Auctions Design with Private Costs of Valuation Discovery *

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Abstract This paper considers auctions design in a general independent private value (IPV) setting where each potential bidder has a valuation discovery cost which is his private information. First, the revenue-maximizing auction generally involves individual entry fee for each bidder which equals the hazard rate of his entry cost distribution, evaluated at the optimal entry-threshold for him. The second-price auction with no entry fee and a reserve price equal to seller's valuation remains the ex ante efficient auction. Second, even for the symmetric setting, it is in general an auction implementing asymmetric rather than symmetric entry across bidders that maximizes the total expected surplus or the seller's expected revenue. This result holds no matter the entry costs are the bidders' private information or common knowledge. Third, if the distributions of entry costs are degenerated (common-knowledge costs), there is no loss of generality in considering entry patterns where every bidder participates in probability of either 0 or 1, for the revenue-maximizing (also total-expected-surplus-maximizing) auction. The corresponding revenue-maximizing auction generally employs positive entry fees to extract the surplus of the participants.

Keywords: Auctions Design, Endogenous Participation, Valuation Discovery Cost.
JEL classifications: D44, D82.

1 Introduction

The impact of valuation discovery costs on bidders' entry decision and auctions design, has been extensively studied by Johnson (1979), Milgrom (1981), French and McCormick (1984), McAfee and McMillan (1987), Engelbrecht-Wiggans (1987, 1993), Harstad (1990), Levin and Smith (1994) and Ye (2004) among others. Here, valuation discovery costs refer to the costs for bidders to discover their valuations of the auctioned object.[1] All these previous studies assume that bidders' valuation discovery costs are common knowledge.[2] In this paper, we allow the valuation discovery costs to be bidders' private information in a general independent private value (IPV) framework allowing asymmetry across bidders, and provide a path to show the auctions that maximize the total expected surplus and seller's revenue, respectively. The case of common-knowledge costs corresponds to the setting that the distributions of entry costs are degenerated. The methodology developed for private-information costs case is also applied in this paper to the case of common-knowledge discovery costs. This exercise provides new insights by allowing full flexibility in bidders' entry pattern.

We first review the literature and introduce our new findings for the common-knowledge discovery costs setting. McAfee and McMillan (1987) study an IPV framework with infinite number of potential bidders. They show that the optimal reserve price for seller is his own valuation.[3] By allowing the optimal number of entrants to be a non-integer, they show that a first-price auction with a reserve price equal to seller's valuation and a positive entry fee is the revenue-maximizing auction. Since non-integer number of entrants

[1] There are many studies that focus on other entry costs that are incurred even when bidders know their valuations. These studies include Green and Laffont (1984), Samuelson (1985), Stegeman (1996), Menezes and Monteiro (2000) and Lu (2004a, 2004b, 2005).

[2] Matthews (1984) and Tan (1992) consider the case where the information quality depends on the amount of investment chosen by the bidders.

[3] Engelbrecht-Wiggans (1987) independently finds a similar result.

is not implementable,[4] McAfee and McMillan (1987) further establish that a first-price auction with a reserve price equal to seller's valuation and zero entry fee leads to the closest discrete number of participants to the unrestricted optimum. Engelbrecht-Wiggans (1993) further examines the revenue-maximizing auction in a setting allowing for affiliation among bidders' private valuations while restricting the number of participants to an integer. In all these works, it is assumed that bidders participate in either probability of 1 or 0. Levin and Smith (1994) and Ye (2004) look at the symmetric mixed strategy entry equilibrium in a symmetric setting, i.e., every potential bidder participates in a common probability that falls in the interval $(0, 1)$. Levin and Smith (1994) show that the revenue-maximizing auction must also induce socially optimal entry. Particularly, for the private value case, this optimal entry occurs when seller charges no entry fee and a reserve price equal to seller's valuation. Ye (2004) shows a similar result in a private value framework, which differs from Levin and Smith (1994) in assuming that after incurring an entry cost, each entrant not only observes his own valuation, but also observes information that updates his belief on the distributions of other bidders' valuations.

In this paper, we further allow each bidder to participate in any probability in the interval $[0, 1]$. This results in new findings. First, there is no loss of generality in considering the entry patterns where every bidder participates in probability of 0 or 1 for the revenue-maximizing (meanwhile ex ante efficient) auction. Second, for symmetric settings considered in Levin and Smith (1994), we explicitly show that the optimal entry pattern maximizing the seller's expected revenue (meanwhile the total expected surplus) is generally an asymmetric corner solution rather than the symmetric inner solution in terms of the participation probabilities. This insight explains why seller's expected revenue may drop with the number of potential bidders in Levin and Smith (1994) when they restrict the entry pattern to be symmetric. Third, the highest total expected surplus is still im-

[4] Although the expected number of entrants can be a non-integer if bidders participate in mixed strategy, the number of entrants is still random.

plemented through a second-price auction with no entry fee and a reserve price equal to the seller's valuation, even if the optimal entry pattern is asymmetric across bidders. However, generally there exist other entry fees that are also ex ante efficient. Fourth, individual entry fees for each bidder generally are essential to extract all the expected surplus of the participants at the revenue-maximizing entry pattern. In other words, if asymmetric participation probabilities across bidders are allowed, then zero entry fee remains ex ante efficient although it is not unique, however, generally it is not revenue-maximizing. An interesting question to ask would be to what extent these results still hold if the entry costs are private information of bidders.

For the setting with private information entry costs, we first establish a convenient way to write the optimal total expected surplus and seller's expected revenue as a function of the vector of participation thresholds in terms of the bidders' entry costs. This allows us to establish useful connections between the first order conditions characterizing the optimal entry thresholds and the expected surplus of these threshold types in a second-price auction with no entry fee and a reserve price equal to seller's valuation. Based on these connections, we find that a second-price auction with no entry cost and a reserve price equal to seller's valuation remains the ex ante efficient auction. However, the seller's optimal expected revenue generally cannot be implemented through such an auction. In addition, the entry pattern maximizing the seller's expected revenue generally no longer coincides with the one maximizing the total expected surplus. The entry fees in the revenue-maximizing auction are generally positive and different across bidders. Specifically, the optimal entry fees for different bidders are respectively the hazard rates of the individuals' entry cost distribution evaluated at the bidders' participation thresholds. In the case where bidders' valuation discovery costs are common knowledge, the auction maximizing the seller's expected revenue must also maximize the total expected surplus of both the seller and bidders, since the seller can extract all the expected surplus of the participants using entry fees. However, in the case that the entry costs are private information of bidders, the seller can no longer extract all the expected surplus of participants. This leads to the discrepancy between the problems of maximizing the seller's expected revenue and maximizing the total expected surplus. This is why the entry pattern maximizing the seller's expected revenue generally does not maximize total expected surplus, and the revenue-maximizing auction is different from the ex ante efficient auction.

Interestingly, we find that even for symmetric setting with private-information of entry costs, it could be an auction implementing asymmetric entry rather than symmetric entry that maximizes the total expected surplus or the seller's expected revenue. For example, let us consider a setting where there are 2 potential bidders. Bidders' private values follow the uniform distribution on $[0, 1]$, and bidders' entry costs follow the uniform distribution on $[0.4, 0.5]$. The seller's valuation is assumed to be zero. In this setting, the total expected surplus takes the maximum of 0.05 when the participation thresholds of entry costs for the 2 bidders are set at 0.5 and 0.4, respectively; seller's expected revenue takes the maximum of 0.025 when the participation thresholds for the 2 bidders are set at 0.45 and 0.4, respectively. If we restrict the participation thresholds to be the same across the two bidders, then the total expected surplus takes the maximum of 0.023 when the common participation threshold is set at 0.4231, and seller's expected revenue takes the maximum of 0.01875 when the common participation threshold is set at 0.4187. The intuition for the asymmetric optimum is explained as follows. Suppose the symmetric optimum is an inner solution where every bidder participates in a common probability in $(0, 1)$. The fundamental reason for asymmetric optimum lies in the fact that the marginal contribution of an additional participant's valuation to the total expected surplus or seller's expected revenue strictly decreases with the number of other participants. This fact implies that given the summation of the participating probabilities of any two bidders, the marginal contribution of their valuations to the total surplus or the seller's revenue is maximized when the difference between their participating probabilities is maximized. It follows that the optimum must be asymmetric unless the symmetric optimum is a corner solution, provided that the above difference-maximizing adjustment in the entry probabilities of the two bidders does not substantially change the expected entry costs.[5]

This paper is organized as follows. In section 2, we consider a general IPV setting where potential bidders have different distributions on both valuations and valuation discovery costs. The ex ante efficient auction and revenue-maximizing auction are established. In section 3, we focus on the symmetric IPV setting, where potential bidders share identical distributions on valuations and valuation discovery costs. We show that even in symmetric setting, it could be an auction implementing asymmetric entry rather than symmetric entry that maximizes the total expected surplus or the seller's expected revenue. Section 4 revisits the common-knowledge entry costs case. The findings in section 4 echo those in sections 2 and 3, and provide new insights in auctions design with common-knowledge entry costs. Section 5 concludes.

2 Auctions Design under General IPV Setting

There are N potential bidders who are interested in a single item auctioned, where N is public information. Denote this group of potential bidders by $\mathcal{N} = \{1, 2, ..., N\}$. The seller's valuation is v_0, which is public information. Without loss of generality, we normalize v_0 to be zero. Bidder i has to incur an entry cost of c_i in order to enter the auction.[6] After incurring c_i, he observes his private value v_i. Both c_i and v_i are assumed to be private information of bidder i. The cumulative distribution function of c_i is $G_i(c_i)$ with

[5]This condition clearly holds when entry costs are common knowledge. In general, it holds when the ranges of private entry costs are small.

[6]This assumption is adopted by most of the literature. However, this assumption precludes the possibility that a bidder may simply submit a bid equal to the unconditional expected valuation.

density function of $g_i(c_i)$, while the cumulative distribution function of v_i is $F_i(v_i)$ with density function of $f_i(v_i)$. The support of $G_i(c_i)$ is $[\underline{c}_i, \overline{c}_i]$, and the support of $F_i(v_i)$ is $[\underline{v}_i, \overline{v}_i]$. We assume $g_i(\cdot) > 0$ on its support. The distributions of c_i and v_i, $i \in \mathcal{N}$ are assumed to be common knowledge. For simplicity, we assume c_i and v_j, $\forall i, j \in \mathcal{N}$ are mutually independent. The seller and bidders are assumed to be risk neutral.

The timing of the auction is as follows.

Time 0: The group of potential bidders $\mathcal{N}$ and the seller's valuation v_0 are revealed by Nature as public information. Every bidder i observes his private cost c_i, $i \in \mathcal{N}$.

Time 1: The auctioneer announces the rule of the auction.

Time 2: The bidders simultaneously and confidentially make their entry decisions. If they do not enter, they take the outside option which gives them zero payoff. If they enter, they incur their private entry costs and observe their private values.

Time 3: Every participant bids.[7]

Time 4: The payoffs of the auctioneer and all the participating bidders are determined according to the announced rule at time 1.

We study the auction rules announced at time 1, which maximizes the total expected surplus and the seller's expected revenue, respectively. Hereafter, the auction maximizing the total expected surplus of seller and bidders is called the **ex ante efficient auction**; the auction maximizing the expected revenue of the seller is called the **revenue-maximizing auction**. We use $\mathcal{A}_\mathbf{0}$ to denote the second-price auction with no entry fee and a reserve price equal to the seller's valuation v_0.

Before we proceed to consider auctions design, we first characterize the feasible entry patterns.

Lemma 1: *Any equilibrium entry pattern can be described through a vector of entry thresholds* $\mathbf{C_c} = (c_c^{(1)}, ..., c_c^{(N)})$ *satisfying the following properties: (i)* $c_c^{(i)} \in [\underline{c}_i, \overline{c}_i]$, $\forall i \in \mathcal{N}$;

[7]Every participant may or may not observe the other participants. The auctions designed later work in both cases.

(ii) if $c_i < c_c^{(i)}$, bidder i participates with probability of 1; if $c_i > c_c^{(i)}$, bidder i participates with probability of 0.

Proof: Let us consider any entry equilibrium $\mathcal{E}$ implemented by a feasible auction rule. If all bidders other than i adopt the equilibrium entry strategy in $\mathcal{E}$, the bidder i's equilibrium entry strategy in $\mathcal{E}$ must be his best entry strategy. Given all bidders other than i adopt the equilibrium entry strategy in $\mathcal{E}$, there must exist an entry threshold $c_c^{(i)} \in [\underline{c}_i, \overline{c}_i]$ such that bidder i's best entry strategy is described by property (ii) in Lemma 1. This is true because the expected payoff of bidder i from participating in any given auction decreases strictly with his entry cost, given all bidders other than i adopt the equilibrium entry strategy in $\mathcal{E}$. $\square$

Given a threshold-vector $\mathbf{C_c}$ where $c_c^{(i)} \in [\underline{c}_i, \overline{c}_i], \ \forall i \in \mathcal{N}$, for simplicity we assume (i) if $c_c^{(i)} > \underline{c}_i$, bidder i participates if and only if $c_i \leq c_c^{(i)}$; (ii) if $c_c^{(i)} = \underline{c}_i$, no type of bidder i participates. This simplification is a reasonable one, because if $c_c^{(i)} > \underline{c}_i$ bidder i with cost $c_c^{(i)}$ at least weakly prefers participation, and if $c_c^{(i)} = \underline{c}_i$ then bidder i with cost $c_c^{(i)}$ at least weakly prefers nonparticipation. Moreover, this simplification only further specifies the participation of the threshold type $c_c^{(i)}$ of bidder i. The total expected surplus and seller's expected revenue are not affected.

Next, we establish the following results regarding the restricted efficient auction and revenue-maximizing auction, which implements a given entry threshold-vector $\mathbf{C_c} = (c_c^{(1)}, ..., c_c^{(N)})$, where $c_c^{(i)} \in [\underline{c}_i, \overline{c}_i], \ \forall i \in \mathcal{N}$.

Lemma 2: *Among all auctions implementing any given participation threshold vector $\mathbf{C_c} = (c_c^{(1)}, ..., c_c^{(N)})$, a second-price auction with a reserve price equal to seller's valuation and appropriately set entry fee (or subsidy) for each bidder provides the highest seller's expected revenue as well as the highest total expected surplus. The entry fees (or subsidies) are charged upon entry at time 2 before the valuations are learned by the entrants, and are set at levels such that the threshold-type entrants get zero expected surplus.*

Proof: Let us first consider auction $\mathcal{A}_{\mathbf{0}}$. Suppose all bidders other than i participate in auction $\mathcal{A}_{\mathbf{0}}$ according to thresholds $\mathbf{C_c} = (c_c^{(1)}, ..., c_c^{(N)})$. Denote bidder i's expected surplus by $S_i(c_c^{(i)}; \mathbf{C_c})$ if he participates in $\mathcal{A}_{\mathbf{0}}$ while his entry cost is $c_c^{(i)}$. Set a time-2 entry fee (or subsidy) for bidder i as $e_i = S_i(c_c^{(i)}; \mathbf{C_c}),\ \forall i \in \mathcal{N}$. Clearly, for a second-price auction with entry fee (or subsidy) e_i for bidder i and a reserve price equal to seller's valuation, bidder i's expected surplus is $c_c^{(i)} - c_i$ if he participates and his entry cost is c_i. Hence, the above auction with entry fee e_i for bidder i implements participation thresholds $\mathbf{C_c} = (c_c^{(1)}, ..., c_c^{(N)})$. Note that for any auction implementing participation thresholds $\mathbf{C_c} = (c_c^{(1)}, ..., c_c^{(N)})$, the total expected entry costs are the same. Thus the above designed auction achieves the highest possible total expected surplus among the class of auctions implementing $\mathbf{C_c}$, as the auction always awards the item to the participant (including the seller) with the highest valuation.

Moreover, for any auction implementing entry threshold-vector $\mathbf{C_c} = (c_c^{(1)}, ..., c_c^{(N)})$, the expected surplus of bidder i with entry cost $c_i \leq c_c^{(i)}$ can not be smaller than $c_c^{(i)} - c_i$. This is due to the fact that a type c_i of bidder i can always mimic a type $c_c^{(i)}$ of bidder i, and by doing so he gets at least $c_c^{(i)} - c_i$. Recall that in a second-price auction with entry fee e_i for bidder i and a reserve price equal to seller's valuation, bidder i's expected surplus is exactly $c_c^{(i)} - c_i$ if he participates and his entry cost is c_i. As a result, this auction achieves the highest possible seller's expected revenue among all auctions implementing any given entry threshold-vector $\mathbf{C_c} = (c_c^{(1)}, ..., c_c^{(N)})$. $\square$

For a given entry threshold-vector $\mathbf{C_c}$ where $c_c^{(i)} \in [\underline{c}_i, \overline{c}_i],\ \forall i \in \mathcal{N}$, we denote the highest total expected surplus attainable by $TES(\mathbf{C_c})$, and the highest seller's revenue attainable by $SER(\mathbf{C_c})$. We next introduce a convenient way of writing $TES(\mathbf{C_c})$ and $SER(\mathbf{C_c})$.

Define set $\mathcal{K} = \{(k_1, k_2, ..., k_N) | k_i \in \{0, 1\},\ i \in \mathcal{N}\}$, where k_i denotes bidder i's entry status. Specifically, $k_i = 1$ stands for the participation of bidder i, while $k_i = 0$ represents the non-participation of bidder i. In addition, let $k_0 = 1$ to symbolize the participation of the seller for convenience. For any $\mathbf{k} = (k_1, k_2, ..., k_N) \in \mathcal{K}$, use $v_{\mathbf{k}}$ to denote the highest valuation of all participants including the seller. Then $v_{\mathbf{k}}$ can be written as the following

$$v_{\mathbf{k}} = \max_{\{k_j=1, 0 \leq j \leq N\}} \{v_j\}.$$

We use $F_{\mathbf{k}}(v_{\mathbf{k}})$ and $f_{\mathbf{k}}(v_{\mathbf{k}})$ to denote the cumulative and density function of $v_{\mathbf{k}}$ respectively. Furthermore, we use $V_{\mathbf{k}}$ to denote the expectation of $v_{\mathbf{k}}$.[8] $TES(\mathbf{C_c})$ and $SER(\mathbf{C_c})$ can then be written as the following

$$\begin{aligned} TES(\mathbf{C_c}) &= \sum_{\{\mathbf{k} \in \mathcal{K}\}} \Big(V_{\mathbf{k}} \prod_{i \in \mathcal{N}} (G_i(c_c^{(i)})^{k_i} (1 - G_i(c_c^{(i)}))^{1-k_i}) \Big) - \sum_{i \in \mathcal{N}} \int_{\underline{c}_i}^{c_c^{(i)}} c_i g_i(c_i) dc_i, \quad (1) \\ SER(\mathbf{C_c}) &= \sum_{\{\mathbf{k} \in \mathcal{K}\}} \Big(V_{\mathbf{k}} \prod_{i \in \mathcal{N}} (G_i(c_c^{(i)})^{k_i} (1 - G_i(c_c^{(i)}))^{1-k_i}) \Big) - \sum_{i \in \mathcal{N}} c_c^{(i)} G_i(c_c^{(i)}). \quad (2) \end{aligned}$$

The first summation term in (1) is the contribution of the valuations of all participants including the seller to the total expected surplus if potential bidders participate according to threshold-vector $\mathbf{C_c}$. The second term is the contribution (negative) of the entry costs of participants to the total expected surplus if the potential bidders participate according to threshold-vector $\mathbf{C_c}$. Following the proof of Lemma 2, we know that the expected information rent of bidder i is $\int_{\underline{c}_i}^{c_c^{(i)}} (c_c^{(i)} - c_i) g_i(c_i) dc_i$. This leads to (2) giving the seller's expected revenue, which is the difference between the total expected surplus and the bidders' expected information rent.

2.1 Ex Ante Efficient Auction

We derive the ex ante efficient auction through two steps. First, we characterize the first order conditions for the optimal threshold-vector $\mathbf{C_c^*} = (c_c^{*(1)}, ..., c_c^{*(N)})$, which maximizes $TES(\mathbf{C_c})$. Second, we show that auction $\mathcal{A}_{\mathbf{0}}$ implements $\mathbf{C_c^*}$.

[8] Define $w_{\mathbf{k}} = \max\{v_0, \max_{\{k_j=1, j \in \mathcal{N}\}} \{\overline{v}_j\}\}$, we have $V_{\mathbf{k}} = v_0 F_{\mathbf{k}}(v_0) + \int_{v_0}^{w_{\mathbf{k}}} v_{\mathbf{k}} f_{\mathbf{k}}(v_{\mathbf{k}}) dv_{\mathbf{k}}$.

First, we characterize the first order conditions for the optimal threshold-vector $\mathbf{C}_{\mathbf{c}}^*$ maximizing $TES(\mathbf{C}_{\mathbf{c}})$. Let us consider bidder i's optimal threshold $c_c^{*(i)}$, $\forall i \in \mathcal{N}$. The 2^N components in $\sum_{\{\mathbf{k}\in\mathcal{K}\}} \left(V_{\mathbf{k}} \prod_{i\in\mathcal{N}}(G_i(c_c^{(i)})^{k_i}(1-G_i(c_c^{(i)}))^{1-k_i})\right)$ can be divided into 2 groups. In group 1, there are 2^{N-1} components in which $k_i = 1$, while in group 2 there are other 2^{N-1} components in which $k_i = 0$. Note that for each component in group 1, there is a corresponding component in group 2 satisfying that all k_j, $\forall j \neq i$ are same. This feature leads to the following result.

Define $\mathcal{K}_{-i} = \{(k_1, ..., k_{i-1}, k_{i+1}, ..., k_N)|k_j \in \{0,1\},\ j \neq i.\}$, $\forall i \in \mathcal{N}$. For any $\mathbf{k}_{-i} = (k_1, ..., k_{i-1}, k_{i+1}, ..., k_N) \in \mathcal{K}_{-i}$, define vector $\tilde{\mathbf{k}}_1(\mathbf{k}_{-i}) = (\tilde{k}_j^1) \in \mathcal{K}$, in which $\tilde{k}_j^1 = k_j$, $\forall j \neq i$, and $\tilde{k}_i^1 = 1$; define vector $\tilde{\mathbf{k}}_0(\mathbf{k}_{-i}) = (\tilde{k}_j^0) \in \mathcal{K}$, in which $\tilde{k}_j^0 = k_j$, $\forall j \neq i$, and $\tilde{k}_i^0 = 0$. We then have

$$\frac{\partial TES(\mathbf{C}_{\mathbf{c}})}{\partial c_c^{(i)}} = g_i(c_c^{(i)}) \sum_{\{\mathbf{k}_{-i}\in\mathcal{K}_{-i}\}} [(V_{\tilde{\mathbf{k}}_1(\mathbf{k}_{-i})} - V_{\tilde{\mathbf{k}}_0(\mathbf{k}_{-i})} - c_c^{(i)}) \prod_{j\neq i} G_j(c_c^{(j)})^{k_j}(1 - G_j(c_c^{(j)}))^{1-k_j}]. \quad (3)$$

Suppose $\mathbf{C}_{\mathbf{c}}^*$ maximizes $TES(\mathbf{C}_{\mathbf{c}})$, then we have the following characterization for $\mathbf{C}_{\mathbf{c}}^*$. For all $i \in \mathcal{N}$,

$$\frac{\partial TES(\mathbf{C}_{\mathbf{c}}^*)}{\partial c_c^{(i)}} = \begin{cases} 0, & \text{if } \ c_c^{*(i)} \in (\underline{c}_i, \overline{c}_i), \\ \geq 0, & \text{if } \ c_c^{*(i)} = \overline{c}_i, \\ \leq 0, & \text{if } \ c_c^{*(i)} = \underline{c}_i. \end{cases} \quad (4)$$

Define $TS_i(\mathbf{C}_{\mathbf{c}}) = \sum_{\{\mathbf{k}_{-i}\in\mathcal{K}_{-i}\}}[(V_{\tilde{\mathbf{k}}_1(\mathbf{k}_{-i})} - V_{\tilde{\mathbf{k}}_0(\mathbf{k}_{-i})} - c_c^{(i)}) \prod_{j\neq i} G_j(c_c^{(j)})^{k_j}(1-G_j(c_c^{(j)}))^{1-k_j}]$. This term in (3) is the marginal contribution of bidder i with entry cost $c_c^{(i)}$ to the total expected surplus, given that other bidders participate in auction $\mathcal{A}_{\mathbf{0}}$ according to $\mathbf{C}_{\mathbf{c}}$.

On the other hand, $TS_i(\mathbf{C}_{\mathbf{c}})$ is the expected payoff of bidder i with entry cost $c_c^{(i)}$ from participating in auction $\mathcal{A}_{\mathbf{0}}$, provided that other bidders participate according to $\mathbf{C}_{\mathbf{c}}$. This can be seen from the following arguments. Note that the economic meaning of $V_{\tilde{\mathbf{k}}_1(\mathbf{k}_{-i})} - V_{\tilde{\mathbf{k}}_0(\mathbf{k}_{-i})} - c_c^{(i)}$ is the marginal contribution of bidder i with cost $c_c^{(i)}$ to the total expected surplus if he participates in auction $\mathcal{A}_{\mathbf{0}}$, and all other participants are those bidders with $k_j = 1,\ j \neq i$ in vector $\mathbf{k}_{-i}$. Hence, $V_{\tilde{\mathbf{k}}_1(\mathbf{k}_{-i})} - V_{\tilde{\mathbf{k}}_0(\mathbf{k}_{-i})} - c_c^{(i)}$ can be alternatively written as

$$\begin{aligned} V_{\tilde{\mathbf{k}}_1(\mathbf{k}_{-i})} - V_{\tilde{\mathbf{k}}_0(\mathbf{k}_{-i})} - c_c^{(i)} = \int_{v_0}^{\overline{v}_i} [(v_i - v_0) F_{\tilde{\mathbf{k}}_0(\mathbf{k}_{-i})}(v_0) \\ + \int_{v_0}^{v_i} (v_i - v_{\tilde{\mathbf{k}}_0(\mathbf{k}_{-i})}) f_{\tilde{\mathbf{k}}_0(\mathbf{k}_{-i})}(v_{\tilde{\mathbf{k}}_0(\mathbf{k}_{-i})}) dv_{\tilde{\mathbf{k}}_0(\mathbf{k}_{-i})}] f_i(v_i) dv_i - c_c^{(i)}, \end{aligned} \tag{5}$$

where $F_{\tilde{\mathbf{k}}_0(\mathbf{k}_{-i})}(\cdot)$ and $f_{\tilde{\mathbf{k}}_0(\mathbf{k}_{-i})}(\cdot)$ are the cumulative distribution function and density function of $v_{\tilde{\mathbf{k}}_0(\mathbf{k}_{-i})}$, respectively. In addition, note that the right hand side of (5) can also be interpreted as the expected payoff of bidder i with cost $c_c^{(i)}$ when he participates in auction $\mathcal{A}_\mathbf{0}$, if all other participants are those bidders with $k_j = 1,\ j \neq i$ in vector $\mathbf{k}_{-i}$. The result then follows that $TS_i(\mathbf{C_c})$ is the expected payoff of bidder i with cost $c_c^{(i)}$ when he participates in auction $\mathcal{A}_\mathbf{0}$, if all other potential bidders participate according to $\mathbf{C_c}$. This insight together with (3) and (4) lead to the following proposition which addresses the ex ante efficient auction.

Proposition 1: *The second-price auction $\mathcal{A}_\mathbf{0}$ with a reserve price equal to seller's valuation and no entry fee is ex ante efficient.*

Proof: Since $g_i(\cdot) > 0$, it is clearly a Nash equilibrium that every bidder participates in auction $\mathcal{A}_\mathbf{0}$ according to $\mathbf{C}_\mathbf{c}^*$, when (4) is satisfied for $\mathbf{C}_\mathbf{c}^*$. □

From Proposition 1, the result that auction $\mathcal{A}_\mathbf{0}$ is ex ante efficient holds in a much more general environment than the symmetric IPV setting with common-knowledge costs studied in Levin and Smith (1994). Furthermore, the result does not depend on the restriction of symmetry on entry pattern across bidders. Moreover, Proposition 1 accommodates the flexibility of an optimal corner solution, as indicated by (4). An example of corner solution is provided in the following symmetric setting, where $v_0 = 0$, $N = 2,\ F_i(v_i) = v_i,\ \forall v_i \in [0, 1]$, and $G_i(c_i) = 10(c_i - 0.4),\ \forall c_i \in [0.4, 0.5]$. In this setting, $TES(\mathbf{C_c})$ takes the maximum of 0.05 when $c_c^{(1)} = 0.5$ and $c_c^{(2)} = 0.4$. This means that one bidder always participates, while the other one never participates.

Although zero entry fee in a second-price auction with reserve price equal to seller's valuation is indeed ex ante efficient, the same may apply to other entry fees. This is always true when $c_c^{*(i)}$ is not an inner solution. If $c_c^{*(i)} = \overline{c}_i$, any entry fee for bidder i which is smaller than or equal to $TS_i(\mathbf{C}_\mathbf{c}^*)$, is also ex ante efficient. If $c_c^{*(i)} = \underline{c}_i$, any entry fee for bidder i which is greater than or equal to $TS_i(\mathbf{C}_\mathbf{c}^*)$ is also ex ante efficient.

2.2 Revenue-Maximizing Auction

We now study the revenue-maximizing auction. From (1), (2) and (3), we have

$$\frac{\partial SER(\mathbf{C_c})}{\partial c_c^{(i)}} = g_i(c_c^{(i)}) \sum_{\{\mathbf{k}_{-i} \in \mathcal{K}_{-i}\}} \Big\{ \big(\prod_{j \neq i} G_j(c_c^{(j)})^{k_j} (1 - G_j(c_c^{(j)}))^{1-k_j} \big) \cdot \big(V_{\tilde{\mathbf{k}}_1(\mathbf{k}_{-i})} - V_{\tilde{\mathbf{k}}_0(\mathbf{k}_{-i})} - c_c^{(i)} - \frac{G_i(c_c^{(i)})}{g_i(c_c^{(i)})} \big) \Big\}. \tag{6}$$

Suppose that $\mathbf{C}_\mathbf{c}^+ = (c_c^{+(1)}, ..., c_c^{+(N)})$ maximizes $SER(\mathbf{C_c})$, then we have the following characterization for $\mathbf{C}_\mathbf{c}^+$. For all $i \in \mathcal{N}$,

$$\frac{\partial SER(\mathbf{C}_\mathbf{c}^+)}{\partial c_c^{(i)}} = \begin{cases} 0, & \text{if } \ c_c^{+(i)} \in (\underline{c}_i, \overline{c}_i), \\ \geq 0, & \text{if } \ c_c^{+(i)} = \overline{c}_i, \\ \leq 0, & \text{if } \ c_c^{+(i)} = \underline{c}_i. \end{cases} \tag{7}$$

Denote $\sum_{\{\mathbf{k}_{-i} \in \mathcal{K}_{-i}\}} \Big\{ \big(\prod_{j \neq i} G_j(c_c^{(j)})^{k_j} (1 - G_j(c_c^{(j)}))^{1-k_j} \big) \big(V_{\tilde{\mathbf{k}}_1(\mathbf{k}_{-i})} - V_{\tilde{\mathbf{k}}_0(\mathbf{k}_{-i})} - c_c^{(i)} - \frac{G_i(c_c^{(i)})}{g_i(c_c^{(i)})} \big) \Big\}$ by $BS_i(\mathbf{C_c})$. Suppose that all other potential bidders participate according to $\mathbf{C}_\mathbf{c}^+$. Based on the arguments in section 2.1, we have that $BS_i(\mathbf{C}_\mathbf{c}^+)$ is the expected payoff of bidder i with cost $c_c^{(i)}$, if he participates in a second-price auction with a reserve price equal to v_0 and a time-2 entry fee of $\frac{G_i(c_c^{+(i)})}{g_i(c_c^{+(i)})}$ for bidder i.[9]

Based on this insight, we obtain from (6) and (7) the following proposition that addresses the revenue-maximizing auction.

[9] As pointed in Lemma 2, the entry fees (or subsidies) are charged upon entry at time 2 before the valuations are learned by the entrants

Proposition 2: *Suppose that* $\mathbf{C}_{\mathbf{c}}^{+}$ *maximizes* $SER(\mathbf{C}_{\mathbf{c}})$*, then a second-price auction with reserve price equal to seller's valuation and time-2 entry fees* e_i *for bidder* i *defined below leads to the seller achieving the highest expected surplus.*[10] *The time-2 optimal entry fees* $e_i,\ i \in \mathcal{N}$ *are defined as*

$$e_i = \begin{cases} \frac{G_i(c_c^{+(i)})}{g_i(c_c^{+(i)})}, & \text{if} \quad c_c^{+(i)} \in (\underline{c}_i, \overline{c}_i), \\ TS_i(\mathbf{C}_{\mathbf{c}}^{+}) \geq \frac{1}{g_i(\overline{c}_i)}, & \text{if} \quad c_c^{+(i)} = \overline{c}_i, \\ \textit{any number} \geq TS_i(\mathbf{C}_{\mathbf{c}}^{+}), & \text{if} \quad c_c^{+(i)} = \underline{c}_i. \end{cases} \tag{8}$$

Furthermore, no other entry fees works.

Proof: Since $g_i(\cdot) > 0$, it is clearly a Nash equilibrium that every bidder participates in the above defined auction according to $\mathbf{C}_{\mathbf{c}}^{+}$, when (7) and (8) are satisfied for $\mathbf{C}_{\mathbf{c}}^{+}$. Moreover, if e_i is defined differently from (8), then it is clear that either the entry threshold $c_c^{+(i)}$ can not be implemented or the surplus of the entrant i with $c_c^{+(i)}$ can not be extracted completely. $\square$

From Proposition 2, if the entry cost is private information of bidders, then generally the revenue-maximizing auction involves individual entry fees. The second-price auction $\mathcal{A}_{\mathbf{0}}$ with a reserve price equal to seller's valuation and no entry fee provides the seller the highest expected revenue if and only if $c_c^{+(i)} = \underline{c}_i$ and $TS_i(\mathbf{C}_{\mathbf{c}}^{+}) \leq 0$, $\forall i \in \mathcal{N}$. This is a degenerate case where it is inefficient for any bidder to participate in any chance, i.e., $\int_{v_0}^{\overline{v}_i} (v_i - v_0) f_i(v_i) dv_i \leq \underline{c}_i,\ \forall i \in \mathcal{N}$.

Propositions 1 and 2 show that if the entry costs are private information of the bidders, then the optimal entry patterns that maximize the expected total surplus and the seller's expected revenue are generally different. Thus, the ex ante efficient auction and the revenue-maximizing auction are generally different. In contrast, as will be revealed in Section 4.1, if the entry costs are public information of the bidders, then the optimal

[10]Here and hereafter, the entrants pay their entry fees or receive their entry subsidies before their valuations are learned.

entry patterns that maximize the expected total surplus and the seller's expected revenue are the same, although the optimal entry fees in the ex ante efficient auction and revenue-maximizing auction can be different. The intuition behind this difference is explained as follows. If the entry costs are public information, the seller can always extract all the surplus of the participants. Thus the entry pattern maximizing total expected surplus also maximizes the seller's expected revenue. However, if the entry costs are private information of the bidders, then the seller has no way to extract all the surplus of the participants as shown in the proof of Lemma 2. It is this special feature that leads to the discrepancy between the problems of maximizing seller's expected revenue and maximizing the total expected surplus. It follows that generally the entry pattern maximizing the seller's expected revenue generally does not maximize total expected surplus, and the revenue-maximizing auction must generally be different from the ex ante efficient auction.

3 Further Issues in Symmetric IPV Setting

In section 2, we consider the unrestricted ex ante efficient and revenue-maximizing auctions in a general IPV setting where bidders have private-information entry costs. The setting considered in this section is identical to that of section 2 except the following further symmetry restrictions on the distributions of the bidders' valuations and entry costs. The cumulative distribution functions of all $c_i,\ i \in \mathcal{N}$ are $G(\cdot)$ with density function of $g(\cdot)$. The cumulative distribution functions of all $v_i,\ i \in \mathcal{N}$ are $F(\cdot)$ with density function of $f(\cdot)$. The support of $G(\cdot)$ is $[\underline{c}, \overline{c}]$, and the support of $F(\cdot)$ is $[\underline{v}, \overline{v}]$. We assume $g(\cdot) > 0$ on its support. One reason for further considering the symmetric setting lies in that people are usually interested in the symmetric equilibrium in this setting.

Clearly, all the findings in section 2 apply to the above specified symmetric IPV setting. In this section, we investigate some special issues which are unique to the above symmetric setting. First, we show that even for the symmetric setting, the entry patterns maximizing the total expected surplus and the seller's revenue are generally asymmetric, i.e., the optimal participation thresholds are different across bidders. Second, on one hand, discriminating symmetric bidders may not always be allowed; on the other hand, people usually are interested in the symmetric entry in symmetric setting. We then derive the restricted ex ante efficient auction and revenue-maximizing auction within the class of auctions implementing symmetric entry across bidders.

3.1 Asymmetry in the Optimal Participation Thresholds

We begin from the following example. Consider a symmetric setting where $v_0 = 0$, $N = 2,\ F(v) = v,\ \forall v \in [0, 1]$, and $G(c) = 10(c - 0.4),\ \forall c \in [0.4, 0.5]$. Direct calculations using (1) and (2) give the following results. $TES(\mathbf{C_c})$ takes the maximum of 0.05 when $c_c^{(1)} = 0.5$ and $c_c^{(2)} = 0.4$, and $SER(\mathbf{C_c})$ takes the maximum of 0.025 when $c_c^{(1)} = 0.45$ and $c_c^{(2)} = 0.4$. If we restrict $c_c^{(1)} = c_c^{(2)}$, then we have $TES(\mathbf{C_c})$ takes the maximum of 0.023 when $c_c^{(1)} = c_c^{(2)} = 0.4231$, and $SER(\mathbf{C_c})$ takes the maximum of 0.01875 when $c_c^{(1)} = c_c^{(2)} = 0.4187$. In other words, the optimal participation patterns maximizing the total expected surplus and the seller's expected revenue are asymmetric.

The above example immediately shows that "symmetric" entry is generally restrictive for auctions design. The intuitions behind this result is as follows. Define $W_n,\ n \geq 0$ as the expectation of the highest among the valuations of the seller and n symmetric bidders. For simplicity, let us consider the case with 2 potential bidders (N=2). From (1) and (2), the first common component of $TES(c_c^{(1)}, c_c^{(2)})$ and $SER(c_c^{(1)}, c_c^{(2)})$ can be written as

$$\begin{aligned}
&G(c_c^{(1)})G(c_c^{(2)})W_2 + [G(c_c^{(1)})(1 - G(c_c^{(2)})) + G(c_c^{(2)})(1 - G(c_c^{(1)}))]W_1 \\
&\qquad\qquad +(1 - G(c_c^{(1)}))(1 - G(c_c^{(2)}))W_0 \\
&= (G(c_c^{(1)}) + G(c_c^{(2)}))(W_1 - W_0) \\
&\qquad\qquad +\frac{1}{4}[(G(c_c^{(1)}) + G(c_c^{(2)}))^2 - (G(c_c^{(1)}) - G(c_c^{(2)}))^2][(W_2 - W_1) - (W_1 - W_0)].
\end{aligned}$$

Note that $W_2 - W_1 > W_1 - W_0$ as $W_{n+1} - W_n$ (the contribution of the valuation of an additional bidder if there are already n bidders) decreases with n. Thus for given $G(c_c^{(1)}) + G(c_c^{(2)})$, we want to maximize $G(c_c^{(1)}) - G(c_c^{(2)})$ in order to maximize the above component. Therefore, if the symmetric optimum is an inner solution where both bidders participate in a common probability in $(0, 1)$, we must have that the unrestricted optimal thresholds maximizing the above common component be asymmetric. The above arguments can be generalized for $N > 2$ by focusing on the entry probabilities of any two bidders while assuming the entry probabilities of all other bidders are fixed. It is clear that if $\underline{c}$ and $\overline{c}$ are close enough, maximizing the difference between the entry probabilities of any 2 bidders while keeping the sum of entry probabilities unchanged will lead to higher total expected surplus and seller's expected revenue because the second terms in (1) and (2) do not change much. This is especially true when entry costs are common knowledge as in section 4.2, because the second term in (16) does not change if the sum of entry probabilities remains unchanged.

Although the optimal entry patterns are generally asymmetric even in the symmetric setting, a seller may not be allowed to discriminate against some bidders. In addition, more attention is paid to the auctions implementing symmetric entry across symmetric bidders in the literature. We thus emphasize the restricted ex ante efficient auction and revenue-maximizing auction within the class of auctions implementing symmetric entry across bidders.

3.2 Ex Ante Efficient Auction in Symmetric-Entry Class

Symmetric entry across bidders implies that the thresholds $c_c^{(i)}$ are same across all potential bidders. Suppose $c_c^{(i)} = c_c \in [\underline{c}, \overline{c}], \forall i \in \mathcal{N}$. For notational simplicity, we define $TES(c_c) = TES(\mathbf{C_c})$, where $c_c^{(i)} = c_c$ in vector $\mathbf{C_c}$, $\forall i \in \mathcal{N}$.

Under this restriction, we have

$$\frac{dTES(c_c)}{dc_c} = \sum_{\{i\in\mathcal{N}\}} \frac{\partial TES(\mathbf{C_c})}{\partial c_c^{(i)}} = N\frac{\partial TES(\mathbf{C_c})}{\partial c_c^{(i)}},\ \forall i \in \mathcal{N}. \tag{9}$$

Equation (9) leads to

$$\frac{\partial TES(\mathbf{C_c})}{\partial c_c^{(i)}} = \frac{dTES(c_c)}{dc_c}/N,\ \forall i \in \mathcal{N}. \tag{10}$$

Suppose that c_c^{**} maximizes $TES(c_c)$. Use $\mathbf{C_c^{**}}$ to denote the threshold-vector in which every element equals c_c^{**}. Then we have the following characterization for c_c^{**} from (10). For all $i \in \mathcal{N}$,

$$TS_i(\mathbf{C_c^{**}}) = \frac{\partial TES(\mathbf{C_c^{**}})}{\partial c_c^{(i)}}/g(c_c^{**}) = \begin{cases} 0, & \text{if} \quad c_c^{**} \in (\underline{c}, \overline{c}), \\ \geq 0, & \text{if} \quad c_c^{**} = \overline{c}, \\ \leq 0, & \text{if} \quad c_c^{**} = \underline{c}. \end{cases} \tag{11}$$

Equation (11) means that $\mathbf{C_c^{**}}$ must be a local optimal solution of $TES(\mathbf{C_c})$, if it is not globally optimal. [11]

Note that $TS_i(\mathbf{C_c^{**}})$ is the expected payoff of bidder i with cost c_c^{**} when he participates in auction $\mathcal{A}_\mathbf{0}$, if all other potential bidders participate according to threshold c_c^{**}. Thus, equation (11) leads to the following proposition.

Proposition 3: *In a symmetric IPV setting with private-information valuation-discovery costs for bidders, the second-price auction $\mathcal{A}_\mathbf{0}$ with a reserve price equal to seller's valuation and no entry fee is ex ante efficient among the symmetric-participation class.*

Proof: Since $g(\cdot) > 0$, clearly it is a Nash equilibrium that every bidder participates in auction $\mathcal{A}_\mathbf{0}$ according to threshold $\mathbf{C_c^{**}}$, when (11) is satisfied for $\mathbf{C_c^{**}}$. □

Although zero entry fee in a second-price auction with reserve price equal to seller's valuation is indeed ex ante efficient among the symmetric-entry class, other entry fees

[11]When v_i and c_i follow the uniform distribution on $[0, 1]$, we can verify that the symmetric optimal solution is globally optimal.

could also be ex ante efficient. Similar to the asymmetric setting, this is always true when c_c^{**} is not an inner solution.

3.3 Revenue-Maximizing Auction In Symmetric-Entry Class

Again for notational simplicity, define $SER(c_c) = SER(\mathbf{C_c})$, where $c_c^{(i)} = c_c$ in $\mathbf{C_c}$, $\forall i \in \mathcal{N}$. Under this restriction, we have

$$\frac{dSER(c_c)}{dc_c} = \sum_{\{i \in \mathcal{N}\}} \frac{\partial SER(\mathbf{C_c})}{\partial c_c^{(i)}} = N \frac{\partial SER(\mathbf{C_c})}{\partial c_c^{(i)}}, \ \forall i \in \mathcal{N}. \tag{12}$$

Equation (12) leads to

$$\frac{\partial SER(\mathbf{C_c})}{\partial c_c^{(i)}} = \frac{dSER(c_c)}{dc_c}/N, \ \forall i \in \mathcal{N}. \tag{13}$$

Suppose that c_c^{++} maximizes $SER(c_c)$. Define threshold-vector $\mathbf{C_c^{++}} = (c_c^{++}, ..., c_c^{++})$. Then based on equation (13), we have the following characterization for c_c^{++}. For all $i \in \mathcal{N}$,

$$BS_i(\mathbf{C_c^{++}}) = \frac{\partial SER(\mathbf{C_c^{++}})}{\partial c_c^{(i)}}/g(c_c^{++}) = \begin{cases} 0, & \text{if} \quad c_c^{++} \in (\underline{c}, \overline{c}), \\ \geq 0, & \text{if} \quad c_c^{++} = \overline{c}, \\ \leq 0, & \text{if} \quad c_c^{++} = \underline{c}. \end{cases} \tag{14}$$

Thus $\mathbf{C_c^{++}}$ must be a local optimal solution of $SER(\mathbf{C_c})$, if it is not globally optimal.[12]

Note that $BS_i(\mathbf{C_c^{++}})$ is the expected payoff of bidder i with cost c_c^{++} when he participates in a second price auction with a time-2 entry fee of $\frac{G(c_c^{++})}{g(c_c^{++})}$ and a reserve price equal to v_0, if all other potential bidders participate according to c_c^{++}. Thus, equation (14) leads to the following proposition.

Proposition 4: *Suppose c_c^{++} maximizes $SER(c_c)$, then a second-price auction with a reserve price equal to seller's valuation and a time-2 entry fee, e, defined below maximizes*

[12]Similar to the case of $TES(c_c)$, when v_i and c_i follow the uniform distribution on $[0, 1]$, we can verify that the symmetric optimal solution is globally optimal.

the seller's expected revenue among the symmetric-entry class. The time-2 optimal entry fee, e, is defined as

$$e = \begin{cases} \frac{G(c_c^{++})}{g(c_c^{++})}, & \text{if } c_c^{++} \in (\underline{c}, \overline{c}), \\ TS_i(\mathbf{C}_\mathbf{c}^{++}) \geq \frac{1}{g(\overline{c})}, & \text{if } c_c^{++} = \overline{c}, \\ \textit{any number} \geq TS_i(\mathbf{C}_\mathbf{c}^{++}), & \text{if } c_c^{++} = \underline{c}. \end{cases} \tag{15}$$

Furthermore, no other entry fee works.

Proof: Since $g(\cdot) > 0$, clearly it is a Nash equilibrium that every bidder participate in the above defined auction according to threshold c_c^{++}, when (14) and (15) are satisfied for c_c^{++}. Moreover, if the e is defined differently from (15), then it is clear that either the symmetric entry threshold c_c^{++} can not be implemented or the surplus of the entrants with the threshold entry costs can not be extracted completely. □

From Proposition 4, if the entry cost is private information of bidders, then generally the revenue-maximizing auction involves individual entry fee. The necessary and sufficient conditions for auction $\mathcal{A}_\mathbf{0}$ with a reserve price equal to seller's valuation and no entry fee to maximize the seller's expected revenue among the symmetric-entry class are that $c_c^{++} = \underline{c}$ and $TS_i(\mathbf{C}_\mathbf{c}^{++}) \leq 0$. As mentioned before in section 2.2, this case is a degenerate one where it is inefficient for any bidder to participate in any chance.

Propositions 3 and 4 show that restricting the entry to be symmetric across bidders does not change the essence of the conclusion. Specifically, if the entry costs are private information of the bidders, then the optimal symmetric entry patterns that maximize the expected total surplus and the seller's expected revenue are generally different. In addition, the revenue-maximizing auction must generally be different from the ex ante efficient auction. In contrast, if the entry costs are public information, Levin and Smith (1994) show that optimal symmetric participation probability that maximize the expected total surplus also is optimal for the seller's expected revenue. Moreover, if this participation probability belongs to $(0, 1)$, then the revenue-maximizing auction is same as the ex ante efficient auction $\mathcal{A}_{\mathbf{0}}$.

4 Further Results with Common-Knowledge Entry Costs

In this section, we keep all the assumptions in section 2, except that the distributions of the entry costs are assumed to be degenerate. In other words, we have common-knowledge entry costs $c_i \geq 0$ for bidder i, $\forall i \in \mathcal{N}$. Those costs c_i are allowed to be asymmetric across potential bidders. This setting is more general than the IPV setting studied in Levin and Smith (1994) since asymmetry among bidders is allowed. Levin and Smith (1994) focus on symmetric mixed strategy (strictly) entry equilibrium in settings where this type of entry equilibria exist. We instead allow full flexibility in bidders' entry probabilities even when we consider the symmetric setting later. This flexibility is natural for asymmetric setting, and it does enlarge the freedom for auctions design in the symmetric settings.

4.1 Ex Ante Efficient Auction and Revenue-Maximizing Auction

Use $\mathbf{P} = (p_1, p_2, ..., p_N)$ to denote the entry pattern, where $p_i \in [0,1]$ is the entry probability of bidder i. Similar to Lemma 2, we have the following result. The proof is omitted, since it is similar to that of Lemma 2.

Lemma 3: *Among all auctions implementing any given entry pattern* $\mathbf{P} = (p_1, p_2, ..., p_N)$, *a second-price auction with a reserve price equal to the seller's valuation and appropriately set individual entry fee (or subsidy) for each bidder provides the highest seller's expected revenue as well as the highest total expected surplus. The entry fees (or subsidies) are charged upon entry at time 2, and are set at the levels making bidders' expected surplus* *to be zero when they participate in the auction.*

According to Lemma 3, at the optimum the expected surplus of bidders is zero. Thus the optimal seller's expected revenue equals to the optimal total expected surplus. Denote this optimal total expected surplus by $TES(\mathbf{P})$. Then $TES(\mathbf{P})$ can be written as follows.

$$TES(\mathbf{P}) \quad = \sum_{\{\mathbf{k}\in\mathcal{K}\}} \Big(V_{\mathbf{k}} \prod_{i\in\mathcal{N}} p_i^{k_i}(1-p_i)^{1-k_i}\Big) - \sum_{i\in\mathcal{N}} p_i c_i. \tag{16}$$

Suppose $\mathbf{P}^* = (p_1^*, ..., p_N^*)$ maximizes $TES(\mathbf{P})$, where $\mathbf{P}^*$ is allowed to be corner solution. Following closely the procedure adopted in section 2.1, we have the following results.

Proposition 5: *(i) A second-price auction with a reserve price equal to seller's valuation and no entry fee maximizes the total expected surplus. (ii) A second-price auction with a reserve price equal to seller's valuation and time-2 entry fee e_i for bidder i defined below maximizes the seller's expected revenue. The time-2 entry fee e_i, $\forall i \in \mathcal{N}$ are defined as following,*

$$e_i = \begin{cases} 0, & \text{if} \quad p_i^* \in (0,1), \\ TS_i(\mathbf{P}^*) \geq 0, & \text{if} \quad p_i^* = 1, \\ \textit{any number} \geq TS_i(\mathbf{P}^*), & \text{if} \quad p_i^* = 0, \end{cases} \tag{17}$$

where $TS_i(\mathbf{P}^*) = \sum_{\{\mathbf{k}_{-i}\in\mathcal{K}_{-i}\}} [(V_{\tilde{\mathbf{k}}_1(\mathbf{k}_{-i})} - V_{\tilde{\mathbf{k}}_0(\mathbf{k}_{-i})}) \prod_{j\neq i} p_i^{*k_j}(1-p_i^*)^{1-k_j}] - c_i$ is the expected payoff of bidder i with cost c_i when he participates in auction $\mathcal{A}_{\mathbf{0}}$, if all other potential bidders participate according to entry probability vector $\mathbf{P}^*$.

Clearly, no entry fees other than those defined in (17) can implement the participation pattern $\mathbf{P}^*$ while extracting all the surplus of the entrants. Therefore, the revenue-maximizing entry fees are uniquely defined in (17). For the same reason as pointed in sections 2.1 and 3.2, the zero entry fee is not uniquely ex ante efficient when the entry probabilities are corner solutions.

4.2 Optimality of $\{0, 1\}$ Entry Decisions

In the section 3.1 setting with private information cost, we have shown that the optimal participation patterns which maximize the total expected surplus and the seller's expected revenue are generally asymmetric, even if the bidders are symmetric. When the distributions of the entry costs are degenerated (i.e., common-knowledge entry cost), the same result should still hold naturally. The following proposition further establishes a stronger result that in a general setting (with symmetric or asymmetric bidders) with common-knowledge entry costs, there is no loss of generality to consider only the entry probability vector in which each element is either 0 or 1.

Proposition 6: *There must exist an entry probability vector* $\mathbf{P}^*$ *where* $p_i^* \in \{0, 1\}, \forall i \in \mathcal{N}$*, which maximizes* $TES(\mathbf{P})$*. In other words, it is not restrictive to optimize* $TES(\mathbf{P})$ *by considering only the entry probability vector in which each element is either 0 or 1.*

Proof: Suppose $\mathbf{P}^{*(\mathbf{0})} = \{p_1^{*(0)}, ..., p_N^{*(0)}\}$ maximizes $TES(\mathbf{P})$, where $p_i^{*(0)} \in [0, 1], \forall i \in \mathcal{N}$. If $p_i^{*(0)} \in \{0, 1\}, \forall i \in \mathcal{N}$, then the proof is completed. Otherwise, there must exist $i_0 \in \mathcal{N}$ satisfying $p_{i_0}^{*(0)} \in (0, 1)$. Since $p_{i_0}^{*(0)}$ is an inner solution, we must have $\frac{\partial TES(\mathbf{P}^{*(\mathbf{0})})}{\partial p_{i_0}} = TS_{i_0}(\mathbf{P}^{*(\mathbf{0})}) = 0$. Thus setting $p_{i_0}^{*(0)}$ to be either 0 or 1 will not change the total expected surplus. This change in $p_{i_0}^{*(0)}$ leads to a new entry probability vector $\mathbf{P}^{*(\mathbf{1})}$ maximizing $TES(\mathbf{P})$. The only difference between $\mathbf{P}^{*(\mathbf{0})}$ and $\mathbf{P}^{*(\mathbf{1})}$ lies in the values that the i_0th elements in the two vectors take. If now $p_i^{*(1)} \in \{0, 1\}, \forall i \in \mathcal{N}$, then the proof is completed. Otherwise, the above process can continue until every element in the entry probability vector becomes either 0 or 1. $\square$

While Proposition 5 shows that zero entry fee in a second-price auction with reserve price equal to seller's valuation is indeed ex ante efficient, Proposition 6 implies that there are other entry fees which are also ex ante efficient. This is true, because p_i^* is generally a corner solution. If $p_i^* = 1$, any entry fee for bidder i which is less than or equal to $TS_i(\mathbf{P}^*)$, is also ex ante efficient. If $p_i^* = 0$, any entry fee for bidder i which is more than or equal to $TS_i(\mathbf{P}^*)$, is also ex ante efficient.

4.3 Special Issues in Symmetric Setting

We further restrict the entry costs and distributions of values to be symmetric across bidders. Suppose $c_i = c$ and $F_i(\cdot) = F(\cdot)$, $\forall i \in \mathcal{N}$. This is the symmetric IPV setting studied in Levin and Smith (1994). First of all, the findings in sections 4.1 and 4.2 naturally hold in the above specified symmetric IPV setting. The result regarding market thickness in Levin and Smith (1994) implies that even for the symmetric IPV setting with common-knowledge costs, the ex ante efficient entry could be asymmetric across symmetric bidders. Proposition 6 further shows that there is no loss of generality in considering the entry patterns where each bidder participates in probability of either 0 or 1, for the revenue-maximizing auction.

A further characterization of all the entry patterns that maximizes the total expected surplus as well as the seller's expected revenue for this setting is provided below. Suppose $\mathbf{P}^* = (p_1^*, p_2^*, ..., p_N^*)$ maximizes total expected surplus $TES(\mathbf{P})$. Without loss of generality, we assume that $p_i^* \geq p_j^*$ if $N \geq i > j \geq 1$. We then have $TS_i(\mathbf{P}^*) \geq TS_j(\mathbf{P}^*)$ if $N \geq i > j \geq 1$. Define $\mathcal{I}_1 = \{i | TS_i(\mathbf{P}^*) > 0,\ i \in \mathcal{N}\}$, $\mathcal{I}_0 = \{i | TS_i(\mathbf{P}^*) = 0,\ i \in \mathcal{N}\}$, $\mathcal{I}_{-1} = \{i | TS_i(\mathbf{P}^*) < 0,\ i \in \mathcal{N}\}$. Then we have $\frac{\partial TES(\mathbf{P}^*)}{\partial p_i} > 0$ if $i \in \mathcal{I}_1$, $\frac{\partial TES(\mathbf{P}^*)}{\partial p_i} = 0$ if $i \in \mathcal{I}_0$, and $\frac{\partial TES(\mathbf{P}^*)}{\partial p_i} < 0$ if $i \in \mathcal{I}_{-1}$. Thus we must have that $p_i^* = 1$, $\forall i \in \mathcal{I}_1$, $p_1^* = 0$, $\forall i \in \mathcal{I}_{-1}$, and for $\forall i \in \mathcal{I}_0$, the p_i^* must take a same value in $[0, 1]$.

The following examples illustrate the insights stated in both Proposition 6 and the previous paragraph. Assume $v_0 = 0$, $c = 0.1$ and $N = 3$. Assume $F(v_i) = v_i^\alpha,\ i \in \mathcal{N}$ where $\alpha > 0$. Based on (16), direct calculation shows the following results. If $\alpha = 0.3$, $\mathbf{P}^* = (1, 1, 0)$ maximizes $TES(\mathbf{P})$ and $TES(\mathbf{P}^*) = 0.175$. If $\alpha = 1.0$, $\mathbf{P}^* = (1, 1, 0)$ maximizes $TES(\mathbf{P})$ and $TES(\mathbf{P}^*) = 0.4667$. If $\alpha = 2.0$, $\mathbf{P}^* = (1, 1, 0)$ maximizes $TES(\mathbf{P})$ and $TES(\mathbf{P}^*) = 0.60$. More interestingly, if $\alpha = 0.5$, $\mathbf{P}^* = (1, 1, p)$ where p is any real number in $[0, 1]$ maximizes $TES(\mathbf{P})$ and $TES(\mathbf{P}^*) = 0.30$. If $c = 0.5$, $\alpha = 2.0$ and $N = 2$, then $\mathbf{P}^* = (1, 0)$ maximizes $TES(\mathbf{P})$ and $TES(\mathbf{P}^*) = 0.1667$.

Define $MS(N) = TS_i(\mathbf{P}^*)$ where $\mathbf{P}^* = (p_1^*, p_2^*, ..., p_N^*) = (1, 1, ..., 1)$. $MS(N)$ is then the marginal contribution of any participant to the total expected surplus if all N potential bidders participate in auction $\mathcal{A}_\mathbf{0}$. Clearly, $MS(N)$ decreases with N. If $MS(1) < 0$, define $N^* = 0$, otherwise define $N^* = \max_{MS(N)\geq 0}\{N\}$. Based on Proposition 6, integer $N^* \geq 0$ must satisfy the following property. If the number of potential bidders $N \leq N^*$, it must be optimal that all bidders participate with a probability of 1. On the other hand, if the number of potential bidders $N \geq N^*$, it must be optimal that any N^* bidders participate with a probability of 1, and other bidders participate with a probability of zero. Furthermore, this entry pattern is implemented through a second-price auction with reserve price equal to seller's valuation and zero entry fee. If no restriction is imposed on the entry pattern, N^* is the optimal number of bidders if the number of potential bidders $N > N^*$. In other words, additional potential bidders beyond N^* will neither increase nor decrease the seller's optimal expected revenue. This observation provides new insights into the market thickness puzzle raised by Levin and Smith (1994). In Levin and Smith (1994), the result that total expected surplus decreases with the number of potential bidders is due to the fact that the entry patterns are restricted to be symmetric across bidders. Let us look at the example in the previous paragraph where $c = 0.1$, $\alpha = 2.0$. Calculation shows that when $N = 2$, $\mathbf{P}^* = (1, 1)$ is the optimal solution, which gives $TES(\mathbf{P}^*) = 0.60$. Note that it is a symmetric solution. When $N = 3$, $\mathbf{P}^* = (1, 1, 0)$ maximizes $TES(\mathbf{P})$ and $TES(\mathbf{P}^*) = 0.60$. In addition, $\mathbf{P}^* = (.82759, .82759, .82759)$ is the optimal symmetric participation, which gives $TES(\mathbf{P}^*) = 0.57017$. In this example, the above defined N^* is 2. This example also illustrates that $TES(\mathbf{P}^*)$ decreases with N when $N > N^*$ if the entry probabilities are restricted to be symmetric across bidders. Clearly, N^* is the maximum number of potential bidders which supports a symmetric entry probability of 1 in auction $\mathcal{A}_{\mathbf{0}}$.

We now restrict the entry probabilities to be symmetric across bidders, i.e., $p_i = p \in [0,1]$. Note that p is allowed to be corner solutions such as 0 or 1. Once again, for notational simplicity, define $TES(p) = TES(\mathbf{P})$, where vector $\mathbf{P} = (p_i)$ with $p_i = p,\ i \in \mathcal{N}$. Suppose p^* maximizes $TES(p)$ where p^* could be 0 or 1. Following similar procedures in section 3.2, we obtain the following result.

Proposition 7: *For a symmetric IPV setting with common-knowledge entry costs, (i) a second-price auction with a reserve price equal to the seller's valuation and no entry fee maximizes the total expected surplus among the class of auctions implementing symmetric entry across bidders; (ii) a second-price auction with a reserve price equal to seller's valuation and a time 2 entry fee, e, defined below maximizes the seller's expected revenue. The time-2 entry fee, e, is defined as*

$$e = \begin{cases} 0, & \text{if} \quad p^* \in (0,1), \\ TS_i(1) \geq 0, & \text{if} \quad p^* = 1, \\ \textit{any number} \geq TS_i(0), & \text{if} \quad p^* = 0, \end{cases} \tag{18}$$

where $TS_i(p^*) = \sum_{\{\mathbf{k}_{-i} \in \mathcal{K}_{-i}\}} [(V_{\tilde{\mathbf{k}}_1(\mathbf{k}_{-i})} - V_{\tilde{\mathbf{k}}_0(\mathbf{k}_{-i})}) \prod_{j \neq i} p^{*k_j}(1-p^*)^{1-k_j}] - c$ is the expected payoff of bidder i with entry cost c from participating in auction $\mathcal{A}_{\mathbf{0}}$, if all other potential bidders participate in probability p^*.

The result for p^* falling in $(0,1)$ corresponds to that obtained by Levin and Smith (1994) through a different path. From Propositions 5 and 7, we see that a second-price auction with a reserve price equal to seller's valuation and a zero entry fee does not in general maximize the seller's expected revenue. However, similar to section 3.2, restricted optimal solution $p_i = p^*,\ i \in \mathcal{N}$ is at least locally optimal in the symmetric setting.

5 Conclusion

This paper first presented an endogenous entry model for single-object auction in a general independent private value (IPV) framework, where each bidder has a valuation discovery cost that is his private information. This framework allows asymmetry across bidders in the distributions of their entry costs and private valuations. The framework covers as a special case the IPV setting with common-knowledge costs that has been considered in the literature.

We then developed a path to show the auctions maximizing the total expected surplus and the seller's expected revenue respectively in this general IPV setting with private-information entry costs. A key connection is discovered between the first order conditions characterizing the optimal entry thresholds and the expected surplus of the threshold types for a second-price auction with zero entry fee and a reserve price equal to seller's valuation. Our major results are based on this connection.

Unlike the case of common-knowledge costs, bidders enjoy information rents when entry costs are their private information. Due to these information rents, discrepancy appears between entry patterns maximizing the total expected surplus and the seller's expected revenue. This further leads to the divergence between the auctions maximizing the total expected surplus and the seller's expected revenue for the setting with private-information entry costs. The ex ante efficient entry is always implemented through a second-price auction with no entry fee and a reserve price equal to seller's valuation. However, the seller's optimal expected revenue generally cannot be implemented through such an auction. We find that the auction maximizing the seller's optimal expected revenue generally involves individual entry fees for bidders. The optimal entry fee for each bidder is given by the hazard rate of his entry cost distribution, evaluated at the optimal threshold entry cost of the bidder.

These findings hold when we restrict the entries to be symmetric across bidders in the symmetric IPV setting. Interestingly, even for symmetric setting with private-information of entry costs, it could be an auction implementing asymmetric entry rather than symmetric entry that maximizes the total expected surplus or seller's expected revenue, provided that the symmetric optimum is an inner solution where every bidder participates in a common probability in $(0, 1)$. The intuition for asymmetric optimum lies in the fact that the marginal contribution of an additional participant's valuation to the total surplus or seller's revenue strictly decreases with the number of other participants. This fact leads to that given the summation of the participating probabilities of any two bidders, the marginal contribution of their valuations to the total surplus or seller's revenue is maximized when the difference between their participating probabilities is maximized. It then follows that the optimum must be asymmetric unless the symmetric optimum is a corner solution, if the above difference-maximizing adjustment in the entry probabilities does not change much the total expected entry costs. This is especially true when entry costs are common knowledge.

A further look at the case with common-knowledge costs provides new insights. A second-price auction with a reserve price equal to the seller's valuation and zero entry fee is shown to be ex ante efficient in more general settings than that considered in the existing literature. However, generally this auction does not lead to the highest seller's expected revenue as some bidders may enjoy strictly positive expected surplus in such an auction. Thus, individual entry fees are necessary to extract the bidders' surplus. Furthermore, when the discovery costs are common knowledge, we show that there is no loss of generality in considering the entry patterns where each bidder participates in probability of 0 or 1, for the optimal auction. This result provides new insights into the market thickness puzzle raised by Levin and Smith (1994).

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