On the Non-Robustness of Nash Implementation *

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Abstract. I consider the implementation problem under complete information and employ Nash equilibrium as a solution concept. This paper asks the maximal amount of incomplete information which allows the canonical game form of Maskin (1999) to be robust. I establish a general impossibility result on robust Nash implementation. To be exact, under some mild condition on the social choice rules, one can construct a canonical perturbation of the complete information structure under which a sequence of Nash equilibria of the *Maskinian* game form converges to a *non-Nash* equilibrium outcome in the limit. Therefore, there is a precise sense in which the Maskinian game form is not robust to a very small amount of incomplete information.

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1 Introduction

To be added

2 The Setup

2.1 The Environment

There is a finite set $N = \{1, \ldots, n\}$ of players. Let $f : \Theta \to A$ be a social choice function, where Θ denotes the set of *payoff* states, and A denotes the set of outcomes. Associated with each state θ is a preference profile $\succeq^\theta$, which is a list $(\succeq_1^\theta, \ldots, \succeq_n^\theta)$ where $\succeq_i^\theta$ is player i's state θ preference relation over A. I read $a \succeq^\theta a^{'}$ as "a is at least as good as $a^{'}$ in state θ." I read $a \succ^\theta a^{'}$ as "a is strictly preferred to $a^{'}$ in state θ.

Players do not observe the state directly but are informed of the state via signals. Player i's signal set is S_i which I set $|S_i| = |\Theta|$ for each $i \in N$. A signal profile is an element $s = (s_1, \ldots, s_n) \in S = \times_{i \in N} S_i$. Let μ be a prior probability over $\Theta \times S$. I designate s^θ to be the signal profile in which each player's signal s_i^θ corresponds to the state θ. *Complete information* refers to the environment in which $\mu(\theta, s) = 0$ whenever $s \neq s^\theta$. This specification of the signal space is without loss of generality as long as I only consider environments with complete information.

Given a *game form* $\Gamma = (M, g)$, a designer is interested in the set of Nash equilibrium (NE) outcomes under complete information. Here $M \equiv \times_{i \in N} M_i$, M_i is player i's *message space* and $g : M \to A$ is the *outcome function*. The designer, however, entertains the possibility that players face uncertainty about payoffs. Then, he has to take into account the set of "nearby" incomplete information structures in which the original complete information structure is subsumed. The designer then employs (Bayesian) Nash equilibrium (BNE) as a solution concept for those nearby incomplete information games. He wants to ascertain that his prediction about players' strategic behavior changes *continuously* with respect to some topology.

2.2 Definitions and Notations

Mathematically, the coarser the topology chosen, the larger the set of continuous correspondences with respect to it and therefore, the harder the achievement of robust implementation with respect to it. By the same token, the finer the topology chosen, the less it demands that the BNE correspondence be continuous with respect to it. In order to talk about the topology, I have to explicitly expound the topological structure of the domain of the BNE correspondence into which the players' belief structure is embedded. Let $(\Theta \times S, \mu)$ be a complete information structure. I shall construct the set of states of the world, called Ω, that is consistently extended from a given complete information environment. I want to keep track of the complete information environment and let it be embedded in Ω so that we are able to coherently discuss the epistemic states of all players when the environment is subject to incomplete information. [3]

Definition 1 *A space $\Theta \times S$* ***is (algebraically) immersed in*** *Ω if there exists a one-to-one correspondence $h : \Theta \times S \to \Omega$. h is said to be an immersion of $\Theta \times S$ into Ω.*

Definition 2 *Assume that $\Theta \times S$ is immersed in Ω. A probability space (Ω, Σ, P^*) is a* ***consistent extension*** *from the complete information structure $(\Theta \times S, \mu)$ if (Ω, Σ, P^*) is equivalent to the probability space $\left(\Theta \times S, 2^{\Theta \times S}, \mu\right)$, where $P^* \equiv \mu \circ h^{-1}$.*

Let (Ω, Σ, P^*) be a probability space consistently extended from the complete information structure $(\Theta \times S, \mu)$. I fix a measurable space (Ω, Σ) throughout the argument while I change the probability distributions. In particular, I am interested in a net of probability distributions $\{P^k\}_{k=1}^{\infty}$ for which $P^k \to P^*$ as $k \to \infty$ in "a certain" property preserving way. The very issue is that I have to be specific about with respect to which topology the two probability distributions are deemed to be close. Let $\mathscr{P}$ be the space of all probability distributions over Ω. Let $\mathscr{F}$ be the space of all partitions of Ω, the elements of which are in Σ. Let $\mathscr{F}^*$ be the subset of $\mathscr{F}$ such that for any $\Pi \in \mathscr{F}^*$, $\omega \in \Pi(\omega)$ for each $\omega \in \Omega$. An element of the form $\Pi = (\Pi_i)_{i \in N} \in (\mathscr{F}^*)^n$ is called a *partition structure*. Let $\Pi^* : \Omega \to 2^{\Omega}$ be the finest possibility correspondence that is coarser than Π_i for each $i \in N$. An event E is said to be *common knowledge* at ω if $\Pi^*(\omega) \subset E$. I fix a partition structure $\Pi = (\Pi_i)_{i \in N} \in (\mathscr{F}^*)^n$ for the moment. Player i's strategy in the game $\Gamma(P)$ is a function $\sigma_i : \Omega \to M_i$ which is Π_i-measurable. Let σ be a strategy profile in a game $\Gamma(P)$.

I take for granted Nash equilibrium as a reasonable solution in a game with complete information. Therefore, players are assumed to choose their strategy *independently* of other players' choice provided there is common knowledge about what game being played. This implies that the amount of incomplete information I allow for the robustness analysis should not contradict the use of Nash equilibrium under complete information. The formalization of this is summarized by the following measurability condition on possible strategy profiles.

Definition 3 *Let a partition structure $\Pi \in \mathscr{F}^n$ and the associated game $\Gamma(P)$ be given. A strategy profile σ is* ***consistent*** *with the complete information structure if, for any $\omega, \omega' \in \Omega$, whenever there exists a profile $(\theta, s^{\theta}) \in \Theta \times S$ such that $h(\theta, s^{\theta}) = \tilde{\omega}$ for any $\tilde{\omega} \in \Pi^*(\omega) \cup \Pi^*(\omega')$, then we have $\sigma_i(\omega) = \sigma_i(\omega')$ for each $i \in N$. When σ is consistent with the complete information structure, we simply say that it is a consistent strategy profile.*

[3] Players' epistemic state dictates what they know or believe about the game and about each other's actions, knowledge, and beliefs.

Remember that s^θ denotes the signal profile in which each player's signal corresponds to the state θ. By the focus only on consistent strategy profiles, I exclude the extent of correlation of equilibrium strategies which invalidates the original equilibrium analysis under complete information. Otherwise, this robustness analysis on Nash implementation becomes trivial. Because then the use of Nash equilibrium under complete information is far from appropriate *from the beginning*. This is my stance consistent with the use of Nash equilibrium. [4]

In this paper, I take ordinal preferences over the set of (pure) outcomes as a primitive. In order to analyze players' behavior under uncertainty, I shall extend the ordinal preferences into preferences over "acts." An act is a mapping $\alpha : \Omega \to A$. A belief is a probability distribution β on Ω. I read "$\alpha \succ_i^\beta \alpha^{'}$" as "player i prefers α to $\alpha^{'}$ under his belief β. I assume that the set of axioms is imposed on possible acts enough to obtain the subjective expected utility representation. Furthermore, I assume that any representation for preferences over acts is permissible as long as it is consistent with the ordinal preferences over pure outcomes. The act α_σ^Γ induced by σ under Γ is defined by $\alpha_\sigma^\Gamma(\omega) = g(\sigma(\omega))$ for any $\omega \in \Omega$. With these notations, I shall define Nash equilibrium (NE) and Bayesian Nash equilibrium (BNE), respectively.

Definition 4 *Let a partition structure $\Pi = \times_{i \in N} \Pi_i \in (\mathscr{F}^*)^n$ and the associated game $\Gamma(P)$ be given. A consistent strategy profile σ is a* ***Bayesian Nash equilibrium*** *(BNE) of $\Gamma(P)$ if σ_i is Π_i-measurable for each $i \in N$, and for each $i \in N$, state ω with $P(\Pi_i(\omega)) > 0$, and strategy $\sigma_i^{'}$ which is Π_i-measurable, we have $\alpha_\sigma^\Gamma \succeq_i^{P(\cdot|\Pi_i(\omega))} \alpha^\Gamma_{\sigma_i^{'},\sigma_{-i}}$.*

The above definition suffices for σ to be a Nash equilibrium of the game $\Gamma(P)$ if P is a complete information prior. Define the set of acts $\mathscr{A} \equiv A^\Omega$. I define $\psi_\Gamma^{BNE} : \mathscr{P} \to \mathscr{A}$ as the Bayesian Nash equilibrium correspondence associated with the game form Γ. Here, $\mathscr{A}$ is endowed with product topology. I shall introduce a topology which enables us to determine how close any two probability distributions are. To define such topologies, I need some definitions and notations.

Monderer and Samet (1989) introduced the concept of "*common p-belief*" as an approximation to common knowledge, which is common 1-belief. Let $B_i^q(E) \equiv \{\omega \in \Omega | P(E|\Pi_i(\omega)) \geq q\}$ denote the set of states in which player i assigns probability at least q to the event E. I call this player i's *q-belief operator*. In particular, when $q = 1$, I call B_i^1 player i's 1-*belief operator* corresponding to player i's knowledge operator. [5] An event E is said to be *q-evident* if $E \subset B_i^q(E)$ for all $i \in N$. This means that whenever E is true, everyone believes with probability at least q that E is true. An event E is said to be *common q-belief* at ω if there exists a q-evident event

[4] Moreover, I restrict my attention only to *pure strategy* Nash equilibrium. This is consistent with almost all papers in the implementation literature. Admittedly, it has a substantive effect on implementability.

[5] Here we define "knowledge" as belief with probability 1.

F such that $\omega \in F \subset \bigcap_{i\in N} B_i^q(E)$. I will loosely say that an event E is *approximate common knowledge* at ω if E is common q-belief at ω, for q close to 1.

Consider the notion of the closeness of probability distributions. Define d_0 by the rule

$$d_0(P, P^{'}) = \sup_{E\subset\Omega} |P(E) - P^{'}(E)|.$$

Let P^* be the complete information prior. I will require extra conditions on conditional probabilities. Define as $\mathscr{G}(\eta)$ the set of all states in which there is a common $(1-\eta)$-belief about what game being played as follows:

$$\mathscr{G}(\eta) = \left\{\omega \in \Omega \middle| \ \exists\theta \in \Theta \text{ such that } \Gamma(\theta) \text{ is common } (1-\eta)\text{-belief at } \omega\right\}.$$

Let

$$d_1(P) = \inf\{\eta \mid P\left(\mathscr{G}(\eta)\right) = 1\},$$

and

$$d^*(P, P^{'}) = \max\{d_0(P, P^{'}), d_1(P), d_1(P^{'})\}.$$

Note that $d_1(P^*) = 0$ by definition. By construction, d^* is non-negative and symmetric, and $d^*(P, P^{'}) = 0$ if and only if $P = P^{'}$ and both p and $P^{'}$ are complete information priors. d^* generates a topology in the following sense: a net $\{P^k | \, k \in K\}$, where K is a directed index set with partial order $\succ$, converges to P^* if and only if for any $\varepsilon > 0$, there is a $\bar{k} \in K$ such that $k \succ \bar{k}$ implies $d^*(P^k, P^*) < \varepsilon$. The reader is referred to Kunimoto (2006) for the topology induced by d^*.

Definition 5 *Let Γ be a game form. ψ_Γ^{BNE} is* ***upper hemi-continuous*** *at a complete information prior P^* with respect to the topology induced by d^* if, $\psi_\Gamma^{BNE}(P^k) \to \psi_\Gamma^{NE}(P^*)$ as $k \to \infty$ whenever $d^*(P^k, P^*) \to 0$ as $k \to \infty$. Here $\psi_\Gamma^{NE}(P^*)$ denotes the set of NE outcomes of the game $\Gamma(P^*)$.*

The next result is given as a corollary of Theorem 1 in Kunimoto (2006).

Theorem 1 (Kunimoto (2006)) *Let Γ be a game form. Then, the Bayesian Nash equilibrium correspondence associated with Γ is upper hemi-continuous at any complete information prior with respect to the topology induced by d^*.*

I want to apply the above result to the notion of robust Nash implementation. In so doing, I need some definitions. Let $NE(\Gamma(\theta)) \subset M$ be the entire set of (pure strategy) Nash equilibria of the complete information game $\Gamma(\theta)$. First, I shall define the notion of Nash implementation.

Definition 6 *A social choice function f is* ***implementable*** *in Nash equilibrium if there exists a game form $\Gamma = (M, g)$ such that for any $\theta \in \Theta$, (1) there is a Nash equilibrium m^θ of the game $\Gamma(\theta)$ such that $g(m^\theta) = f(\theta)$ and (2) $g(NE(\Gamma(\theta)) = f(\theta)$.*

Condition (1) of Nash implementation requires that there be always a Nash equilibrium outcome which coincides with the socially desirable outcome in each state. Condition (2) of Nash implementation requires that *every* Nash equilibrium outcome coincide with the socially desirable outcome in each state. Having defined Nash implementation, I introduce the concept of *robust* Nash implementation as follows.

Definition 7 *A social choice function f is **robustly** implementable relative to d^* (d^{**}) if (1) it is implementable in Nash equilibrium and (2) the Bayesian Nash equilibrium correspondence associated with some Nash implementing game form is upper hemi-continuous at any complete information prior with respect to the topology induced by d^* (d^{**}).*

In particular, Theorem 1 implies that *all* Nash implementing game forms are robust relative to d^*. To strengthen this robustness result on Nash implementation, I introduce a slightly coarser topology than that induced by d^*. Let

$$\tilde{d}_1(P) = \inf \left\{ \eta \middle| P\left(\mathscr{G}(\eta)\right) \geq 1 - \eta \right\}.$$

Define $d^{**}(P, P^{'})$ as follows:

$$d^{**}(P, P^{'}) = \max \left\{ d_0(P, P^{'}), \tilde{d}_1(P), \tilde{d}_1(P^{'}) \right\}.$$

The reader is referred to Kunimoto (2006) for the topology induced by d^{**}.

Definition 8 *Let Γ be a game form. ψ_Γ^{BNE} is **upper hemi-continuous** at a complete information prior P^* with respect to the topology induced by d^{**} if, $\psi_\Gamma^{BNE}(P^k) \to \psi_\Gamma^{NE}(P^*)$ as $k \to \infty$ whenever $d^{**}(P^k, P^*) \to 0$ as $k \to \infty$. Here $\psi_\Gamma^{NE}(P^*)$ denotes the set of NE outcomes of the game $\Gamma(P^*)$.*

3 The Maskinian Game Form

In his "classical" paper, Maskin (1999) showed that *monotonicity* is a necessary condition for Nash implementation. [6]

Definition 9 *A social choice function f is **monotonic** if for every pair of states θ and $\theta^{'}$ such that for each player i,*

$$a \succ_i^{\theta^{'}} f(\theta) \Rightarrow a \succ_i^{\theta} f(\theta),$$

we have $f(\theta^{'}) = f(\theta)$.

Maskin (1999) also provided sufficient conditions for Nash implementation when there are at least three players. Before stating his result, I need one more definition.

[6] Maskin's original paper had been circulated as a MIT working paper since 1977.

Definition 10 *A social choice function f satisfies* **no veto power** *if, for all $\theta \in \Theta$ and all $a \in A$, whenever there exists $i \in N$ such that, for all $j \neq i$ and all $b \in A$, if $a \succeq_i^\theta b$, then $f(\theta) = a$.*

No veto power says that if an outcome is at the top of $n-1$ players' preference orderings, then the last player cannot prevent this outcome from being the social optimum.

Theorem 2 (Maskin (1999)) *Let the number of players be at least three, i.e., $n \geq 3$. Then, any monotonic social choice function f satisfying no veto power is implementable in Nash equilibrium.*

The proof of Theorem 2 is based on the construction of a canonical game form which I call the *Maskinian* game form. Define the Maskinian game form $\Gamma = (M, g)$, in which each message $m_i \in M_i$ allowed to player i consists of an alternative, a payoff state and a nonnegative integer. Thus, a typical message sent by player i is denoted

$$m_i = (a^i, \theta^i, z^i),$$

where $a^i \in A$, $\theta^i \in \Theta$ and $z^i \in \{0, 1, 2, \dots\}$. The outcome function g of the Maskinian game form Γ is defined with the following rules, where $m = (m_1, \dots, m_n)$:

1. If all players announce the same message, $m_i = (a^i, \theta^i, z^i) = (a, \theta, 0)$ for all $i \in N$, and $f(\theta) = a$, then $g(m) = a$.

2. If all players but one announce the same message, that is, $m_j = (a, \theta, 0)$ for all $j \neq i$ with $f(\theta) = a$ and player i announces $m_i = (a^i, \theta^i, z^i) \neq (a, \theta, 0)$, then we can have three cases:

$$g(m) = \begin{cases} a^i & \text{if } a \succ_i^\theta a^i \\ a & \text{if } a^i \succ_i^\theta a \\ a(i, \theta) & \text{if } a \sim_i^\theta a^i \end{cases}$$

   where $a(i, \theta)$ is the worst outcome for player i in state θ.

3. In all other cases, an *integer game* is played. That is,

$$g(m) = a^{i^*},$$

   where i^* is the lowest index among those who announce the highest integer, i.e., $z^{i^*} = \max_j z^j$.

This paper made a slight modification on Rule 2 of the original game form in Maskin (1999). Rule 2 of the original canonical game form is as follows:

$$g(m) = \begin{cases} a^i & \text{if } a \succeq_i^\theta a^i \\ a & \text{if } a^i \succ_i^\theta a \end{cases}$$

It is important to note that regardless of this modification, Maskin's sufficiency result (Theorem 2) continues to be valid.

4 An Impossibility Theorem

In this section, I rather pursue an impossibility theorem on robust Nash implementation. To establish this impossibility theorem, I shall impose two assumptions on the social choice functions.

Assumption 1 *A social choice function f satisfies the following two conditions:*

1. *There are two states $\theta, \theta^{'}$ such that $f(\theta^{'}) \succ_i^{\theta} f(\theta)$ and $f(\theta^{'}) \succ_i^{\theta^{'}} f(\theta)$ for some player $i \in N$.*

2. *$f(\theta) \succ_i^{\theta} a(i, \theta)$ for each $i \in N$ and $\theta \in \Theta$.*

Assumption 1-1 says that the social choice function f describes the objective of a society in which there is some minimal degree of congruence of interests across *some* states for *some* player. Namely, some player i prefers $f(\theta^{'})$ to $f(\theta)$ across θ and $\theta^{'}$. Assumption 1-2 says that the social choice function f never give the worst outcome for *any* player in *any* state. Most importantly, Assumption 1-2 with Rule 2 of the Maskinian game form ensures that the truth telling Nash equilibrium m^{θ} is a *strict* Nash equilibrium of the game $\Gamma(\theta)$ for any $\theta \in \Theta$. I also impose some topological structure on the set of outcomes A.

Assumption 2 *The set of outcomes A is compact.*

Since the objective of this section is to establish the impossibility result, the compactness of A is a fairly innocuous assumption. The next theorem is the main result of the paper.

Theorem 3 *Let the number of players be at least three, i.e., $n \geq 3$. Let a social choice function f satisfy Assumption 1. Let $\Gamma = (M, g)$ be the Maskinian game form which implements f in Nash equilibrium under complete information. Assume further that the set of outcomes A satisfies Assumption 2. Then, for any complete information prior P^*, there is a net of probability distributions $\{P^k\}_{k=1}^{\infty}$ with the property that $d^{**}(P^k, P^*) \to 0$ as $k \to \infty$ for which the Bayesian Nash equilibrium correspondence associated with Γ exhibits a **discontinuity** at P^*.*

Proof of Theorem 2: Let θ and $\theta^{'}$ be two payoff states satisfying Assumption 1-1. Suppose that m^{θ} is the truth telling Nash equilibrium of the game $\Gamma(\theta)$ where $m_i^{\theta} = (f(\theta), \theta, 0)$ for each $i \in N$ and $m^{\theta^{'}}$ is the truth telling Nash equilibrium of the game $\Gamma(\theta^{'})$ where $m_i^{\theta^{'}} = (f(\theta^{'}), \theta^{'}, 0)$ for each $i \in N$. Let us consider a canonical perturbation of the complete information structure. This canonical perturbation is constructed so as to preserve the complete information assumption in any state other than θ and $\theta^{'}$. Therefore, without loss of generality, we may continue the rest of the argument as if there were only two payoff states θ and $\theta^{'}$. Define as follows:

$$\begin{aligned} p &= \mu(\theta, s^{\theta} | \{\theta, \theta^{'}\}) \\ 1 - p &= \mu(\theta^{'}, s^{\theta^{'}} | \{\theta, \theta^{'}\}) \end{aligned}$$

We define $\Omega = \{(0,0),(1,0),(1,1),(2,1),(2,2)\}$. Let us define h as a mapping from $\Theta \times S$ to Ω.

$$\begin{aligned} h^{-1}(0,0) &= (\theta', s_i^{\theta'}, s_{-i}^{\theta'}), \\ h^{-1}(1,0) &= (\theta, s_i^{\theta}, s_{-i}^{\theta'}), \\ h^{-1}(1,1) = h^{-1}(2,1) = h^{-1}(2,2) &= (\theta, s_i^{\theta}, s_{-i}^{\theta}). \end{aligned}$$

Therefore, h is an immersion from $\Theta \times S$ into Ω. We consider the following probability distribution $P^\varepsilon \in \Omega$:

- $P^\varepsilon(0,0) = 1-p$;
- $P^\varepsilon(1,0) = p\varepsilon$;
- $P^\varepsilon(1,1) = p\varepsilon(1-\varepsilon)$;
- $P^\varepsilon(2,1) = p\varepsilon(1-\varepsilon)^2$;
- $P^\varepsilon(2,2) = p(1-\varepsilon)^3$.

Players' partition structure Π is given as follows:

- $\Pi_i(0,0) = \{(0,0)\}, \Pi_i(1,0) = \Pi_i(1,1) = \{(1,0),(1,1)\}, \Pi_i(2,1) = \Pi_i(2,2) = \{(2,1),(2,2)\}$;
- $\Pi_j(0,0) = \Pi_j(1,0) = \{(0,0),(1,0)\}, \Pi_j(1,1) = \Pi_j(2,1) = \{(1,1),(2,1)\}$, $\Pi_j(2,2) = \{(2,2)\}$ for all $j \neq i$.

Fix this partition structure Π throughout the rest of the argument. The matrix below displays the type space for the nearby incomplete information games we have explained. The row is i's signal and the column is everybody else's signal.

		j's signal ($\forall j \neq i$)		
		0	1	2
i's signal	0	$1-p$	0	0
	1	$p\varepsilon$	$p\varepsilon(1-\varepsilon)$	0
	2	0	$p\varepsilon(1-\varepsilon)^2$	$p(1-\varepsilon)^3$

In the appendix, we show that this perturbation of the complete information structure is characterized by a convergent net of probability distributions with respect to the topology induced by d^{**}. Namely, $d^{**}(P^\varepsilon, P^*) \to 0$ as $\varepsilon \to 0$. [7] For strategy profile σ to be consistent with complete information, any strategy

[7] This perturbation is first used in a simple example in Kunimoto (2005) for achieving a different objective.

profile $\sigma : \Omega \to M$ in the Bayesian game $\Gamma(P^\varepsilon)$ must satisfy the property that $\sigma_j(2,1) = \sigma_j(2,2)$ for any $j \neq i$. The next proposition explicitly constructs a Bayesian Nash equilibrium σ^* of the game $\Gamma(P^\varepsilon)$ for any $\varepsilon > 0$ sufficiently small. Let m_i^θ denote player i's message for which $m_i^\theta = (f(\theta), \theta, 0)$ for all $i \in N$.

Proposition 1 *There exists sufficiently small $\bar{\varepsilon} > 0$ such that for any $\varepsilon \in (0, \bar{\varepsilon}]$ there exists a Bayesian Nash equilibrium σ^* of the game $\Gamma(P^\varepsilon)$ with the following properties:*

- $\sigma_i^*(0,0) = m_i^{\theta'}$
- $\sigma_i^*(1,0) = \sigma_i^*(1,1) = m_i^*$
- $\sigma_i^*(2,1) = \sigma_i^*(2,2) = m_i^\theta$
- $\sigma_j^*(0,0) = \sigma_j^*(1,0) = m_j^{\theta'}$ *for any* $j \neq i$
- $\sigma_j^*(1,1) = \sigma_j^*(2,1) = \sigma_j^*(2,2) = m_j^\theta$ *for any* $j \neq i$

Proof of Proposition 1: Until the end of the proof, we keep assuming $m_i^* = m_i^{\theta'}$. It will turn out that this assumption can be modified without changing all the conclusions. First, we claim that for any $j \neq i$, $\sigma_j^*(0,0) = \sigma_j^*(1,0) = m_j^{\theta'}$ is a best response to $\sigma_{-j}^*(0,0) = \sigma_{-j}^*(1,0) = m_{-j}^{\theta'}$ if ε is sufficiently small.

Fix any player $j \neq i$. Playing $m_j^{\theta'}$ conditional upon $\{(0,0),(1,0)\}$ gives the following lottery:

- $g(m^{\theta'})$ with probability $(1-p)/(1-p+p\varepsilon)$ at $(0,0)$ in $\Gamma(\theta')$
- $g(m^{\theta'})$ with probability $p\varepsilon/(1-p+p\varepsilon)$ at $(1,0)$ in $\Gamma(\theta)$

Consider any other message m_j. Playing m_j conditional upon $\{(0,0),(1,0)\}$ gives the following lottery:

- $g(m_j, m_{-j}^{\theta'})$ with probability $(1-p)/(1-p+p\varepsilon)$ at $(0,0)$ in $\Gamma(\theta')$
- $g(m_j, m_{-j}^{\theta'})$ with probability $p\varepsilon/(1-p+p\varepsilon)$ at $(1,0)$ in $\Gamma(\theta)$

Because $m^{\theta'}$ is the truth telling Nash equilibrium of the game $\Gamma(\theta')$ and if we apply Rule 2 of the Maskinian game form Γ with Assumption 1-2, we have

$$g(m^{\theta'}) \succ_j^{\theta'} g(m_j, m_{-j}^{\theta'}) \ \forall m_j \neq m_j^{\theta'}.$$

By continuity of preferences over acts, we show that $m_j^{\theta'}$ is a best response to the strategies of other players specified above for ε sufficiently small.

Second we claim that $\sigma_j^*(1,1) = \sigma_j^*(2,1) = \sigma_j^*(2,2) = m_j^\theta$ for any $j \neq i$ is a best response to $\sigma_i^*(1,1) = m_i^{\theta'}$ and $\sigma_i^*(2,1) = \sigma_i^*(2,2) = m_i^\theta$ and $\sigma_k^*(1,1) = \sigma_k^*(2,1) = \sigma_k^*(2,2) = m_k^\theta$ for any $k \notin \{i,j\}$. The partition structure Π requires us to satisfy $\sigma_j^*(1,1) = \sigma_j^*(2,1)$. Moreover, by consistency of strategy profiles and the partition structure Π, we must have $\sigma_j^*(1,1) = \sigma_j^*(2,1)$ for any $j \neq i$. In sum, we must satisfy $\sigma_j^*(1,1) = \sigma_j^*(2,1) = \sigma_j^*(2,2)$ for any $j \neq i$. Note that m^θ constitutes the truth telling Nash equilibrium of the game $\Gamma(\theta)$.

Fix any player $j \neq i$. Playing m_j^θ conditional upon $\{(1,1),(2,1),(2,2)\}$ gives the following lottery:

- $g(m_i^{\theta'}, m_{-i}^\theta)$ with probability ε at $(1,1)$ in $\Gamma(\theta)$
- $g(m^\theta)$ with probability $\varepsilon(1-\varepsilon)$ at $(2,1)$ in $\Gamma(\theta)$
- $g(m^\theta)$ with probability $(1-\varepsilon)^2$ at $(2,2)$ in $\Gamma(\theta)$

Consider any other message m_j. Playing m_j conditional upon $\{(1,1),(2,1),(2,2)\}$ gives the following lottery:

- $g(m_i^{\theta'}, m_j, m_{-\{i,j\}}^\theta)$ with probability ε at $(1,1)$ in $\Gamma(\theta)$
- $g(m_j, m_{-j}^\theta)$ with probability $\varepsilon(1-\varepsilon)$ at $(2,1)$ in $\Gamma(\theta)$
- $g(m_j, m_{-j}^\theta)$ with probability $(1-\varepsilon)^2$ at $(2,2)$ in $\Gamma(\theta)$

Because m^θ is the truth telling Nash equilibrium of the game $\Gamma(\theta)$ and if we apply Rule 2 of the Maskinian game form Γ with Assumption 1-2, we have

$$g(m^\theta) \succ_j^\theta g(m_j, m_{-j}^\theta) \ \forall m_j \neq m_j^{\theta'}.$$

By continuity of preferences over acts, we show that m_j^θ is a best response to the strategies of other players specified above for ε sufficiently small.

Third, we claim that $\sigma_i^*(1,0) = \sigma_i^*(1,1) = m_i^{\theta'}$ "can" be a best response to $\sigma_{-i}^*(0,0) = \sigma_{-i}^*(1,0) = m_{-i}^{\theta'}$ and $\sigma_{-i}^*(1,1) = \sigma_{-i}^*(2,1) = \sigma_{-i}^*(2,2) = m_j^\theta$. At states $\{(1,0),(1,1)\}$, player i knows that the game $\Gamma(\theta)$ is played. However, player i does not know if all other players know that the game $\Gamma(\theta)$ is played. Conditional on $\{(1,0),(1,1)\}$, player i believes with probability z that all other players do not know which game is being played and with probability $1-z$ that all other players know the game $\Gamma(\theta)$ is played. We can calculate z as follows:

$$z = \frac{\varepsilon}{\varepsilon + \varepsilon(1-\varepsilon)} = \frac{1}{2-\varepsilon} > 1/2 \quad \text{as long as } \varepsilon > 0$$

Playing $m_i^{\theta'}$ conditional upon $\{(1,0),(1,1)\}$ gives the following lottery:

- $g(m^{\theta'})$ with probability $z > 1/2$ at $(1,0)$
- $g(m_i^{\theta'}, m_{-i}^{\theta})$ with probability $1 - z < 1/2$ at $(1,1)$

Playing m_i^{θ} conditional upon $\{(1,0),(1,1)\}$ gives the following lottery:

- $g(m_i^{\theta}, m_{-i}^{\theta'})$ with probability $z > 1/2$ at $(1,0)$
- $g(m^{\theta})$ with probability $1 - z < 1/2$ at $(1,1)$

Applying Rule 2 of the Maskinian game form Γ with Assumption 1-1, we can rewrite the above expressions:

Playing $m_i^{\theta'}$ conditional upon $\{(1,0),(1,1)\}$ gives the following lottery:

- $g(m^{\theta'})$ with probability $z > 1/2$ at $(1,0)$
- $g(m_i^{\theta'}, m_{-i}^{\theta})$ with probability $1 - z < 1/2$ at $(1,1)$

Playing m_i^{θ} conditional upon $\{(1,0),(1,1)\}$ gives the following lottery:

- $g(m^{\theta})$ with probability $z > 1/2$ at $(1,0)$
- $g(m^{\theta})$ with probability $1 - z < 1/2$ at $(1,1)$

Since m^{θ} is the truth telling Nash equilibrium of the game $\Gamma(\theta)$ and due to Rule 2 of the Maskinian game form with Assumption 1-2, we have $g(m^{\theta}) \succ_i^{\theta} g(m_i^{\theta'}, m_{-i}^{\theta})$. We assign the following utility value to each outcome as follows:

$$\begin{aligned} u_i(g(m^{\theta});\theta) = u_i(g(m_i^{\theta}, m_{-i}^{\theta'});\theta) &= 0, \\ u_i(g(m^{\theta'});\theta) &= 3, \text{ and} \\ u_i(m_i^{\theta'}, m_{-i}^{\theta};\theta) &= -1. \end{aligned}$$

Playing m_i^{θ} conditional upon $\{(1,0),(1,1)\}$ gives the following expected utility:

$$0 \times z + 0 \times (1 - z) = 0$$

Playing $m_i^{\theta'}$ conditional upon $\{(1,0),(1,1)\}$ gives the following expected utility:

$$3 \times z - 1 \times (1 - z) = 4z - 1 > 1 \quad \because z > 1/2$$

Therefore, $m_i^{\theta'}$ is a "better" response to the belief specified above than m_i^{θ}. In other words, m_i^{θ} is "not" a best response. Note that the assignment of utilities here is not important. That is, even if we perturb the utilities slightly, the same argument goes through. If $m_i^{\theta'}$ is indeed a *best* response, this shows $\sigma_i^*(1,0) = \sigma_i^*(1,1) = m_i^{\theta'}$

can be a candidate for part of the Bayesian Nash equilibrium. Suppose, to the contrary, that $m_i^{\theta'}$ is not a best response. Assume that $u_i(\cdot;\theta) : A \to \mathbb{R}$ is continuous. Since Rule 2 of the Maskinian game form is only relevant here, the choice of messages is equivalent to the choice of outcomes. Since A is compact by Assumption 2, by Waierstrass theorem, there must exist $m_i^* \neq m_i^{\theta'}$ such that

$$m_i^* \in \arg\max_{\tilde{m}_i \in M_i} z \times u_i\left(g(\tilde{m}_i, m_{-i}^{\theta'});\theta\right) + (1-z) \times u_i\left(g(\tilde{m}_i, m_{-i}^{\theta});\theta\right)$$

In this case, all we have to do is to replace $m_i^{\theta'}$ with m_i^* and check that player j's best responses are unchanged at all states. We claim that for any $j \neq i$, $\sigma_j^*(0,0) = \sigma_j^*(1,0) = m_j^{\theta'}$ is a best response to $\sigma_i^*(0,0) = m_i^{\theta'}, \sigma_i^*(1,0) = \sigma_i^*(1,1) = m_i^*$ and $\sigma_k^*(0,0) = \sigma_k^*(1,0) = m_k^{\theta'}$ for any $k \notin \{i,j\}$ if ε is sufficiently small.

Fix any player $j \neq i$. Playing $m_j^{\theta'}$ conditional upon $\{(0,0),(1,0)\}$ gives the following lottery:

- $g(m^{\theta'})$ with probability $(1-p)/(1-p+p\varepsilon)$ at $(0,0)$ in $\Gamma(\theta')$
- $g(m_i^*, m_{-i}^{\theta'})$ with probability $p\varepsilon/(1-p+p\varepsilon)$ at $(1,0)$ in $\Gamma(\theta)$

Consider any other message m_j. Playing m_j conditional upon $\{(0,0),(1,0)\}$ gives the following lottery:

- $g(m_j, m_{-j}^{\theta'})$ with probability $(1-p)/(1-p+p\varepsilon)$ at $(0,0)$ in $\Gamma(\theta')$
- $g(m_i^*, m_j, m_{-\{i,j\}}^{\theta'})$ with probability $p\varepsilon/(1-p+p\varepsilon)$ at $(1,0)$ in $\Gamma(\theta)$

Because $m^{\theta'}$ is the truth telling Nash equilibrium of the game $\Gamma(\theta')$ and if we apply Rule 2 of the Maskinian game form Γ with Assumption 1-2, we have

$$g(m^{\theta'}) \succ_j^{\theta'} g(m_j, m_{-j}^{\theta'}) \; \forall m_j \neq m_j^{\theta'}.$$

By continuity of preferences over acts, we show that $m_j^{\theta'}$ is a best response to the belief specified above for ε sufficiently small.

We claim that $\sigma_j^*(1,1) = \sigma_j^*(2,1) = \sigma_j^*(2,2) = m_j^{\theta}$ for any $j \neq i$ is a best response to $\sigma_i^*(1,1) = m_i^*$ and $\sigma_i^*(2,1) = \sigma_i^*(2,2) = m_i^{\theta}$ and $\sigma_k^*(1,1) = \sigma_k^*(2,1) = \sigma_k^*(2,2) = m_k^{\theta}$ for any $k \notin \{i,j\}$.

Fix any player $j \neq i$. Playing m_j^{θ} conditional upon $\{(1,1),(2,1),(2,2)\}$ gives the following lottery:

- $g(m_i^*, m_{-i}^\theta)$ with probability ε at $(1,1)$ in $\Gamma(\theta)$
- $g(m^\theta)$ with probability $\varepsilon(1-\varepsilon)$ at $(2,1)$ in $\Gamma(\theta)$
- $g(m^\theta)$ with probability $(1-\varepsilon)^2$ at $(2,2)$ in $\Gamma(\theta)$

Consider any other message m_j. Playing m_j conditional upon $\{(1,1),(2,1),(2,2)\}$ gives the following lottery:

- $g(m_i^*, m_j, m_{-\{i,j\}}^\theta)$ with probability ε at $(1,1)$ in $\Gamma(\theta)$
- $g(m_j, m_{-j}^\theta)$ with probability $\varepsilon(1-\varepsilon)$ at $(2,1)$ in $\Gamma(\theta)$
- $g(m_j, m_{-j}^\theta)$ with probability $(1-\varepsilon)^2$ at $(2,2)$ in $\Gamma(\theta)$

Because m^θ is the truth telling Nash equilibrium of the game $\Gamma(\theta)$ and if we apply Rule 2 of the Maskinian game form Γ with Assumption 1-2, we have

$$g(m^\theta) \succ_j^\theta g(m_j, m_{-j}^\theta) \ \forall m_j \neq m_j^{\theta'}.$$

By continuity of preferences over acts, we show that m_j^θ is a best response to the strategies of other players specified above for ε sufficiently small.

Finally, we check that at state $(0,0)$, $m_i^{\theta'}$ is a best response to $\sigma_{-i}^*(0,0) = \sigma_{-i}^*(1,0) = m_{-i}^{\theta'}$. Since $m_{\theta'}$ is the truth telling Nash equilibrium of the game $\Gamma(\theta')$ and if we apply Rule 2 of the Maskinian game form with Assumption 1-2, we have

$$g(m^{\theta'}) \succ_i^{\theta'} g(m_i, m_{-i}^{\theta'}) \ \ \forall m_i \neq m_i^{\theta'}.$$

This completes the proof of Proposition 1. ■

To complete the proof of the theorem, it remains to check that a non-Nash equilibrium outcome is supported by the Bayesian Nash equilibrium σ^*. In fact, we have the following:

$$g(\sigma^*(1,1)) = g(m_i^*, m_{-i}^\theta) \neq g(NE(\Gamma(\theta))) = g(m^\theta),$$

where $h(\theta, s^\theta) = (1,1)$. Because if we apply Rule 2 of the Maskinian game form with Condition 2 of Assumption 1, we have

$$g(m^\theta) \succ_i^\theta g(m_i^*, m_{-i}^\theta).$$

This implies the failure of the upper hemi-continuity of the Bayesian Nash equilibrium correspondence at any complete information prior. This completes the proof of Theorem 2. ■

When I combine Theorem 2 with Maskin's sufficiency result on Nash implementation, called Theorem 1 in this paper, I have the following corollary.

Corollary 1 *Let the number of players be at least three, i.e., $n \geq 3$. Let the set of outcomes A satisfy Assumption 2. Suppose that f is a monotonic social choice function satisfying no veto power and Assumption 1. Then, f is **not** robustly implementable relative to d^{**} by the Maskinian game form.*

The implication of the above corollary is as follows: If I am concerned with achieving robust implementation relative to d^{**}, I have to devise another canonical game form which must be different from the Maskinian game from. One open question this paper has not addressed is to ask what additional modifications must be imposed on the Maskinian game form such that the modified Maskinian game form is robust to incomplete information.

5 Concluding Remarks

To be added

6 Appendix

In this appendix, I will show that the canonical perturbation of the complete information structure is characterized by the topology induced by d^{**}.

Lemma 1 *There is approximate common knowledge at $(0,0)$ and $(2,2)$ about which game being played.*

Proof: Since player i knows that the game $\Gamma(\theta')$ is played at $(0,0)$, we denote $B_i^1(\{(0,0)\}) = \{(0,0)\}$, i.e., $\{(0,0)\}$ is the state in which player i assigns probability 1 to event $\{(0,0)\}$. *A fortiori*, we must have $B_i^q(\{(0,0)\}) = \{(0,0)\}$ for any $q < 1$. Since all other players do not know which game being played at $(0,0)$, we want to know any such j's ($j \neq i$) q-belief operator $B_j^q(\{(0,0)\})$. Set

$$q(\varepsilon) = \frac{1-p}{1-p+p\varepsilon}.$$

Note that $q(\varepsilon) \to 1$ as $\varepsilon \to 0$. Consider j's 1-belief operator. Then we have $B_j^1(\{(0,0)\}) = \emptyset$. But we have for j's q-belief operator that $B_j^q(\{(0,0)\}) = \{(0,0),(1,0)\}$ for $q = q(\varepsilon)$. Note that $\{(0,0)\}$ is $q(\varepsilon)$-evident because $\{(0,0)\} \subset B_k^{q(\varepsilon)}(\{(0,0)\})$ for $k \in N$. Since $B_i^q(\{(0,0)\}) \cap \bigcap_{j\neq i} B_j^q(\{(0,0)\}) = \{(0,0)\}$ for $q = q(\varepsilon)$, we claim that when ε is sufficiently small, it is a common $q(\varepsilon)$-belif at $(0,0)$ that the game $\Gamma(\theta')$ is played.

Consider the event $\{(2,2)\}$. Since all other players but i are able to distinguish between $\{(2,2)\}$ and others, we have $B_j^1(\{(2,2)\}) = \{(2,2)\}$ for any $j \neq i$. Furthermore, we have that $B_j^q(\{(2,2)\}) = \{(2,2)\}$ for any $q < 1$. Next consider player i. We have i's 1-belief operator as $B_i^1(\{(2,2)\}) = \emptyset$. Next consider i's q-belief operator. We have that $B_i^q(\{(2,2)\}) = \{(2,1),(2,2)\}$ for $q = 1-\varepsilon$. Since $\{(2,2)\} \subset B_k^q(\{(2,2)\})$ for each $k \in N$ and for $q = 1-\varepsilon$, the event $\{(2,2)\}$ is $(1-\varepsilon)$-evident. Note that $B_i^q(\{(2,2)\}) \cap \bigcap_{j\neq i} B_j^q(\{(2,2)\}) = \{(2,2)\}$ for $q = 1-\varepsilon$. Hence, when ε is sufficiently small, it is a common $(1-\varepsilon)$-belief at state $\{(2,2)\}$ that the game $\Gamma(\theta)$ is played. ■

Corollary 2 *$d^{**}(P^\varepsilon, P^*) \to 0$ as $\varepsilon \to 0$. That is, there exists a sequence $\{q^k\}_{k=1}^\infty$ converging to 1 such that with probability at least q^k, there is a common q^k-belief about which game being played.*

Proof: Define $\{\varepsilon(k)\}_{k=1}^\infty$ as a seququence such that $\varepsilon(k) \to 0$ as $k \to \infty$. Let

$$q_I(\varepsilon) = 1-\varepsilon.$$

Then, there is a common $q_I(\varepsilon)$-belief at $(2,2)$ that the game $\Gamma(\theta)$ being played. Let

$$q_J(\varepsilon) = \frac{1-p}{1-p+p\varepsilon}.$$

Then, there is common $q_J(\varepsilon)$-belief at $(0,0)$ that the game $\Gamma(\theta')$ being played. Define $q^*(\varepsilon)$ as follows:

$$q^*(\varepsilon) = \min\{1-p+p(1-\varepsilon)^3, q_I(\varepsilon), q_J(\varepsilon)\}.$$

We set $q^k \equiv q^*(\varepsilon(k))$ for each k. Then, with probability at least q^k, there is a common q^k-belief about which game being played for each k. Define $\{P^k\}_{k=1}^\infty \equiv \{P^{\varepsilon(k)}\}_{k=1}^\infty$. Therefore, we can confirm that $\{P^k\}_{k=1}^\infty$ is a convergent net of probability distributions converging to the complete information prior P^* according to the topology induced by d^{**}. ■

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