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Ideals in Sequential Bargaining Structures †

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ABSTRACT

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This note suggests possible extensions of the baseline Rubinstein sequential bargaining structure - applied to the negotiation of stationary infinitely termed contracts - that incorporate a direct reference to the "ideal" utilities of the players. This is a feature of the Kalai-Smorodinsky cooperative solution – even if not of the generalized Nash maximand; it is usually not encountered in non-cooperative equilibria.

Firstly, it is argued that different bargaining protocols than conventionally staged are able to incorporate temporary all-or(and)-nothing splits of the pie. We advance scenarios where such episodes are interpreted either as – out of bargaining - war or unilateral appropriation events, or free experience contracts.

Secondly, we experiment with some modifications to the Rubinstein infinite horizon paradigm allowing for mixed strategies under alternate offers, and matching or synchronous decisions in a simultaneous (yet, discrete) bargaining environment. We derive solutions where the reference to the winner-takes-it-all outcome arises as a parallel – out-of-the-protocol - outside option to the *status quo* point.

In some cases, we thrived to derive the limiting maximand for instantaneous bargaining.

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Ideals in Sequential Bargaining Structures

> *"This is the sign of the covenant which I make between me and you and every living creature that is with you, for all future generations…"*
> In *Genesis* 9: 12.

Introduction

Rubinstein's (1982) structure has become a major reference in game theory and wage bargaining literature, possessing in its most well-known form the agreeable characteristic of generating – under perfect information, rational players and a realistic bargaining protocol – an immediate settlement and a unique equilibrium with no time loss, illustrating both the first-mover and patience advantages. Moreover, it provided, after Binmore, Rubinstein and Wolinsky (1986), a rationale for the cooperative solution implied by the generalized Nash (1950, 1953) maximand. In its solution, no direct reference to the maximal utility that each player can attain is usually found – a feature possessed by the Kalai-Smorodinsky (1975) result. Elsewhere – Martins (2004) -, we attempted an inclusion of such reference in plausible cooperative maximands; in this research, we search for bargaining structures where it could also be accommodated.

In all scenarios, at stake is the split of an infinite flow of benefits, periodically available at subsequent, equally distant, discrete points in time. For simplicity, it is assumed that only stationary divisions of the cake are contractually acceptable, and enforceable, once agreed upon, *ad infinitum* – a context akin to wage bargaining, but also realistic for other settings, namely rental - tenancy and leasing – agreements, and barter of capital or durable goods.

In a first class of models, we advance bargaining protocols where the party receiving the offers has the right to a full periodic "cake" in case of acceptance. Such settings may reproduce promotion policies by a seller, conceding free initial consumption to a potential buyer. And conversely, we contemplate the opposite case where the making of an acceptable proposition entitles/requires the full consumption of the periodic bargained endowment by the individual – possibly an incumbent, with some "hold-up" rights; the scenario can apply to long-term labor contracts with an initial apprenticeship period with very low pay (even if we could argue that it is accompanied by exchanges in kind – in human capital).

In a second setting, the possibility of one party actually "stealing" the periodic bargained object – and attaining the maximum utility – is addressed. This stage has two methodological interests: it allows for an immediate visualization of negotiations break-down; the underlying model encompasses the negotiation-bargaining vs. war-stealing decision internally.

The basic context of negotiations has been enlarged to encompass outside options – see Haller and Holden (1990) and Fernandez and Glazer (1991) - under which some of the standard equilibrium properties are lost. To some extent, the second type of bargaining models inspected in this research subscribe to the outside-option literature, even if in our case the "outside option" may be an out-of-the-protocol take-over of the "inside opportunities". We derive the properties of a stationary equilibrium allowing the players to decide whether to negotiate or not, and also to play mixed (i.e., random) strategies. These are known to exist for familiar bargaining games, even if not necessarily called for to assure equilibrium [1]; given the complexity introduced by the additional alternative, they become crucial to allow for interior equilibrium solutions under particular parameter ranges.

Two types of sequential bargaining protocols were inspected, both with the salient property of generating measurable expected waiting times until agreement is reached. A first one, where a simple extension of the Rubinstein structure is introduced: at a player's turn to "make a move", he can chose not to make any, enjoying a pay-off different from the one he gets when refusing an offer, and wait for next period's decision of the opponent concerning the same choice. In a second class of models, an equilibrium with simultaneous (but still sequential) bargaining – i.e., with simultaneous offer exchange - is forwarded.

Simultaneous bargaining has been studied in the literature to obviate the dependence of Rubinstein's results on the order and timing of offers. Usually, it is staged in a sequential set-up where time is assumed to be continuous and minimum delays between offers to exist - see Perry and Reny (1993) and Sákovics (1993). Instead, we keep the discrete time and forward the notion of "matching" or synchronous equilibrium, defined relying on each player using a mixed (probabilistic) strategy conditional on (and statistically independent of) the other player's action; a probabilistic measure of the likelihood of each player being the first to make offers can be generated with it.

Comparisons to the baseline Rubinstein structure under linear utility functions are provided. Inspection of potentially applicable instantaneous maximands to each case is also carried out for some of the models.

The exposition proceeds as follows: Rubinstein bargaining framework is forwarded in section 1. Potential maximal temporary payoffs for each of the players are introduced as an initial exchange in section 2. Uncertain attainment of the ideal utility is introduced as a permanent outside option under an alternate offers setup in section 3. Mixed strategies under an alternate sequential bargaining structure where a recurrent outside option is available till contract closure are introduced in section 4. In section 5, properties of simultaneous, yet sequential, equilibria are inspected. The exposition ends with some concluding remarks.

[1] See Admati and Perry (1985), for example.

1. Notation: Rubinstein (R) Sequential Bargaining Structure

. A description of the sequential bargaining prototype can be found in several articles. We lie down the structure of an infinite horizon, two-sided offer game with a simple recurrent contract being bargained [2]: let a "pie" of fixed size, normalized to 1, be made available to the two individuals every period, each with per period utility function – a discrete, well-behaved, "felicity" function - $u_i(z)$, with z denoting the share obtained by i, which they discount at factor δ_i.

For simplicity, assume that the bargaining protocol is such that an agreement on the share of the pie is binding for eternity: if an agreement is reached about the split of the pie for a particular period, the same split will hold forever [3]. Also, individuals make offers – proposals – on alternating periods, in which the opponent either accepts – and the split is settled – or rejects; in this case, the pie is lost and the "haggling" reinitiates next period with the opponent making the offer.

A (the) sub-game perfect equilibrium is derived as follows:

Denote by x the share accruing to player 1 when he is the first to make an offer – in which case (1 – x) is the share going to 2; and by y, the share accruing to player 1 when 2 is the first to propose – in which case (1 – y) is the share going to 2.

Assume individual 1 receives an offer y this period. He may accept, in which case he will obtain an accumulated discounted utility of:

$$\sum_{i=0}^{\infty} d_1^t\, u_1(y) \;=\; \frac{u_1(y)}{1-d_1}$$

If he rejects the offer, this period he will get $u_1(0)$ or, in general, d_1; if he makes an acceptable proposition x* next period, he will obtain the stream $u_1(x^*)$ of utilities from next period on, i.e., accumulating as:

$$d_1 + \sum_{i=1}^{\infty} d_1^t\, u_1(x^*) \;=\; d_1 + \frac{d_1\, u_1(x^*)}{1-d_1}$$

Player 1 will only accept the offer y iff:

[2] See Manzini (1998), for a survey of similar structures and results, including finite horizon games. Also, Busch and Wen (1995), Muthoo (1995) and Muthoo (1999).

[3] We will contrast some implications with those of other type of contracts forwarded in the bargaining literature below in this section. This condition/assumption restricts the relevant strategies to the players to be stationary in the long-run.

$$\sum_{i=0}^{\infty} d_1^t\, u_1(y) \;=\; \frac{u_1(y)}{1-d_1} \;\geq\; d_1 + \sum_{i=1}^{\infty} d_1^t\, u_1(x^*) \;=\; d_1 + \frac{d_1\, u_1(x^*)}{1-d_1}$$

or $$u_1(y) - d_1 \;\geq\; \delta_1\, [u_1(x^*) - d_1]$$

Comparisons with other hypothetical later settlements is useless: every two periods while an agreement is not reached, each player's position repeats itself and would generate the same solution for a contract settlement – only delay with loss of "cake" would have been passed.

Likewise, when player 2 ponders an offer (1-x) of individual 1, he will accept it iff

$$\frac{u_2(1-x)}{1-d_2} \;\geq\; d_2 + \frac{d_2\, u_2(1-y^*)}{1-d_2}$$

or $$u_2(1-x) - d_2 \;\geq\; \delta_2\, [u_2(1-y^*) - d_2]$$

If player 1 does not accept today's offer, both players wait for tomorrow's reasoning of player 2 according to the last expression and vice-versa.

The first to play – to make an offer - will always push the offer in his own favor till the other player's inequality barely holds – but strictly holds, to guarantee acceptance. He is constrained by the fact that, as the second to play is – with such strategy - indifferent between acceptance or not, the latter can wait for tomorrow and replicate the reasoning towards the first player. Then, the mutual acquiescent solution will entail a pair (x*, y*), x* being the share accruing to 1 if he is the first to make an offer, y* if 2 is the first to make the offers – or rather, (x*-ε, y*+ε'), where ε and ε' are infinitesimal quantities, guaranteeing that the accepting part will, in fact, accept and not wait for the other's turn; or assume that in case of indifference, the game ends -, such that the two previous conditions are met as equalities:

(1) $$u_1(y^*) - d_1 \;=\; \delta_1\, [u_1(x^*) - d_1]$$

and

(2) $$u_2(1-x^*) - d_2 \;=\; \delta_2\, [u_2(1-y^*) - d_2]$$

It must be the case that in that solution: $u_1(x^*)$, $u_1(y^*) > d_1$; $u_2(1-x^*)$, $u_2(1-y^*) > d_2$. This will also guarantee that the first to play is better-off making the offer than waiting to be made one, i.e.:

$$\frac{u_1(x^*)}{1-d_1} \;>\; d_1 + \frac{d_1\, u_1(y^*)}{1-d_1} \;=\; d_1 + \delta_1\, [d_1 + \frac{d_1\, u_1(x^*)}{1-d_1}]$$

and

$$\frac{u_2(1-y^*)}{1-d_2} \;>\; d_2 + \frac{d_2\, u_2(1-x^*)}{1-d_2} \;=\; d_2 + \delta_2\, [d_2 + \frac{d_2\, u_2(1-y^*)}{1-d_2}]$$

No delay in reaching the settlement is observed and no loss of “cake” either and, in general, the two equations possess a unique solution pair. From the first (second) equality, (1) ((2)), given that $\delta_1 < 1$ ($\delta_2 < 1$), y* < x* (1-x* < 1-y*) and the share accruing to 1 (2) is larger when he is the first to make the offer: the problem exhibits a “first-mover advantage”.

Under linear utility functions $u_1(z) = z$; without loss of generality, let d_i, an alternative in case no agreement is reach in the period, be available to each player i. Then:

$$(3) \qquad x^* = \frac{d_2(1-d_1)d_1 + (1-d_2)(1-d_2)}{1-d_1 d_2} \quad ; \quad y^* = \frac{(1-d_1)d_1 + d_1(1-d_2)(1-d_2)}{1-d_1 d_2}$$

$$1 - x^* = \frac{d_2(1-d_1)(1-d_1) + (1-d_2)d_2}{1-d_1 d_2} \quad ; \quad 1 - y^* = \frac{(1-d_1)(1-d_1) + d_1(1-d_2)d_2}{1-d_1 d_2}$$

For an interior solution to be valid:

$$(4) \qquad x^*, y^* > d_1;\ 1 - x^*, 1 - y^* > d_2 \quad \text{which require} \quad d_1 + d_2 < 1$$

In interior solutions (provided $d_1 + d_2 < 1$), 1’s shares in either case decrease with his impatience – increase with δ_1 – and increase with the opponent’s impatience – decrease with δ_2.

Also, player i’s shares increase with his “outside option”, d_i, and decrease with the opponent’s, d_j: having alternatives, improves a player’s bargaining position.

Under equal discount rates:

$$(5) \qquad x^* = \frac{d d_1 + (1-d_2)}{1+d} \quad ; \quad y^* = \frac{d_1 + d(1-d_2)}{1+d}$$

If the alternatives to the bargaining yield no utility, i.e., $d_i = u_i(0) = 0$, the “pure split” bargaining problem is established. In general, this is the benchmark case with which we will want to compare our solutions: while no agreement is reached, the periodic “pie” is completely lost – and no “crumbs” accrue to any of the players: x and y are shares of a surplus the generation of which also depends upon the players reaching (previous) agreement over its division. Then:

$$(6) \qquad x^* = \frac{1-d_2}{1-d_1 d_2} \quad ; \quad y^* = \frac{d_1(1-d_2)}{1-d_1 d_2}$$

The general conclusions do not change, but there will always be an interior solution. And under equal discount factors:

(7) $$x^* = \frac{1}{1+d} \quad ; \quad y^* = \frac{d}{1+d}$$

. In standard games, the size of the pie is traditionally normalized to 1. The hypothesis does not alter the general conclusions under most contexts. Say the periodic bun has size D and define x and y as the proportions of D allocated to 1: D x is the parcel accruing to player 1 when he is the first to make an offer – in which case D (1 – x) is the part going to 2; and D y is the parcel accruing to player 1 when 2 is the first to propose – in which case D (1 – y) is the one going to 2. One could develop the equilibrium solution for linear utility functions and conclude that in (3), d_i , i=1,2, would just be replaced by $\frac{d_i}{D}$: general comparative statics for the players' shares relative to any parameter but size D itself would remain unaltered – and therefore, indeed without loss of generality, we will stick to the normalized case. We could then write:

(8) $$x^* = \frac{d_2(1-d_1)d_1 + (1-d_2)(D-d_2)}{(1-d_1 d_2)D} \quad ; \quad y^* = \frac{(1-d_1)d_1 + d_1(1-d_2)(D-d_2)}{(1-d_1 d_2)D}$$

$$1 - x^* = \frac{d_2(1-d_1)(D-d_1) + (1-d_2)d_2}{(1-d_1 d_2)D} \quad ; \quad 1 - y^* = \frac{(1-d_1)(D-d_1) + d_1(1-d_2)d_2}{(1-d_1 d_2)D}$$

Interestingly, x* rises with D iff $\frac{1-d_2}{d_2}$ $d_2 > (1 - \delta_1)\, d_1$; 1 – y* rises with D iff $\frac{1-d_1}{d_1}$ $d_1 > (1 - \delta_2)\, d_2$. Hence, being i the first player, his *relative* share rises with the size of the pie iff:

(9) $$(1 - \delta_i)\, d_i < \frac{1-d_j}{d_j}\, d_j$$

Consequently, under identical (and fixed absolute) alternatives and discount factors, the size of the pie benefits relatively more the first mover. In the general case, a large discount factor (patience) and a small alternative will make it more likely that a player's (relative) share will increase with the size of the bargained item.

Finally, if the available alternatives after disagreement are indexed to D and i receives D d_i (instead of d_i) in such case, or if they are inexistent and d_i = 0, i=1,2 (a commonly used assumption), we revert to the normalized solution (3) – and the relative shares x* and y* are independent of the size of the pie.

. A note should be added with respect to the interplay with the capital market. The previous conclusions were derived under the assumption that lending or borrowing is absent. Assume perfect capital markets and that the traders can switch part of their equilibrium settlement z_i*, say, Δ, by a perpetuity at the market interest rate mr_i, then maximizing:

$$\underset{\Delta}{Max} \quad u_i(z_i^* - \Delta) + \frac{d_i\, u_i(z_i^* + mr_i\,\Delta)}{1 - d_i}$$

F.O.C. with respect to Δ would then imply – replacing δ_i by $\frac{1}{1+r_i}$ - that:

$$u_i'(z_i^* - \Delta) = \frac{mr_i}{r_i}\, u_i'(z_i^* + mr_i\, \Delta)$$

If a player is risk-averse and the player's felicity function is concave – insuring that S.O.C. hold -, he is a perpetual lender/renter – $\Delta^* > 0$ – iff $mr_i > r_i$, that is, he discounts the future utility less intensely than the market discounts future revenues. In the opposite case – if $mr_i < r_i$ –, $\Delta^* < 0$ and he will be a perpetual borrower.

One could advance or hypothesize that under perfect capital markets, the solution of the game between the players – whatever their utility functions are – would be represented instead by the linear utility solution where the discount rates are replaced by the market interest rates that they face, mr_i, provided that this solution yields higher utility streams when borrowing is then allowed for; such behavior of the players would then mimic maximization of the net present value of the game for them. The subject is left for further inspection.

. The previous bargaining protocol posited offers as binding long-term – infinite-term - splits. Indeed, labor contracts, for example, are set for several periods. Nevertheless, a one period "pie" could be at stake from one point in time till eternity [4]. Under such circumstances, one can claim that 1 accepts an offer y if the gains he expects by accepting it today are larger than those he expects if he does not accept and makes a reasonable proposition tomorrow:

$$u_1(y) - d_1 \geq 0 + \delta_1 [u_1(x^*) - d_1]$$

In other words, it is as if he weights consuming $u_1(y)$ today and d_1 tomorrow, in which case he gets $u_1(y) + \delta_1 d_1$, with enjoying d_1 today and $u_1(x^*)$ tomorrow (at best), getting $d_1 + \delta_1 u_1(x^*)$ – and, in both cases, d_1 ever since the day after tomorrow [5]. For 2:

$$u_2(1 - x) + \delta_2 d_2 \geq d_2 + \delta_2 u_2(1 - y^*)$$

[4] See, for example, Kennan and Wilson (1989), modelling labor contract settlements.

[5] Hence, comparing accumulated utility flows for the same time horizon.

The equilibrium split is guaranteed by the same x* and y* as above, provided the same d_1 and d_2 apply. One cannot, however, superimpose easily a sequence of alternating games in which the conditions of this problem would replicate eternally. We will therefore stick to the long-term contract structure.

. Binmore *et al* (1986) suggest that the solution to the game above if both δ_1 and δ_2 tend to 1 due to a decreasing time interval between offers, can be seen to yield the Nash maximand. Let $\delta_i = e^{-r_i\Delta}$, where r_i denotes the instantaneous interest rate (yet, measured per unit of time) at which individual i discounts "felicity" and Δ the time length of the interval between offers [6]; let Δ tend to 0. The two conditions above become:

$$u_1(y^*) - d_1 = e^{-r_1\Delta}\,[u_1(x^*) - d_1]$$

and

$$u_2(1-x^*) - d_2 = e^{-r_2\Delta}\,[u_2(1-y^*) - d_2]$$

For small Δ, a first-order approximation to δ_i is $(1 - r_i\,\Delta)$ and is legitimate. Then, we can approximate:

$$u_1(y^*) - d_1 = (1 - r_1\,\Delta)\,[u_1(x^*) - d_1]$$

and

$$u_2(1-x^*) - d_2 = (1 - r_2\,\Delta)\,[u_2(1-y^*) - d_2]$$

Re-arranging

$$u_1(y^*) - u_1(x^*) = -\,r_1\,\Delta\,[u_1(x^*) - d_1]$$

$$u_2(1-x^*) - u_2(1-y^*) = -\,r_2\,\Delta\,[u_2(1-y^*) - d_2]$$

Then

$$\frac{u_1(y^*) - u_1(x^*)}{u_2(1-x^*) - u_2(1-y^*)} = \frac{r_1}{r_2}\,\frac{u_1(x^*) - d_1}{u_2(1-y^*) - d_2}$$

The left hand-side can be written as

$$\frac{\dfrac{u_1(y^*) - u_1(x^*)}{y^* - x^*}}{\dfrac{u_2(1-x^*) - u_2(1-y^*)}{1 - x^* - (1-y^*)}}$$

As Δ tends to 0, x* tends to y*, and the expression tends to the ratio of marginal utilities evaluated at x* and 1 – x* in the numerator and denominator respectively. The previous condition becomes:

[6] We follow Cahuc and Zylberberg (2001) exposition. A formal interpretation of discrete time interval bargaining of continuous time optimizing players – and the justification of the replacement of δ_i by $e^{-r_i\Delta}$ in the equilibrium conditions - is developed in Appendix 1.

$$(10) \qquad \frac{u'_1(x^*)}{u'_2(1-x^*)} = \frac{r_1}{r_2}\, \frac{u_1(x^*)-d_1}{u_2(1-x^*)-d_2}$$

coinciding with the optimal F.O.C. conditions derived under the generalized Nash maximand.

Under an egalitarian possibility set, say G, with respect to utilities, i.e., of the form $u_1 + u_2 \leq \overline{U}$, the solution of

$$u^{NG} = \underset{u \in G}{\text{Arg Max}} \ (u_1 - d_1)^{\gamma_1} \ (u_2 - d_2)^{\gamma_2}$$

where $\frac{g_i}{g_1+g_2}$ represents bargaining power of player i and equals $\frac{\frac{1}{r_i}}{\frac{1}{r_1}+\frac{1}{r_2}}$ is:

$$(11) \qquad u_1^{NG} - d_1 = \frac{g_1}{g_1+g_2}\ (\overline{U} - d_1 - d_2) \qquad \text{or} \qquad u_1^{NG} = \frac{g_1(\overline{U}-d_2)+g_2 d_1}{g_1+g_2}$$

Of course, the features of this solution with respect to utilities translate immediately to the arguments for the case of linear utilities. This form – with $\overline{U}$ = 1 - can be seen as consistent with (3) if we allow – in the solutions x* and y* of (3) - for a first-order approximation to δ_i as (1 - $r_i \Delta$) and consider that Δ goes to 0.

2. Bargaining Structures with Compensating Protocols

In certain bargaining situations proposals may only be forwarded after inspection of the product. That is, for a recurrent buyer to accept a long-term supplying contract, he may wish or (should) be entitled to inspect a one period trial. On other cases, to make a reasonable proposition, the proposing part may need to study/consume the bargained object. In this section we include such type of hypothesis in protocol design. In the labor market, both can reproduce a long-term contract with initial “gift-exchange”; indeed initial low pay is known to occur in some professions – taking the form of training or apprenticeship period, hence with other claimed functionalities; part of these may, however be attributed to bargaining arrangements.

In some instances, the scenario may reproduce the effect of “hold-up” rights. And/or in others, the existence of an outside option; or yet potential bribe exchange between/to the players.

2.1. Experience Rights

Consider that the individual who is assessing the proposal is entitled to a one-period free bonus in case he accepts the stream of proposed gains; that is, in that period if he accepts he is entitled to the periodic "ideal" utility $\bar{u}_i$. If he does not accept the bargain, he must return the product with no left-over utility (say, he must pay the experience at his own expense) over the *status quo* utility d_i.

When 1 is assessing 2's proposal he compares:

$$\bar{u}_1 + \sum_{i=1}^{\infty} d_1^t \, u_1(y) \;=\; \bar{u}_1 + \delta_1 \frac{u_1(y)}{1-d_1} \geq$$

$$\geq\; d_1 + \delta_1 d_1 + \sum_{i=2}^{\infty} d_1^t \, u_1(x^*) \;=\; d_1 + \delta_1 d_1 + \delta_1^{\,2} \frac{u_1(x^*)}{1-d_1}$$

or:

$$\frac{(1-d_1)\bar{u}_1 + d_1 u_1(y)}{1-d_1} \;\geq\; d_1 + \delta_1 \frac{(1-d_1)d_1 + d_1 u_1(x^*)}{1-d_1}$$

Even if he makes an acceptable proposal tomorrow, if the latter becomes accepted, 1 must give the one period experience to player 2 in that next period as well.

For 2, an identical reasoning is made when he receives an offer 1-x. The sub-game perfect equilibrium will generate:

(12) $$(1 - \delta_1)(\bar{u}_1 - d_1) + \delta_1 [u_1(y^*) - d_1] = \delta_1^{\,2} [u_1(x^*) - d_1]$$

and

(13) $$(1 - \delta_2)(\bar{u}_2 - d_2) + \delta_2 [u_2(1-x^*) - d_2] = \delta_2^{\,2} [u_2(1-y^*) - d_2]$$

It must be the case that in that solution: $d_1 + \delta_1 d_1 + \delta_1^{\,2} \frac{u_1(x^*)}{1-d_1} > \frac{d_1}{1-d_1}$, and symmetrically for player 2. Hence: $u_1(x^*) > d_1$; $u_2(1-y^*) > d_2$.

In equilibrium, if player 1 is the first-mover, he will get:

(14) $$Z_1^* = d_1 + \delta_1 \frac{u_1(x^*)}{1-d_1}$$

If he is the second-mover, and it is 2's turn to propose:

(15) $$V_1^* = \bar{u}_1 + \delta_1 \frac{u_1(y^*)}{1-d_1}$$

The first equation describing the equilibrium solves for $V_1^* = d_1 + \delta_1 Z_1^*$ and - as $\delta_1 < 1$ and both $Z_1^*(1 - \delta_1)$ and $V_1^*(1 - \delta_1)$ must exceed d_1 – the problem always exhibits a first-mover advantage in accumulated utility flows, i.e., $Z_1^* > V_1^*$. As $\bar{u}_1 > d_1$, - from (14) and (15) - that implies that $u_1(x^*) > u_1(y^*)$ and a first-mover advantage is also present for the periodic share.

Also, $V_1^* > \dfrac{d_1}{1-d_1}$, which requires that: $(1 - \delta_1)\,\bar{u}_1 + \delta_1 u_1(y^*) > d_1$.

. Let $u_i(z) = z$, i =,1,2 and d_i and $\bar{u}_i$ rest undefined. Then, in line with (3), we can derive:

$$(16)\quad x^* = \frac{d_1 d_2(1-d_2) + d_2^2(1-d_1^2)d_1 - d_1(1-d_2^2)d_2 - d_2^2(1-d_1)\bar{u}_1 + d_1(1-d_2)\bar{u}_2}{d_1 d_2(1-d_1 d_2)} \;;$$

$$y^* = \frac{d_1^2 d_2(1-d_2) + d_2(1-d_1^2)d_1 - d_1^2(1-d_2^2)d_2 - d_2(1-d_1)\bar{u}_1 + d_1^2(1-d_2)\bar{u}_2}{d_1 d_2(1-d_1 d_2)}$$

As for the long-term accumulated flows:

$$(17)\quad Z_1^* = d_1 + \delta_1 \frac{u_1(x^*)}{1-d_1} =$$

$$= \frac{d_1}{1-d_1}\,\frac{d_1 d_2(1-d_2) + d_2(1+d_2)(1-d_1)d_1 - d_1(1-d_2^2)d_2 - d_2^2(1-d_1)\bar{u}_1 + d_1(1-d_2)\bar{u}_2}{d_1 d_2(1-d_1 d_2)}$$

$$V_1^* = \bar{u}_1 + \delta_1 \frac{u_1(y^*)}{1-d_1} =$$

$$= \frac{d_1}{1-d_1}\,\frac{d_1^2 d_2(1-d_2) + d_2(1-d_1^2)d_1 - d_1^2(1-d_2^2)d_2 - d_1 d_2^2(1-d_1)\bar{u}_1 + d_1^2(1-d_2)\bar{u}_2}{d_1 d_2(1-d_1 d_2)}$$

Player i's share, as accumulated utility, increases with his "outside option", d_i, and decreases with the other's, d_j. It decreases with the perceived utility obtained with the initial down-payment he gets with acceptance, $\bar{u}_i$, and increases with the opponent's $\bar{u}_j$; this would seem counter-intuitive but can be explained by the equilibrium mechanics: the opportunity cost of rejecting an offer today rises with the size of the own bonus, it decreases with the opponent's once we will only pay it if we reject today's offer. To accept a bribe to close a contract is not worthwhile for a player; that the opponent does, is. In other words, one is better off paying a bribe to close a long-term contract than receiving it.

Under equal discount rates, the previous expressions simplify to:

$$(18)\quad x^* = \frac{d + d(1+d)d_1 - (1+d)d_2 - d\bar{u}_1 + \bar{u}_2}{d(1+d)} \;;$$

$$y^* = \frac{d^2 + (1+d)d_1 - d(1+d)d_2 - \bar{u}_1 + d\bar{u}_2}{d(1+d)}$$

$$Z_1^* = \frac{d}{1-d} \; \frac{d + (1+d)d_1 - (1+d)d_2 - d\bar{u}_1 + \bar{u}_2}{d(1+d)}$$

$$V_1^* = \frac{d}{1-d} \; \frac{d^2 + (1+d)d_1 - d(1+d)d_2 - d^2\bar{u}_1 + d\bar{u}_2}{d(1+d)}$$

The impact of his own outside option d_i on the share or accumulated utility of a first-mover is never more pronounced (in absolute value) than the effect of a rise in the opponent's; the opposite occurs for the second to play. We encounter the same relative magnitudes with respect to the effect of the outside option $\bar{u}_i$.

In absolute terms, the effect of $\bar{u}_i$ is always smaller than that of d_i (not unexpectedly, once the loss of the latter is anterior to any closing contract) on the share of any player.

The expressions become quite cumbersome to interpret; allow for simplicity that the alternatives are $d_i = u_i(0) = 0$ and the initial period transfer/gift is complete, i.e., $\bar{u}_i = u_i(1) = 1$. Then, it is straight-forward to show that:

(19) $$x^* = \frac{d_1 - d_2^2}{d_1 d_2(1-d_1 d_2)} \quad ; \quad y^* = 1 - \frac{d_2 - d_1^2}{d_1 d_2(1-d_1 d_2)}$$

$$Z_1^* = \frac{d_1}{1-d_1} \; \frac{d_1 - d_2^2}{d_1 d_2(1-d_1 d_2)} \quad ; \quad V_1^* = \frac{d_1}{1-d_1} \; \frac{d_1(d_1 - d_2^2)}{d_1 d_2(1-d_1 d_2)}$$

Patience is rewarded: x* as y* rise with δ_1 and decrease with δ_2.

Comparing (19) with (6) of the standard model of section 3 we can confirm that:

$$\frac{d_1 - d_2^2}{d_1 d_2(1-d_1 d_2)} > \frac{1-d_2}{1-d_1 d_2} \quad \text{iff} \quad \delta_1(1-\delta_2) > \delta_2^2(1-\delta_1)\text{, i.e., } \delta_1 > \frac{d_2^2}{1-d_2+d_2^2}$$

$$1 - \frac{d_2 - d_1^2}{d_1 d_2(1-d_1 d_2)} < \frac{d_1(1-d_2)}{1-d_1 d_2} \quad \text{iff} \quad \delta_2(1-\delta_1) > \delta_1^2(1-\delta_2)\text{, i.e., } \delta_2 > \frac{d_1^2}{1-d_1+d_1^2}$$

That is, if $\delta_1 = \delta_2 = \delta$ (or provided the two discount rates are close) the first-mover is has a higher periodic share than if no experience rights existed – provided that the opponent is not much more patient, having a high discount factor (for given δ_1, the higher δ_2 the less likely the first condition will hold, once $\frac{d_2^2}{1-d_2+d_2^2}$ increases with δ_2); the second player has a lower share – unless he is very patient, having a high discount factor ($\frac{d_1^2}{1-d_1+d_1^2}$ increases with δ_1: for given δ_2, the higher δ_1 the less likely the condition will hold). The first part of the proposition is somewhat counter-intuitive, and can be reasoned (again) in terms of opportunity costs: in case of rejection, the individual assessing the proposal will not only lose this period's free experience as he will have to postpone consumption to make an acceptable proposition next period, when he must also offer a free gift to the opponent.

Yet, the accumulated flows should provide the appropriate welfare comparisons: Z_1^* should compare with $\frac{u_1(x^*)}{1-d_1}$ of the standard problem; V_1^* with $\frac{u_1(y^*)}{1-d_1}$. Then

$$\frac{d_1}{1-d_1}\ \frac{d_1-d_2^2}{d_1 d_2(1-d_1 d_2)} > \frac{1}{1-d_1}\ \frac{1-d_2}{1-d_1 d_2} \qquad \text{iff} \quad \delta_1 > \delta_2$$

$$\frac{d_1}{1-d_1}\ \frac{d_1(d_1-d_2^2)}{d_1 d_2(1-d_1 d_2)} > \frac{1}{1-d_1}\ \frac{d_1(1-d_2)}{1-d_1 d_2} \qquad \text{iff} \quad \delta_1 > \delta_2$$

The more patient player, whether he will move first or second, is better-off with the current protocol.

This justifies *promotion policies* based on a one period *initial* experiment or gift – that if rejected can be expected to be followed by a rise in price (the "return gift" to the seller): the trader with lower interest rate will have an advantage in such protocol. Interestingly, the conclusion is based on the properties of the implied bargaining equilibrium and not on uncertain information or knowledge about the (quality of) the good being bartered.

Under $\delta_1 = \delta_2 = \delta$:

$$\text{(20)} \qquad x^* = \frac{1}{d(1+d)} \quad ; \quad y^* = 1 - \frac{1}{d(1+d)}$$

$$Z_1^* = \frac{1}{1-d^2} \quad ; \quad V_1^* = \frac{d}{1-d^2}$$

Even with equal discount rates, x*, the first mover periodic share, is now higher than if no free bonus is offered at acceptance; that is, the first mover has double advantage in this case. Yet, with equal impatience of the players, accumulated flows are the same than in the standard protocol.

. Considering the replacement of δ_i by $e^{-r_i\Delta}$ [7], manipulation of (12) and (13) and letting Δ tend to 0 to represent instantaneous bargaining, we can arrive at:

$$\text{(21)} \qquad \frac{u'_1(x^*)}{u'_2(1-x^*)} = \frac{r_1}{r_2}\ \frac{\bar{u}_1 - d_1 + u_1(x^*) - d_1}{\bar{u}_2 - d_2 + u_2(1-x^*) - d_2}$$

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[7] One can show that such replacement will continue to be valid if the players maximize a continuous-time utility function and bargaining offers take place at discrete time intervals of length Δ analogously to the setting described in Appendix 1.

The maximand originating such first-order conditions will be:

(22) $$u^W = \underset{u \in G}{\text{Arg Max}} \left[u_1 - (2d_1 - \overline{u}_1)\right]^{g_1} \left[u_2 - (2d_2 - \overline{u}_2)\right]^{g_2}$$

Again, $\frac{g_i}{g_1 + g_2}$ represents bargaining power of player i and equals $\frac{\frac{1}{r_i}}{\frac{1}{r_1} + \frac{1}{r_2}}$. It maximizes the product of the utility gains relative to the difference between twice the *status quo* and maximum utility achieved by the player, powered by a factor proportional to the inverse of the individual's interest rate.

Graphically, for equal bargaining strength of the players, the new hyperbolas are such that the original reference center is shifted to $(2d_1 - \overline{u}_1,\ 2d_2 - \overline{u}_2)$ - hence, relative to the standard Nash maximand, in favor of the player with lower maximal utility.

We can alternatively write [8]:

$$u^W = \underset{u \in G}{\text{Arg Max}} \left[u_1 - d_1 + \overline{u}_1 - d_1\right]^{g_1} \left[u_2 - d_2 + \overline{u}_2 - d_2\right]^{g_2}$$

Under an egalitarian possibility set with respect to utilities, $u_1 + u_2 = \overline{U}$:

$$u_1^W - d_1 = \frac{g_1(\overline{U} + \overline{u}_2 - 2d_2 - d_1) + g_2(d_1 - \overline{u}_1)}{g_1 + g_2} \quad \text{or} \quad u_1^W = \frac{g_1(\overline{U} + \overline{u}_2 - 2d_2) + g_2(2d_1 - \overline{u}_1)}{g_1 + g_2}$$

If we assume that $\overline{u}_1 = \overline{u}_2 = \overline{U}$:

$$u_1^W - d_1 = \frac{(2g_1 - g_2)\overline{U} - 2g_1 d_2 - (g_1 - g_2)d_1}{g_1 + g_2} \quad \text{or} \quad u_1^W = \frac{(2g_1 - g_2)\overline{U} - 2(g_1 d_2 - g_2 d_1)}{g_1 + g_2}$$

In case $\gamma_1 = \gamma_2$, the solution, $u_1^W = \frac{\overline{U} - 2(d_2 - d_1)}{2}$, differs from the Nash solution, $u_1^N = \frac{\overline{U} - d_2 + d_1}{2}$, even if it also possesses similar the properties. the equilibrium favors more the player with higher "threat" utility than the Nash solution does.

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[8] Technically, the mathematical arrangement of the maximand is found in some finance literature. See Verchenko (2004).

If $\overline{u}_i = \overline{U}$ - d_j:

$$u_1^W - d_1 = \frac{(2g_1 - g_2)(\overline{U} - d_1 - d_2)}{g_1 + g_2} \quad \text{or} \quad u_1^W = \frac{(2g_1 - g_2)(\overline{U} - d_2) + (2g_2 - g_1)d_1}{g_1 + g_2}$$

The utility surplus (relative to the status quo) accruing now to 1 is added of the difference between 1 and 2's bargaining strengths relative to the Nash solution.

Summarizing:

Proposition 1: Assume an alternating offers protocol as above where a first period bonus is conceded upon acceptance of an offer.

1.1. The sub-game perfect equilibrium will possesses a first mover advantage (both in accumulated utility as in the periodic share) and reward patience as usual (under linear utilities).

1.2. (With linear utilities) The more patient player is now in a better position than in the traditional framework; the first-mover is expected to enjoy a higher periodic share than in the standard model.

1.3. A player's periodic share and accumulated utility decrease with the utility he derives from the bonus he obtains when accepting an offer, and increase with the other player's.

1.4. The underlying instantaneous maximand is of the multiplicative Nash form, but weights – equally - in each factor both the incremental utility allowed by bargaining as the maximal incremental utility of each player.

2.2. Proposing Party Entitlements

Suppose that making proposals is costly in terms of time and one period with loss of the cake must pass before an offer - with the same traits as before: an infinite-time binding settlement on the share accruing to each party - can be made again by either side; simultaneously, the individual making an acceptable proposition is entitled to one trial – a "prize" - when his offer becomes accepted.

When 1 is assessing 2's proposal y he compares:

$$d_1 + \sum_{i=1}^{\infty} d_1^t u_1(y) = d_1 + \delta_1 \frac{u_1(y)}{1-d_1} \geq$$

$$\geq d_1 + \delta_1 d_1 + \delta_1^2 \overline{u}_1 + \sum_{i=3}^{\infty} d_1^t u_1(x^*) = d_1 + \delta_1 d_1 + \delta_1^2 \overline{u}_1 + \delta_1^3 \frac{u_1(x^*)}{1-d_1}$$

that is: $$\frac{u_1(y)}{1-d_1} \geq d_1 + \delta_1 \overline{u}_1 + \delta_1^2 \frac{u_1(x^*)}{1-d_1} = d_1 + \delta_1 \frac{d_1 u_1(x^*) + (1-d_1)\overline{u}_1}{1-d_1}$$

For 2, an identical assessment is made when he receives an offer 1-x. The sub-game perfect equilibrium will generate:

$$(23) \qquad [u_1(y^*) - d_1] + \delta_1^2 [\bar{u}_1 - u_1(x^*)] = \delta_1 (\bar{u}_1 - d_1)$$

and

$$(24) \qquad [u_2(1-x^*) - d_2] + \delta_2^2 [\bar{u}_2 - u_2(1-y^*)] = \delta_2 (\bar{u}_2 - d_2)$$

In equilibrium, each player may be in any of four points. Consider player 1. He may be the first-mover and either be before or after the preparation of the proposal; he will get:

(25) if before, $Z_{1b}^* = d_1 + \delta_1 \bar{u}_1 + \delta_1^2 \frac{u_1(x^*)}{1-d_1}$; if after, $Z_{1a}^* = \bar{u}_1 + \delta_1 \frac{u_1(x^*)}{1-d_1}$ and $Z_{1a}^* > Z_{1b}^*$.

If he is the second-mover, and it is 2's turn to propose, either

(26) $V_{1a}^* = d_1 + \delta_1 \frac{u_1(y^*)}{1-d_1}$ or $V_{1b}^* = d_1 + \delta_1 d_1 + \delta_1^2 \frac{u_1(y^*)}{1-d_1}$ and $V_{1a}^* > V_{1b}^*$

The first equation describing the equilibrium solves for $V_{1a}^* = d_1 + \delta_1 Z_{1b}^*$; as $\delta_1 < 1$ and both $Z_{1b}^*(1 - \delta_1)$ and $V_{1a}^*(1 - \delta_1)$ must exceed d_1 – the problem always exhibits a first-mover advantage, i.e., $Z_{1a}^* > Z_{1b}^* > V_{1a}^* > V_{1b}^*$.

The relevant measures may either be Z_{1b}^* and V_{1b}^* or Z_{1a}^* and V_{1a}^* – according to whether, in the game we might want to replicate, a first period delay is required to make the initial proposition or not.

. Let $u_i(z) = z$, i =,1,2. Then:

$$(27) \qquad x^* = \frac{(1-d_2^2) + d_2^2(1-d_1)d_1 - (1-d_2)d_2 + d_1 d_2^2(1-d_1)\bar{u}_1 - d_2(1-d_2)\bar{u}_2}{1-d_1^2 d_2^2};$$

$$y^* = \frac{d_1^2(1-d_2^2) + (1-d_1)d_1 - d_1^2(1-d_2)d_2 + d_1(1-d_1)\bar{u}_1 - d_1^2 d_2(1-d_2)\bar{u}_2}{1-d_1^2 d_2^2}$$

One can also solve for:

$$(28) \qquad Z_{1a}^* = \bar{u}_1 + \delta_1 \frac{u_1(x^*)}{1-d_1} =$$

$$= \frac{d_1(1-d_2^2)+d_1d_2^2(1-d_1)d_1-d_1(1-d_2)d_2+(1-d_1)\bar{u}_1-d_1d_2(1-d_2)\bar{u}_2}{(1-d_1)(1-d_1^2d_2^2)}$$

$$Z_{1b}^* = d_1 + \delta_1 Z_{1a}^* =$$

$$= \frac{d_1^2(1-d_2^2)+(1-d_1)d_1-d_1^2(1-d_2)d_2+d_1(1-d_1)\bar{u}_1-d_1^2d_2(1-d_2)\bar{u}_2}{(1-d_1)(1-d_1^2d_2^2)}$$

$$V_{1a}^* = d_1 + \delta_1 \frac{u_1(y^*)}{1-d_1} =$$

$$= \frac{d_1^3(1-d_2^2)+(1-d_1)(1+d_1-d_1^2d_2^2)d_1-d_1^3(1-d_2)d_2+d_1^2(1-d_1)\bar{u}_1-d_1^3d_2(1-d_2)\bar{u}_2}{(1-d_1)(1-d_1^2d_2^2)}$$

$$V_{1b}^* = d_1 + \delta_1 V_{1a}^* =$$

$$\frac{d_1^4(1-d_2^2)+(1-d_1)(1+d_1+d_1^2-d_1^2d_2^2-d_1^3d_2^2)d_1-d_1^4(1-d_2)d_2+d_1^3(1-d_1)\bar{u}_1-d_1^4d_2(1-d_2)\bar{u}_2}{(1-d_1)(1-d_1^2d_2^2)}$$

As usual, player i's accumulated utility as the periodic share increase with his "outside option", d_i, and decrease with the other's. Interestingly – and in opposite direction to what was obtained for the scenario of the previous section, but as one might expect –, player i's welfare increases with the initial extra payment he can obtain with acceptance of his offer, and decreases with what the other player can get: it "pays" to make reasonable propositions.

Under equal discount rates the previous expressions simplify to:

(29) $$x^* = \frac{1+d+d^2d_1-d_2+d^3\bar{u}_1-d\bar{u}_2}{(1+d)(1+d^2)} \qquad y^* = \frac{d^2(1+d)+d_1-d^2d_2+d\bar{u}_1-d^3\bar{u}_2}{(1+d)(1+d^2)}$$

$$Z_{1a}^* = \frac{d(1+d)+d^3d_1-dd_2+\bar{u}_1-d^2\bar{u}_2}{1-d^4} \quad ;$$

$$Z_{1b}^* = \frac{d^2(1+d)+d_1-d^2d_2+d\bar{u}_1-d^3\bar{u}_2}{1-d^4}$$

$$V_{1a}^* = \frac{d^3(1+d)+(1+d-d^4)d_1-d^3d_2+d^2\bar{u}_1-d^4\bar{u}_2}{1-d^4}$$

$$V_{1b}^* = \frac{d^4(1+d)+(1+d+d^2-d^4-d^5)d_1-d^4d_2+d^3\bar{u}_1-d^5\bar{u}_2}{1-d^4}$$

The impact of his own outside option d_i on the periodic share of a first-mover is less pronounced (in absolute value) than the effect of a rise in the opponent's; the same occurs with respect to the outside option $\bar{u}_i$. For the second to play, the opposite relative magnitudes of those effects are observed.

In absolute terms – and as before -, the effect of $\bar{u}_i$ is always smaller than that of d_i on the periodic share of any player.

(Given the delay implied with the current structure, it is difficult to justify comparisons on accumulated flows; the previous regularities are not totally preserved for them.)

Take again the simplest case where $d_i = 0$ and $\bar{u}_i = u_i(1) = 1$. Then:

(30)
$$x^* = 1 - \frac{d_2}{1+d_1 d_2} \quad ; \quad y^* = \frac{d_1}{1+d_1 d_2}$$
$$Z_{1a}^* = \frac{1}{(1-d_1)(1+d_1 d_2)} \quad ; \quad Z_{1b}^* = \frac{d_1}{(1-d_1)(1+d_1 d_2)}$$
$$V_{1a}^* = \frac{d_1^2}{(1-d_1)(1+d_1 d_2)} \quad ; \quad V_{1b}^* = \frac{d_1^3}{(1-d_1)(1+d_1 d_2)}$$

Patience is still rewarded. $x^* > y^*$ and a first-mover advantage is also present for the periodic share.

One can easily show, confronting the periodic share with that of the standard model that:

$$1 - \frac{d_2}{1+d_1 d_2} < \frac{1-d_2}{1-d_1 d_2} \quad \text{iff} \quad \delta_2 < \frac{1}{2-d_1}$$
$$\frac{d_1}{1+d_1 d_2} > \frac{d_1(1-d_2)}{1-d_1 d_2} \quad \text{iff} \quad \delta_1 < \frac{1}{2-d_2}$$

That is, the first-mover – provided the opponent is less or not much more patient than him [9] - is now expected to enjoy a worse periodic share than if no accepted proposal rewards nor time losses between proposals existed. The second mover – unless he is much more patient than the opponent [10] – will be in a better periodic situation.

Comparing Z_{1a}^* with $\frac{u_1(x^*)}{1-d_1}$ of the standard problem and likewise V_{1a}^* with $\frac{u_1(y^*)}{1-d_1}$, we see that the inequalities switch signs:

$$\frac{1}{(1-d_1)(1+d_1 d_2)} > \frac{1}{1-d_1}\,\frac{1-d_2}{1-d_1 d_2} \quad \text{iff} \quad \delta_1 < \frac{1}{2-d_2}$$
$$\frac{d_1^2}{(1-d_1)(1+d_1 d_2)} < \frac{1}{1-d_1}\,\frac{d_1(1-d_2)}{1-d_1 d_2} \quad \text{iff} \quad \delta_1 (1 - \delta_1 \delta_2) < (1 - \delta_2)(1 + \delta_1 \delta_2)$$

If an immediate solution entails no “first” time loss, it is the first to propose that receives the benefits of the bonus. The second mover – unless he is much more patient than the opponent – will be in a worse position than in the standard game.

Yet, comparing Z_{1b}^* with $\frac{u_1(x^*)}{1-d_1}$ of the standard problem and likewise V_{1b}^* with $\frac{u_1(y^*)}{1-d_1}$, we conclude that:

[9] The first condition implies that: if $\delta_1 = 0.8$, then $\delta_2 < 0.83$; if $\delta_1 = 0.5$, then $\delta_2 < 0.67$.

[10] The second condition implies that: if $\delta_2 = 0.8$, then $\delta_1 < 0.83$; if $\delta_2 = 0.5$, then $\delta_1 < 0.67$.

$$\frac{d_1}{(1-d_1)(1+d_1 d_2)} < \frac{1}{1-d_1}\ \frac{1-d_2}{1-d_1 d_2} \quad \text{iff} \quad \delta_1 (1 - \delta_1 \delta_2) < (1 - \delta_2)(1 + \delta_1 \delta_2)$$

$$\frac{d_1^3}{(1-d_1)(1+d_1 d_2)} < \frac{1}{1-d_1}\ \frac{d_1(1-d_2)}{1-d_1 d_2} \quad \text{iff} \quad \delta_1^2 (1 - \delta_1 \delta_2) < (1 - \delta_2)(1 + \delta_1 \delta_2)$$

In this case, the current game yields smaller welfare than the standard problem, once now both players are penalized by the unavoidable one (even first) period delay – for the preparation of the proposal.

Under $\delta_1 = \delta_2 = \delta$,

(31) $$x^* = 1 - \frac{d}{1+d^2} \quad ; \quad y^* = \frac{d}{1+d^2}$$

$$Z_{1a}^* = \frac{1}{(1-d)(1+d^2)} \quad ; \quad Z_{1b}^* = \frac{d}{(1-d)(1+d^2)}$$

$$V_{1a}^* = \frac{d^2}{(1-d)(1+d^2)} \quad ; \quad V_{1b}^* = \frac{d^3}{(1-d)(1+d^2)}$$

The one period implied loss of cake between offers has the interesting effect of factoring twice the discount factor in the denominator and switching the general aspect of the x* and y* solutions, relative to the standard case. In fact, that switch is what is partly achieved when the proposing party is rewarded with a full "pie" – even if x* is still larger than y*.

. Considering the replacement of δ_i by $e^{-r_i\Delta}$ [11] in (23) and (24) and subsequent manipulation to represent instantaneous bargaining, we can arrive at:

(32) $$\frac{u'_1(x^*)}{u'_2(1-x^*)} = \frac{r_1}{r_2}\ \frac{u_1(x^*) - \frac{\bar{u}_1 + d_1}{2}}{u_2(1-x^*) - \frac{\bar{u}_2 + d_2}{2}}$$

With concave utility functions, one would want to:

[11] One can show that such replacement will still be valid if the players maximize a continuous-time utility function and relevant bargaining periods initiate at discrete time intervals of length Δ analogously to the setting described in Appendix 1.

$$(33) \qquad u^W = \underset{u \in G}{\text{Arg Max}} \left[u_1 - \frac{d_1 + \bar{u}_1}{2}\right]^{g_1} \left[u_2 - \frac{d_2 + \bar{u}_2}{2}\right]^{g_2}$$

We maximize the product of the gains relative to average expected utility powered by a factor proportional to the inverse of each player's interest rate. Alternatively, (32) could be generated by:

$$u^W = \underset{u \in G}{\text{Arg Max}} \left[(u_1 - d_1) - (\bar{u}_1 - u_1)\right]^{g_1} \left[(u_2 - d_2) - (\bar{u}_2 - u_2)\right]^{g_2}$$

weighing equally and symmetrically in each of the Cobb-Douglas function factors the distance to the threat point as to the "ideal" status achieved by the player [12].

Of course, for the maximand to have meaning, $\left(\frac{d_1 + \bar{u}_1}{2}, \frac{d_2 + \bar{u}_2}{2}\right)$ must belong to set G. If it does not, one could generate the F.O.C. (32) requiring that

$$u^W = \underset{u \geqslant G \text{ (and } u \in G)}{\text{Arg Min}} \left[\frac{d_1 + \bar{u}_1}{2} - u_1\right]^{g_1} \left[\frac{d_2 + \bar{u}_2}{2} - u_2\right]^{g_2} =$$

$$= \underset{u \geqslant G \text{ (and } u \in G)}{\text{Arg Min}} \left[(\bar{u}_1 - u_1) - (u_1 - d_1)\right]^{g_1} \left[(\bar{u}_2 - u_2) - (u_2 - d_2)\right]^{g_2}$$

or, in general (provided the arguments of the absolute value brackets have the same sign):

$$\underset{u \geqslant G \text{ (and } u \in G)}{\text{Arg Min}} \left|(\bar{u}_1 - u_1) - (u_1 - d_1)\right|^{g_1} \left|(\bar{u}_2 - u_2) - (u_2 - d_2)\right|^{g_2}$$

where $u \geqslant G$ stands for u is to the northeast or on the northeastern boundary of set G - in space (u_1, u_2).

For instance, under an egalitarian possibility set with respect to utilities, and $u_1 + u_2 = \bar{U}$:

$$u_1^W - d_1 = \frac{g_1(2\bar{U} - 2d_1 - d_2 - \bar{u}_2) + g_2(\bar{u}_1 - d_1)}{2(g_1 + g_2)} \quad \text{or} \quad u_1^W = \frac{g_1(2\bar{U} - d_2 - \bar{u}_2) + g_2(d_1 + \bar{u}_1)}{2(g_1 + g_2)}$$

[12] See Martins (2004) for cooperative or axiomatic bargaining forms also including both distances.

Assuming $\bar{u}_i = \overline{U}$, as $(\frac{d_1+\overline{U}}{2}, \frac{d_2+\overline{U}}{2})$ does not belong to set G, using the implied interior F.O.C. – identical to the condition from which we, nevertheless, departed - deriving:

$$u_1^W - d_1 = \frac{(g_1+g_2)(\overline{U}-d_1)-g_1(d_2+d_1)}{2(g_1+g_2)} = \frac{(g_1+g_2)\overline{U}-(2g_1+g_2)d_1-g_1d_2}{2(g_1+g_2)}$$

$$\text{or} \quad u_1^W = \frac{(g_1+g_2)\overline{U}-(g_1d_2-g_2d_1)}{2(g_1+g_2)}$$

will not originate a maximum for the proposed objective function (33). In case $\gamma_1 = \gamma_2$, the solution hence derived, $u_1^W = \frac{\overline{U}-\frac{d_2-d_1}{2}}{2}$, differs again from the Nash solution, $u_1^N = \frac{\overline{U}-d_2+d_1}{2}$, but now in the opposite direction: the equilibrium favors less the player with higher "threat" utility than the Nash solution does.

If $\bar{u}_i = \overline{U}$ - d_j, even under asymmetric bargaining strength of the two players we arrive at the egalitarian split of the surplus:

$$u_1^W - d_1 = \frac{\overline{U}-d_1-d_2}{2} \qquad \text{or} \qquad u_1^W = \frac{\overline{U}-d_2+d_1}{2}$$

Proposition 2: Assume an alternating offers protocol as above where a first period bonus is conceded to the maker of an acceptable(ed) an offer and a one period loss is implied between offers.

2.1. The sub-game perfect equilibrium will possess a first mover advantage and reward patience as usual (under linear utilities).

2.2. (With linear utilities) The first-mover is expected to have a worse periodic share than in the traditional framework – even if he is the one receiving the bonus – but to enjoy a better welfare position if the starting period delay is avoidable.

2.3. A player's accumulated utility and periodic share increases with the utility he can derive from the bonus he obtains with acceptance of his offer, and decreases with the other player's.

2.4. The underlying instantaneous maximand is of the multiplicative Nash form, but in each factor the incremental utility is measured relative to the mid-point between the *status quo* and maximal potential utility of each player.

3. Alternating Offers with Uncertain Outside Options

. Assuming the possibility that in case of rejection of an offer, the player can enjoy the full periodic pie with probability p_i, generates the context of the traditional model. d_i is just replaced by

(34) $$\overline{w}_i = [p_i \, \overline{u}_i + (1 - p_i) \, d_i]$$

in the equations of the previous section. $\overline{w}_i$ has the status of a recurrent *outside option* always available to any offer made to individual i until contract closure.

Consider instead a setting where player i may reject a proposition, in which case he gets d_i, but implying engaging in uncertain war – with per period utility $\overline{w}_i$ - thereafter. If player 1 is listening to an offer, he will accept 2's proposition y iff:

$$\sum_{i=0}^{\infty} d_1^t \, u_1(y) = \frac{u_1(y)}{1-d_1} \geq d_1 + \sum_{i=1}^{\infty} d_1^t \, [p_1 \overline{u}_1 + (1-p_1) d_1] = d_1 + \frac{d_1 [p_1 \overline{u}_1 + (1-p_1) d_1]}{1-d_1}$$

or $$u_1(y) - d_1 \geq \delta_1 \, p_1 \, [\overline{u}_1 - d_1]$$

Knowing this, if player 2 is making an offer, he will propose the lowest y such that the inequality holds; hence y* will be such that equality is observed. (The condition will prevail if, in case of rejection, 1 can still make an acceptable offer - provided that $p_1 \, [\overline{u}_1 - d_1] > u_1(x^*) - d_1$, where x* refers the solution for x in the conventional model.)

Likewise, if player 1 is proposing, he will choose the highest x such that:

$$u_2(1\text{-}x) - d_2 \geq \delta_2 \, p_2 \, [\overline{u}_2 - d_2]$$

That is, x* will be such that the equality will hold.

Proceed now to the evaluation of the players' relative increment utilities under both circumstances, replacing δ_i by $e^{-r_i\Delta}$ [13] in the two conditions in equality. They will translate into:

$$u_1(y^*) - d_1 = e^{-r_1\Delta} \, p_1 \, [\overline{u}_1 - d_1]$$

and $$u_2(1\text{-}x^*) - d_2 = e^{-r_2\Delta} \, p_2 \, [\overline{u}_2 - d_2]$$

[13] One can show that such replacement will still be valid if the players maximize a continuous-time utility function and relevant bargaining periods initiate at discrete time intervals of length Δ analogously to the setting described in Appendix 1.

As Δ tends to 0, we may conjecture that if x* tends to y* - or rather, that only y* and 1 – x* can occur and both players' position is equally likely:

(35)
$$\frac{u_1(x^*)-d_1}{u_2(1-x^*)-d_2} \rightarrow \frac{p_1}{p_2}\,\frac{\overline{u}_1-d_1}{\overline{u}_2-d_2}$$

Then a generalization of the Kalai-Smorodinsky solution - weighted by the relative probability of unilateral win ratio - is found. Note that under the egalitarian possibility set, if $\overline{u}_i = \overline{U} - d_j$:

(36)
$$\frac{u_1(x^*)-d_1}{u_2(1-x^*)-d_2} \rightarrow \frac{p_1}{p_2}$$

and we recover an optimal solution implying the ratio of the bargaining gains equal to the relative win probabilities.

However, it is still possible, subtracting $u_1(x^*)$ from both sides of the first equation and $u_2(1\text{-}y^*)$ from the second and, assuming that x* and y* will tend to coincide, to arrive at:

$$\frac{u'_1(x^*)}{u'_2(1-x^*)} \rightarrow \frac{p_1(\overline{u}_1-d_1)-u_1(x^*)+d_1}{p_2(\overline{u}_2-d_2)-u_2(1-x^*)+d_2} = \frac{u_1(x^*)-[p_1\overline{u}_1+(1-p_1)d_1]}{u_2(1-x^*)-[p_2\overline{u}_2+(1-p_2)d_2]},$$

We recover an unweighted Nash maximand, with factors defined relative to the effectively perceived threat points:

$$u^N = \underset{u\in G}{\text{Arg Max}}\ \{u_1-[p_1\overline{u}_1+(1-p_1)\,d_1]\}\,\{u_2-[p_2\overline{u}_2+(1-p_2)\,d_2]\}$$

The conjecture that as Δ tends to 0 x* tends to y* is, nevertheless, improbable.

4. Mixed Strategies under Alternating Offers

. Consider instead a more flexible bargaining structure and that we allow for mixed strategies of the players. The turn to propose alternates between the two players, with each player having the right to propose – make an offer - or not. That is, strategies involve not only a (potential) share proposal – to hold eternally -, but also a probability attached to its actual transmission when it is a player's turn to make offers; and an eventual acceptance probability of the partner's proposition.

Denote by r_i the probability with which player i makes (decides to make) an offer when it is his turn to make it, and $(1-r_i)$ the probability with which he does not make an offer. r_i can (or will) only differ from either 0 or 1 in equilibrium iff i is indifferent between making an offer or not.

If the player making the offers does not make one, it is understood that he engages in theft/war, obtaining expected utility in the period $\overline{w}_i$ = [p_i $\overline{u}_i$ + (1 – p_i) d_i]. Also, i engages in war - or in general, consumes that outside option yielding utility $\overline{w}_i$ - if he does not receive an offer when it is due. That is, when no offer is exchanged, both players attain their "war" utility per period, $\overline{w}_i$.

When a player receives an offer, he can either accept it or not- we will denote by s_i the probability with which i accepts a proposition made to him by player j. If an offer is received by i, then "war utility" is not an immediate possibility, but only acceptance or rejection yielding utility d_i in the period for i - while j receives $\overline{w}_j > d_j$[14]; upon rejection, next period, it is i's turn to make or not offers.

Let V_i be the value of the game – the accumulated discounted expected utility – for player i when he is the first to receive offers (i.e., the second to propose, or the second to play), or rather, at a moment where he is waiting for a proposal from j; Z_i is the value of the game at a point where it is i's turn to make offers.

Equilibria will entail the following properties [15]:

i) For player i to use mixed strategies, he will balance the two alternatives: s_i (r_i) will (or can) differ from either 0 or 1 in equilibrium iff i is indifferent between acceptance (making an offer) or not: they must generate the same *extended* value of the game – otherwise, i would use pure strategies.

ii) Nevertheless, the extended value of a two-branch decision node may still equalize with pure strategies (if not, it will be the maximum of the extended value of the two routes; then the node does not provide an equality for the equilibrium identification): meaning that the other player is forcing one's hand when that is beneficial to him.

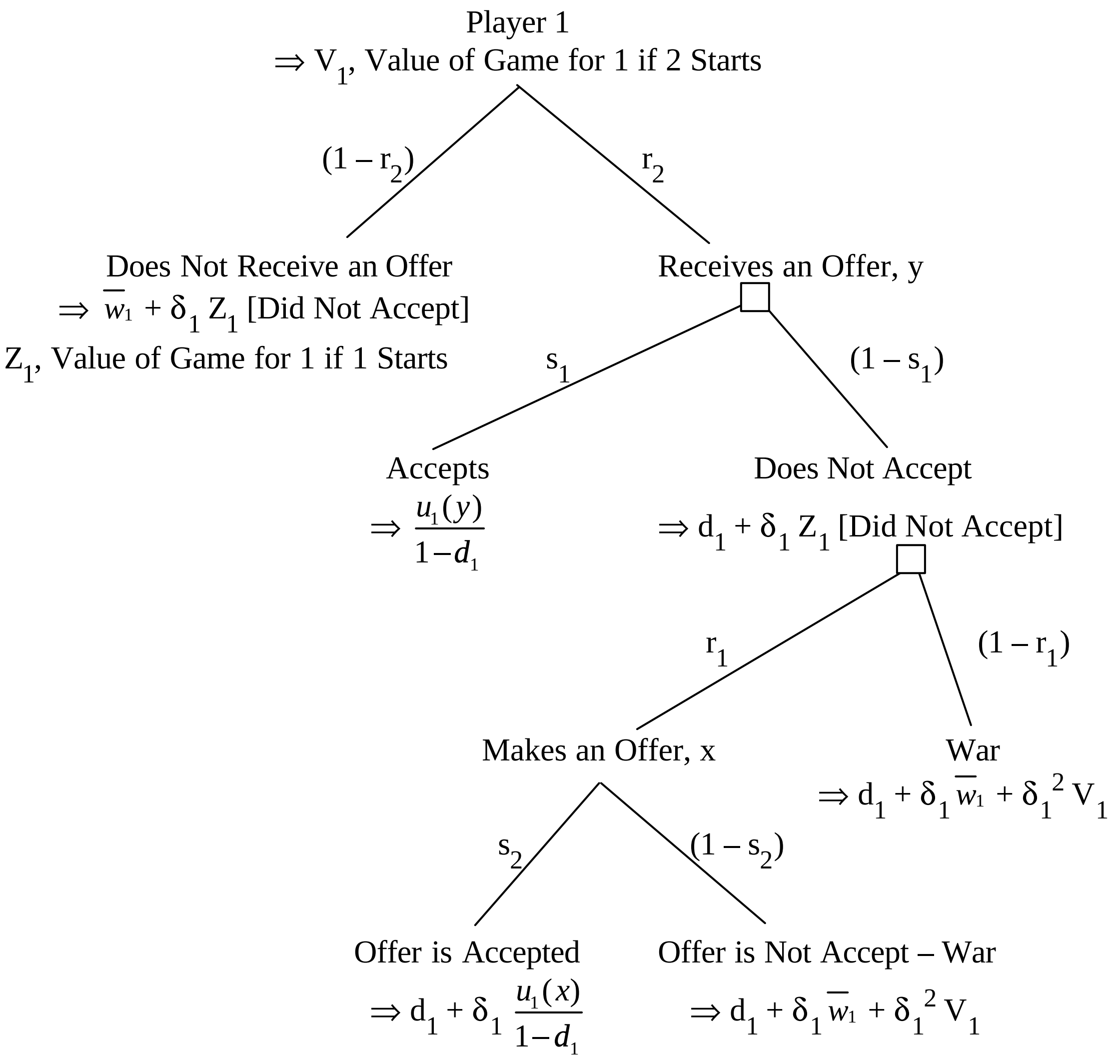
If beneficial, a player can push the opponent to the most of his advantage. If i uses mixed strategies of one variable or is indifferent, the extended value of the game for j at the corresponding junction may be pushed to the worst outcome (for j) by the use of i's probability: it will not alter player i's outcome, but it may the opponent's. If it does, the threat to the opponent may secure i an infinitesimal benefit. One can therefore add:

[14] The equilibrium would not change if j received d_j in this case, provided $\overline{w}_j > d_j$. That a player may receive more upon rejection of a proposition that he made than by rejecting one that was made to him may be realistic.

[15] Non-convexity of felicity functions should insure satisfactory performance of the statements. Given the relative independence of the pie distribution deducted before, the general linearity of the remaining problem should suffice.

iii) If i uses mixed strategies of one variable or is indifferent relative to it in the equilibrium solution, the extended value of the game for j at the corresponding (non-decisional) junction at a sub-game perfect equilibrium – but not necessarily at a Nash equilibrium - will be the largest of j's alternatives (which he cannot choose from…) at that point (up to an infinitesimum).

Let us describe the game structure from the point of view, as a decision tree of Player 1 – 2's is depicted in Appendix 2. Square marks denote decision nodes of the player. For purposes of interpreting the equilibrium, as in the pure strategies case, let us position the decision tree at the point where 2 is due to start the game, i.e., proposing offers; to determine the equilibrium, the convenient "background" tree for 2 will therefore be the one in which 1 is due to start:



The first stage represents whether individual 2 makes (made) an offer – which he does when it is his turn to propose with probability r_2 - or not.

If 2 does not make an offer, 1 gets the outside option $\overline{w}_1$ and the one period discounted value utility of not accepting an offer – modeled in the right hand-side of the tree: if he is not made an offer, next period he is in the same situation as if he had not accepted one today: he is then the first mover, with value of the game Z_1.

If 2 makes an offer, y, 1 is not "allowed" to make war – he is listening to propositions. He can either accept it – and he gets the sum of the corresponding discounted utilities - or not. If not, this period he gets d_1; next period, it is his turn to decide whether to make an offer – which he makes with probability r_1 – or war.

2 accepts an offer from 1 with probability or frequency s_2.

We will look for equilibria where – and make the assumption, common in the literature, that if both alternatives yield the same utility stream, an acceptance is strictly preferred to an income stream with either war or *status quo* - $s_i^* = 1$ (or 0), i = 1,2, and where acceptable propositions are made (or not at all). That is, under indifference, the game resumes – with a closing contract. In fact, several $0 < s_i^* < 1$ will be compatible with interior equilibrium given that if 1's proposal is not accepted, 1 obtains the same payoff than if he had not made one. A "free" – interior - s_i^* would require that the payoff *to i* from accepting an offer would have to be equal to the expected payoff of not accepting one. If a rejection of 1's proposal implied a payoff $\bar{s}_1$ for 1 as well, - that is $d_1 + \delta_1 \bar{s}_1 + \delta_1^2 V_1$ in the last right hand-side branch of the tree [16] -, then a $0 < s_2^* < 1$ might be possible and its determination necessary; because 2 is indifferent between accepting or not, he could then choose to harm – "threat" - player 1 with mixed strategies. As for the standard game, the dilemma can be solved requiring the solution pair to be ($x^*-\varepsilon$, $y^*+\varepsilon'$), where ε and ε' are infinitesimal quantities, $x^*-\varepsilon$ being the share accruing to 1 if he is the first to decide to make an offer, $y^*+\varepsilon'$ if 2 is the first, x^* and y^* being such that the equilibrium conditions (below) are met [17]. Or imposing that in case of indifference, a player ends the game - by ruling out mixed strategies in what offer acceptance is concerned.

[16] We inspect the solutions of games with multiple alternatives in the second and third parts of Appendix 3.

[17] Allowing for mixed strategies in the model of section 1, we see that if such advantage is not conceded to the second mover, who is indifferent as whether to start at 0 or at 1, an equilibrium solution is possible, with the second mover enjoying its standard expected value and using mixed strategies, alternating offers – rather, also rejecting with a probability larger than 0 once settlements are eternal - with the *status quo* and pushing the first-mover – who, conversely to the second-mover is better-off starting at 0… - also to a second-mover expected value solution (if 2 is the first to play, by 1 using $s_1^* = \dfrac{d_2}{1+d_2}$, solving $s_1 \dfrac{u_2(1-y^*)}{1-d_2} + (1 - s_1)\,[d_2 + \delta_2 \dfrac{u_2(1-x^*)}{1-d_2}] = \dfrac{u_2(1-x^*)}{1-d_2}$ and using the fact that $u_2(1-x^*)-d_2 = d_2[u_2(1-y^*)-d_2]$). Note, however, that only an infinitesimal is required to insure that the second-mover is no longer indifferent towards accepting at 0 – then, credible threat of further use of mixed strategies disappears. Also, we are assuming that the first-mover only considers pure strategies…

A Nash equilibrium with mixed offer strategies will consist of a four-tuple (x*, y*, r_1*, r_2*) such that, for player 1:

- He will be indifferent between accepting or not offer y* if it arrives:

(37) $$\frac{u_1(y^*)}{1-d_1} = d_1 + \delta_1 Z_1^*$$

For 1 to accept, the left hand-side must be not smaller than the right hand-side; player 2 will pick the smallest y*, hence establishing equality. (This would be required for an interior free s_1*.)

- He will be indifferent between making an acceptable offer x* or war:

(38) $$\frac{u_1(x^*)}{1-d_1} = \overline{w}_1 + \delta_1 V_1^*$$

For an r_1* between 0 and 1, equality is required between the yield of the two alternatives of the decision node. (This will guarantee that player 1's offer x* is the lowest that he will accept – 2 will push it to that limit with an opposite potential inequality.)

As noted, if a rejection of 1's proposal implied a payoff d_1 for 1 as well, - that is $d_1 + \delta_1 d_1 + \delta_1^2 V_1$ in the last right hand-side branch of the tree -, then, the condition above would be replaced by:

$$s_2^* \frac{u_1(x^*)}{1-d_1} + (1 - s_2^*)(d_1 + \delta_1 V_1^*) = \overline{w}_1 + \delta_1 V_1^*$$

An s_2* = 0 could be possible, as well as values between 0 and 1.

- The equilibrium value of the game if 1 starts, Z_1*, can be written as:

One could argue then that equilibrium would be guaranteed by $\frac{u_1(y'^*)}{1-d_1} = d_1 + \delta_1 \frac{u_1(x'^*)}{1-d_1}$ and $\frac{u_2(1-y'^*)}{1-d_2} = d_2 + \delta_2 \frac{u_2(1-x'^*)}{1-d_2}$ where y' is today's share of 1 and x is tomorrow's. Under linear utilities, the system solves for y'* = $\frac{(1-d_2)(1-d_2)-(1-d_1)d_1}{d_1-d_2}$ (the effective split) and x'* = $\frac{d_1(1-d_2)(1-d_2)-d_2(1-d_1)d_1}{d_1-d_2}$, but not under equal discount rates or alternatives – and requiring 0 < y'*, x'* < 1...

(39) $$Z_1^* = r_1^* \frac{u_1(x^*)}{1-d_1} + (1 - r_1^*)(\overline{w}_1 + \delta_1 V_1^*) = \frac{u_1(x^*)}{1-d_1}$$

The preceding comment still applies.
- The equilibrium value of the game for 1 if 2 starts, V_1^*:

(40) $$V_1^* = (1 - r_2^*)(\overline{w}_1 + \delta_1 Z_1^*) + r_2^* \frac{u_1(y^*)}{1-d_1} =$$

$$= (1 - r_2^*)\{\overline{w}_1 + [\frac{u_1(y^*)}{1-d_1} - d_1]\} + r_2^* \frac{u_1(y^*)}{1-d_1} = \frac{u_1(y^*)}{1-d_1} + (1 - r_2^*)(\overline{w}_1 - d_1)$$

that given (39) – or (37) and (39) -, can be written as the last expression.

And symmetrically for player 2. We have 8 – linearly independent - equations and 8 unknowns – x*, y*, V_1^*, V_2^*, Z_1^*, Z_2^*, r_1^* and r_2^* - and in general the system will possess a unique solution exhibiting acceptance and interior solutions for the r_i's. Two things should be highlighted: firstly, with 8 equations – and even if the immediate payoff for not making an offer and for not having an offer accepted differ between themselves and from the one received when no offer occurs - we can never expect to solve for mixed strategies in both r_i's and both s_i's.

Secondly, if making a rejectable proposition yields the same utility as not making any, only one offer will be relevant. Suppose that the two decisions yielded a different utility; then, as long as rejecting generates a unique immediate payoff, d_i – even if different from that of not making an offer – only one offer, say k, can be accepted by a player with interior s_i^k – see Appendix 3. It cannot be the higher proposition from his point of view, because then the other player would not want to make it, or will not consider making no offers. If the lower, we could not solve for all variables (with the method…); if just equalization for i of expected values at s_i^k and $(1 - s_i^k)$, some inconsistency would be generated at j's emission decision nodes.

Exploring the properties of the equilibrium solution:

Conditions (37) and (39) lead to: $$\frac{u_1(y^*)}{1-d_1} = d_1 + \delta_1 \frac{u_1(x^*)}{1-d_1}$$

generating:

(41) $$u_1(y^*) - d_1 = \delta_1 [u_1(x^*) - d_1]$$

Hence, we will recover the same optimal offers than in the no war game.

With (41), (38) implies that:

(42) $$V_1^* = \frac{u_1(x^*) - (1-d_1)\overline{w}_1}{d_1(1-d_1)} = \frac{u_1(y^*) - (1-d_1)(d_1 + d_1\overline{w}_1)}{d_1^2(1-d_1)}$$

where (x*, y*) are the Rubinstein's splits. Player's 1 value of the game if he starts is:

(43) $$Z_1^* = \frac{u_1(x^*)}{1-d_1}$$

once, by (38) equality of the two weighted terms must be observed for $0 < r_1^* < 1$. Then the first-mover is as well-off as in the standard game.

For the interior solution for r_1 to be possible, if 2 starts, the value of the game for player 1, $V_1^* > \frac{d_1 + d_1\overline{w}_1}{1-d_1^2}$, the utility he would get alternating rejection or obtaining no offer, with making no offers - a path always open to a player and the minimum that he can secure by himself - which implies:

(44) $$u_1(x^*) > \frac{\overline{w}_1 + d_1 d_1}{1+d_1} \quad \text{or} \quad u_1(y^*) > \frac{d_1 + d_1\overline{w}_1}{1+d_1}$$

If 1 starts, $Z_1^* = \frac{u_1(x^*)}{1-d_1} > \frac{\overline{w}_1 + d_1 d_1}{1-d_1^2}$ what 1 can get with making no offer and rejecting the other player's offers – guaranteeing that, as a first-mover, he would actually make a proposition -, which also implies

(45) $$u_1(x^*) > \frac{\overline{w}_1 + d_1 d_1}{1+d_1} \quad \text{or} \quad u_1(y^*) > \frac{d_1 + d_1\overline{w}_1}{1+d_1}$$

Comparing (42) and (43), $Z_1^* > V_1^*$ and the first-mover advantage is still present. Note that $\frac{u_1(x^*)}{1-d_1}$ was the value of the game if 1 starts in the general set-up; hence, he makes no gain nor loss for having an outside option under the current structure; but – see (40) - $V_1^* > \frac{u_1(y^*)}{1-d_1}$ as long as $\overline{w}_1 > d_1$ and $r_2^* < 1$, i.e., $u_1(x^*) > \frac{\overline{w}_1 + d_1 d_1}{1+d_1}$ required by (45).

Yet, given that $\overline{w}_1$ does not intervene in the determination of x* nor y*, (42) implies that in an interior solution with mixed strategies the value of the(is) game for a second-mover in fact becomes reduced with the size of his outside options: creating/improving alternatives is only good for a "second-mover" if it gets the player out of the bargaining table, otherwise, the first-mover will always be able to anticipate it and force the potential gains of a second-mover down[18].

Finally, with (40) and (42) we infer

$$(46)\qquad 1 - r_2^* = \frac{(1-d_1)V_1^* - u_1(y^*)}{(\bar{w}_1 - d_1)(1-d_1)} = \frac{u_1(x^*) - \bar{w}_1 - d_1[u_1(y^*) - \bar{w}_1]}{(1-d_1)d_1(\bar{w}_1 - d_1)} =$$

$$= \frac{(1+d_1)u_1(x^*) - \bar{w}_1 - d_1 d_1}{d_1(\bar{w}_1 - d_1)} = \frac{(1+d_1)u_1(y^*) - d_1\bar{w}_1 - d_1}{d_1^2(\bar{w}_1 - d_1)}$$

$$= \frac{V_1^* - (d_1 + d_1 Z_1^*)}{(\bar{w}_1 - d_1)} \quad \text{and}$$

$$r_2^* = \frac{(1+d_1)[\bar{w}_1 - u_1(x^*)]}{d_1(\bar{w}_1 - d_1)} = \frac{(1+d_1)[d_1(\bar{w}_1 - d_1) - u_1(y^*) + d_1]}{d_1^2(\bar{w}_1 - d_1)}$$

r_2^* increases with $\bar{w}_1$ iff $u_1(x^*) > d_1$; it is independent of $\bar{w}_2$.

Provided $\bar{w}_1 > d_1$:

$$(47)\qquad r_2^* > 0 \text{ iff } \quad \bar{w}_1 > u_1(x^*) = \frac{u_1(y^*) - (1-d_1)d_1}{d_1} \quad \text{or}$$

$$u_1(y^*) < (1 - \delta_1)\, d_1 + \delta_1 \bar{w}_1$$

$$r_2^* < 1 \text{ iff } \quad \bar{w}_1 < (1+d_1)u_1(x^*) - d_1 d_1 = u_1(x^*) + d_1[u_1(x^*) - d_1] =$$

$$= \frac{(1+d_1)u_1(y^*) - d_1}{d_1}$$

$$\text{or} \quad u_1(x^*) > \frac{\bar{w}_1 + d_1 d_1}{1+d_1} \quad \text{or} \quad u_1(y^*) > \frac{d_1 + d_1\bar{w}_1}{1+d_1}$$

Then, $r_2^* < 1$ will guarantee (44) or (45): interestingly, the failure of $r_2^* < 1$ implies that if in pure strategies player 1 would not make a first-mover offer.

Player 2 will face symmetrical constraints. Then:

$$(48)\qquad u_2(1-x^*) - d_2 = \delta_2\, [u_2(1-y^*) - d_2]$$

$$(49)\qquad V_2^* = \frac{u_2(1-y^*) - (1-d_2)\bar{w}_2}{d_2(1-d_2)}$$

$$(50)\qquad Z_2^* = (1 - r_2^*)\, [\bar{w}_2 + \delta_2 V_2^*] + r_2^* \frac{u_2(1-y^*)}{1-d_2} = \frac{u_2(1-y^*)}{1-d_2}$$

And

$$(51)\qquad 1 - r_1^* = \frac{(1-d_2)V_2^* - u_2(1-x^*)}{(1-d_2)(\bar{w}_2 - d_2)} = \frac{u_2(1-y^*) - \bar{w}_2 - d_2[u_2(1-x^*) - \bar{w}_2]}{(1-d_2)d_2(\bar{w}_2 - d_2)}$$

[18] There is no inconsistency with the last statement of the previous paragraph: the structure of the standard game is completely different than that of the current stage.

$$= \frac{(1+d_2)u_2(1-y^*) - \bar{w}_2 - d_2 d_2}{d_2(\bar{w}_2 - d_2)} = \frac{(1+d_2)u_2(1-x^*) - d_2\bar{w}_2 - d_2}{d_2^2(\bar{w}_2 - d_2)} \quad \text{and}$$

$$r_1^* = \frac{(1+d_2)[\bar{w}_2 - u_2(1-y^*)]}{d_2(\bar{w}_2 - d_2)} = \frac{(1+d_2)[d_2(\bar{w}_2 - d_2) - u_2(1-x^*) + d_2]}{d_2^2(\bar{w}_2 - d_2)}$$

r_1^* increases with $\bar{w}_2$ iff $u_2(1-y^*) > d_2$; it is independent of $\bar{w}_1$.

Provided $\bar{w}_2 > d_2$:

(52) $r_1^* > 0$ iff $\bar{w}_2 > u_2(1-y^*) = \dfrac{u_2(1-x^*) - (1-d_2)d_2}{d_2}$ or

$$u_2(1-x^*) < (1 - \delta_2)\, d_2 + \delta_2\, \bar{w}_2$$

$r_1^* < 1$ iff $\bar{w}_2 < (1+d_2)u_2(1-y^*) - d_2 d_2 = \dfrac{(1+d_2)u_2(1-x^*) - d_2}{d_2}$

or $u_2(1-y^*) > \dfrac{\bar{w}_2 + d_2 d_2}{1+d_2}$ or $u_2(1-x^*) > \dfrac{d_2 + d_2\bar{w}_2}{1+d_2}$

In general, each player may have equal chance to be the first to start the game. The value of the game for player i is, thus,

$$(53)\quad \frac{1}{2}V_1^* + \frac{1}{2}Z_1^* = \frac{(1+d_1)u_1(x^*) - (1-d_1)\bar{w}_1}{2d_1(1-d_1)} = \frac{(1+d_1)u_1(y^*) - (1-d_1^2)\,d_1 - d_1(1-d_1)\bar{w}_1}{2d_1^2(1-d_1)}$$

$$\frac{1}{2}V_2^* + \frac{1}{2}Z_2^* = \frac{(1+d_2)u_2(1-y^*) - (1-d_2)\bar{w}_2}{2d_2(1-d_2)} = \frac{(1+d_2)u_2(1-x^*) - (1-d_2^2)d_2 - d_2(1-d_2)\bar{w}_2}{2d_2^2(1-d_2)}$$

The value of the game for player i decreases with $\bar{w}_i$.

The probability that player i makes an offer in the first period is $\frac{1}{2}$ r_i^*; that an offer is made in the first period, $\frac{1}{2}$ $(r_1^* + r_2^*)$. The probability that player i is the first to advance an offer ever is:

$$(54)\quad \frac{1}{2}\{ r_i^* + (1 - r_i^*)(1 - r_j^*)\, r_i^* + [(1 - r_i^*)(1 - r_j^*)]^2\, r_i^* + \ \dots \} +$$

$$+ \frac{1}{2}\{(1 - r_j^*)\, r_i^* + (1 - r_j^*)(1 - r_i^*)(1 - r_j^*)\, r_i^* + (1 - r_j^*)\,[(1 - r_i^*)(1 - r_j^*)]^2\, r_i^* + \ \dots \} =$$

$$= \frac{1}{2}\ \frac{r_i^*}{r_i^* + r_j^* - r_i^* r_j^*} + \frac{1}{2}\ \frac{r_i^*(1-r_j^*)}{r_i^* + r_j^* - r_i^* r_j^*} = \frac{1}{2}\ \frac{r_i^*(2-r_j^*)}{r_i^* + r_j^* - r_i^* r_j^*}$$

The probability that an offer will ever be made is, in the interior solution of this game, 1 – the probability that there will never be an offer is 0. The probability that 1 makes the (first) offer is larger than the one that 2 does, i.e., $\frac{1}{2}\frac{r_1*(2-r_2*)}{r_1*+r_2*-r_1*r_2*} > \frac{1}{2}\frac{r_2*(2-r_1*)}{r_1*+r_2*-r_1*r_2*}$, iff $r_1* > r_2*$.

Finally, the expected waiting time until a first offer is made if i is the first to play is

$$(55)\quad 1\,r_i* + 2\,(1-r_i*)\,r_j* + 3\,(1-r_i*)\,(1-r_j*)\,r_i* + 4\,(1-r_i*)^2\,(1-r_j*)\,r_j* + 5\,(1-r_i*)^2\,(1-r_j*)^2\,r_i* + 6\,(1-r_i*)^3\,(1-r_j*)^2\,r_j* + \ldots = \frac{r_i*[1+(1-r_i*)(1-r_j*)]+2r_j*(1-r_i*)}{(r_i*+r_j*-r_i*r_j*)^2}$$

The expected waiting time until an offer is made is, thus, $\frac{\frac{r_1*+r_2*}{2}[1+(1-r_i*)(1-r_j*)]+r_j*(1-r_i*)+r_i*(1-r_j*)}{(r_i*+r_j*-r_i*r_j*)^2} = \frac{\frac{r_1*+r_2*}{2}[3+(1-r_i*)(1-r_j*)]-2r_1*r_2*}{(r_i*+r_j*-r_i*r_j*)^2}$.

$r_j* < 1$ guarantees that in the interior solution conditions (44) hold: that at the point of making offers, i is better-off accepting than rejecting and then not proposing, which yields $\frac{d_i+d_i\overline{w_i}}{1-d_i^2}$, and at the point of listening to j's offers, than $\frac{\overline{w_i}+d_id_i}{1-d_i^2}$.

Notice that $r_j* > 0$ implies that in any interior solution, i would be better-off even if he got the welfare of a first player and therefore, if the game was not played – i.e., if neither he nor the opponent made offers. That is, the Nash equilibrium with mixed strategies is not optimal. Technically, it can occur – it is a Nash equilibrium: each player is doing his best given that the other player made an offer with more than probability 0 – because (provided $r_j* < 1$) it still insures that each player is better-off than what he can secure on his own - $\frac{d_i+d_i\overline{w_i}}{1-d_i^2}$ or $\frac{\overline{w_i}+d_id_i}{1-d_i^2}$; ultimately, because during the game, any player is forced to listen and while it does, it cannot secure more than d_i. It is easy to show, however, that such equilibrium is not unique and that $r_i* = 0$ for i = 1,2 is, in that parameter range, also a Nash equilibrium – and the unique sub-game perfect equilibrium [19].

The range conditions also shed light on equilibria outside the parameter range.

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[19] Multiple Nash equilibria in which one is inferior for both players and cannot be discarded without refinements such as sub-game perfection are known in pure strategies: equilibrium in price strategies is – can be systematically - better (worse) for firms than in quantity strategies in duopolies of substitute (complement) goods – see Singh and Vives (1984) using linear demands -, but both provide Nash equilibria. The same can occur with wages relative to employment strategies of union substitute (complement) labor types - see examples in Martins (2005).

Firstly, one can establish that the other several possible equilibrium types are five: $r_i^* = 0$, $0 < r_j^* < 1$; $r_i^* = 1$, $0 < r_j^* < 1$; $r_i^* = 0$, $r_j^* = 1$; $r_i^* = 0$, $r_j^* = 0$; $r_i^* = 1$, $r_j^* = 1$. The first two stage mixed strategies of one player and pure of the other, the last two, equal pure offer strategies of both players - the last one, the conventional exchange. We will inspect –superficially - the vicinity of the doubly interior solutions.

Given that the interior solution in mixed strategies originates an inefficient result – related to – and the opposite significance found for each bound (of type (44)) relative to optimality and lower alternative, the equilibrium solutions when, say, r_i^* (but not r_j^*) only crosses one of the limits (may) suffer discontinuities – and reversals.

Let us assume that the mixed strategy equilibrium probability $r_i^* > 1$ for a player but not for the other. If the player for which $r_i^* > 1$ made an (the…) offer with $r_i^* = 1$, the other would better respond with $r_j^* = 0$; but with $r_j^* = 0$, $r_i^* = 0$ too leaves both players in the better solution than (x*, y*) does: the game "stops" at the - preferable for both - no-offer equilibrium.

(Note: If $r_i^* = 0$ for a player but not for the other, for i to accept an offer from j, he is going to press j to the felicity bound $u_j(1-z^*) = \overline{w_j}$ - j will obtain $V_j^* = \frac{\overline{w_j}}{1-d_j}$. Additionally, r_j^* is determined through the expected value definition of the game of player i (because z* was determined); if then $r_j^* > 0$ mixed strategies are available; otherwise, $r_j^* = 1$ could be required.)

If $r_i^* < 0$, (but $0 < r_j^* < 1$) j would be obtaining $u_j(1-z^*) > \overline{w_j}$ but i $u_i(z^*) < \overline{w_i}$. If j is the first to play, he makes the Rubinstein offer provided the other is not better off rejecting it. If i is the first to play, he may make an offer z* such that $u_j(1-z^*) = \overline{w_j}$ if $u_i(z^*) > \overline{w_i}$, otherwise, he makes no offers and allows j to be the first to play and offer him z* or the Rubinstein split. Multiple equilibria may still occur.

. Consider linear utility functions for both players and the cake normalized to 1. The optimal x* and y* are still the same as in (3) and the same comments apply:

$$x^* = \frac{d_2(1-d_1)d_1 + (1-d_2)(1-d_2)}{1-d_1 d_2} \quad ; \quad y^* = \frac{(1-d_1)d_1 + d_1(1-d_2)(1-d_2)}{1-d_1 d_2}$$

Then:

$$V_1^* = \frac{d_2(1-d_1)d_1 + (1-d_2)(1-d_2) - (1-d_1 d_2)(1-d_1)\overline{w_1}}{d_1(1-d_1)(1-d_1 d_2)} \tag{56}$$

and

$$Z_1^* = \frac{d_2(1-d_1)d_1 + (1-d_2)(1-d_2)}{(1-d_1)(1-d_1 d_2)} \tag{57}$$

$$\frac{1}{2}V_1^* + \frac{1}{2}Z_1^* = \frac{(1+d_1)[d_2(1-d_1)d_1 + (1-d_2)(1-d_2)] - (1-d_1 d_2)(1-d_1)\overline{w_1}}{2d_1(1-d_1)(1-d_1 d_2)} \tag{58}$$

(59) $$r_2^* = (1+\delta_1)\,\frac{(1-d_1 d_2)\overline{w}_1 - d_2(1-d_1)d_1 - (1-d_2)(1-d_2)}{d_1(1-d_1 d_2)(\overline{w}_1 - d_1)}$$

r_2^* increases with d_2 iff $\overline{w}_1 > d_1$; it will decrease with d_1 iff $\overline{w}_1 + d_2 < 1$.
Provided $\overline{w}_1 > d_1$:

(60) For $r_2^* > 0$: $$\overline{w}_1 > \frac{d_2(1-d_1)d_1 + (1-d_2)(1-d_2)}{1-d_1 d_2}$$

For $r_2^* < 1$: $$\overline{w}_1 < \frac{(d_2-d_1)d_1 + (1+d_1)(1-d_2)(1-d_2)}{1-d_1 d_2}$$

Under equal discount rates we recover (5):

$$x^* = \frac{d d_1 + (1-d_2)}{1+d} \quad ; \quad y^* = \frac{d_1 + d(1-d_2)}{1+d}$$

(61) $$V_1^* = \frac{d d_1 + 1 - d_2 - (1-d^2)\overline{w}_1}{d(1-d^2)}$$

(62) $$Z_1^* = \frac{d d_1 + (1-d_2)}{1-d^2}$$

(63) $$\frac{1}{2}V_1^* + \frac{1}{2}Z_1^* = \frac{d d_1 + 1 - d_2 - (1-d)\overline{w}_1}{2d(1-d)}$$

and

(64) $$r_2^* = \frac{(1+d)\overline{w}_1 - d d_1 - 1 + d_2}{d(\overline{w}_1 - d_1)}$$

Provided that $\overline{w}_1 > d_1$:

(65) For $r_2^* > 0$: $\overline{w}_1 > \dfrac{d d_1 + 1 - d_2}{1+d}$ For $r_2^* < 1$: $\overline{w}_1 < 1 - d_2$

(65) highlights that in this case, in interior solutions, $\overline{w}_i$ can never be larger than 1 (d_j is non-negative) – the alternative cannot be larger than the pie for each individual - and never smaller than $\dfrac{d d_i + 1 - d_j}{1+d}$ and therefore, if d_i*'s are negligible, than 0.5. $\overline{w}_1 + \overline{w}_2$ will sum more than 1 (provided only that $d_1 + d_2 < 1$) – and therefore, than the "pie"; yet, it is not exchangeable between the players -, but less than 2.

To simplify the comparisons, let $d_i = u_i(0) = 0$ and $\overline{w}_i = p_i$ (potentially, $\overline{w}_i = p_i\, u_i(1) + (1-p_i)\, u_i(0) = p_i\, u_i(1) + (1-p_i)\, d_i$ and represent the expected periodic war pay-off - equal to p_i with linear utilities and for $d_i = 0$). The optimal x* and y* are again:

$$x^* = \frac{1-d_2}{1-d_1 d_2} \quad ; \quad y^* = \frac{d_1(1-d_2)}{1-d_1 d_2}$$

Now:

(66) $$V_1^* = \frac{1-d_2-(1-d_1)(1-d_1d_2)p_1}{d_1(1-d_1d_2)(1-d_1)} = \frac{1-d_2}{d_1(1-d_1d_2)(1-d_1)} - \frac{p_1}{d_1}$$

(67) $$Z_1^* = \frac{1-d_2}{(1-d_1 d_2)(1-d_1)}$$

(68) $$\frac{1}{2}V_1^* + \frac{1}{2}Z_1^* = \frac{(1+d_1)(1-d_2)-(1-d_1)(1-d_1d_2)p_1}{2d_1(1-d_1d_2)(1-d_1)}$$

(69) $$1 - r_2^* = \frac{(1-d_1)V_1^* - u_1(y^*)}{(1-d_1)(\overline{w}_1 - d_1)} = \frac{(1-d_2)(1+d_1)-(1-d_1d_2)p_1}{d_1(1-d_1d_2)p_1}$$

$$r_2^* = \frac{(1+d_1)[(1-d_1d_2)p_1-(1-d_2)]}{d_1(1-d_1d_2)p_1} = \frac{(1+d_1)[d_2(1-d_1p_1)-(1-p_1)]}{d_1(1-d_1d_2)p_1}$$

r_2^* increases with δ_2, with patience of player 2.

(70) For $0 < r_2^* < 1$: $$\frac{1-d_2}{1-d_1d_2} < p_1 < 1 - \frac{d_2-d_1}{1-d_1d_2} = \frac{(1-d_2)(1+d_1)}{1-d_1d_2}$$

If 1 starts the game, 2's expected payoff will be:

(71) $$V_2^* = \frac{1-d_1-(1-d_2)(1-d_1d_2)p_2}{d_2(1-d_1d_2)(1-d_2)}$$

(72) If 2 starts: $$Z_2^* = \frac{1-d_1}{(1-d_1 d_2)(1-d_2)}$$

(73) $$\frac{1}{2}V_2^* + \frac{1}{2}Z_2^* = \frac{(1+d_2)(1-d_1)-(1-d_2)(1-d_1d_2)p_2}{2d_2(1-d_1d_2)(1-d_2)}$$

(74) $$1 - r_1^* = \frac{(1-d_2)V_2^* - u_2(1-x^*)}{(1-d_2)(\overline{w}_2 - d_2)} = \frac{(1-d_1)(1+d_2)-(1-d_1d_2)p_2}{d_2(1-d_1d_2)p_2}$$

$$r_1^* = \frac{(1+d_2)[(1-d_1d_2)p_2-(1-d_1)]}{d_2(1-d_1d_2)p_2} = \frac{(1+d_2)[d_1(1-d_2p_2)-(1-p_2)]}{d_2(1-d_1d_2)p_2}$$

r_1^* increases with δ_1, with patience of player 1.

(75) For $0 < r_1^* < 1$: $$\frac{1-d_1}{1-d_1d_2} < p_2 < 1 - \frac{d_1-d_2}{1-d_1d_2} = \frac{(1-d_1)(1+d_2)}{1-d_1d_2}$$

We depict below the range of admissible pairs (δ_i, δ_j) – i=1,2, j=2,1 -, for interior solutions of r_j^* - applying if j is the first-mover - after formulas (70) and (75), for p_i = 0.4 and 0.75. Such admissible pairs are in the lenses formed between the upper and lower curve of each graph: above the lim Inf lines ($\delta_i < (1 - \frac{1-d_j}{p_i})\frac{1}{d_j}$ and) $r_j^* > 0$, below lim Sup, $r_j^* < 1$. In the upper north-west region to the lens, $r_j^* > 1$, in the lower south-east, $r_j^* < 0$.

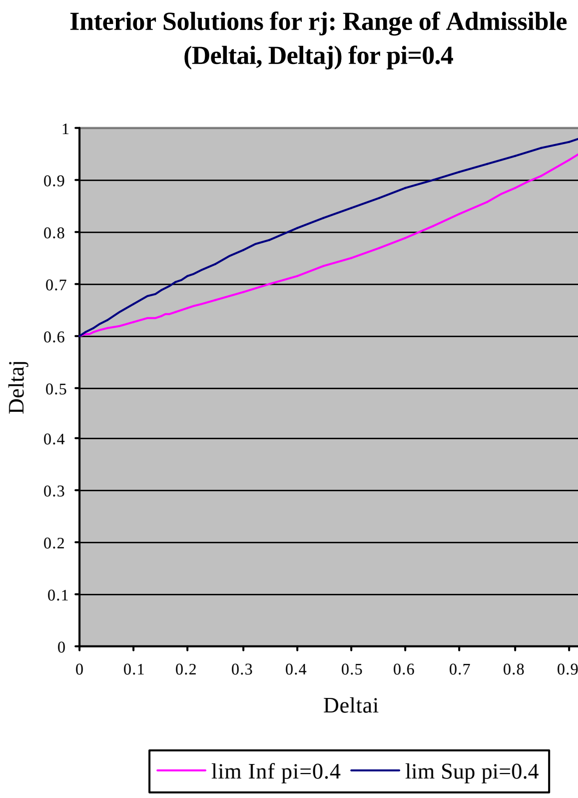


Figure 1. Admissible (δ_i, δ_j), p_i = 0.4

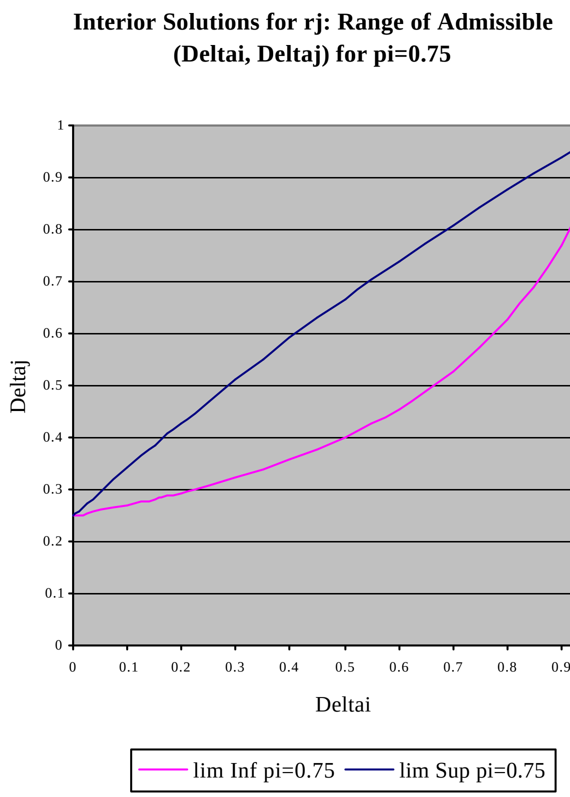


Figure 2. Admissible (δ_i, δ_j), p_i = 0.75

Apparently, as p_i rises [20], the admissible area for interior solutions, i.e., mixed strategies, for player j to be possible increases. The area of ranges implying $r_j^* < 0$ decreases.

Also,

(76) $$r_1^* > r_2^* \quad \text{iff} \quad \frac{d_1(1-d_1)(1+d_2)}{p_2} - \frac{d_2(1-d_2)(1+d_1)}{p_1} < (\delta_1 - \delta_2)\,(1-d_1 d_2)$$

At given discount rates, the lower p_1 and the higher p_2, the more likely will 1 be the first to go forward.

. Under equal discount rates, we still recover (6):

$$x^* = \frac{1}{1+d} \qquad ; \qquad y^* = \frac{d}{1+d}$$

Now:

(77) $$V_1^* = \frac{1-(1-d^2)p_1}{d(1-d^2)}$$

(78) $$Z_1^* = \frac{1}{1-d^2}$$

(79) $$\frac{1}{2}V_1^* + \frac{1}{2}Z_1^* = \frac{1-(1-d)p_1}{2d(1-d)}$$

(80) $$1 - r_2^* = \frac{1}{d}\,\frac{1-p_1}{p_1} \quad ; \quad r_2^* = \frac{(1+d)p_1 - 1}{d\,p_1}$$

$$1 - r_1^* = \frac{1}{d}\,\frac{1-p_2}{p_2} \quad ; \quad r_1^* = \frac{(1+d)p_2 - 1}{d\,p_2}$$

Both r_i's increase with δ: patience promotes the development of the game into (acceptable) proposition exchange.

(81) $r_1^* > r_2^*$ iff $p_2 > p_1$. For r_i^* larger than 0: $p_i > \frac{1}{1+d}$.

With equal impatience, the player with lower alternatives will more likely move first.

Proposition 3: Assume an alternating offers protocol with a general outside option (implying higher utility for a player than the one he derives while listening and

[20] Note that p was depicted for ranges around 0.5…

refusing an offer) described above, admitting mixed strategies for both players.
For an adequate range of outside options, a Nash equilibrium will exist, originating the same exchanged offer levels as in the standard model and where the players will make equilibrium offers with a lower than 1 frequency, alternating their proposals with the (upper) outside option.

3.1. In the interior solution, the first mover obtains the same utility as
in the standard model; but - even if a first-mover advantage is still present – the "second mover" obtains higher utility in the presence of the outside option. Yet, his welfare decreases with that option's size.

3.2. Under linear utilities:

- the expected utility of the second mover decreases with the (exogenous) subjective probability with which he perceives he can unilaterally achieve a total victory.
- the frequency with which a party makes war instead of an offer when it is due decreases with the subjective probability with which the opponent assesses achieving a total victory – his outside option in case of discontentment; decreases with the perceived utility obtained when he rejects an offer; increases with that accruing to the opponent.

3.3. The Nash equilibrium in mixed strategies will not be unique and (both) players would be better "off the game" – in the persistent outside option. The latter is then the sub-game perfect equilibrium.

. If $d_i > \overline{w}_i$, to secure an interior solution in mixed strategies for player 2, it had to be the case that:

For $r_2^* > 0$, iff $\overline{w}_1 < u_1(x^*) = \dfrac{u_1(y^*) - (1-d_1)d_1}{d_1}$ or

$$(1 - \delta_1)\, d_1 + \delta_1 \overline{w}_1 > u_1(y^*)$$

For $r_2^* < 1$, iff $\overline{w}_1 > (1+d_1)u_1(x^*) - d_1 d_1 = u_1(x^*) + d_1[u_1(x^*) - d_1] = \dfrac{(1+d_1)u_1(y^*) - d_1}{d_1}$

$$\text{or} \quad u_1(x^*) < \frac{\overline{w}_1 + d_1 d_1}{1+d_1} \quad \text{or} \quad u_1(y^*) < \frac{d_1 + d_1 \overline{w}_1}{1+d_1}$$

and similarly for player 1. But now, any player would be better-off if both kept alternating rejectable offers… (No offers would just secure the lower alternative $\overline{w}_i$, worse for a first player).

For $\overline{w_1}$ < $u_1(x^*)$ < $\frac{\overline{w_1}+d_1d_1}{1+d_1}$ (and $d_1 > \overline{w_1}$), $\frac{\overline{w_1}+d_1d_1}{1+d_1}$ < d_1 and therefore $u_1(x^*) < d_1$ – and also $u_1(y^*) < d_1$; the Rubinstein formula could still apply – but now the second mover would be in advantage.

In mixed strategies: $u_1(x^*) < \frac{\overline{w_1}+d_1d_1}{1+d_1}$ guarantees that $Z_1^* < \frac{\overline{w_1}+d_1d_1}{1-d_1^2}$, and also insures that $V_1^* < \frac{d_1+d_1\overline{w_1}}{1-d_1^2}$: both players are better-off exchanging rejectable offers.

One can show that in pure strategies:

- as a first mover, player 1 would rather not to make an offer *or* make a rejectable offer – and receive $\overline{w_1}$ - and wait for the other to make a "Rubinstein" offer y* - provided that $u_1(x^*) < \frac{\overline{w_1}+d_1d_1}{1+d_1}$.

- as a second-mover, he would be better-off rejecting, getting d_1 and waiting for next period's either $\overline{w_1}$ when his offer would be turned down (or he would not make an offer) - securing $\frac{d_1+d_1\overline{w_1}}{1+d_1}$ ($u_1(y^*) < \frac{d_1+d_1\overline{w_1}}{1+d_1}$ and $u_1(x^*) < u_1(y^*)$).

Both players would exchange rejectable offers – and the first (a…) mover would just had to "agree" to make a rejectable one rather than no offer… Such arrangement would always seem possible (no offer or forwarding a rejectable offer yields $\overline{w_i}$ in the period for a player) – and the interior solution in mixed strategies would not occur.

5. Simultaneous Sequential Bargaining with Mixed Strategies (and Outside Options)

. The previous model suffers from the first mover indeterminacy of the traditional sequential bargaining model. In this section, relying on mixed strategies, we model simultaneous bargaining contemplating the possibility of synchronous mutual offers being made and try to infer how robust the previous conclusions are to the alternate bargaining protocol assumption.

As before, even if mixed – hence, probabilistic or random - strategies are allowed, the model is still one of perfect information of the players. The decision tree of player 1 – player 2's will be symmetric and depicted in Appendix 2 – is depicted below.

At a given decision point in time – which take place at discrete (unit) time intervals - while the game is being played and has not ended in the closing of a contract, the players decide to make an offer – the share rule for all future pies - or not. If an offer is exchanged, the player receiving it may accept it or not; s_i is the frequency with which i chooses to accept a contract when he is offered one.

We admit two general options to closing a contract: if i's proposal is not accepted, i gets $\overline{w_i}$ in the period which, for i, is always an available alternative to making an offer. If i receives an offer, he either accepts it, or gets d_i - that may or not differ from $\overline{w_i}$ - in the current period. When a proposal is not accepted, the game reverts next period to its initial state, of value V_i for player i.

The equilibrium concept applicable is going to be one which has a matching nature, being associated to either a proposal of 1 or a proposal of 2 being made with a given probability. We denote by r_2 the frequency with which 2 makes an offer; by r_1 the frequency with which 1 makes an offer. Neither r_1 nor r_2 are the observed probabilities with which an acceptable offer is made, nor the probability that a player is the first to make offers, but the subjective frequencies with which each player chooses to make them. There will be an offer made by 1 (2) in the period if 1 (2) makes an offer and 2 (1) does not.

Synchronicity of the exchange offer decision is guaranteed by assuming:

- independence with a two-by-two initial branch of each player's game tree and with the probability of the player - 1 in the tree below - making an offer forced to be the same whether he received or not an offer.

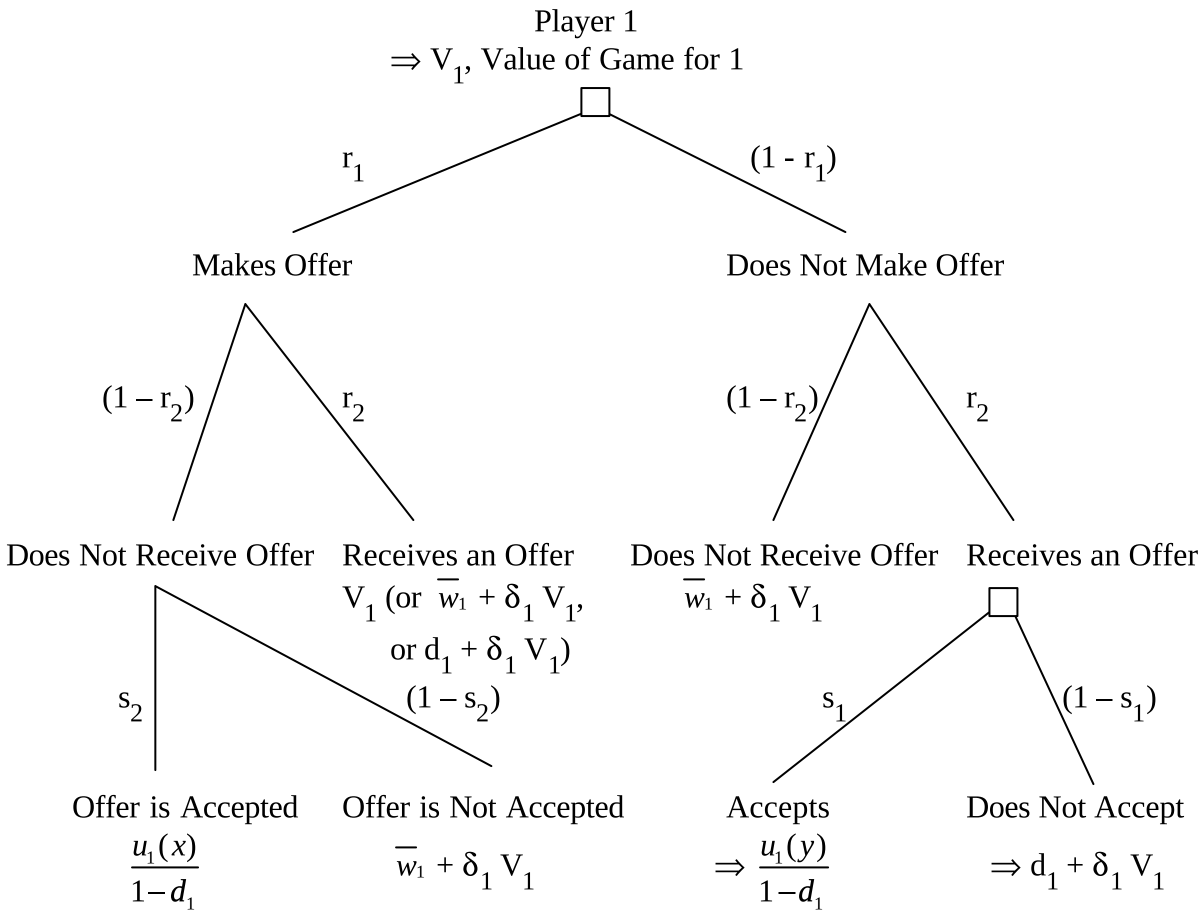
- identical frequencies in the symmetric nodes of both opponents' decision trees.

We can now postulate that if both make offers, hence, disagreeing, either:

- they both get the outside option $\overline{w_i}$ (or d_i) and the game restarts next period.

- they get another "try", being as if the game started again. This rules out in the ex-post observed decisions the case of simultaneous offers.

Of course, allowing for an exogenous probability of ending in either of the two previous solutions in case both players make offers would be possible.



Let us consider the second alternative only. Then, the probability that player 1 effectively makes an offer in the first period – in each period while the game has not finished in a contract - is:

$$(82)\ r_1 (1 - r_2) + r_1 r_2 r_1 (1 - r_2) + (r_1 r_2)^2 r_1 (1 - r_2) + (r_1 r_2)^3 r_1 (1 - r_2) + ... = \frac{r_1(1-r_2)}{1-r_1 r_2}$$

For player 2, we would have symmetrically:

$$(83)\qquad \frac{r_2(1-r_1)}{1-r_2 r_1}$$

Still, $\frac{r_1(1-r_2)}{1-r_1 r_2} > \frac{r_2(1-r_1)}{1-r_2 r_1}$ iff $r_1 > r_2$.

The probability that there will be no offer exchange in the period – and of both players getting $\overline{w}_i + \delta_i V_i$ -, denoted by p is:

$$(84)\ p = (1-r_1)(1-r_2) + r_1 r_2 (1-r_1)(1-r_2) + (r_1 r_2)^2(1-r_1)(1-r_2) + (r_1 r_2)^3(1-r_1)(1-r_2) + ... =$$

$$= \frac{(1-r_1)(1-r_2)}{1-r_1 r_2} = 1 - \frac{r_1(1-r_2)}{1-r_1 r_2} - \frac{r_2(1-r_1)}{1-r_2 r_1}$$

Of course, *ex-post*, the probabilities that either 1 or 2 are the first to propose will only be meaningful for acceptable offers. All other choices would be equivalent and "added" to this no offer exchange situation.

In fact, the settings of the game could be such that decisions are finalized at discrete time intervals but moves are prolonged in continuous time; p is the probability that a decision was not and will not be reached, that discussion time elapses without a conclusion – say, both want to make offers -, having the same consequences for the players as if no offer is made.

The expected number of periods till an offer is made is:

(85) $$1\,(1-p) + 2\,p\,(1-p) + 3\,p^2\,(1-p) + 4\,p^3\,(1-p) + \ldots = \frac{1}{1-p} = \frac{1-r_1 r_2}{r_1+r_2}$$

It will, thus, move in the same direction as p.

. We still assume that an acceptable offer is strictly preferred to any other state if yielding the same utility stream. And as before, we will inspect equilibria where offers are accepted when made; hence $s_i^* = 1$, $i = 1,2$. Or, the equilibrium (an equilibrium ending in acceptance) is a solution pair ($x^*-\varepsilon$, $y^*+\varepsilon'$), where ε and ε' are infinitesimal quantities, $x^*-\varepsilon$ being the share accruing to 1 when he makes the offer, $y^*+\varepsilon'$ if 2 makes the offer, x* and y* being such that the conditions (below) are satisfied.

The Nash equilibrium will consist of a four-tuple (x^*, y^*, r_1^*, r_2^*) such that the extended value of the game in the two immediate branches of any decision node equalize and satisfying a definition of the value of the game. Then, for player 1:

- He will be indifferent between accepting or not offer y* if it arrives:

(86) $$\frac{u_1(y^*)}{1-d_1} = d_1 + \delta_1\ V_1^*$$

Player 2 will push y* down till equality holds.

- He will be indifferent between making an acceptable offer x* or not (using also the last conclusion). As the decision to make an offer is simultaneous to the other player's, and as we are searching for acceptable offers (If the rejection of player i's offer implied a payoff for i of $d_i < \overline{w}_i$, the conditions would still hold.):

(87) $$(1-r_2^*)\ \frac{u_1(x^*)}{1-d_1} + r_2^*\ V_1^* = (1-r_2^*)\ (\overline{w}_1 + \delta_1\ V_1^*) + r_2^*\ \frac{u_1(y^*)}{1-d_1}$$

$$= (1 - r_2^*)\ \overline{w}_1 + r_2^* d_1 + \delta_1 V_1^*$$

This is required if an interior solution for r_1^* is going to hold.

- The equilibrium value of the game, given the previous indifference, is the previous equality:

(88) $$V_1^* = (1 - r_2^*)\ \overline{w}_1 + r_2^* d_1 + \delta_1 V_1^* = \frac{\overline{w}_1 - r_2^*(\overline{w}_1 - d_1)}{1 - d_1}$$

Then, using (87):

(89) $$V_1^* = \frac{u_1(x^*)}{1 - d_1}$$

From (86):

(90) $$V_1^* = \frac{u_1(y^*) - (1 - d_1)d_1}{(1 - d_1)d_1}$$

Now, equating (89) to (90) we recover the standard Rubinstein condition:

(91) $$u_1(y^*) - d_1 = \delta_1 [u_1(x^*) - d_1]$$

Then (90) implies that interior solutions will (must…) guarantee to both players the expected utility of a first-mover in the conventional alternate offers game.

Comparing the value of this game, given by (89), with that of the alternate offers game (47), which involves the same (x*, y*), the synchronous setting will imply larger payoff:

(92) $$\frac{u_1(x^*)}{1 - d_1} > \frac{(1 + d_1)u_1(x^*) - (1 - d_1)\overline{w}_1}{2d_1(1 - d_1)} \quad \text{iff} \quad u_1(x^*) < \overline{w}_1$$

Using (88), (89) and (86):

(93) $$r_2^* = \frac{\overline{w}_1 - u_1(x^*)}{\overline{w}_1 - d_1} = \frac{d_1\overline{w}_1 + (1 - d_1)d_1 - u_1(y^*)}{d_1(\overline{w}_1 - d_1)} \quad \text{and}$$

$$1 - r_2^* = \frac{u_1(x^*) - d_1}{\overline{w}_1 - d_1} = \frac{u_1(y^*) - d_1}{d_1(\overline{w}_1 - d_1)}$$

r_2^* increases with $\overline{w}_1$ iff $u_1(x^*) > d_1$.

From (93), provided $\overline{w}_1 > d_1$:

(94) For $r_2^* > 0$: $\overline{w}_1 > u_1(x^*)$. For $r_2^* < 1$: $u_1(x^*) > d_1$

If $\overline{w}_1 < d_1$, which can occur, once the player cannot get out of the game, the opposite signs hold.

The symmetric conditions for player 2 would close the game.

From (91) – or (1), even if we cannot talk about a first mover in the present context – x* > y* iff $d_1 < u_1(x^*)$; then, (93) implies that for r_2^* in a relevant range, $d_1 < u_1(y^*) < u_1(x^*) < \overline{w}_1$. Also, if x* > y*, 1 – y* > 1 – x*; then an internal solution requires likewise that also $\overline{w}_2 > u_2(1-y^*) > u_2(1-x^*) > d_2$.

Given that the player can face future war (no proposal) in case he rejects today's offer from the opponent, it is possible, in spite of (1), that $u_1(y^*) < d_1$, and hence, y* > x*, which occurs for $\overline{w}_1 < u_1(x^*) < u_1(y^*) < d_1$. If war, or rather, do not receive nor make propositions implies more loss than rejecting a proposal – an also likely case, in realistic terms -, i.e., $d_1 > \overline{w}_1$ and $d_2 > \overline{w}_2$, it is better to be made than to make an acceptable offer.

The probability that player 1 makes the first offer in the period is:

(95) $$\frac{(1-r_2^*)r_1^*}{1-r_2^*r_1^*} = \frac{[u_1(x^*)-d_1][\overline{w}_2-u_2(1-y^*)]}{(\overline{w}_1-d_1)(\overline{w}_2-d_2)-[\overline{w}_1-u_1(x^*)][\overline{w}_2-u_2(1-y^*)]}$$

The probability that no offer will be made in the period is:

(96) $$p^* = \frac{(1-r_1^*)(1-r_2^*)}{1-r_2^*r_1^*} = \frac{[u_1(x^*)-d_1][u_2(1-y^*)-d_2]}{(\overline{w}_1-d_1)(\overline{w}_2-d_2)-[\overline{w}_1-u_1(x^*)][\overline{w}_2-u_2(1-y^*)]}$$

Provided $u_2(1-y^*) > d_2$, this probability decreases with $\overline{w}_1$. Provided $u_1(x^*) > d_1$, this probability decreases with $\overline{w}_2$.

As in the alternate offers game of section 4, $r_j^* > 0$ implies that in any interior solution, i would be better-off if the game was not played – i.e., if neither he nor the opponent made offers. That is, the Nash equilibrium with mixed strategies is not optimal. Again, it is a Nash equilibrium: because ($r_j^* < 1$) it still insures that each player is better-off than what he can secure on his own – now, d_i. As before, that such equilibrium is not unique and that $r_i^* = 0$ for i = 1,2 is, in that parameter range, also a Nash equilibrium – and the unique sub-game perfect equilibrium.

Again, the range conditions (94) suggests equilibria outside the parameter range.

If the mixed strategy equilibrium probability $r_i^* < 0$ for a player but not for the other, j the equilibrium in mixed strategies is lost. If the player, say i, for which $r_i^* < 0$ does not make offers, i.e., $r_i^* = 0$, for it to accept an offer from j, he is going to press j to the felicity bound $u_j(1-z^*) = \overline{w}_j$ - j will obtain $V_j^* = \dfrac{\overline{w}_j}{1-d_j}$. Additionally, r_j^* is determined through the expected value definition of the game of player i; if then $r_j^* > 0$, which only occurs if i can obtain a higher expected value of the game than $\frac{\overline{w}_i}{1-d_i}$, mixed strategies are available; otherwise, $r_j^* = 1$ could be required. Most likely, $r_i^* = 1$ and $r_j^* = 0$, with j, in any case, pushed to $V_j^* = \frac{\overline{w}_j}{1-d_j}$.

. Specify now the linear utility functions. We will have the same offer level as before in (3):

$$x^* = \frac{d_2(1-d_1)d_1 + (1-d_2)(1-d_2)}{1-d_1 d_2} \quad ; \quad y^* = \frac{(1-d_1)d_1 + d_1(1-d_2)(1-d_2)}{1-d_1 d_2}$$

Then:

(97) $$V_1^* = \frac{d_2(1-d_1)d_1 + (1-d_2)(1-d_2)}{(1-d_1)(1-d_1 d_2)}$$

and

(98) $$r_2^* = \frac{(1-d_1 d_2)\overline{w}_1 - d_2(1-d_1)d_1 - (1-d_2)(1-d_2)}{(1-d_1 d_2)(\overline{w}_1 - d_1)}$$

r_2^* increases with d_2 iff $\overline{w}_1 > d_1$. It decreases with d_1 iff $\overline{w}_1 + d_2 < 1$.
r_2^* decreases with δ_1 iff $d_1 + d_2 < 1$. It increases with δ_2 iff $d_1 + d_2 < 1$.
Provided $\overline{w}_1 > d_1$:

(99) For $r_2^* > 0$: $\overline{w}_1 > \frac{d_2(1-d_1)d_1 + (1-d_2)(1-d_2)}{1-d_1 d_2}$

For $r_2^* < 1$: $d_1 + d_2 < 1$

Note that this type of range condition arises, for example, in Ponsatí and Sákovics (1998) and we refer the reader to them for further justifications. We showed that the above conditions insure existence of a “matching equilibrium” – under which, a synchronous exchange offer decision along with mixed strategies broaden the range of possibilities.

Under equal discount rates we recover (5):

$$x^* = \frac{d d_1 + (1-d_2)}{1+d} \quad ; \quad y^* = \frac{d_1 + d(1-d_2)}{1+d}$$

(100) $$V_1^* = \frac{d d_1 + 1 - d_2}{1-d^2}$$

and

(101) $$r_2^* = \frac{(1+d)\overline{w}_1 - d d_1 - (1-d_2)}{(1+d)(\overline{w}_1 - d_1)}$$

Provided that $\overline{w}_1 > d_1$:

(102) For $r_2^* > 0$: $\overline{w}_1 > \frac{dd_1 + 1 - d_2}{1+d}$. Always, $r_2^* < 1$.

(102) highlights that in this case, in interior solutions, $\overline{w}_i$ can never be smaller than $\frac{dd_i + 1 - d_j}{1+d}$ and therefore, if d_i*'s are negligible, than 0.5; then, $\overline{w}_1 + \overline{w}_2$ will sum more than 1 (provided only that $d_1 + d_2 < 1$) – and therefore, than the "pie". Yet, the $\overline{w}_i$'s can now be larger than 1.

. For simplicity, let $d_i = 0$ and specify – as in the previous section - $\overline{w}_i = p_i$. Then:

$$x^* = \frac{1-d_2}{1-d_1 d_2} \qquad ; \qquad y^* = \frac{d_1(1-d_2)}{1-d_1 d_2}$$

Now:

(103) $$V_1^* = \frac{1-d_2}{(1-d_1)(1-d_1 d_2)}$$

(104) $$r_2^* = \frac{(1-d_1 d_2)p_1 - (1-d_2)}{(1-d_1 d_2)p_1} \quad \text{or} \quad 1 - r_2^* = \frac{1-d_2}{(1-d_1 d_2)p_1}$$

Always, $r_2^* < 1$. For $r_2^* > 0$:

(105) $$p_1 > \frac{1-d_2}{1-d_1 d_2} \quad \text{or} \quad \delta_2 > \frac{1-p_1}{1-d_1 p_1}$$

We depict below the separating line assuring an internal solution for cases of $p_i = 0.4$ and $p_i = 0.75$. Above the line(s), $r_j^* > 0$ and we have an interior solution; below, $r_j^* < 0$.

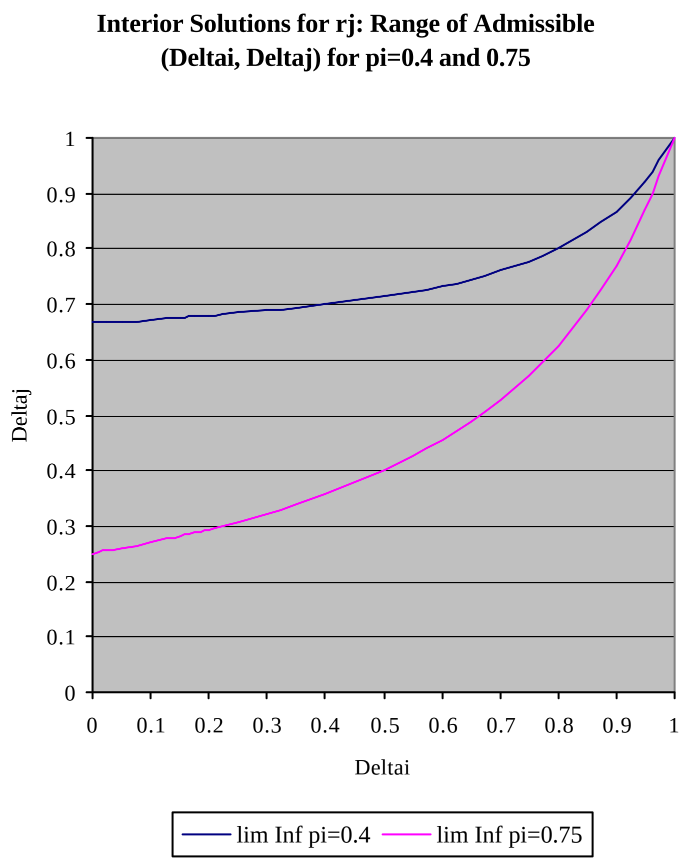


Figure 3. Admissible (δ_i, δ_j), $p_i = 0.4$ and 0.75

Again, a rise in seems to broaden the admissible area for interior solutions to be possible.

(106) $r_1^* > r_2^*$ iff $p_2 (1 - \delta_2) > p_1 (1 - \delta_1)$

The smaller p_1 and the higher δ_1 – the worse his alternatives and the more patient or lower interest rate player 1 has - the more likely player 1 will be the "first" player - the one to "move" first.

The probability that no offer will be exchanged in the period is:

(107) $$p^* = \frac{(1-d_1)(1-d_2)}{(1-d_1)(1-d_1 d_2) p_1 + (1-d_2)(1-d_1 d_2) p_2 - (1-d_1)(1-d_2)}$$

It decreases with either p_i; for it to be positive: $\frac{(1-d_1 d_2) p_1}{1-d_2} + \frac{(1-d_1 d_2) p_2}{1-d_1} > 1$. To be smaller than 1:

$$\frac{(1-d_1 d_2) p_1}{1-d_2} + \frac{(1-d_1 d_2) p_2}{1-d_1} > 2$$

Hence this second condition prevails.
The expected number of periods till an offer is made:

(108) $$\frac{1-r_1*r_2*}{r_1*+r_2*} = \frac{1}{1-d_1 d_2} \frac{(1-d_1 d_2)[(1-d_1)p_1+(1-d_2)p_2]-(1-d_1)(1-d_2)}{2(1-d_1 d_2)p_1 p_2-(1-d_1)p_1-(1-d_2)p_2}$$

Under equal discount rates:

$$x* = \frac{1}{1+d} \quad ; \quad y* = \frac{d}{1+d}$$

(109) $$V_1* = \frac{1}{1-d^2}$$

(110) $$r_2* = \frac{(1+d)p_1-1}{(1+d)p_1} \quad \text{or} \quad 1 - r_2* = \frac{1}{(1+d)p_1}$$

(111) Always, $r_2* < 1$. For $r_2* > 0$: $p_1 > \frac{1}{1+d}$

(112) $r_1* > r_2*$, iff $p_1 < p_2$.

The probability that no offer will be exchanged in the period is:

(113) $$p* = \frac{1}{(1+d)(p_1+p_2)-1}$$

It decreases either δ – with the patience of the players; for it to be smaller than 1:

$$p_1 + p_2 > \frac{2}{1+d}$$

which will also guarantee it will be positive.
The expected time till an offer is made (that moves in the same direction as p*) is:

(114) $$\frac{1-r_1*r_2*}{r_1*+r_2*} = \frac{(1+d)(p_1+p_2)-1}{(1+d)(p_1+p_2)-2}$$

Proposition 4: Assume a simultaneous offers protocol with asymmetric outside options described above, admitting mixed strategies for both players and assuming in 4.1 and 4.2 that turning down an offer or not making one implies higher utility for a player than the one he derives while listening and refusing an offer (d_i).

For an adequate range of outside options, a Nash equilibrium will exist, originating the same exchanged offer levels as in the standard model and where the players will make equilibrium offers with a lower than 1 frequency, alternating their proposals with the (upper) outside option.

4.1. In the internal solutions, the expected welfare of each player is that of a first-mover of the Rubinstein game. The player with higher periodic equilibrium share perspective and/or the one with lower alternatives will more likely be the first-mover.

4.2. Under linear utilities, in the internal solutions:

- the more patient player and/or the one with lower (outside) alternatives or probability of success in a war will more likely be the first-mover.

- the frequency with which a party makes war instead of an offer decreases with the subjective probability with which the opponent assesses achieving a total victory – his outside option in case of discontentment; decreases with the perceived utility obtained when he simply rejects an offer; increases with that accruing to the opponent.

- (yet) the frequency with which no offer exchange is expected to be observed decreases with the outside options - the alternative to making an offer, also available in case of being rejected. Under equal discount rates, it decreases with the patience of the players.

4.3. 3.3 holds.

. Yet, now, if we switch the relative magnitude of $\overline{w_i}$ and d_1, we suggest a possible unique sub-game perfect equilibrium: (97)-(98) apply, in (99) the inequality signs switch: now, $d_1 + d_2 > 1$; under equal discount rates, (100) and (101) hold. But sensitivity to the different parameters also switches.

Notice that, because $d_1 > u_1(x^*)$ and $d_2 > u_2(1-y^*)$, the second-mover is in a better position than a first-mover; also, d_i is not attainable simultaneously by both players and is only observed in case of rejection – not if offers are not exchanged. $u_1(x^*) > \frac{\overline{w_1} + d_1 d_1}{1+d_1}$ guarantees that player 1 would still rather be a first-mover than not to make an offer – receiving $\overline{w_1}$ - and wait for the other player to make a Rubinstein offer. Then, two interesting cases can occur:

- $\overline{w_1} < \frac{\overline{w_1} + d_1 d_1}{1+d_1} < u_1(x^*) < u_1(y^*) < d_1$ (Note that if $\frac{\overline{w_1} + d_1 d_1}{1+d_1} < u_1(x^*)$ – replacing the periodic split formula -, $\frac{d_1 + d_1 \overline{w_1}}{1+d_1} < u_1(y^*)$ and a second mover is also better off alternating rejectable offers…) and likewise for 2, $\overline{w_2} < \frac{\overline{w_2} + d_2 d_2}{1+d_2} < u_2(1-y^*) < u_2(1-x^*) < d_2$. Then one of the players would be better-off being a second mover in a Rubinstein split – but, who is going to agree to be the first mover rather than the – now in advantage - second one?

- $\overline{w}_1 < u_1(x^*) < \frac{\overline{w}_1 + d_1 d_1}{1+d_1} < d_1$ and $\overline{w}_2 < u_2(1-y^*) < \frac{\overline{w}_2 + d_2 d_2}{1+d_2} < d_2$. Then both players would be better-off alternating rejectable offers. Yet, on the one hand, they would have then to "commit" to such an alternate protocol… On the other, they would also have to agree on who is the first-mover: the second mover, with welfare $\frac{d_i + d_i \overline{w}_i}{1-d_i^2}$, as $d_i > \overline{w}_i$, is still better-off…

If we now let $\overline{w}_i = 0$, only r_2^* changes relative to (97)-(99), with (98) becoming:

$$(115) \qquad r_2^* = \frac{d_2(1-d_1)d_1 + (1-d_2)(1-d_2)}{(1-d_1 d_2)\, d_1} \quad ; \quad 1 - r_2^* = \frac{(1-d_2)(d_1 + d_2 - 1)}{(1-d_1 d_2)\, d_1}$$

For $r_2^* > 0$, $(1 - \delta_2)(d_1 + d_2 - 1) < (1 - \delta_1 \delta_2)\, d_1$. For $r_2^* < 1$, $d_1 + d_2 > 1$ – yet, d_1 and d_2 cannot occur simultaneously…

r_2^* now decreases with d_2. It will also decrease with d_1 iff $d_2 < 1$.

$$(116) \qquad r_1^* > r_2^* \text{ iff } \qquad \delta_1 + (1-\delta_1)\frac{1-d_1}{d_2} > \delta_2 + (1-\delta_2)\frac{1-d_2}{d_1}$$

r_i^* is expected to be larger for the more patient player (of higher δ_i) and (for similar δ_1 and δ_2) for the one with lower "inside" alternative d_i (provided these are higher than 0.5).

p* and the expected time till an offer is made (that moves in the same direction as p*) are also derivable but become messy expressions.

With equal discount factors:

$$(117) \qquad r_2^* = \frac{d d_1 + (1-d_2)}{(1+d)\, d_1} \quad ; \quad 1 - r_2^* = \frac{d_1 + d_2 - 1}{(1+d)\, d_1}$$

$$(118) \qquad p^* = \frac{d_1 + d_2 - 1}{1 + 2d}$$

The expected waiting time:

$$(119) \qquad \frac{1 - r_1^* r_2^*}{r_1^* + r_2^*} = \frac{1+2d}{2(1+d) - (d_1 + d_2)}$$

We can now summarize, applying when $d_i > \overline{w}_i$:

Proposition 5: Assume a simultaneous offers protocol with asymmetric outside options described above, admitting mixed strategies for both players and assuming in 5.2 and 5.3 that turning down an offer or not making one implies lower utility for a player than the one he derives while listening and refusing an offer (d_i).

5.1. For an adequate range of outside options, a Nash equilibrium will exist, originating the same exchanged offer levels as in the standard model - but these referred to a higher valued alternative - and where the players will make equilibrium offers with a lower than 1 frequency, alternating their proposals with the (upper) outside option.

5.2. In the internal solutions, the expected welfare of each player is that of a first-mover of the Rubinstein game – but both players' shares now increase with the alternative they obtain when they reject an offer. The player with lower periodic equilibrium share perspective and/or the one with higher alternative when rejecting an offer will more likely be the first-mover.

5.3. Under linear utilities, in the internal solutions:

- the more patient player and/or the one with lower ("inside") alternative to acceptance will more likely be the first-mover.

- the frequency with which a party makes war instead of an offer increases with his outside option in case of discontentment; increases with the perceived utility obtained when he simply rejects an offer; increases with that accruing to the opponent.

- (yet) the frequency with which no offer exchange is expected to be observed increases with the outside options - the alternative to making an offer, also available in case of being rejected. Under equal discount rates, it decreases with the patience of the players and increases with the "inside" rejection pay-offs.

5.4. The mixed strategies equilibrium may now be pareto-optimal – if alternate protocols are not available…

The reason why this equilibrium arises - and did not in section 4 - is that one can never secure the "rejection" by oneself: a player also has to be made a proposition and now, without the alternating protocol, the other player is also competing for it.

Finally, "rejection" by i – when $d_i > \overline{w}_i$ - could as well represent a war with j, d_i standing for the probability of player i attaining the full pie if he engages in such a war instead of closing a contract – $\overline{w}_j$ representing the expected utility simultaneously accruing to j; if we impose $d_i + \overline{w}_j < 1$, no "outside" source other than the pie is needed for the equilibrium to exist (even if we can always argue they represent subjective probabilities of winning the pie)...

Conclusion

The maximum utility each player may possibly achieve in a given negotiation may be a more important reference point to the players than conventional cooperative as non-cooperative games benchmark solutions imply. This research presented several non-cooperative bargaining setups in which infinite-term contracts are negotiated, under which the equilibrium solution possesses such mathematical property. In some, a direct reference to a potential unilateral appropriation – out of the bargaining table – of the surplus being bargained over was incorporated. In others, the implied cooperative maximand was also inspected.

A first type of scenarios invoked the possibility of a one (initial) period "gift" of the total surplus to one of the players at contract closing. In general, it implied a higher periodic share of the first mover than in the conventional model when the accepting party receives a full period bonus, lower when the proposing party receives the bonus – also, an advantage to the more patient player when the gift is given by the proposing party; and a maximand of Stone-Geary form – hence hyperbolic pseudo-social indifference curves – but with a different "origin" than that implied by Nash solution.

A second type of models, with more complex bargaining protocols, where the choice to make or not offers is formally modeled as a (random) strategy target, generated "outside option" type of results. Under reasonable assumptions of alternate sequential bargaining games, such possibility does not condition the optimal split of an (one…) internal equilibrium – even if, of course, it does its existence – and sub-game perfection.

Finally, a simultaneous bargaining game was staged, entailing a synchronous equilibrium at the offer exchange decision level. The modeling stands for its methodological interest, definitely solving the first mover dilemma in sequential bargaining setups – perfecting the attempt of the mixed strategies allowance in the previous alternate bargaining structure; as before, visualization of the decision tree of each player became useful in the construction as interpretation of the implied equilibrium.

As a final assessment of the mixed strategies models results, it was found that: the more patient player and/or the one with lower alternatives to making an offer within the protocol will more likely be the first-mover. However, interior solutions of mixed strategy frequencies of both players always imply multiple Nash equilibria and that both players would be better off in pure strategies in an alternate offers protocol; and in the simultaneous offer game if the no offer option is the highly valued; the inefficient equilibrium arises because the players are forced to listen.

In the range of inefficient mixed strategy equilibria: the frequency with which a party is expected to make an (acceptable) offer increases with the opponent's alternative in case he is rejected; increases with the perceived utility obtained when he simply rejects an offer; decreases with that accruing to the opponent. The size of the outside options as, under equal discount rates, patience of the players decreases the no offer exchange equilibrium frequency. Yet, the outside option accrues as an alternative to making an offer, and as a response to a rejection (requiring that an offer is made to be available through this channel) – this conditions the registered sign of its impact. Most change for the (almost…) efficient simultaneous game – with high "inside" (of own rejection) option, but low outside one. Rubinstein's optimal periodic division in a closed contract remained robust to most of the settings.

Bibliography and References.

Admati, Anat R. and Motty Perry. (1987) "Strategic Delay in Bargaining." *Review of Economic Studies*, 54, 345-364.

Binmore, K. Ariel Rubinstein and A. Wolinsky. (1986) "The Nash Solution in Economic Modelling." *Rand Journal of Economics.* Vol 17(2): 176-188.

Busch, Lutz-Alexander and Quan Wen. (1995) "Perfect Equilibria in a Negotiation Model." *Econometrica.* Vol 63(3): 545-565.

Cahuc, Pierre and André Zylberberg. (2001) *Le Marché du Travail*. Bruxelles: De Boeck & Larcier.

Fernandez, Raquel and Jacob Glazer. (1991) "Striking for a Bargaining Between Two Completely Informed Agents." *American Economic Review*, 81, N. 1, pp. 240-252.

Haller, Hans and Steiner Holden. (1990) "A Letter to the Editor on Wage Bargaining." *Journal of Economic Theory*, 52, 232-236.

Kalai, Ehud. (1977) "Proportional Solutions to Bargaining Situations: Interpersonal Utility Comparisons." *Econometrica.* Vol 45(7): 1623-1630.

Kalai, E. and M. Smorodinsky. (1975) "Other Solutions to Nash's Bargaining Problem." *Econometrica.* Vol 43: 513-518.

Kennan, John and Robert Wilson. (1989) "Strategic Bargaining Models and Interpretation of Strike Data." *Journal of Applied Econometrics.* Vol 4: S87-S111.

Manzini, Paola. (1998) "Game Theoretic Models of Wage Bargaining." *Journal of Economic Surveys.* Vol 12(1): 1-41.

Martins, Ana Paula. (2004) "Reconciling the Nash and Kalai-Smorodinsky Cooperative Solutions: Generalized Maximands of CES Form." Universidade Católica Portuguesa, unpublished manuscript.

Martins, Ana Paula. (2005) "Union Duopoly with Heterogeneous Labor." *International Journal of Social Economics.* Vol. 32 (4), p. 339-383.

Muthoo, Abhinay. (1995) "Bargaining in a Long - Term Relationship with Endogenous Termination." *Journal of Economic Theory.* Vol 66: 590-598.

Muthoo, Abhinay. (1999) *Bargaining Theory with Applications.* Cambridge University Press.

Nash, John. (1950) "The Bargaining Problem." *Econometrica.* Vol 18(1): 155-162.
Nash, John. (1953) "Two-person Cooperative Games." *Econometrica.* Vol 21(1): 128-140.
Osborne, M. and Ariel Rubinstein. (1990) *Bargaining and Markets.* Academic Press.
Perry, Motty and Philip J. Reny. (1993) "A Non-cooperative Bargaining Model with Strategically Timed Offers." *Journal of Economic Theory*, 59, 50-77.
Ponsatí, Clara and József Sákovics. (1998) "Rubinstein Bargaining with Two-Sided Outside Options." *Economic Theory*. Vol 11(3): 667-672.
Rubinstein, Ariel. (1982) "Perfect Equilibrium in a Bargaining Model." *Econometrica.* Vol 50: 97-109.
Sákovics, József. (1993) "Delay in Bargaining Games with Complete Information." *Journal of Economic Theory*, 59, 78-95.
Singh, Nirvikar and Xavier Vives. (1984) "Price and Quantity Competition in a Differentiated Duopoly." *Rand Journal of Economics*, Vol 15: 546-54.
Verchenko, P.I. (2004) "Multiplicative Approach in Efficiency Characteristics Estimation and Risk Pricing." In *Proceedings* of the Scientific School MA SR – 2004, St. Petersburg: p. 368-373

Appendix 1

. The conclusions and optimal split would not change if the players were continuous time optimizers but forced/allowed to bargain at discrete time intervals.

Say individuals maximize an utility function of the type:

(A.1) $$U_i = \int_0^{\infty} e^{-r_i u} u_i(z_u)\, du$$

$u_i(z_u)$ is the instantaneous felicity function (measured in a per unit of time basis) accruing to i when he obtains the share z_u of the "pie" at time u – pies are continuously being offered to the players at the rate of 1 per period. r_i is the interest rate – measured per unit of (the same) time - at which the felicity function is discounted.

Assume that between two offers, Δ units of time elapse and that offers must consist of constant settlements for that interval; that is:

$$z_u = z_t \qquad \text{for} \quad \Delta\, t < u < \Delta\,(t+1) \quad , \quad t = 0, 1, 2, \ldots$$

Then, the "chunk" of utility at stake in the t-th round of negotiations would have current (present, i.e., at moment u=0) value:

(A.2) $$\int_{\Delta t}^{\Delta(t+1)} e^{-r_i u}\, u_i(z_u)\, du \;=\; u_i(z_t) \int_{\Delta t}^{\Delta(t+1)} e^{-r_i u}\, du \;=\; \frac{u_i(z_t)}{r_i} \left[e^{-r_i \Delta t} - e^{-r_i \Delta (t+1)} \right] \;=$$

$$= \frac{u_i(z_t)}{r_i}\; e^{-r_i \Delta t} \left[1 - e^{-r_i \Delta} \right]$$

Total lifetime utility at time 0 will be:

(A.3) $$U_i = \sum_{t=0}^{\infty} \frac{u_i(z_t)}{r_i}\; e^{-r_i \Delta t} \left[1 - e^{-r_i \Delta} \right]$$

. Suppose that contracts at stake consist of stationary splits so that z_t = z. Then, if 1 is assessing an offer y at time 0, he accepts iff it has higher value than rejecting and making an acceptable offer, x*, in the next round. That is:

(A.4) $$\sum_{t=0}^{\infty} \frac{u_1(y)}{r_1}\; e^{-r_1 \Delta t} \left[1 - e^{-r_1 \Delta} \right] = \frac{u_1(y)}{r_1} \;\geq$$

$$\geq \frac{u_1(0)}{r_1} \left[1 - e^{-r_1 \Delta} \right] + \sum_{t=1}^{\infty} \frac{u_1(x^*)}{r_1}\; e^{-r_1 \Delta t} \left[1 - e^{-r_1 \Delta} \right] =$$

$$= \frac{u_1(0)}{r_1} \left[1 - e^{-r_1 \Delta} \right] + \frac{u_1(x^*)}{r_1}\; e^{-r_1 \Delta}$$

Re-arranging and denoting $u_1(0)$ by d_1:

(A.5) $$u_1(y) - d_1 \;\geq\; e^{-r_1 \Delta}\; [u_1(x^*) - d_1]$$

Likewise, when 2 is assessing an offer x (or 1-x), he accepts it iff:

(A.6) $$u_2(1-x) - d_2 \;\geq\; e^{-r_2 \Delta}\; [u_2(1-y^*) - d_2]$$

The equilibrium will require equality of the two expressions at x=x* and y=y*.

. The theory departs from the assumption that the "units of time" are the ones for which $u_i(x)$ is an appropriate measure of felicity – which is theoretically legitimate -, and correspondence is assumed with the rate at which one unit of manna is poured into the system.

We could however assume that felicity should indeed refer to interval of length h in the (a particular…) observed empirical units measure, not coinciding with those at which the continuous depreciation takes place at the discount rate r_i. Then $u_i(z_u)$ should be replaced by

$\frac{u_i(hz_u)}{h}$ in the expressions. h and Δ may not coincide. If it is legitimate to admit that h tends to 0, using the l'Hôpital's rule we can infer that:

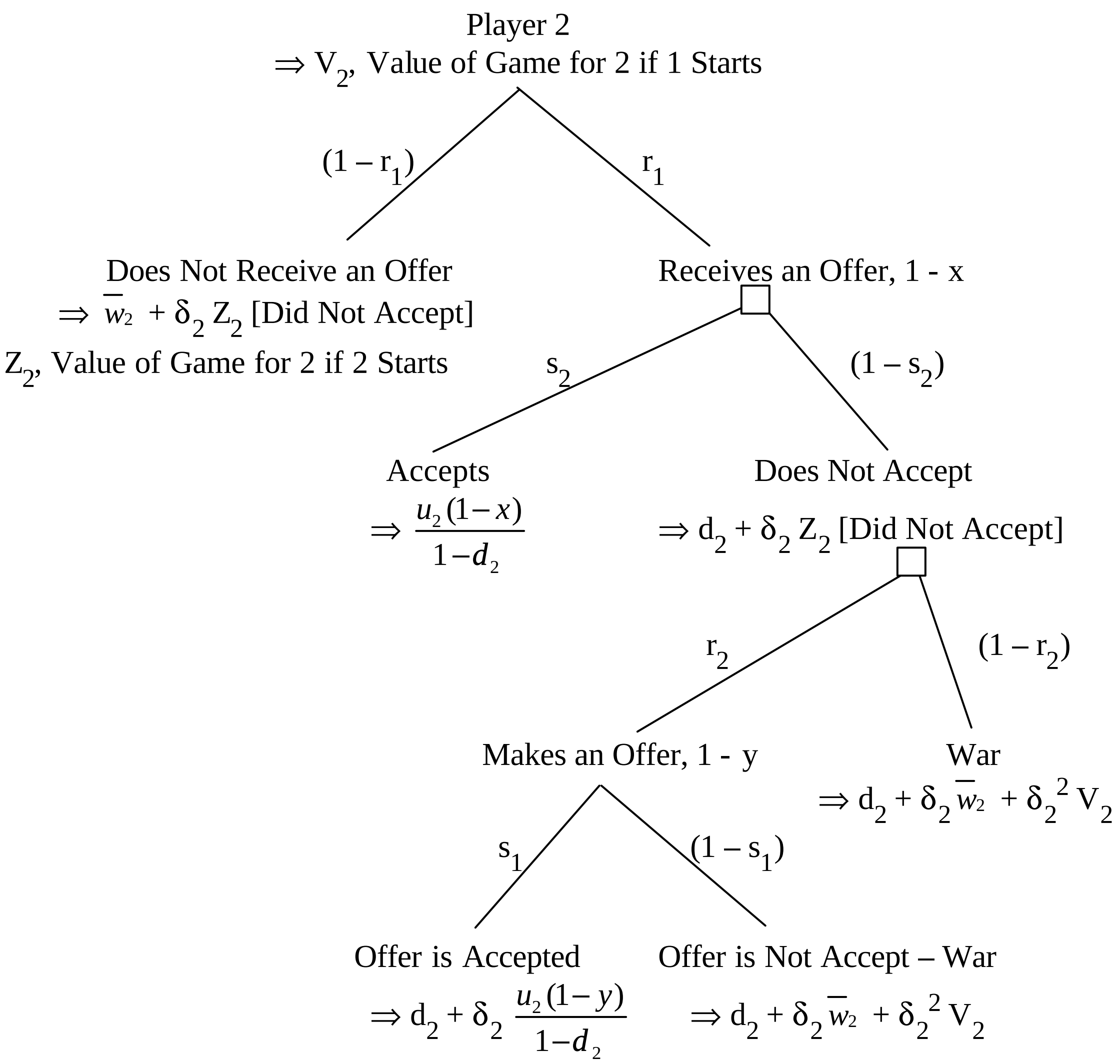
$$\lim_{h \to 0} \frac{u(hz_u)}{h} = z_u \, u_i'(0)$$

Under the above manipulations we would arrive at similar equilibrium conditions as before but as if derived for linear felicity functions, i.e., $u_i(z_u) = z_u$.

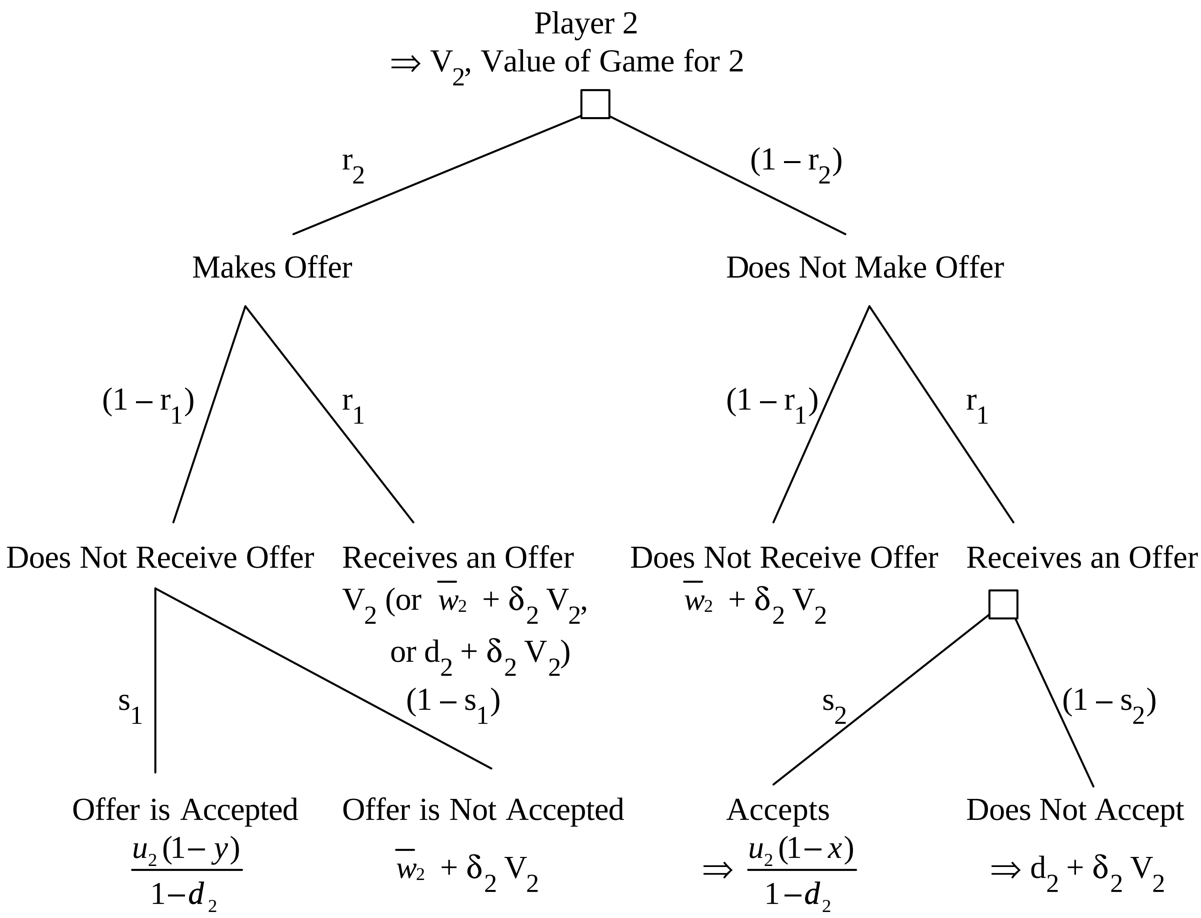
Appendix 2

Player 2's decision trees with outside options:

Alternate offers game:



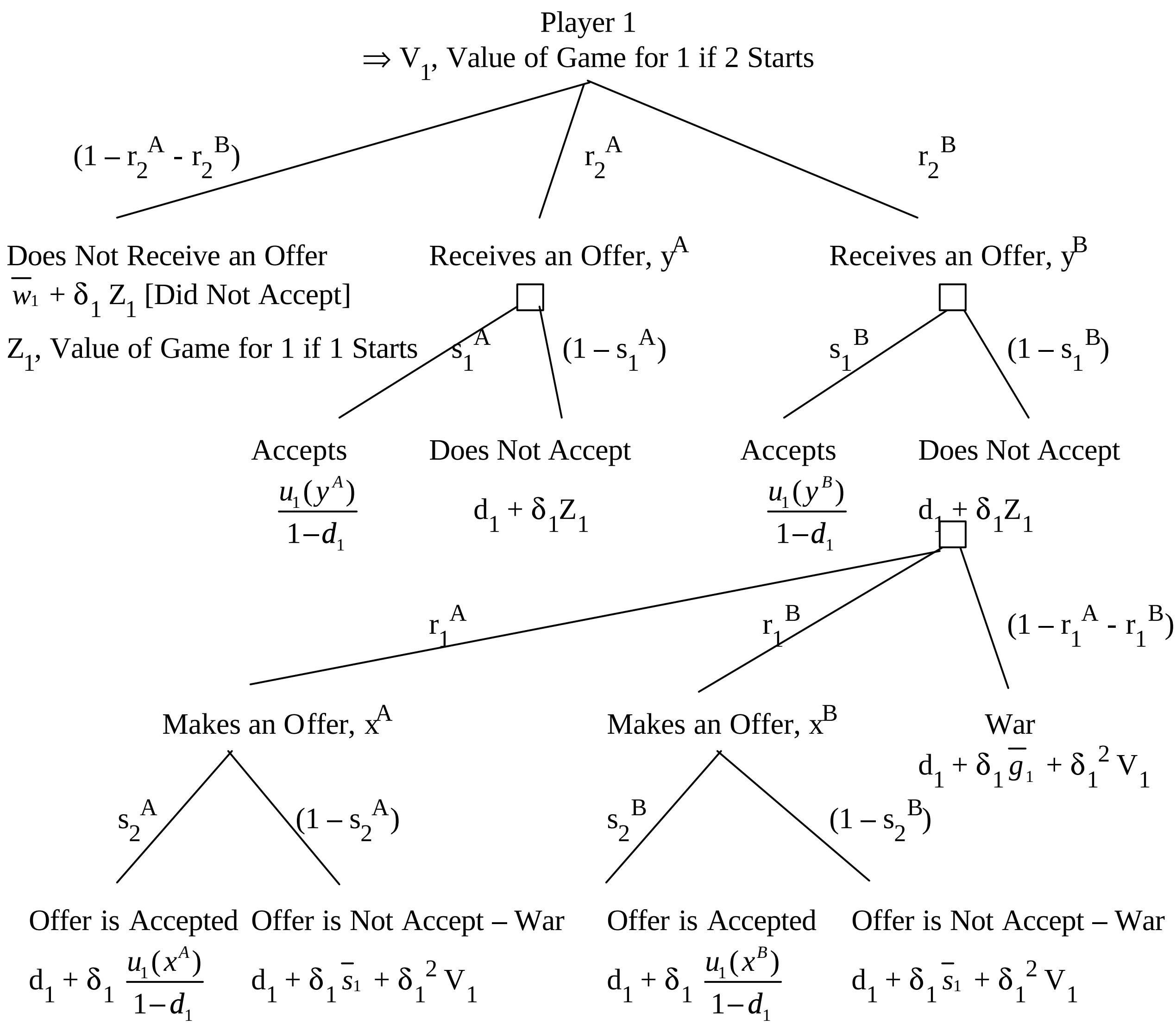
Simultaneous offers game:



Appendix 3

Multiplicity of Alternatives in the Alternate Offers Game:

We depict be low a decision tree of Player 1 that admits, on the one hand, that periodic payoffs are branch-specific and allow for eventual multiple offers – we just draw 2, A and B – to be randomly forwarded by each player.



We will study three situations: possibility of effective multiple offers to be exchanged; with the usual "simple" offers, an equilibrium with mixed strategies in s_i but not in r_i; again with "simple" offers, the equilibrium – equivalent to that of the text - with mixed strategies in r_i:

. Let us inspect the possibility of two offers A and B to be considered by each player i.

Lemma 1: s_1^A and s_1^B (s_2^A and s_2^B) cannot both exhibit interior solutions. Then, if one of the offers, say B, exhibits an interior solution (or is the more stringent one),

- for the other, A, to be acceptable(ed) in equilibrium and $s_1^A = 1$, $y^A > y^B$ ($1 - x^A > 1 - x^B$).

- for the other, A, not to be accepted in equilibrium and $s_1^A = 0$, $y^A < y^B$ ($1 - x^A < 1 - x^B$).

Proof: If they both exhibited interior solutions, in equilibrium, $\frac{u_1(y^A)}{1-d_1} = d_1 + \delta_1 Z_1 = \frac{u_1(y^B)}{1-d_1}$: y^A and y^B would have to equalize – but then the two offers would be indistinguishable. The rest of the lemma therefore follows.

Lemma 2: $s_2^A = 0$ and $s_2^B > 0$ ($s_1^A = 0$ and $s_1^B > 0$) cannot occur.

Proof: If they could, using lemma 1, one would have that $x^A > x^B$. Interior solutions of r_1^A and r_1^B would require $s_2^B \frac{u_1(x^B)}{1-d_1} + (1 - s_2^B)(\bar{s}_1 + \delta_1 V_1) = \bar{s}_1 + \delta_1 V_1 = \bar{g}_1 + \delta_1 V_1$. The first equality would imply $s_2^B = 0$, which would be a contradiction; the second equality would be impossible.

This lemma would make intuitive sense: to make an unacceptable offer would be tantamount to make none.

Lemma 3: For $s_2^A = 1$ and $s_2^B > 0$ ($s_1^A = 1$ and $s_1^B > 0$): $s_2^B < 1$ and $\bar{g}_1 > \bar{s}_1$. $\bar{g}_1 = \bar{s}_1$ cannot occur.

Proof: Using lemma 1, for $s_2^A = 1$ and $s_2^B > 0$, $x^A < x^B$. Interior solutions of r_1^A and r_1^B would require $s_2^B \frac{u_1(x^B)}{1-d_1} + (1 - s_2^B)(\bar{s}_1 + \delta_1 V_1) = \frac{u_1(x^A)}{1-d_1} = \bar{g}_1 + \delta_1 V_1$. The first equality would imply that $s_2^B < 1$ or would have to equalize; it would also imply that for $x^A < x^B$, $\frac{u_1(x^B)}{1-d_1} > \bar{s}_1 + \delta_1 V_1$; for the second equality to be possible then, $\frac{u_1(x^B)}{1-d_1} > \bar{g}_1 + \delta_1 V_1 > \bar{s}_1 + \delta_1 V_1$ – and then $\bar{g}_1 > \bar{s}_1$. And for $s_2^B > 0$, and x^A to differ from x^B, $\bar{s}_1 = \bar{g}_1$ cannot happen.

A possible equilibrium would require then:

(A.1) $$\frac{u_1(y^B)}{1-d_1} = d_1 + \delta_1 Z_1$$

(A.2) $$s_2^B \frac{u_1(x^B)}{1-d_1} + (1 - s_2^B)(\bar{s}_1 + \delta_1 V_1) = \frac{u_1(x^A)}{1-d_1} = \bar{g}_1 + \delta_1 V_1$$

(A.3) $$Z_1 = \bar{g}_1 + \delta_1 V_1$$

(A.4) $$V_1 = (1 - r_2^A - r_2^B)(\bar{w}_1 + \delta_1 Z_1) + r_2^A \frac{u_1(y^A)}{1-d_1} + r_2^B \frac{u_1(y^B)}{1-d_1}$$

and similarly for player 2. We have 10 equations and 14 unknowns… Exploration of this model is beyond the scope of this work and left for further study – because we focused on the case for which $\bar{s}_1 = \bar{g}_1$...

Likewise, a solution with $r_i^A + r_i^B = 1$ would also merit attention. Lemma 3 would partly apply, and one would expect then that at best $s_i^A = 1$ and $0 < s_i^B < 1$; If they were both possible, would require investigation.

. One can further show in such game that an interior solution for s_i's but not for the r_i would generate an inferior equilibrium. We would have that, for player 1:

(A.5) $$\frac{u_1(y)}{1-d_1} = d_1 + \delta_1 Z_1$$

(A.6) $$s_2 \frac{u_1(x)}{1-d_1} + (1 - s_2)(\bar{s}_1 + \delta_1 V_1) = \bar{g}_1 + \delta_1 V_1$$

(A.7) $$Z_1 = \bar{g}_1 + \delta_1 V_1$$

(A.8) $$V_1 = \frac{u_1(y)}{1-d_1}$$

Using (A.5), (A.8) and (A.7), $V_1 = d_1 + \delta_1 Z_1 = d_1 + \delta_1 (\bar{g}_1 + \delta_1 V_1)$. Then:

(A.9) $$V_1^* = \frac{d_1 + d_1 \bar{g}_1}{1-d_1^2}$$

(A.10) $$Z_1^* = \frac{\bar{g}_1 + d_1 d_1}{1-d_1^2}$$

and

(A.11) $$u_1(y^*) = \frac{\delta_1 + \delta_1 \bar{g}_1}{1+\delta_1}$$

If $\bar{g}_1 = \bar{w}_1$ (> d_1), we are definitely in a worst solution than in the solution of the main text – mixed strategies of s_i would not be applied for they would leave the two players in the lower bound, obtained when rejectable offers are exchanged. From player 2's problem, also V_2^*, Z_2^* and x* would be determined – of course, now, y* and x* are not the usual periodic splits. Then:

(A.12) $$s_2^* = \frac{(1-\delta_1^2)(\bar{g}_1 - \bar{s}_1)}{(1+\delta_1)u_1(x^*) - (1-\delta_1^2)\bar{s}_1 - \delta_1(\delta_1 + \delta_1 \bar{g}_1)}$$

Suppose $\bar{g}_1 > \bar{s}_1$.

For $s_2^* > 0$: $(1 + \delta_1)\, u_1(x^*) > (1 - \delta_1^2)\, \bar{s}_1 + \delta_1 (d_1 + \delta_1\, \bar{g}_1) = \bar{s}_1 + \delta_1\, d_1 + \delta_1^2 (\bar{g}_1 - \bar{s}_1)$.

For $s_2^* < 1$: $(1 + \delta_1)\, u_1(x^*) > \delta_1\, d_1 + \bar{g}_1$.

If $\bar{g}_1 > \bar{s}_1$, $\bar{s}_1 + \delta_1\, d_1 + \delta_1^2 (\bar{g}_1 - \bar{s}_1) < \delta_1\, d_1 + \bar{g}_1$. Then, if $\bar{g}_1 > \bar{s}_1$, interior solutions just require that $\frac{u_1(x^*)}{1-\delta_1} > \frac{\delta_1 + \delta_1 \bar{g}_1}{1-\delta_1^2} = V_1^*$.

One could likewise show that if $\bar{g}_1 < \bar{s}_1$, interior solutions would require $\frac{u_1(x^*)}{1-\delta_1} < \frac{\delta_1 + \delta_1 \bar{g}_1}{1-\delta_1^2} = V_1^*$.

. Finally, interior solutions for the r_i's only with the new multiplicity of alternative would generate an equilibrium similar to the one developed in the text. For player 1:

(A.13) $$\frac{u_1(y^*)}{1-\delta_1} = d_1 + \delta_1 Z_1^*$$

(A.14) $$\frac{u_1(x^*)}{1-\delta_1} = \bar{g}_1 + \delta_1 V_1^*$$

(A.15) $$Z_1^* = \frac{u_1(x^*)}{1-\delta_1}$$

(A.16) $$V_1^* = r_2 \frac{u_1(y^*)}{1-\delta_1} + (1 - r_2)(\bar{w}_1 + \delta_1 Z_1^*)$$

(A.13) and (A.15) still imply that the equilibrium split is the traditional one and that the first-mover is as well-off as in the standard game. We could further develop (A.16) to generate:

(A.17) $$V_1^* = \frac{u_1(y^*)}{1-d_1} + (1 - r_2)(\overline{w}_1 - d_1)$$

It now implies, using (A.14):

(A.18) $$V_1^* = \frac{u_1(x^*) - (1-d_1)\overline{g}_1}{d_1(1-d_1)} = \frac{u_1(y^*) - (1-d_1)(d_1 + d_1\overline{g}_1)}{d_1^2(1-d_1)}$$

and

(A.19) $$1 - r_2^* = \frac{(1+d_1)u_1(x^*) - \overline{g}_1 - d_1 d_1}{d_1(\overline{w}_1 - d_1)} = \frac{(1+d_1)u_1(y^*) - d_1\overline{g}_1 - d_1}{d_1^2(\overline{w}_1 - d_1)}$$

$$r_2^* = \frac{\overline{g}_1 + d_1\overline{w}_1 - (1+d_1)u_1(x^*)}{d_1(\overline{w}_1 - d_1)} = \frac{(1-d_1^2)d_1 + d_1(\overline{g}_1 + d_1\overline{w}_1) - (1+d_1)u_1(y^*)}{d_1^2(\overline{w}_1 - d_1)}$$

If $\overline{w}_1 > d_1$, $0 < r_2^* < 1$ would require $\frac{\overline{g}_1 + d_1 d_1}{1+d_1} < u_1(x^*) < \frac{\overline{g}_1 + d_1\overline{w}_1}{1+d_1}$ or/and $\frac{d_1 + d_1\overline{g}_1}{1+d_1} <$ $u_1(y^*) < \frac{(1-d_1^2)d_1 + d_1(\overline{g}_1 + d_1\overline{w}_1)}{(1+d_1)}$. Then two cases are possible:

If $\overline{w}_1 < \overline{g}_1$, $r_2^* > 0$ implies that a first-mover is better-off exchanging rejectable offers. (But he may or not be better-off than without offer exchange).

If $\overline{w}_1 > \overline{g}_1$, $r_2^* > 0$ would imply $u_1(x^*) < \frac{\overline{g}_1 + d_1\overline{w}_1}{1+d_1} < \overline{w}_1$ and then a first player would (even) be better-off if nobody makes offers.