2}(q,\theta_2)} = \frac{1-\tilde{\alpha}(\theta_1)}{1-\tilde{\alpha}(\theta_2)}. \tag{A18}$$

Using the expression of $q(\cdot)$ as an interim efficient outcome coming from (32), we get also:

$$\frac{\frac{\partial q}{\partial \theta_1}}{\frac{\partial q}{\partial \theta_2}} = \frac{\dot{b}(\theta_1)}{\dot{b}(\theta_2)}. \tag{A19}$$

Identifying (A18) and (A19), we get:

$$\frac{\dot{b}(\theta_1)}{1-\tilde{\alpha}(\theta_1)} = \frac{\dot{b}(\theta_2)}{1-\tilde{\alpha}(\theta_2)} \tag{A20}$$

which must be true for any pairs (θ_1, θ_2). This leads to set:[31]

$$\dot{b}(\theta_1) = 1-\tilde{\alpha}(\theta_1). \tag{A21}$$

This yields the following differential equation solved by $\tilde{\alpha}(\cdot)$:

$$\tilde{\alpha}(\theta) = \frac{d}{d\theta}\left(\frac{1-F(\theta)}{f(\theta)}(1-\tilde{\alpha}(\theta))\right).$$

Setting $z(\theta) = \theta - b(\theta) = \frac{1-F(\theta)}{f(\theta)}(1-\tilde{\alpha}(\theta))$, we have $z(\underline{\theta}) = z(\bar{\theta}) = 0$ and

$$\dot{z}(\theta) = 1 - \frac{f(\theta)}{1-F(\theta)}z(\theta). \tag{A22}$$

Finally, taking into account the initial conditions, the solution writes as:

$$z^*(\theta) = (1-F(\theta))\int_{\underline{\theta}}^{\theta}\frac{dx}{1-F(x)} \text{ and } b^*(\theta) = \theta - (1-F(\theta))\int_{\underline{\theta}}^{\theta}\frac{dx}{1-F(x)}. \tag{A23}$$

This leads to find:

$$\frac{\partial p}{\partial \theta}(q,\theta) = 1-\tilde{\alpha}(\theta) = 1 - f(\theta)\int_{\underline{\theta}}^{\theta}\frac{dx}{1-F(x)}. \tag{A24}$$

From which, we derive

$$p(q,\theta) = b^*(\theta) + \phi(q) \tag{A25}$$

where $\phi(\cdot)$ is a constant of integration. Using the fact that

$$p(q(\theta_1,\theta_2),\theta_1) + p(q(\theta_1,\theta_2),\theta_2) = C'(q(\theta_1,\theta_2)) = b(\theta_1) + b(\theta_2)$$

[31] Alternatively, we may set $\mu\dot{b}(\theta_1) = 1-\tilde{\alpha}(\theta_1)$ for some $\mu > 0$. The proof of the impossibility is similar.

we deduce that $\phi(q(\theta_1,\theta_2)) = 0$ for all (θ_1,θ_2). Hence, we obtain

$$p(q,\theta) = b^*(\theta). \tag{A26}$$

From writing principal 1's best-response, we get for any pair (θ_1,θ_2):

$$\theta_1 - p(q(\theta_1,\theta_2),\theta_1) = (1-F(\theta_2))\int_{\underline{\theta}}^{\theta_2}\frac{dx}{1-F(x)}.$$

But, using (A26), this gives: $(1-F(\theta_2))\int_{\underline{\theta}}^{\theta_2}\frac{dx}{1-F(x)} = (1-F(\theta_1))\int_{\underline{\theta}}^{\theta_1}\frac{dx}{1-F(x)}$, which cannot hold for all (θ_1,θ_2). ■

Proof of Proposition 6: The expression of the social weights is obtained by identifying (25) and (32). This yields: $\frac{1-F(\theta)}{f(\theta)}\frac{1}{1+\beta} = \frac{1-F(\theta)}{f(\theta)}(1-\tilde{\alpha}(\theta))$ so the result. ■

Proof of Proposition 7: First, we now provide an alternative description of the equilibrium allocations. This characterization provides a direct way of checking whether a given output schedule arises at equilibrium or not.[32]

Proposition 9 *: An output schedule $q(\cdot)$ is a symmetric equilibrium of the common agency game only if there exists a function $\tilde{p}(\theta_1,\theta_2)$ such that the following system holds:*

$$\tilde{p}(\theta_1,\theta_2) = \frac{\theta_1\left(\frac{1-F(\theta_1)}{f(\theta_1)}\right)\frac{\partial q}{\partial\theta_1} + (C'(q(\theta_1,\theta_2))-\theta_2)\left(\frac{1-F(\theta_2)}{f(\theta_2)}\right)\frac{\partial q}{\partial\theta_2}}{\left(\frac{1-F(\theta_1)}{f(\theta_1)}\right)\frac{\partial q}{\partial\theta_1} + \left(\frac{1-F(\theta_2)}{f(\theta_2)}\right)\frac{\partial q}{\partial\theta_2}}, \tag{A27}$$

and

$$\frac{\partial\tilde{p}}{\partial\theta_1} - \frac{\partial\tilde{p}}{\partial\theta_2}\frac{\frac{\partial q}{\partial\theta_1}}{\frac{\partial q}{\partial\theta_2}} = \frac{(\theta_1+\theta_2-C'(q(\theta_1,\theta_2)))\frac{\partial q}{\partial\theta_1}}{\left(\frac{1-F(\theta_1)}{f(\theta_1)}\right)\frac{\partial q}{\partial\theta_1} + \left(\frac{1-F(\theta_2)}{f(\theta_2)}\right)\frac{\partial q}{\partial\theta_2}}. \tag{A28}$$

Proof: From the Envelope Theorem, we have

$$\frac{\partial U(\theta_1,\theta_2)}{\partial\theta_1} = \frac{\partial t}{\partial\theta_1}(q(\theta_1,\theta_2),\theta_1).$$

Differentiating again and using Schwarz Lemma for $U(\cdot)$ twice differentiable, we obtain:

$$\frac{\partial^2 U(\theta_1,\theta_2)}{\partial\theta_1\partial\theta_2} = \frac{\partial^2 t}{\partial q\partial\theta_1}(q(\theta_1,\theta_2),\theta_1)\frac{\partial q(\theta_1,\theta_2)}{\partial\theta_2} = \frac{\partial^2 t}{\partial q\partial\theta_2}(q(\theta_1,\theta_2),\theta_2)\frac{\partial q(\theta_1,\theta_2)}{\partial\theta_1}$$

or using previous notations:

$$\frac{\partial p}{\partial\theta_1}(q(\theta_1,\theta_2),\theta_1)\frac{\partial q(\theta_1,\theta_2)}{\partial\theta_2} = \frac{\partial p}{\partial\theta_2}(q(\theta_1,\theta_2),\theta_2)\frac{\partial q(\theta_1,\theta_2)}{\partial\theta_1}. \tag{A29}$$

[32] In particular, it is easy to check on (A27) and (A28) that the first-best output cannot be implemented as an equilibrium.

Using

$$\theta_1 - p(q(\theta_1,\theta_2),\theta_1) = \frac{1-F(\theta_2)}{f(\theta_2)}\frac{\partial p}{\partial \theta_2}(q(\theta_1,\theta_2),\theta_2), \tag{A30}$$

we get

$$\left(\frac{\theta_1 - p(q(\theta_1,\theta_2),\theta_1)}{\frac{1-F(\theta_2)}{f(\theta_2)}}\right)\frac{\partial q(\theta_1,\theta_2)}{\partial\theta_1} = \left(\frac{\theta_2 - p(q(\theta_1,\theta_2),\theta_2)}{\frac{1-F(\theta_1)}{f(\theta_1)}}\right)\frac{\partial q(\theta_1,\theta_2)}{\partial\theta_2}. \tag{A31}$$

Using now that

$$p(q(\theta_1,\theta_2),\theta_1) + p(q(\theta_1,\theta_2),\theta_2) = C'(q(\theta_1,\theta_2)) \tag{A32}$$

we obtain that $p(q(\theta_1,\theta_2),\theta_1) = \tilde{p}(\theta_1,\theta_2)$ solves (A27).

Still using (A27) and (A30), we get

$$\theta_1 - p(q(\theta_1,\theta_2),\theta_1) = \frac{(\theta_1+\theta_2 - C'(q(\theta_1,\theta_2)))\frac{1-F(\theta_2)}{f(\theta_2)}\frac{\partial q}{\partial\theta_2}}{\left(\frac{1-F(\theta_1)}{f(\theta_1)}\right)\frac{\partial q}{\partial\theta_1} + \left(\frac{1-F(\theta_2))}{f(\theta_2)}\right)\frac{\partial q}{\partial\theta_2}} = \frac{1-F(\theta_2)}{f(\theta_2)}\frac{\partial p}{\partial\theta_2}(q(\theta_1,\theta_2),\theta_2).$$

But, simple algebra shows that

$$\frac{\partial p}{\partial\theta_2}(q(\theta_1,\theta_2),\theta_2) = \frac{\partial\tilde{p}}{\partial\theta_2}(\theta_2,\theta_1) - \frac{\partial\tilde{p}}{\partial\theta_1}(\theta_2,\theta_1)\frac{\frac{\partial q}{\partial\theta_1}}{\frac{\partial q}{\partial\theta_2}}$$

which, permuting θ_1 and θ_2 and using the symmetry of $q(\cdot)$, yields (A28). ■

Take now $q(\cdot)$ defined by (35). Using (A27), this output corresponds to $\tilde{p}(\theta_1,\theta_2) = \theta_1 - \lambda$ which satisfies (A28).

To derive the expression of the social weights, note that solving $\lambda = \frac{1-F(\theta)}{f(\theta)}(1-\tilde{\alpha}(\theta))$ leads to $\tilde{\alpha}(\theta) = 0$.

Let us now suppose that there exists another interim efficient allocation corresponding to a function $b(\cdot)$. Using (A27) and substituting into (A28) yields (after manipulations) the following partial differential equation in $q(\cdot)$:

$$\begin{aligned}&(\theta_1+\theta_2-2\lambda-q)\frac{\partial q}{\partial\theta_1}\frac{\partial q}{\partial\theta_2}\left(\frac{\partial q}{\partial\theta_1}+\frac{\partial q}{\partial\theta_2}\right) \\ = \;& (\theta_1+\theta_2-q)\left(-2\frac{\partial q}{\partial\theta_1}\frac{\partial q}{\partial\theta_2}\frac{\partial^2 q}{\partial\theta_1\partial\theta_2} + \frac{\partial^2 q}{\partial\theta_1^2}\left(\frac{\partial q}{\partial\theta_2}\right)^2 + \frac{\partial^2 q}{\partial\theta_2^2}\left(\frac{\partial q}{\partial\theta_1}\right)^2\right).\end{aligned} \tag{A33}$$

When $C'(q) = q$, interim efficiency implies that $\frac{\partial^2 q}{\partial\theta_1\partial\theta_2} = 0$. Expressed in terms of $b(\cdot)$, (A33) yields

$$\begin{aligned}&(\theta_1+\theta_2-2\lambda-b(\theta_1)-b(\theta_2))\dot{b}(\theta_1)\dot{b}(\theta_2)(\dot{b}(\theta_1)+\dot{b}(\theta_2)) \\ = \;& (\theta_1+\theta_2-b(\theta_1)-b(\theta_2))\left(\ddot{b}(\theta_1)\dot{b}^2(\theta_2)+\ddot{b}(\theta_2)\dot{b}^2(\theta_1)\right).\end{aligned} \tag{A34}$$

Taking $\theta = \theta_1 = \theta_2$, we find that $b(\cdot)$ must solve the following differential equation:

$$(\theta - \lambda - b(\theta))\dot{b}(\theta) = \lambda(\theta - b(\theta))\ddot{b}(\theta). \tag{A35}$$

Inserting this expression for $\ddot{b}(\theta_1)$ and $\ddot{b}(\theta_2)$ into (A34) yields (after manipulations) the condition:

$$\dot{b}(\theta_1)\dot{b}(\theta_2)(\theta_2 - b(\theta_2) - \theta_1 + b(\theta_1))\left(\frac{\dot{b}(\theta_1)}{(\theta_1 - b(\theta_1))} - \frac{\dot{b}(\theta_2)}{(\theta_2 - b(\theta_2))}\right) = 0.$$

A first possibility is to set $\theta - b(\theta) = \mu$ for any θ for some real number μ but using (A35), we have necessarily $\lambda = \mu$ as requested. The second possibility is to set $(\theta - b(\theta))\dot{b}(\theta) = \mu$ for any θ. But, using (A35) yields to a contradiction. ■

Proof of Proposition 8: The proof proceeds with a sequence of lemmata:

Lemma 5 *: Under Assumption 3, $T[p] \in B$, for all $p \in B$.*

Proof: Observe that for all $p \in B$, $E_{\widetilde{\theta}}(G(p(q,\theta) + p(q,\widetilde{\theta}) - C'(q)))$ and $E_{\widetilde{\theta}}(g(p(q,\theta) + p(q,\widetilde{\theta}) - C'(q)))$ are continuous functions of θ because $p(\cdot)$ is continuous and bounded. Moreover, if $E_{\widetilde{\theta}}(g(p(q,\theta) + p(q,\widetilde{\theta}) - C'(q)))$ is null, then $E_{\widetilde{\theta}}(G(p(q,\theta) + p(q,\widetilde{\theta}) - C'(q)))$ is also null. In this case, one may then use the convention $T[p](q,\theta) = \theta$. More generally, it is immediate to see that $T[p](q,\theta) \leq \theta$.

Finally, note that the monotone hazard rate property $\frac{d}{dx}\left(\frac{G(x)}{g(x)}\right) \geq 0$ (to be shown in Remark 1 below) implies that:

$$\theta g(p(q,\theta) + p(q,\widetilde{\theta}) - C'(q)) - G(p(q,\theta) + p(q,\widetilde{\theta}) - C'(q))$$
$$\geq g(p(q,\theta) + p(q,\widetilde{\theta}) - C'(q))\left(\theta - \frac{G(\theta + \bar{\theta} - C'(q))}{g(\theta + \bar{\theta} - C'(q))}\right)$$

from Assumption 3. Taking expectations over $\tilde{\theta}$ yields $T[p](q,\theta) \geq 0$. ■

We want now to show the existence and uniqueness of the fixed-point of $T[\cdot]$. Moreover, we have to prove that the necessary first-order condition (36) is also sufficient.

Consider first the function $L(\cdot|\cdot)$ defined on I^2 as: $L(y|x) = G(y)g'(x) - g(y)g(x)$.

Lemma 6 *: Under Assumptions 1 and 2, we have:*

$$0 \geq L(y|x) \geq L(x|x), \quad \textit{for all } (x,y) \in I^2.$$

Proof: Taking the derivative of $L(y|x)$ with respect to y, we get:

$$\frac{\partial L}{\partial y}(y|x) = g(y)g'(x) - g'(y)g(x).$$

Thus, $\frac{\partial L}{\partial y}(y|x) \geq 0$ if and only if $\frac{g'(x)}{g(x)} \geq \frac{g'(y)}{g(y)}$ and, by Assumption 1, if and only if $y \geq x$. By Assumptions 1 and 2, $L(\bar{\varepsilon}|x) = g'(x) - g(\bar{\varepsilon})g(x) \leq 0$ and $L(0|x) = -g(0)g(x) \leq 0$. Therefore, $0 \geq L(y|x) \geq L(x|x)$. ∎

Remark 1 : *Observe that Lemma 6 implies the monotone hazard rate property:*

$$\frac{d}{dx}\left(\frac{G(x)}{g(x)}\right) = -\frac{L(x|x)}{g(x)^2} \geq 0 \quad \text{for all } x \in I.$$

Lemma 7 : *For all $x \in I$ and $p \in B$, we have:*

$$E_{\widetilde{\theta}}\left[G(p(q,\theta) + p(q,\widetilde{\theta}) - C'(q))\right] g'(x) - E_{\widetilde{\theta}}\left[g(p(q,\theta) + p(q,\widetilde{\theta}) - C'(q))\right] g(x) < 0.$$

Proof: It is just a matter of applying Lemma 6 for each $y = p(q,\theta) + p(q,\widetilde{\theta}) - C'(q)$ and integrating out the inequalities with respect to $\widetilde{\theta}$. ∎

Lemma 8 : *Under Assumptions 1 to 3, $T[\cdot]$ is monotonically decreasing, i.e., $p_0 < p_1$ in B implies $T[p_0] < T[p_1]$.* [33]

Proof: Let $h = p_1 - p_0 > 0$ and define the function $\varphi(t,\theta) = T[p + th](q,\theta)$ for $t \in [0,1]$. Taking the derivative of $\varphi(\cdot)$ w.r.t. t, we get:

$$\partial_t \varphi(t,q,\theta) = \frac{E_{\widetilde{\theta}}\left[G(\cdot)\right] E_{\widetilde{\theta}}\left[g'(\cdot)h(\widetilde{\theta})\right] - E_{\widetilde{\theta}}\left[g(\cdot)\right] E_{\widetilde{\theta}}\left[g(\cdot)h(\widetilde{\theta})\right]}{\left(E_{\widetilde{\theta}}\left[g(\cdot)\right]\right)^2}$$

for all q and $\theta \in \Theta$ and $t \in [0,1]$ such that $\varphi(t,q,\theta) > 0$ (otherwise the derivative is zero, when it exists), where the arguments of the functions are $p_t(q,\theta) + p_t(q,\widetilde{\theta}) - C'(q)$ with $p_t(q,\theta) = p(q,\theta) + th(\theta)$. By Lemma 7, $\partial_t \varphi(t,q,\theta) < 0$ for $h > 0$. Therefore, $T[p_0] > T[p_1]$ ∎

[33] We are using the following convention:

$$\begin{aligned} p_0 &\leq p_1 \text{ if and only if } p_0(\theta) \leq p_1(\theta), \text{ for all } \theta \in \Theta \\ p_0 &< p_1 \text{ if and only if } p_0 \leq p_1 \text{ and } p_0(\theta) < p_1(\theta), \text{ for some } \theta \in \Theta. \end{aligned}$$

Lemma 9 *: Assume that Assumptions 1 to 3 hold and let* 0 *and* I *be respectively the null and the identity function. The following properties hold for all* $n \in \mathbb{N}$*:*

1. $T^{2n}[0] \leq T^{2n}[I]$ *and* $T^{2n+1}[I] \leq T^{2n+1}[0]$*;*
2. $T^{2(n-1)}[0] \leq T^{2n}[0] \leq T^{2n+1}[0] \leq T^{2n-1}[0]$ *and* $T^{2n-1}[I] \leq T^{2n+1}[I] \leq T^{2n}[I] \leq T^{2(n-1)}[I]$.

Proof: 1) This is easily obtained by induction (observing that $0 \leq I$ implies that $T[I] \leq T[0]$ and $T^2[0] \leq T^2[I]$, by Lemma 8).

2) We give the proof only for 0 (the proof for I is similar). Let us first show that the result holds for $n = 1$. We have that $0 \leq T[0], T^2[0]$. Thus, by Lemma 8, $T^2[0] \leq T[0]$ and $T^3[0] \leq T[0]$ and applying $T[\cdot]$ again, we get $T^2[0] \leq T^3[0]$ by Lemma 8. Therefore,

$$0 \leq T^2[0] \leq T^3[0] \leq T[0]$$

which is the claim for $n = 1$. Suppose that the result holds for some $n \in \mathbb{N}$, i.e., $T^{2(n-1)}[0] \leq T^{2n}[0] \leq T^{2n+1}[0] \leq T^{2n-1}[0]$. Applying $T[\cdot]$ to these inequalities and using Lemma 8 we get: $T^{2n}[0] \leq T^{2(n+1)}[0] \leq T^{2n+1}[0] \leq T^{2n-1}[0]$. Doing the same with the first and second inequalities of these last inequalities we get: $T^{2(n+1)}[0] \leq T^{2n+3}[0] \leq T^{2n+1}[0]$. Plugging these inequalities into the previous ones we finally get:

$$T^{2n}[0] \leq T^{2(n+1)}[0] \leq T^{2n+3}[0] \leq T^{2n+1}[0]$$

which is the result for $n + 1$. This concludes our proof by induction. ■

Lemma 10 *: Assume that Assumptions 1 to 3 hold. The sequences* $(T^n[0])_{n\in\mathbb{N}}$ *and* $(T^n[I])_{n\in\mathbb{N}}$ *converge pointwise towards the unique fixed-point of* $T[\cdot]$.

Proof: By 2) in Lemma 9, the sequences $(T^{2n}[0])_{n\in\mathbb{N}}$ and $(T^{2n+1}[I])_{n\in\mathbb{N}}$ converge increasingly pointwise towards $\bar{0}_e$ and $\bar{I}_o$, respectively. Similarly, $(T^{2n+1}[0])_{n\in\mathbb{N}}$ and $(T^{2n}[I])_{n\in\mathbb{N}}$ converge decreasingly to $\bar{0}_o$ and $\bar{I}_e$, respectively. Thus, these functions are fixed-points of T^2. Moreover, by Lemma 9, these functions satisfy

$$\bar{0}_e \leq \bar{I}_e, \quad \bar{I}_o \leq \bar{0}_o, \quad \bar{0}_e \leq \bar{0}_o \text{ and } \bar{I}_o \leq \bar{I}_e.$$

Since, $T^2[\cdot]$ is a strictly increasing operator, these four functions should be the same, say $\bar{p}$. In particular, the sequences $(T^n[0])_{n\in\mathbb{N}}$ and $(T^n[I])_{n\in\mathbb{N}}$ converge pointwisely to $\bar{p}$ (because subsequences of even and odd index have the same adherent point). Therefore, $\bar{p}$ is a fixed-point of $T[\cdot]$ and since every fixed-point of $T[\cdot]$ is in between 0 and I, it should coincide with $\bar{p}$, i.e., $T[\cdot]$ has a unique fixed-point. ■

Lemma 11 : *Assume that Assumptions 1 to 3 hold. The fixed-point $\bar{p}$ of $T[\cdot]$, belongs to B and is a non-decreasing function. Moreover, $\bar{p}$ has for degree of differentiability the minimum of that of g and C' in a neighborhood of θ.*

Proof: Observe that for each $\theta \in \Theta$ such that $\bar{p}(q,\theta) > 0$, the derivative of the function

$$\varphi(x) = x + \frac{E_{\widetilde{\theta}}\left[G(x + \bar{p}(q,\widetilde{\theta}) - C'(q))\right]}{E_{\widetilde{\theta}}\left[g(x + \bar{p}(q,\widetilde{\theta}) - C'(q))\right]}$$

at $x = \bar{p}(q,\theta)$ is given by

$$\varphi'(\bar{p}(q,\theta)) = 1 + \frac{\left(E_{\widetilde{\theta}}\left[g(\cdot)\right]\right)^2 - E_{\widetilde{\theta}}[G(\cdot)]E_{\widetilde{\theta}}\left[g'(\cdot)\right]}{\left(E_{\widetilde{\theta}}\left[g(\cdot)\right]\right)^2} > 0$$

by Lemma 7, where the argument of the functions is $\bar{p}(q,\theta) + \bar{p}(q,\widetilde{\theta}) - C'(q)$. By the Implicit Function Theorem, we have:

$$\frac{\partial \bar{p}}{\partial \theta}(q,\theta) = \left[\varphi'(\bar{p}(q,\theta))\right]^{-1} > 0.$$

Therefore, the result follows. ■

Now, we establish the proof of Proposition 8. The remaining things to prove are: (i) the first-order condition (36) is also sufficient for the principal's problem; (ii) the consistency condition (38).

To prove (i), note that the second-order condition of the principal's objective function amounts to:

$$\frac{E_{\widetilde{\theta}}\left[G(p + \bar{p}(q,\widetilde{\theta}) - C'(q))\right] E_{\widetilde{\theta}}\left[g'(p + \bar{p}(q,\widetilde{\theta}) - C'(q))\right] - 2\left(E_{\widetilde{\theta}}\left[g(p + \bar{p}(q,\widetilde{\theta}) - C'(q))\right]\right)^2}{\left(E_{\widetilde{\theta}}\left[g(p + \bar{p}(q,\widetilde{\theta}) - C'(q))\right]\right)^2} \leq 0$$

for all $p \in \bar{p}(q,\theta)$ for some $\theta \in \Theta$ (where we are using the first-order condition (36)). This is true by Lemma 7.

Observe that (ii) follows from Assumption 3 and the definition of $T[\cdot]$. ■

Non-Differentiable Equilibria: For completeness, we present now a class of non-differentiable equilibria. To analyze those equilibria, the most useful procedure is again based on the *supply profile* due to Wilson (1993).

Consider a principal P_i with type θ_i willing to pay a marginal contribution p for q units of the public good. This principal has to evaluate the probability that principal P_{-i} is also willing to contribute enough so that this amount is produced. Formally, if

P_{-i} follows the (symmetric) strategy $p(q,\cdot)$, the likelihood that q units of public good are produced is

$$\text{proba}\ \left\{p(q,\tilde{\theta}_{-i})+p\geq C'(q)\right\}=1-G(C'(q)-p|q)$$

where $G(\cdot|q)$ is the cumulative distribution of the marginal contribution of principal P_{-i} for q units of the public good. Given that residual supply schedule, P_i chooses a marginal contribution for q units of the public good which solves:

$$p(q,\theta_i)\in\arg\max_p(\theta_i-p)(1-G(C'(q)-p|q)).$$

These best-responses for each θ_i induce a distribution of marginal contributions $G_i(\cdot|q)$ for principal P_i. A symmetric equilibrium of the common agency game is thus a family of distributions (one for each value of q) $G(\cdot|q)$ which are fixed-points for each q.

To find an interesting class of non-differentiable equilibria, it is in fact enough to specify marginal contributions having two steps and a threshold $\theta^*(q)$ such that:

- for $\theta\geq\theta^*(q)$, $p(q,\theta)=\bar{p}(q)$;
- for $\theta\leq\theta^*(q)$, $p(q,\theta)=\underline{p}(q)$ $(<\bar{p}(q))$.

As we will see below, the three functions $\bar{p}(\cdot)$, $\underline{p}(\cdot)$ and $\theta^*(\cdot)$ are linked altogether. Given a function $\theta^*(q)$ (satisfying some properties to be made precise below), one can certainly find a two-step equilibrium (or the marginal contribution associated to it) using those conditions.

For a two-step symmetric equilibrium, let us describe the probability that q units of the public good are produced given a marginal contribution p:

$$\begin{array}{rcll}G(C'(q)-p|q)&=&0&\text{if}\quad p>C'(q)-\underline{p}(q)\\G(C'(q)-p|q)&=&F(\theta^*(q))&\text{if}\quad C'(q)-\bar{p}(q)<p\leq C'(q)-\underline{p}(q)\\G(C'(q)-p|q)&=&1&\text{if}\quad p\leq C'(q)-\bar{p}(q)\end{array}$$

where in first (last) case q units of the public good are (never) produced and the second case there is a probability $1-F(\theta^*(q))$ to be produced.

For each quantity q and type θ_i, P_i's best response is to offer a marginal contribution $\bar{p}(q)=C'(q)-\underline{p}(q)$ whenever

$$\theta_i-C'(q)+\underline{p}(q)\geq\max_{C'(q)-\bar{p}(q)\leq p\leq C'(q)-\underline{p}(q)}(\theta_i-p)(1-F(\theta^*(q)))\qquad\text{(A36)}$$

$$=(\theta_i-C'(q)+\bar{p}(q))(1-F(\theta^*(q))).$$

The set of such types θ_i is thus of the form $[\theta^*(q), \bar{\theta}]$ as requested by the structure postulated for the equilibrium.

At a symmetric two-step equilibrium, the following two conditions must hold:

$$\bar{p}(q) + \underline{p}(q) = C'(q), \tag{A37}$$

and $\theta_i = \theta^*(q)$ solves (A36) as an equality, i.e.,

$$\theta^*(q)F(\theta^*(q)) = \bar{p}(q)(2 - F(\theta^*(q))) - C'(q)(1 - F(\theta^*(q))). \tag{A38}$$

For q such that $\frac{C'(q)}{2} \leq \bar{p}(q) \leq \bar{\theta}$, (A38) defines uniquely $\theta^*(q)$ in $[\underline{\theta}, \bar{\theta}]$. Alternatively, given an increasing schedule $Q^*(\theta)$ which admits an inverse $\theta^*(q)$ which is almost everywhere differentiable, one can reconstruct $\bar{p}(q)$ from (A38) and $\underline{p}(q)$ from (A37). Note that $\bar{p}(q)$ is such that $\bar{p}(q) \geq \frac{C'(q)}{2}$.

Proposition 10 : *There exists a multiplicity of equilibria with two-steps marginal contributions. For each $Q^*(\theta)$ monotonically increasing with $Q^*(\bar{\theta}) = q^*(\bar{\theta}, \bar{\theta})$ and $Q^*(\theta) \leq q^*(\theta, \theta)$, there exists an equilibrium described by (A37) and (A38).*

Proof: The only thing to note is that for $\theta_1 < \theta^*(q) \leq \theta_2$, P_2 offers a marginal contribution $\bar{p}(q)$ whereas P_1 offers $\underline{p}(q)$, leading to the choice of q units. Idem for $\theta_2 \leq \theta^*(q) < \theta_1$ with the identity of the principals being reversed. When $\theta_1 = \theta_2 = \theta^*(q)$, note that both principals are indifferent between paying $\bar{p}(q)$ or $\underline{p}(q)$ at the margin. Break this indifference with a lexicographic order in favor of principal P_1 who pays indeed $\underline{p}(q)$ when both contributions are the same. Then the isoquant for q units cuts the diagonal at $\theta_1 = \theta_2 = \theta^*(q)$. Note that (A38) and $\bar{p}(q) \geq \frac{C'(q)}{2}$ imply that $2\theta^*(q) \geq C'(q)$. ■

It is worth describing the isoquants corresponding to those non-differentiable equilibria. In fact, those curves are the reunion made of the horizontal segment $\{\theta_1 \geq \theta^*(q)\}$ with the vertical segment $\{\theta_2 \geq \theta^*(q)\}$. Those non-differentiable equilibria allow us to describe settings where isoquants are not strictly decreasing ($\psi(q, \cdot)$ being not invertible).